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Relations and Numbers

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Abstract

A different view on the introduction of numbers is opened: working with relations, orders, and lattices as opposed to groups, rings and fields. Several standard ideas of number theory have an intuitive counterpart on the lattice side.

Formulating all this in lifted form, i.e., avoiding point-wise reasoning as well as the use of quantifiers, led to an interesting alternative investigation. It allows also to consider initial finite segments of the naturals in full precision with the possibility to generalize the results thus obtained.

This is a lot of material, but far from what may be considered a book. Most of it has been thoroughly checked over and over again — and seems to be correct.

Keywords relational mathematics, relation algebra, number theory

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1 Introduction

We all admire Weierstraß for his work in Number Theory and for having brought forward infinite series and reasoning with them. This does not mean that research today should continue completely to work in this direction. We know that some problems still remained unsolved, the Riemann conjecture, the Goldbach conjecture, the question whether there are infinitely many prime twins, to mention just a few publicly known ones.

If always using power series and working on convergence of these, one will miss some of the tiny items that are gelatinized via the analytic functions studied. Not least in such situations, the present author hopes, relational — and discrete — methods will in some future day provide additional help. He does not claim to be versed in Number Theory — far from that. His experience reduces to some very trivial facts: In Göttingen, WS 1957/8, he attended Helmut Klingen's lecture on *Elementare Zahlentheorie* and later, in WS 1958/9, Hans-Egon Richert's lecturing on *Dirichletsche Reihen*. That is all the author can report on — apart from mathematical standard lectures.

However, the present author has been working with relations in quite some areas of mathematics and computer science. Among those have been, e.g, program semantics [6, 7, 8], graph theory [11, 12], relational mathematics in general [9], social choice theory [10], and topology [14, 15]. Thus equipped with evidence, he began to also investigate the most elementary parts of Number Theory via relations. This changed his view on the introduction of numbers considerably. He began — to express it in overly simplified terms — to choose as tools *relations, orders, and lattices* in contrast to *groups, rings, and fields*.

Several standard ideas of number theory have an intuitive counterpart on the lattice side. Formulating all this in a "lifted" form, i.e., avoiding point-wise reasoning as well as the use of quantifiers, led to interesting alternative relation-algebraic formulations. Also basics of functions and mappings — in particular binary ones — have been lifted to the relational style. Inverting addition/multiplication so as to obtain subtraction/division turned out to constitute tiny examples of a highly general relational inversion operation, which deserves a study in its own right.

One specific line of argument is as follows: We consider the standard introduction of naturals relationally, thereby initially leaving out just one property, namely the request that the successor relation be totally defined. All the other requirements are postulated. Already then the traditional operations of addition and multiplication may be defined; they may then however be just univalent relations but not necessarily totally defined mappings.

Many properties, expressed as relational implications and equations, may be derived; they all keep track of where addition or multiplication are undefined. So several results of this text may

thus indeed be considered novel ones. Others are hardly more than reformulations and newly — and highly formally — proved known results.

Looking at the text that follows, some specifics will show up.

— It is a follow-up in the author's and his collaborators' developing of a tool kit for work with relations. It continues, thus, in its first section to add on what has already been accumulated with the long list of works [6, 7, 8, 11, 12, 2, 5, 3, 4, 9, 10, 14, 15].

— The books [11], together with [12], and later [9] have been compendia of such intentions – at their time.

— Also here will follow with Chapters 2 and 3 further prerequisites presented in a form, that they may be conceived as general rules. Insofar, these two chapters are in no way a contribution proper to the theory of numbers.

— Starting in Sect. 5.2, it has always been mentioned whether regressive finiteness has to be postulated. This is not a topic in its own right. It shall only serve the purpose to afterwards identify where it has been used.

Concerning the status of proofs: The author was usually satisfied "when he had got it". There will probably exist many possible shortcuts in his proofs or shorter versions that have so far been overlooked.

A traditional way to work in science is to look for examples, to try to get evidence what should be considered a theorem — and in the end to prove it. For the Goldbach problem, formulated in 1742 by a letter to Euler, this approach did not work for centuries. The whole community of mathematicians did not succeed. Whatever great example numbers people had chosen, the assumption proved to be true — but no proof could be given. It seems that results of this text may provide additional help for proof attempts.

A word should be said concerning the author's intentions when writing [15] as well as the present text: In both cases the aim was to further develop relation-algebraic methods. The authors selected Topology as an application field first. The book contains a lot of relation algebra without specific topological context. Rather, topology served as an interesting application field. This combination turned out to be interesting enough, that it finally appeared as volume 2208 in the Lecture Notes in Mathematics.

In quite a similar way Number Theory has been chosen as an application field. The first part of this report is relatively far from number-theoretic content; it is concerned with binary relational operations and the inverses thereof, e.g. In the second part these techniques could indeed be applied fruitfully. That at the end a study of selfsimilarity came close to a method to prove the Goldbach conjecture is an initially not expected — but highly welcome — benefit.

2 Prerequisites

The present text does not require number-theoretic knowledge beyond the initial number-theoretic information in [16].

2.1 Relational basics

We refer to the well-known relational facts that have regularly been published at other occasions, mainly in [11, 12, 2, 9, 14, 15], without repeating them here. There, one will find proof techniques that are here often referred to as shunting, destroy and append, masking, mask shifting, Dedekind rule, Schröder rule.

It is assumed that the following concepts around relations are known:

- Boolean relational operations ". ∪ .", ". ∩ ." and "$\overline{\ .\ }$"
  — together with empty and universal relation $\perp\!\!\!\perp, \top\!\!\!\top$ and possibly identity $\mathbb{I}$
- relational operations proper: transposition "$.^{\intercal}$", and composition "$.\fatsemi.$"
- typing of relations, started with the 1-element set, written as $\mathbb{1}$,
  — explicitly given finite sets
  — product with projections $\pi : X \times Y \longrightarrow X, \quad \rho : X \times Y \longrightarrow Y$
  — sum with injections $\iota : X \longrightarrow X + Y, \quad \kappa : Y \longrightarrow X + Y$
  — powerset construction with "is element of relation" $\varepsilon : X \longrightarrow \mathbf{2}^X$,
  accompanied by singleton injection $\sigma : X \longrightarrow \mathbf{2}^X$,
  powerset ordering $\Omega : \mathbf{2}^X \longrightarrow \mathbf{2}^X$, and powerset negation $\mathcal{N} : \mathbf{2}^X \longrightarrow \mathbf{2}^X$.
- and also the dependent types,
  — natural quotient for an equivalence Ξ with natural projection $\eta : X \longrightarrow X/\Xi$
  for which one has to prove the equivalence property of Ξ
  — extrusion of a subset $v \subseteq X$ with injection $\iota : V \longrightarrow X$
  for which one has to prove that $v \neq \perp\!\!\!\perp$ is a vector.

Together with these, the Kronecker product, strict fork-, and strict join-operators shall be used, as well as the symmetric quotient:

$(A \otimes B) := \pi ; A ; \pi'^{\mathsf{T}} \cap \rho ; B ; \rho'^{\mathsf{T}}$
$(A \oslash B) := A ; \pi^{\mathsf{T}} \cap B ; \rho^{\mathsf{T}}$, provided sources of A and B coincide
$(A \oslash B) := \pi ; A \cap \rho ; B$, provided targets of A and B coincide
$\mathtt{syq}(A, B) := \overline{\overline{A}^{\mathsf{T}} ; B} \cap \overline{A^{\mathsf{T}} ; \overline{B}}$, provided sources of A and B coincide

Assuming an order $E : X \longrightarrow X$, we define (see [15]) **binary meet forming** and **binary join forming** as

$\mathfrak{M}_2 := \mathtt{syq}(\,(E \oslash E)\,, E) = \left[\mathtt{lub}_E(\,(E \oslash E)\,)\right]^{\mathsf{T}}$
$\mathfrak{J}_2 := \mathtt{syq}((E^{\mathsf{T}} \oslash E^{\mathsf{T}}), E^{\mathsf{T}}) = \left[\mathtt{glb}_E((E^{\mathsf{T}} \oslash E^{\mathsf{T}}))\right].^{\mathsf{T}}$

A few newly developed results concerning fork, join and the Kronecker product shall be presented here.

2.1.1 Proposition. i) $(A \oslash B) ; (C \oslash D) \subseteq (A ; C \otimes B ; D)$
$(A \oslash B) ; (C \oslash D) \subseteq A ; C \cap B ; D$

ii) $(A ; \mathbb{T} \oslash B) ; (\mathbb{T} ; C \oslash D) = (A ; \mathbb{T} ; C \otimes B ; D)$

iii) $(A \oslash \mathbb{I}) ; \mathbb{T} = (\mathbb{I} \oslash A^{\mathsf{T}}) ; \mathbb{T}$

$(\rho ; \pi'^{\mathsf{T}} \oslash \mathbb{I}) ; \pi' = (\mathbb{I} \oslash \pi' ; \rho^{\mathsf{T}}) ; \rho$ for projection pairs π, ρ and π', ρ'

iv) $(A \cap \mathbb{T} ; C \oslash B) = (A \oslash B) \cap \mathbb{T} ; C = (A \oslash B \cap \mathbb{T} ; C)$
$(A \cap C ; \mathbb{T} \oslash B) = (A \oslash B) \cap C ; \mathbb{T} = (A \oslash B \cap C ; \mathbb{T})$

v) For an arbitrary homogeneous relation R or A
$\overline{(\mathbb{I} \cap R) ; \mathbb{T}} = (\mathbb{I} \cap \overline{R}) ; \mathbb{T}$ $\qquad$ $\mathbb{I} \cap (\mathbb{I} \cap R) ; \mathbb{T} = \mathbb{I} \cap R$
$(\overline{A} \oslash \mathbb{I}) ; \mathbb{T} = \overline{(A \oslash \mathbb{I}) ; \mathbb{T}}$

vi) If $D \subseteq \mathbb{I}$ is a partial identity, the following will hold:

$D ; [A \cap B] = D ; A \cap B$

$D ; R \subseteq \overline{(\mathbb{I} \cap \overline{D}) ; R}$ $\qquad$ $D ; \mathbb{T} = \overline{(\mathbb{I} \cap \overline{D}) ; \mathbb{T}}$

$\overline{D ; R ; D} = D ; \overline{R} ; D \cup \overline{D ; \mathbb{T} ; D}$ $\qquad$ $D ; R ; D = \overline{D ; \overline{R} ; D} \cap D ; \mathbb{T} ; D$ $\qquad$ $D ; \overline{D ; \mathbb{T} ; D} ; D = \perp\!\!\!\perp$

$\mathtt{Nontrans}(D ; R ; D) \supseteq D ; \mathtt{Nontrans}(R) ; D$ $\qquad$ where $\mathtt{Nontrans}(X) := X \cap \overline{X ; X}$.

Proof: i) $(A \oslash B) ; (C \oslash D) = (\pi ; A \cap \rho ; B) ; (C ; \pi_1^{\mathsf{T}} \cap D ; \rho_1^{\mathsf{T}})$
$\subseteq \pi ; A ; C ; \pi_1^{\mathsf{T}} \cap \rho ; B ; D ; \rho_1^{\mathsf{T}} = (A ; C \otimes B ; D)$

$(A \oslash B) ; (C \oslash D) = (A ; \pi^{\mathsf{T}} \cap B ; \rho^{\mathsf{T}}) ; (\pi ; C \cap \rho ; D) \subseteq A ; C \cap B ; D$

---

ii) $(A ; \mathbb{T} \oslash B) ; (\mathbb{T} ; C \oslash D) = (\pi ; A ; \mathbb{T} \cap \rho ; B) ; (\mathbb{T} ; C ; \pi_1^{\mathsf{T}} \cap D ; \rho_1^{\mathsf{T}})$

$= \rho ; B ; (\mathbb{T} ; C ; \pi_1^\intercal \cap D ; \rho_1^\intercal) \cap \pi ; A ; \mathbb{T}$ masking
$= \rho ; B ; D ; \rho_1^\intercal \cap \mathbb{T} ; C ; \pi_1^\intercal \cap \pi ; A ; \mathbb{T}$ masking
$= \rho ; B ; D ; \rho_1^\intercal \cap \pi ; A ; \mathbb{T} ; C ; \pi_1^\intercal$ masking
$= (A ; \mathbb{T} ; C \otimes B ; D)$

---

iii) $(A \circledcirc \mathbb{I}) ; \mathbb{T} = (\pi ; A \cap \rho) ; \mathbb{T}$
$= (\pi ; A ; \rho^\intercal \cap \mathbb{I}) ; \rho ; \mathbb{T}$ destroy and append reverted
$= (\rho ; A^\intercal ; \pi^\intercal \cap \mathbb{I}) ; \mathbb{T}$ transposing a partial identity; ρ is total
$= (\rho ; A^\intercal ; \pi^\intercal \cap \mathbb{I}) ; \pi ; \mathbb{T}$ totality of π
$= (\rho ; A^\intercal \cap \pi) ; \mathbb{T}$ destroy and append
$= (\mathbb{I} \circledcirc A^\intercal) ; \mathbb{T}$

$(\rho ; \pi'^\intercal \circledcirc \mathbb{I}) ; \pi' = (\pi_1 ; \rho ; \pi'^\intercal \cap \rho_1) ; \pi' = \pi_1 ; \rho \cap \rho_1 ; \pi' = (\pi_1 \cap \rho_1 ; \pi' ; \rho^\intercal) ; \rho = (\mathbb{I} \circledcirc \pi' ; \rho^\intercal) ; \rho$

---

iv) $(A \cap \mathbb{T} ; C \circledcirc B) = \pi ; (A \cap \mathbb{T} ; C) \cap \rho ; B = \pi ; A \cap \rho ; B \cap \mathbb{T} ; C$ since π is total
$= (A \circledcirc B) \cap \mathbb{T} ; C$

---

v) From $\mathbb{T} = \mathbb{I} ; \mathbb{T} = \big[\mathbb{I} \cap (R \cup \overline{R})\big] ; \mathbb{T} = \big[(\mathbb{I} \cap R) \cup (\mathbb{I} \cap \overline{R})\big] ; \mathbb{T} = (\mathbb{I} \cap R) ; \mathbb{T} \cup (\mathbb{I} \cap \overline{R}) ; \mathbb{T}$ follows
$\overline{(\mathbb{I} \cap R) ; \mathbb{T}} \subseteq (\mathbb{I} \cap \overline{R}) ; \mathbb{T}$,
thus proving direction "$\subseteq$". For "$\supseteq$", we use that $\mathbb{I} \cap \overline{R} \subseteq \mathbb{I}$ is injective, prior to applying the Schröder rule:
$(\mathbb{I} \cap \overline{R})^\intercal ; (\mathbb{I} \cap R) ; \mathbb{T} = [(\mathbb{I} \cap \overline{R}) ; \mathbb{I} \cap (\mathbb{I} \cap \overline{R}) ; R] ; \mathbb{T} \subseteq [\mathbb{I} \cap \overline{R} \cap R] ; \mathbb{T} = \bot\!\!\!\bot$

Containment "$\supseteq$" is obvious. For "$\subseteq$" we apply the Dedekind rule:
$(\mathbb{I} \cap R) ; \mathbb{T} \cap \mathbb{I} \subseteq \big[(\mathbb{I} \cap R) \cap \mathbb{I} ; \mathbb{T}^\intercal\big] ; \big[\mathbb{T} \cap (\mathbb{I} \cap R)^\intercal ; \mathbb{I}\big]$
$= (\mathbb{I} \cap R) ; (\mathbb{I} \cap R)^\intercal = (\mathbb{I} \cap R) ; (\mathbb{I} \cap R) \subseteq \mathbb{I} \cap R$ transposing a partial identity

---

$(\overline{A} \circledcirc \mathbb{I}) ; \mathbb{T} = (\pi ; \overline{A} \cap \rho) ; \mathbb{T} = (\pi ; \overline{A} ; \rho^\intercal \cap \mathbb{I}) ; \rho ; \mathbb{T}$ destroy and append in reverse direction
$= \overline{(\pi ; A ; \rho^\intercal \cap \mathbb{I}) ; \mathbb{T}}$ projections π, ρ are mappings
$= \overline{(\pi ; A ; \rho^\intercal \cap \mathbb{I}) ; \mathbb{T}}$ see before
$= \overline{(\pi ; A ; \rho^\intercal \cap \mathbb{I}) ; \rho ; \mathbb{T}} = \overline{(\pi ; A \cap \rho) ; \mathbb{T}}$ destroy and append
$= \overline{(A \circledcirc \mathbb{I}) ; \mathbb{T}}$

---

vi) Direction "$\subseteq$": $D ; [A \cap B] \subseteq D ; A \cap D ; B \subseteq D ; A \cap \mathbb{I} ; B$,
direction "$\supseteq$": $D ; A \cap B \subseteq (D \cap B ; A^\intercal) ; (A \cap D^\intercal ; B)$ using the Dedekind rule
$\subseteq D ; (A \cap \mathbb{I} ; B)$

"$\subseteq$": $D ; R \subseteq \overline{(\mathbb{I} \cap \overline{D}) ; R} \iff$
$D^\intercal ; (\mathbb{I} \cap \overline{D}) ; R = D ; (\mathbb{I} \cap \overline{D}) ; R = (D \cap D ; \overline{D}) ; R \subseteq (D \cap \overline{D}) ; R = \bot\!\!\!\bot \subseteq \overline{R}$
"$\supseteq$" in case $R := \mathbb{T}$:
$(\mathbb{I} \cap \overline{D}) ; \mathbb{T} \cup D ; \mathbb{T} = (\mathbb{I} \cap \overline{D}) ; \mathbb{T} \cup (\mathbb{I} \cap D) ; \mathbb{T} = \big[(\mathbb{I} \cap \overline{D}) \cup (\mathbb{I} \cap D)\big] ; \mathbb{T}$
$= \big[\mathbb{I} \cap (\overline{D} \cup D)\big] ; \mathbb{T} = \big[\mathbb{I} \cap \mathbb{T}\big] ; \mathbb{T} = \mathbb{T}$

For "$\subseteq$" we use $D \subseteq \mathbb{I}$ in

$\overline{D;R;D} \subseteq D;\overline{R};D \cup \overline{D;\top;D} \iff \top = D;R;D \cup D;\overline{R};D \cup \overline{D;\top;D}$, which holds indeed

For "$\supseteq$" we may restrict to considering the first term on the right side:

$\overline{D;R;D} \supseteq D;\overline{R};D \iff \overline{\overline{R};D} \supseteq D;D;R;D = D;R;D$ Schröder rule
$\iff \overline{R};D \subseteq \overline{D;R;D}$ negated
$\iff D;R;D;D \subseteq R$ Schröder rule

The second is like the first, but negated.

We use the second version to obtain the third:

$D;\overline{D;\top;D};D = D;\overline{\overline{D;\top;D};D} \cap D;\top;D = \overline{D;D;\top;D;D} \cap D;\top;D = \overline{D;\top;D} \cap D;\top;D = \mathbb{\perp}$

$\mathtt{Nontrans}(D;R;D) = D;R;D \cap \overline{D;R;D;D;R;D}$ by definition
$= D;R;D \cap \overline{D;R;D;R;D}$ since $D \subseteq \mathbb{I}$
$= D;R;D \cap \left[D;\overline{R;D;R};D \cup \overline{D;\top;D}\right]$ see before
$= \left(D;R;D \cap D;\overline{R;D;R};D\right) \cup \left(D;R;D \cap \overline{D;\top;D}\right)$
$= D;\left(R \cap \overline{R;D;R}\right);D \cup \mathbb{\perp}$
$\supseteq D;\mathtt{Nontrans}(R);D$ because $D \subseteq \mathbb{I}$ □

2.2 Points, vectors, and diagonals

Points (in the relation-algebraic sense) are more or less equivalent to elements of a set. Working truly relationally means, however, to stay completely on the algebraic side and never to go back to elements in the classical sense.

2.2.1 Definition. A **vector** $v : X \longrightarrow Z$ is a row-constant relation, i.e. satisfies $v = v;\top$. Further specializing, a **point** $p : X \longrightarrow Z$ is a vector with additional properties:

i) $p = p;\top$, i.e. p is a row-constant relation,

ii) $\top;p;\top = \top$, meaning nothing else than that $p \neq \mathbb{\perp}$, i.e. p is not the empty relation,

iii) $p;p^\intercal \subseteq \mathbb{I}$, i.e., p is injective. □

Any relation satisfying these requirements comes via the relational mechanisms very close to what we conceive of an element in a set. Points have several important properties showing that they are atoms in the lattice-theoretic sense:

2.2.2 Proposition.

i) Any non-vanishing row-constant relation contained in a point is equal to this point:

$\mathbb{\perp} \neq X = X;\top \subseteq p \implies X = p$

ii) Whenever a point is contained in some union of vectors, it will be fully contained in at least one of these:

$p \subseteq v \cup w \implies p \subseteq v$ **or** $p \subseteq w$

Proof: i) From $\mathbb{L} \neq X$ follows $\mathbb{T} ; X = \mathbb{T} ; X ; \mathbb{T} = \mathbb{T}$, so that with the Dedekind formula
$p = \mathbb{T} ; X \cap p \subseteq (\mathbb{T} \cap p ; X^\mathsf{T}) ; (X \cap \mathbb{T}^\mathsf{T} ; p) \subseteq p ; X^\mathsf{T} ; X \subseteq p ; p^\mathsf{T} ; X \subseteq \mathbb{I} ; X = X.$

---

ii) Assume $p \subseteq v \cup w$ but $\mathbb{L} = p \cap v$. Then
$p = p \cap (v \cup w) = (p \cap v) \cup (p \cap w) = \mathbb{L} \cup (p \cap w) = p \cap w \subseteq w.$
If on the other hand $\mathbb{L} \neq p \cap v$, we apply (i) to the vector
$X := (p \cap v) ; \mathbb{T} \subseteq p ; \mathbb{T} = p$
getting $X = p$, so that finally
$p = (p \cap v) ; \mathbb{T} \subseteq v ; \mathbb{T} = v.$ □

Now a proposition shall be proved that is in some sense trivial, but offers possibilities to be used later.

2.2.3 Proposition. Let be given some point $p : X \longrightarrow Z$ and some vector $v : X \longrightarrow Z$. Then

i) $p \subseteq v \iff p^\mathsf{T} ; \overline{v} = \mathbb{L}$

ii) $p \not\subseteq v \iff p^\mathsf{T} ; \overline{v} = \mathbb{T}$

Proof: i) We use the Schröder rule: $p ; \mathbb{T} = p \subseteq v \iff p^\mathsf{T} ; \overline{v} \subseteq \mathbb{L}$

---

ii) $p \not\subseteq v \iff p ; \mathbb{T} = p \subseteq \overline{v}$ due to Prop. 2.2.2.ii, since certainly $p \subseteq v \cup \overline{v}$. Herefrom we proceed, beginning with the Schröder rule, as follows:
$\iff p^\mathsf{T} ; v \subseteq \mathbb{L} \iff \mathbb{T} \subseteq \overline{p^\mathsf{T} ; v} = p^\mathsf{T} ; \overline{v}$ since p^T is a map □

There is a point concerning points (in the relation-algebraic sense) that should be mentioned in advance. Points may be considered as Boolean vectors or as Boolean matrices of arbitrary "breadth" ≥ 1. When a vector is appropriate, we will use p. Often, however, points are used in combination with relations as in Prop. 5.2.1.ii.b. In such case, we will use $p ; \mathbb{T}$ in order to indicate that "breadth" is required according to the typing mechanism.

If necessary, we will denote the set of points $p : X \longrightarrow \mathbb{1}$ as $\mathtt{pts}(X)$. One will easily see that

$$x, y \in \mathtt{pts}(X) \implies x^\mathsf{T} ; y = \begin{cases} \mathbb{T} & \text{in case } x = y \\ \mathbb{L} & \text{in case } x \neq y \end{cases}$$

with main proof steps using $x = x ; \mathbb{T}$ and $\mathbb{T} ; y^\mathsf{T} = y^\mathsf{T}$ in

$\mathbb{T} \subseteq x^\mathsf{T} ; y \iff \mathbb{T} ; y^\mathsf{T} \subseteq x^\mathsf{T}$ shunting the point y,
$x^\mathsf{T} ; y \subseteq \mathbb{L} \iff x ; \mathbb{T} \subseteq \overline{y}$ Schröder equivalence.

There exist, however, relation algebras in which points do not exist — at least not enough of them. Whenever we assume their existence, we will speak of the **Point Axiom**, which demands that for every relation

$$\mathbb{L} \neq R : X \longrightarrow Y \implies \exists \text{ points } x : X \longrightarrow \mathbb{1} \text{ and } y : Y \longrightarrow \mathbb{1} : x ; y^\mathsf{T} \subseteq R.$$

Normally, we will prove results *relation-algebraically*, that is, based on the quantifier-free algebraic rules recalled earlier. In some rather rare cases, however, it is unavoidable to reason *point-wise*. This will always be mentioned explicitly.

In such cases, we accept that we work with *represented* relations, i.e. indeed with relations $R : X \longrightarrow Z$ and $S : Z \longrightarrow Y$ between sets. Reasoning then reads, e.g.

$$x ; y^{\mathsf{T}} \subseteq R ; S \qquad \Longleftrightarrow \qquad \exists \text{ a point } z : \quad x ; z^{\mathsf{T}} \subseteq R \ \wedge \ z ; y^{\mathsf{T}} \subseteq S$$

In addition to points and vectors — and in very close connection to these — there exist partial identities. With regard to their pictorial appearance they will often be called **diagonals**.

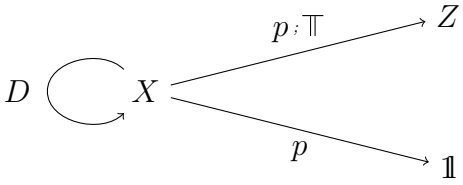


Fig. 2.2.1 Typing diagonals and vectors

2.2.4 Proposition. There are bijections as follows:

i) $p ; \mathbb{T} : X \longrightarrow Z$ a point $\Longrightarrow$ $D_p := p ; p^{\mathsf{T}}$ satisfies $\mathbb{1} \neq D_p = D_p ; \mathbb{T} ; D_p \subseteq \mathbb{I}$

$p_D := D ; \mathbb{T}$ is a point $\Longleftarrow$ $\mathbb{1} \neq D = D ; \mathbb{T} ; D \subseteq \mathbb{I}$

ii) $v : X \longrightarrow Z$ a vector $\Longrightarrow$ $D_v := \mathbb{I} \cap v ; \mathbb{T}$ satisfies $D_v \subseteq \mathbb{I}$

$v_D := D ; \mathbb{T}$ is a vector $\Longleftarrow$ $D \subseteq \mathbb{I}$

Proof: i) $D_p \subseteq \mathbb{I}$ is evident by definition of a point. $D_p ; \mathbb{T} ; D_p = p ; p^{\mathsf{T}} ; \mathbb{T} ; p ; p^{\mathsf{T}} = p ; p^{\mathsf{T}} = D_p$ since a point is surjective by definition.

In reverse direction: p_D is certainly non-empty and row-constant. Injectivity follows from $p_D ; p_D^{\mathsf{T}} = D ; \mathbb{T} ; \mathbb{T} ; D^{\mathsf{T}} = D ; \mathbb{T} ; D \subseteq \mathbb{I}$.

---

ii) The first line as well as the second are absolutely obvious. □

We ought to have these pictorial correspondences between points/vectors and their diagonal versions always in mind.

Without giving formal definitions already here, we mention how we intend to denote primes as diagonals resp. as vectors. One will see that the "breadth" of the vector is more or less immaterial:

$$\texttt{eve} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{ccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} 1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1 \end{pmatrix} \end{array} \subseteq \mathbb{I} \qquad \texttt{eve} : \mathbb{T} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{ccc} W & D & L \end{array} \\ \begin{pmatrix} 1&1&1 \\ 0&0&0 \\ 1&1&1 \\ 0&0&0 \\ 1&1&1 \\ 0&0&0 \\ 1&1&1 \\ 0&0&0 \\ 1&1&1 \end{pmatrix} \end{array} \quad \text{for the set } \{0, 2, 4, 6, 8 \ldots\}$$

$$\texttt{odd} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{ccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} 0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0 \end{pmatrix} \end{array} \subseteq \mathbb{I} \qquad \texttt{odd} : \mathbb{T} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{for the set } \{1, 3, 5, 7 \ldots\}$$

Fig. 2.2.2 Notation intended for `eve`, `odd` diagonals and vectors of various breadths

$$\wp_1 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{cccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \end{array} \subseteq \mathbb{I} \qquad \wp_1 : \mathbb{T} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{cccc} \spadesuit & \heartsuit & \diamondsuit & \clubsuit \end{array} \\ \begin{pmatrix} 0&0&0&0 \\ 1&1&1&1 \\ 1&1&1&1 \\ 1&1&1&1 \\ 0&0&0&0 \\ 1&1&1&1 \\ 0&0&0&0 \\ 1&1&1&1 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix} \end{array} \quad \text{for } \{1, 2, 3, 5, 7, 11 \ldots\}$$

$$\wp_2 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{cccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \end{array} \subseteq \mathbb{I} \qquad \wp_2 : \mathbb{T} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{for } \{2, 3, 5, 7, 11 \ldots\}$$

$$\wp = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{cccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \end{array} \subseteq \mathbb{I} \qquad \wp : \mathbb{T} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{cccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 1&1&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0 \\ 1&1&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0 \\ 1&1&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \end{array} \quad \text{for } \{3, 5, 7, 11 \ldots\}$$

With corresponding extrusions such as, e.g., $\iota_{\wp;\mathbb{T}} : \mathbb{P} \longrightarrow \mathbb{N}$, for $p \in \mathbb{N}$, but $[p\!\!\rightarrow] \in \mathbb{P}$

Fig. 2.2.3 Notation intended for prime diagonals and vectors of various breadths

Even when not demanding the Point Axiom, there exists a correspondence of relations and vectors via a so-called **vectorization**. Assume the following typing where the choice of $Z \neq \mathbb{1}$ is more or less irrelevant.

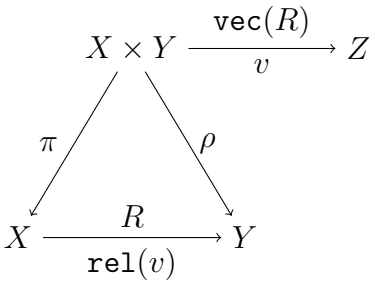


Fig. 2.2.4 Typing vectorization $R \mapsto v := \mathtt{vec}(R)$ and $v \mapsto R := \mathtt{rel}(v)$

2.2.5 Proposition. For any relation $R : X \longrightarrow Y$, we define a vector $v : X \times Y \longrightarrow Z$ by

$R \mapsto v := \mathtt{vec}(R) = (R \between \mathbb{I}) ; \mathbb{T} = (\pi ; R \cap \rho) ; \mathbb{T} = (\rho ; R^\mathsf{T} \cap \pi) ; \mathbb{T} = (\mathbb{I} \between R^\mathsf{T}) ; \mathbb{T}$

and for any vector $v : X \times Y \longrightarrow Z$, we define a relation by

$v \mapsto R := \mathtt{rel}(v) = \pi^\mathsf{T} ; (\rho \cap v ; \mathbb{T})$.

In this configuration the following holds:

i) $\pi ; \mathtt{rel}(v ; \mathbb{T}) \cap \rho = \rho \cap v ; \mathbb{T}$

ii) $\mathtt{vec}(R) ; \mathbb{T} \cap \rho = \pi ; R \cap \rho$

iii) These transitions are inverses of one another, i.e. form a bijection. They satisfy, namely, $\mathtt{vec}(\mathtt{rel}(v ; \mathbb{T})) = v ; \mathbb{T}$ and $\mathtt{rel}(\mathtt{vec}(R)) = R$.

iv) Using the diagonal $D_v := \mathbb{I} \cap v ; \mathbb{T}$, we have $\mathtt{rel}(v) = \pi^\mathsf{T} ; D_v ; \rho$.

Proof: i) $\pi ; \mathtt{rel}(v ; \mathbb{T}) \cap \rho = \pi ; \pi^\mathsf{T} ; (\rho \cap v ; \mathbb{T}) \cap \rho$ by definition
$\subseteq \big(\pi ; \pi^\mathsf{T} \cap \rho ; (\rho \cap v ; \mathbb{T})^\mathsf{T}\big) ; \big((\rho \cap v ; \mathbb{T}) \cap \pi ; \pi^\mathsf{T} ; \rho\big)$ Dedekind rule
$\subseteq (\pi ; \pi^\mathsf{T} \cap \rho ; \rho^\mathsf{T}) ; (\rho \cap v ; \mathbb{T})$
$= \rho \cap v ; \mathbb{T}$ because $\pi ; \pi^\mathsf{T} \cap \rho ; \rho^\mathsf{T} = \mathbb{I}$ for a direct product π, ρ
$\subseteq \pi ; \pi^\mathsf{T} ; (\rho \cap v ; \mathbb{T}) \cap \rho = \pi ; \mathtt{rel}(v ; \mathbb{T}) \cap \rho$

The other versions are easily deduced applying Prop. 2.1.1.iii.

---

ii) $\mathtt{vec}(R) ; \mathbb{T} \cap \rho = (\pi ; R \cap \rho) ; \mathbb{T} \cap \rho$ by definition
$\subseteq \big((\pi ; R \cap \rho) \cap \rho ; \mathbb{T}^\mathsf{T}\big) ; \big(\mathbb{T} \cap (\pi ; R \cap \rho)^\mathsf{T} ; \rho\big)$ Dedekind rule
$\subseteq (\pi ; R \cap \rho) ; \rho^\mathsf{T} ; \rho$
$\subseteq \pi ; R \cap \rho$ since ρ is univalent
$\subseteq (\pi ; R \cap \rho) ; \mathbb{T} \cap \rho$
$= \mathtt{vec}(R) ; \mathbb{T} \cap \rho$

---

iii) $\mathtt{vec}(\mathtt{rel}(v ; \mathbb{T})) = (\pi ; \mathtt{rel}(v ; \mathbb{T}) \cap \rho) ; \mathbb{T}$ by definition
$= (\rho \cap v ; \mathbb{T}) ; \mathbb{T}$ using (i)
$= \rho ; \mathbb{T} \cap v ; \mathbb{T}$ masking
$= v ; \mathbb{T}$ because ρ is total

$\mathtt{rel}(\mathtt{vec}(R)) = \pi^\intercal ; \big(\mathtt{vec}(R) ; \mathbb{T} \cap \rho\big)$ by definition
$= \pi^\intercal ; (\pi ; R \cap \rho)$ using (ii)
$= R \cap \mathbb{T}$ destroy and append
$= R.$

iv) $\pi^\intercal ; D_v ; \rho = \pi^\intercal ; (\mathbb{I} \cap v ; \mathbb{T}) ; \rho = \pi^\intercal ; (\rho \cap v ; \mathbb{T}) = \mathtt{rel}(v)$ □

In addition to just being bijectively related, also some — but not all — operations migrate from one side to the other. These obviously monotonic transitions have been studied in more detail already as Prop. 7.5.5 of [11, 12, 15], where it has been proved that

$\mathtt{vec}(\mathbb{\perp\!\!\!\perp}) = \mathbb{\perp\!\!\!\perp}$	$\mathtt{rel}(\mathbb{\perp\!\!\!\perp}) = \mathbb{\perp\!\!\!\perp}$
$\mathtt{vec}(\mathbb{T}) = \mathbb{T}$	$\mathtt{rel}(\mathbb{T}) = \mathbb{T}$
$\mathtt{vec}(\overline{R}) = \overline{\mathtt{vec}(R)}$	$\mathtt{rel}(\overline{v}) = \overline{\mathtt{rel}(v)}$
$\mathtt{vec}(R \cup S) = \mathtt{vec}(R) \cup \mathtt{vec}(S)$	$\mathtt{rel}(v \cup w) = \mathtt{rel}(v) \cup \mathtt{rel}(w)$
$\mathtt{vec}(R \cap S) = \mathtt{vec}(R) \cap \mathtt{vec}(S)$	$\mathtt{rel}(v \cap w) = \mathtt{rel}(v) \cap \mathtt{rel}(w)$
$R \subseteq S \iff \mathtt{vec}(R) \subseteq \mathtt{vec}(S)$	$v \subseteq w \iff \mathtt{rel}(v) \subseteq \mathtt{rel}(w).$

In total, we have all Boolean relational operations in an appropriately adjusted corresponding form for the respective other side. In the next proposition we will show that also transposition and composition have counterparts on either side.

$$\begin{array}{r} \\ \text{Win} \\ \text{Draw} \\ \text{Loss} \end{array}\!\!\begin{array}{c} \begin{array}{ccccc}\text{Mon}&\text{Tue}&\text{Wed}&\text{Thu}&\text{Fri}\end{array} \\ \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \end{pmatrix} \end{array} \begin{array}{l} = R \\ \\ = \mathtt{rel}(v) \end{array} \qquad \mathtt{vec}(R) = v = \begin{array}{r}(\text{Win,Mon})\\(\text{Win,Tue})\\(\text{Win,Wed})\\(\text{Win,Thu})\\(\text{Win,Fri})\\(\text{Draw,Mon})\\(\text{Draw,Tue})\\(\text{Draw,Wed})\\(\text{Draw,Thu})\\(\text{Draw,Fri})\\(\text{Loss,Mon})\\(\text{Loss,Tue})\\(\text{Loss,Wed})\\(\text{Loss,Thu})\\(\text{Loss,Fri})\end{array}\!\!\begin{pmatrix}\mathbf{0}\\\mathbf{1}\\\mathbf{1}\\\mathbf{0}\\\mathbf{1}\\\mathbf{0}\\\mathbf{1}\\\mathbf{1}\\\mathbf{1}\\\mathbf{0}\\\mathbf{1}\\\mathbf{0}\\\mathbf{0}\\\mathbf{1}\\\mathbf{0}\end{pmatrix}$$

Fig. 2.2.5 Example of a vectorization and its reverse operation

We now recall vectorization properties above this just Boolean level and extend them to matrix formulae with regard to transposition and composition. This requires not least lifting the composition operation as proposed to me by Michael Winter:

$\mathtt{comp}_{XYZ} : (X \times Y) \times (Y \times Z) \longrightarrow X \times Z$
$\mathtt{comp}_{XYZ} := \Big(\mathbb{I}_{(X\times Y)\times(Y\times Z)} \cap \pi_1 ; \rho ; \pi'^\intercal ; \rho_1^\intercal\Big) ; (\pi \otimes \rho')$

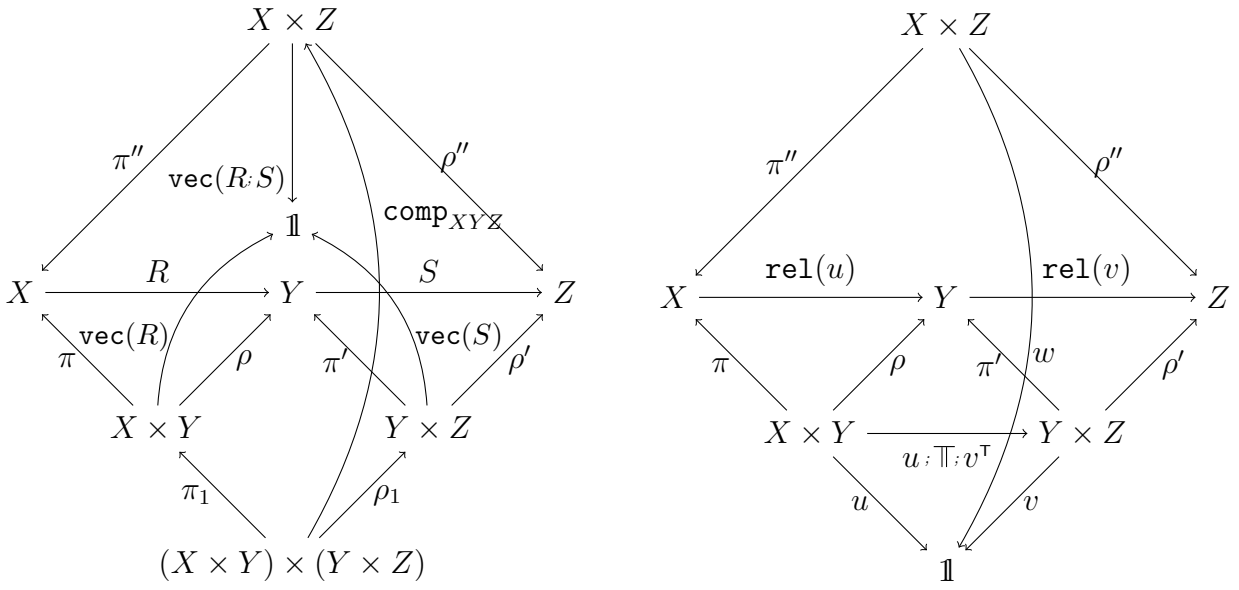


Fig. 2.2.6 Illustrating the proofs of Prop. 2.2.6.iii

In part (iii) of the following proposition, we again face the problem of sharp factorization. It has been presented in full detail at several occasions [11, 12, 15] to which we here refer in order not to report everything from scratch.

2.2.6 Proposition. Transposition and identity are also carried over, but less obviously:

i) $\mathtt{vec}(\mathbb{I}) = (\pi \cap \rho);\mathbb{T} = (\mathbb{I} \circledcirc \mathbb{I});\mathbb{T}$ $\qquad \mathtt{rel}((\pi \cap \rho);\mathbb{T}) = \mathtt{rel}((\mathbb{I} \circledcirc \mathbb{I});\mathbb{T}) = \mathbb{I}$

ii) $\mathtt{vec}(R^\mathsf{T}) = \mathfrak{P};\mathtt{vec}(R)$ $\qquad \mathtt{rel}(\mathfrak{P};v) = \mathtt{rel}(v)^\mathsf{T}$

iii) $\mathtt{vec}(R;S) = (R \circledcirc S^\mathsf{T});\mathbb{T}$ $\qquad \pi^\mathsf{T};(\rho;\pi'^\mathsf{T} \cap u;\mathbb{T};v^\mathsf{T});\rho' = \mathtt{rel}(u);\mathtt{rel}(v)$

$= (R \otimes \mathbb{I});\mathtt{vec}(S)$

$= \mathtt{comp}^\mathsf{T}_{XYZ};(\mathtt{vec}(R) \circledcirc \mathtt{vec}(S))$ see Fig. 2.2.6

iv) $\qquad \rho;\pi'^\mathsf{T} \cap u;\mathbb{T};v^\mathsf{T}$ is difunctional

v) For vectors $v;\mathbb{T}$ and $w;\mathbb{T}$:

$\mathtt{vec}(v;\mathbb{T};w^\mathsf{T}) = (v;\mathbb{T} \circledcirc w;\mathbb{T})$ $\qquad \mathtt{rel}((v;\mathbb{T} \circledcirc w;\mathbb{T})) = v;\mathbb{T};w^\mathsf{T}$

vi) $\iota = \mathtt{Inject}(v;\mathbb{T}) \implies \iota = \iota;(\mathbb{I} \cap v;\mathbb{T})$

Proof: i) $\mathtt{vec}(\mathbb{I}) = (\pi;\mathbb{I} \cap \rho);\mathbb{T} = (\pi \cap \rho);\mathbb{T}$ is trivial. The other follows with Prop. 2.2.5.iii.

---

ii) $\mathfrak{P};\mathtt{vec}(R) = \mathfrak{P};(R \circledcirc \mathbb{I});\mathbb{T}$ by definition

$= (\mathbb{I} \circledcirc R);\mathbb{T}$

$= (R^\mathsf{T} \circledcirc \mathbb{I});\mathbb{T}$ due to Prop. 2.1.1.iii

$= \mathtt{vec}(R^\mathsf{T})$

$\mathtt{rel}(\mathfrak{P};v) = \pi^\mathsf{T};(\rho \cap \mathfrak{P};v;\mathbb{T}) = \pi^\mathsf{T};\mathfrak{P};\mathfrak{P}^\mathsf{T};(\rho \cap \mathfrak{P};v;\mathbb{T})$

$= \rho^\mathsf{T};(\mathfrak{P}^\mathsf{T};\rho \cap \mathfrak{P}^\mathsf{T};\mathfrak{P};v;\mathbb{T}) = \rho^\mathsf{T};(\pi \cap v;\mathbb{T}) = (\rho^\mathsf{T} \cap \mathbb{T};v^\mathsf{T});\pi$ mask shifting

$= [\pi^\mathsf{T};(\rho \cap v;\mathbb{T})]^\mathsf{T} = \mathtt{rel}(v)^\mathsf{T}$

---

iii) $\mathtt{vec}(R;S) = (R;S \oslash \mathbb{I});\mathbb{T}$ by definition
$= (R \otimes \mathbb{I});(S \oslash \mathbb{I});\mathbb{T}$ direction "$\subseteq$" requires sharp factorization!
$= (R \otimes \mathbb{I});\mathtt{vec}(S)$
$= (R \otimes \mathbb{I});(\mathbb{I} \oslash S^{\mathsf{T}});\mathbb{T}$ Prop. 2.1.1.iii
$= (R \oslash S^{\mathsf{T}});\mathbb{T}$ direction "$\supseteq$" again requires sharp factorization!

For the proof of the first occasion of sharp factorization we use points x, y and follow the Intermediate Point Theorem [11, 12, Prop. 2.4.8], a consequence of the Point Axiom: Assume points $x \subseteq (R;S \oslash \mathbb{I});y$. This means

$$x \subseteq (\pi;R;S \cap \rho);y = \pi;R;S;y \cap \rho;y \iff x \subseteq \pi;R;S;y \;\wedge\; x \subseteq \rho;y$$
$$\iff \pi^{\mathsf{T}};x \subseteq R;S;y \;\wedge\; \rho^{\mathsf{T}};x \subseteq y$$

Now we observe that $\pi^{\mathsf{T}};x$ is a point since $\pi^{\mathsf{T}};x \neq \mathbb{\bot}$ and it is obviously also row-constant and injective. The Intermediate Point Theorem then guarantees a point z satisfying

$$\pi^{\mathsf{T}};x \subseteq R;z \text{ and } z \subseteq S;y.$$

Given this with z, y injective, however,

$$x \subseteq \pi;R;z \text{ and } x \subseteq \rho;y \iff x \subseteq \pi;R;z \cap \rho;y = (R;z \oslash y) = (R \otimes \mathbb{I});(z \oslash y)$$

together with

$$(z \oslash y) \subseteq (S;y \oslash y) = (S \oslash \mathbb{I});y.$$

The second part on the left of (iii) may be proved without postulating the possibility of sharp factorization. It uses the technical result[1]

$$\mathtt{vec}(R;S;Q) = (R \otimes Q^{\mathsf{T}});\mathtt{vec}(S),$$

proved as Prop. 7.7 in [9]. We have simply to specialize the last factor to the identity.

Now the right side:

$\mathtt{rel}(u);\mathtt{rel}(v) = \pi^{\mathsf{T}};(\rho \cap u;\mathbb{T});\pi'^{\mathsf{T}};(\rho' \cap v;\mathbb{T})$ expanded
$= \pi^{\mathsf{T}};(\rho;\pi'^{\mathsf{T}} \cap u;\mathbb{T});(\rho' \cap v;\mathbb{T})$ masking
$= \pi^{\mathsf{T}};(\rho;\pi'^{\mathsf{T}} \cap u;\mathbb{T} \cap \mathbb{T};v^{\mathsf{T}});\rho'$ mask shifting
$= \pi^{\mathsf{T}};(\rho;\pi'^{\mathsf{T}} \cap u;\mathbb{T};v^{\mathsf{T}});\rho'$ masking

---

The final part has yet to be elaborated:

$\mathtt{comp}^{\mathsf{T}}_{XYZ};(\mathtt{vec}(R) \oslash \mathtt{vec}(S))$
$= (\pi^{\mathsf{T}} \otimes \rho'^{\mathsf{T}});\Big(\mathbb{I}_{(X\times Y)\times(Y\times Z)} \cap\ \rho_1;\pi';\rho^{\mathsf{T}};\pi_1^{\mathsf{T}}\Big);(\mathtt{vec}(R) \oslash \mathtt{vec}(S))$
$= (\pi^{\mathsf{T}} \otimes \rho'^{\mathsf{T}});\big(\mathbb{I} \cap\ \rho_1;\pi';\rho^{\mathsf{T}};\pi_1^{\mathsf{T}}\big);\big[\pi_1;\mathtt{vec}(R) \cap \rho_1;\mathtt{vec}(S)\big]$
$= (\pi^{\mathsf{T}} \otimes \rho'^{\mathsf{T}});\Big[\big(\mathbb{I} \cap\ \rho_1;\pi';\rho^{\mathsf{T}};\pi_1^{\mathsf{T}}\big);\pi_1;\mathtt{vec}(R) \cap \big(\mathbb{I} \cap\ \pi_1;\rho;\pi'^{\mathsf{T}};\rho_1^{\mathsf{T}}\big);\rho_1;\mathtt{vec}(S)\Big]$ partial identity
$= (\pi^{\mathsf{T}} \otimes \rho'^{\mathsf{T}});\Big[\big(\pi_1 \cap\ \rho_1;\pi';\rho^{\mathsf{T}}\big);\mathtt{vec}(R) \cap \big(\rho_1 \cap\ \pi_1;\rho;\pi'^{\mathsf{T}}\big);\mathtt{vec}(S)\Big]$ destroy and append
$= (\pi^{\mathsf{T}} \otimes \rho'^{\mathsf{T}});\Big[\ (\mathbb{I} \oslash \pi';\rho^{\mathsf{T}});\mathtt{vec}(R) \cap\ (\rho;\pi'^{\mathsf{T}} \oslash \mathbb{I});\mathtt{vec}(S)\Big]$
$= (\pi^{\mathsf{T}} \otimes \rho'^{\mathsf{T}});(\mathbb{I} \oslash \pi';\rho^{\mathsf{T}});\mathtt{vec}(R) \cap\ (\pi^{\mathsf{T}} \otimes \rho'^{\mathsf{T}});(\rho;\pi'^{\mathsf{T}} \oslash \mathbb{I});\mathtt{vec}(S)$ if univalent!
$= \pi'';\pi^{\mathsf{T}};\mathtt{vec}(R) \cap \rho'';\rho'^{\mathsf{T}};\mathtt{vec}(S)$
$= (\pi^{\mathsf{T}};\mathtt{vec}(R) \oslash \rho'^{\mathsf{T}};\mathtt{vec}(S))$

Still incomplete!

---

[1] ... obtained by Britta Kehden

iv) Herein $\rho ; \pi'^{\mathsf{T}} \cap u ; \mathbb{T} ; v^{\mathsf{T}}$ turns out to be difunctional:

$(\rho ; \pi'^{\mathsf{T}} \cap u ; \mathbb{T} ; v^{\mathsf{T}}) ; (\pi' ; \rho^{\mathsf{T}} \cap v ; \mathbb{T} ; u^{\mathsf{T}}) ; (\rho ; \pi'^{\mathsf{T}} \cap u ; \mathbb{T} ; v^{\mathsf{T}}) \subseteq \rho ; \pi'^{\mathsf{T}} \cap u ; \mathbb{T} ; v^{\mathsf{T}}$,

which is proved by these two estimations

$\ldots \subseteq \rho ; \pi'^{\mathsf{T}} ; \pi' ; \rho^{\mathsf{T}} ; \rho ; \pi'^{\mathsf{T}} = \rho ; \mathbb{I} ; \mathbb{I} ; \pi'^{\mathsf{T}} = \rho ; \pi'^{\mathsf{T}}$

$\ldots \subseteq u ; \mathbb{T} ; v^{\mathsf{T}} ; v ; \mathbb{T} ; u^{\mathsf{T}} ; u ; \mathbb{T} ; v^{\mathsf{T}} \subseteq u ; \mathbb{T} ; v^{\mathsf{T}}$

---

v) Just one of the two statements is shown; the other follows by the `rel`–`vec`-bijection of Prop. 2.2.5.iii.

$\mathtt{rel}((v ; \mathbb{T} \otimes w ; \mathbb{T})) = \pi^{\mathsf{T}} ; (\rho \cap (v ; \mathbb{T} \otimes w ; \mathbb{T}))$

$= \pi^{\mathsf{T}} ; (\rho \cap \pi ; v ; \mathbb{T} \cap \rho ; w ; \mathbb{T}) = \pi^{\mathsf{T}} ; (\pi ; v ; \mathbb{T} \cap \rho ; (\mathbb{I} \cap w ; \mathbb{T}))$

$= v ; \mathbb{T} \cap \mathbb{T} ; (\mathbb{I} \cap w ; \mathbb{T})$ destroy and append

$= v ; \mathbb{T} \cap (\mathbb{T} \cap \mathbb{T} ; w^{\mathsf{T}}) ; \mathbb{I}$ mask shifting

$= v ; \mathbb{T} ; w^{\mathsf{T}}$ masking

---

vi) $\iota ; (\mathbb{I} \cap v ; \mathbb{T}) = \iota ; (\iota^{\mathsf{T}} ; \iota)$ since ι extrudes $v ; \mathbb{T}$

$= (\iota ; \iota^{\mathsf{T}}) ; \iota = \iota$ since ι is an injective map □

$$\begin{array}{r} \\ \text{Elizabeth} \\ \text{Florence} \\ \text{Shirley} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{daffodils} & \text{daisies} & \text{roses} & \text{tulips} & \text{violets} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \qquad p = \begin{array}{c} \begin{array}{cccc} \spadesuit & \heartsuit & \diamondsuit & \clubsuit \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$q^{\mathsf{T}} = \begin{array}{r} \spadesuit \\ \heartsuit \\ \diamondsuit \\ \clubsuit \end{array} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$(p ; \mathbb{T} \otimes q ; \mathbb{T}) = \begin{array}{r} \\ \text{(Elizabeth,daffodils)} \\ \text{(Elizabeth,daisies)} \\ \text{(Elizabeth,roses)} \\ \text{(Elizabeth,tulips)} \\ \text{(Elizabeth,violets)} \\ \text{(Florence,daffodils)} \\ \text{(Florence,daisies)} \\ \text{(Florence,roses)} \\ \text{(Florence,tulips)} \\ \text{(Florence,violets)} \\ \text{(Shirley,daffodils)} \\ \text{(Shirley,daisies)} \\ \text{(Shirley,roses)} \\ \text{(Shirley,tulips)} \\ \text{(Shirley,violets)} \end{array} \begin{array}{c} \begin{array}{cccc} \spadesuit & \heartsuit & \diamondsuit & \clubsuit \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Fig. 2.2.7 Relation entry $p ; q^{\mathsf{T}}$ and point pair $(p ; \mathbb{T} \otimes q ; \mathbb{T})$

Also the points take part in vectorization; this shall now be investigated.

2.2.7 Definition. Given any pair p, q of points $p : X \longrightarrow Z$, $q : Y \longrightarrow Z$, there exist two versions of denoting such a compound. We speak — depending on the respective situation — of a

— **relation entry** if $p ; \mathbb{T} ; q^{\mathsf{T}} : X \longrightarrow Y$ or of a

— **point pair** if $(p ; \mathbb{T} \otimes q ; \mathbb{T}) : X \times Y \longrightarrow Z$,

where the latter is itself a point. □

The two are — according to the forthcoming Prop. 2.2.8 — bijectively related. An entry of the relation is more or less an entry of a matrix. In addition to points $p \in \mathtt{pts}(X)$ and $q \in \mathtt{pts}(Y)$, we are going speak of $(p,q) \in \mathtt{ets}(X,Y)$ according to Prop. 2.2.8.ii.

2.2.8 Proposition. Let $p : X \longrightarrow \mathbb{1}$ and $q : Y \longrightarrow \mathbb{1}$ be points. Then

i) $(p \otimes q)$ and $(p \fatsemi \mathbb{T} \otimes q \fatsemi \mathbb{T})$ are again points and elements of $\mathtt{pts}(X \times Y)$.

ii) $\mathtt{rel}(\,(p \fatsemi \mathbb{T} \otimes q \fatsemi \mathbb{T})\,) = p \fatsemi \mathbb{T} \fatsemi q^\intercal \qquad \mathtt{vec}(\,p \fatsemi \mathbb{T} \fatsemi q^\intercal) = (p \fatsemi \mathbb{T} \otimes q \fatsemi \mathbb{T})$

iii) Let $p \fatsemi \mathbb{T} \fatsemi q^\intercal$ be a relation entry; then

$$p \fatsemi \mathbb{T} \fatsemi q^\intercal \subseteq A \cup B \quad \Longleftrightarrow \quad p \fatsemi \mathbb{T} \fatsemi q^\intercal \subseteq A \quad \textbf{or} \quad p \fatsemi \mathbb{T} \fatsemi q^\intercal \subseteq B$$

Proof: i) row-constant:

$$(p \otimes q) \fatsemi \mathbb{T} = (\pi \fatsemi p \cap \rho \fatsemi q) \fatsemi \mathbb{T} = (\pi \fatsemi p \fatsemi \mathbb{T} \cap \rho \fatsemi q \fatsemi \mathbb{T}) \fatsemi \mathbb{T} = \pi \fatsemi p \fatsemi \mathbb{T} \cap \rho \fatsemi q \fatsemi \mathbb{T}$$
$$= (p \fatsemi \mathbb{T} \otimes q \fatsemi \mathbb{T}) = (p \otimes q)$$

injective:

$$(p \otimes q) \fatsemi (p \otimes q)^\intercal = (p \otimes q) \fatsemi (p^\intercal \oslash q^\intercal) \subseteq (p \fatsemi p^\intercal \otimes q \fatsemi q^\intercal) \subseteq (\mathbb{I} \otimes \mathbb{I}) = \mathbb{I} \quad \text{using Prop. 2.1.1.i}$$

non-zero:

$$\mathbb{T} \fatsemi (p \otimes q) = \mathbb{T} \fatsemi (\pi \fatsemi p \cap \rho \fatsemi q) = \mathbb{T} \fatsemi (\pi \fatsemi p \fatsemi \mathbb{T} \cap \rho \fatsemi q \fatsemi \mathbb{T})$$
$$= (\mathbb{T} \cap \mathbb{T} \fatsemi q^\intercal \fatsemi \rho^\intercal) \fatsemi \pi \fatsemi p \fatsemi \mathbb{T} = \mathbb{T} \fatsemi q^\intercal \fatsemi \mathbb{T} \fatsemi p \fatsemi \mathbb{T} = \mathbb{T} \quad \text{mask shifting; } p, q \text{ are points; } \rho^\intercal \fatsemi \pi = \mathbb{T}$$

---

ii) Because points are necessarily vectors, this subsumes to Prop. 2.2.6.v.

---

iii) Proceeding to the respective vectorizations, we arrive at the point pair $(p \fatsemi \mathbb{T} \otimes q \fatsemi \mathbb{T})$ that is itself a point. So we may apply Prop. 2.2.2.ii. □

3 Binary relational operations

In case a relation has a direct product as its source, $R : X \times Y \longrightarrow Z$, we shall sometimes make this "visible" by speaking of a *binary relational operation*. As is tradition for binary mappings, several standard properties will be discussed.

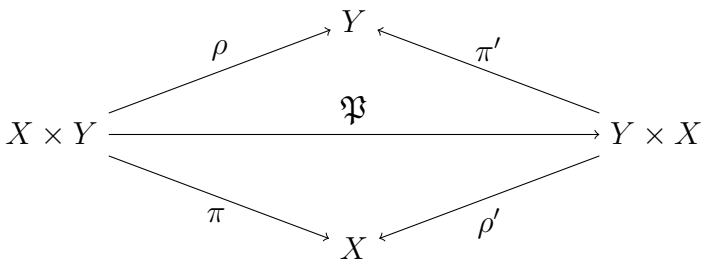
Using such binary relational operations, one will observe several rules that seem highly trivial, but nevertheless need some formal relational proof. These shall here be investigated.

3.1 Standard questions

Whenever a direct product is given with projections π, ρ, there are two possibilities to express an **exchange flip** as shown in Fig. 3.1.1:

$$\mathfrak{P} := (\rho \leftthreetimes \pi) = (\rho' \rightthreetimes \pi') .$$

This $\mathfrak{P}$ is a bijective map satisfying $\mathfrak{P} \,;\, \mathfrak{P}^\intercal = \mathbb{I}$. It is a mere technicality to prove the main properties of $\mathfrak{P}$, namely $\mathfrak{P} \,;\, \pi' = \rho$ and $\mathfrak{P} \,;\, \rho' = \pi$ as well as $\pi^\intercal ; \mathfrak{P} = \rho'^\intercal$ and $\rho^\intercal ; \mathfrak{P} = \pi'^\intercal$, using the destroy and append rule. Only in case $X = Y$, we have $\mathfrak{P}^\intercal = \mathfrak{P}$.



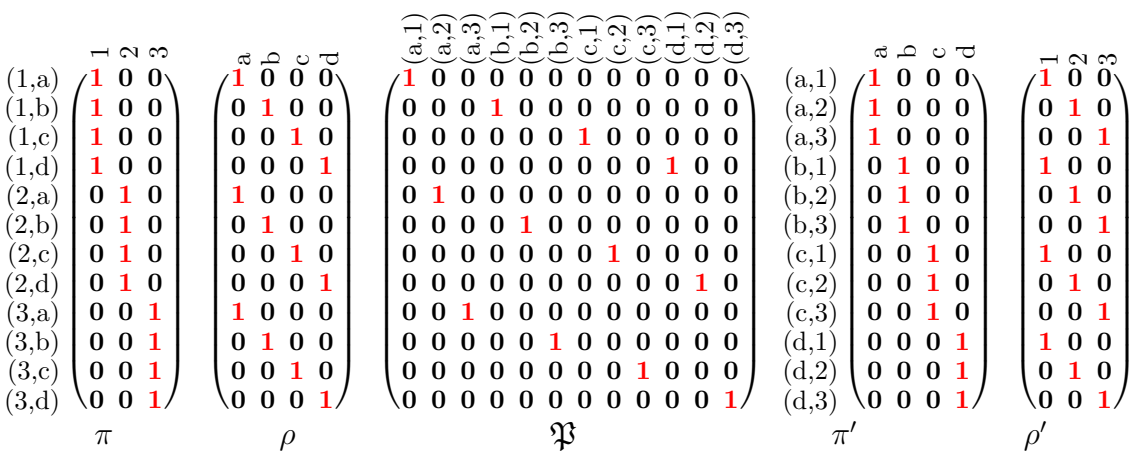


Fig. 3.1.1 Typing the exchange flip $\mathfrak{P}$

3.1.1 Proposition. $\mathfrak{P} \,;\, (R \otimes S) \;=\; (S \otimes R) \,;\, \mathfrak{P}'$ and $(A \oslash B) \,;\, \mathfrak{P} \;=\; (B \oslash A)$.

Proof: omitted in view of Fig. 3.1.2. □

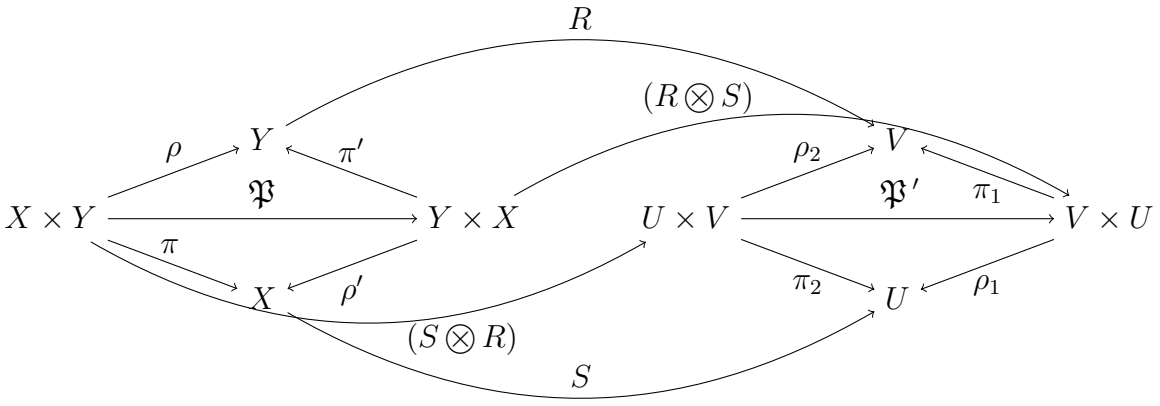


Fig. 3.1.2 Illustrating the property Prop. 3.1.1 of $\mathfrak{P}$

In case $X = Y$, the exchange flip satisfies $\mathfrak{P}^{\mathsf{T}} = \mathfrak{P}$ — as already mentioned earlier. It is then used in particular to express that a function typed $f : X \times X \longrightarrow Z$ is **commutative**, by demanding $\mathfrak{P} \,;\, f = f$.

The following is a highly versatile statement — although easily proved:

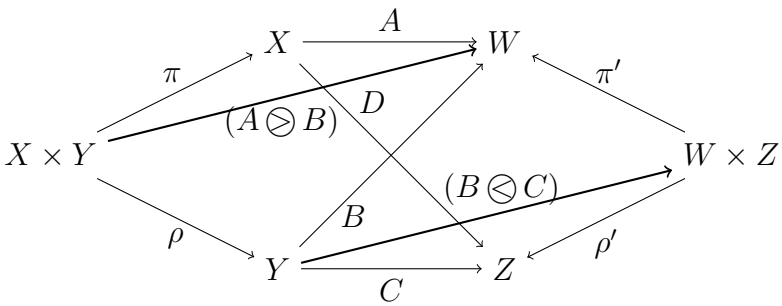


Fig. 3.1.3 Typing the projection migration

3.1.2 Proposition. For the typing given with Fig. 3.1.3, the following formulae hold to which we will refer as **projection migration**:

i) $(\,(A \oslash B) \;\oslash \rho \,;\, C) \;=\; (A \,;\, \pi'^{\mathsf{T}} \oslash (B \oslash C)\,)$

ii) $(\,(P \oslash Q) \oslash R \,;\, \rho^{\mathsf{T}}) \;=\; (\pi' \,;\, P \oslash\; (Q \oslash R)\,)$

iii) $(\,(A \oslash B) \;\oslash \pi \,;\, D) \;=\; (\,(A \oslash D) \oslash B \,;\, \pi'^{\mathsf{T}})$

iv) $(\,(P \oslash Q) \oslash R \,;\, \pi^{\mathsf{T}}) \;=\; (\,(P \oslash R) \;\oslash \pi' \,;\, Q)$

Proof: i) $(\,(A \between B)\ \between \rho\,;C) = (A \between B)\,;\pi'^{\mathsf{T}} \cap \rho\,;C\,;\rho'^{\mathsf{T}} = (\pi\,;A \cap \rho\,;B)\,;\pi'^{\mathsf{T}} \cap \rho\,;C\,;\rho'^{\mathsf{T}}$
$= \pi\,;A\,;\pi'^{\mathsf{T}} \cap \rho\,;B\,;\pi'^{\mathsf{T}} \cap \rho\,;C\,;\rho'^{\mathsf{T}}$
$= \pi\,;A\,;\pi'^{\mathsf{T}} \cap \rho\,;(B\,;\pi'^{\mathsf{T}} \cap C\,;\rho'^{\mathsf{T}}) = \pi\,;A\,;\pi'^{\mathsf{T}} \cap \rho\,;\,(B \between C)$
$= (\,A\,;\pi'^{\mathsf{T}} \between (B \between C)\,)$

---

ii) This is simply the statement (i) transposed, however with A^{T} renamed to P etc.

---

iii) This may be proved in analogy to (i), but we use the occasion to demonstrate how $\mathfrak{P}$ works.
$(\,(A \between B)\ \between \pi\,;D) = (\,\mathfrak{P}\,;\,(B \between A)\ \between \mathfrak{P}\,;\rho\,;D)$
$= \mathfrak{P}\,;\,(\,(B \between A)\ \between \rho\,;D)$ since $\mathfrak{P}$ is univalent
$= \mathfrak{P}\,;\,(B\,;\pi'^{\mathsf{T}} \between (A \between D)\,)$ applying (i)
$= (\,(A \between D)\ \between B\,;\pi'^{\mathsf{T}})$

---

iv) Again, this is (iii) transposed with A^{T} renamed to P etc. □

Routinely, one will also have to change braces when associativity holds. We present the corresponding brace change mappings in the general case, using the diagram of Fig. 3.1.4.

In these formulae, we have used projections and denoted them explicitly. We might instead have used, e.g., $\pi = (\mathbb{I}_Y \between \mathbb{T}_Z)$ and $\rho = (\mathbb{T}_Y \between \mathbb{I}_Z)$, or even without mentioning the set as a subscript, making them nearly unreadable for a human being. The typing and unification mechanisms of TituRel would be able to handle that correctly.

The **brace change shuffle** — that may in case $X = Y = Z$ be used as an **associativity shuffle** — offers two options for the transition, from left to right

$\mathfrak{T}_{12\to 21} : X \times (Y \times Z) \longrightarrow (X \times Y) \times Z$
$\mathfrak{T}_{12\to 21} := (\,(\mathbb{I} \otimes \pi)\ \between \rho'\,;\rho) = (\pi_2^{\mathsf{T}}\,;\,\pi_1^{\mathsf{T}} \between (\rho_2^{\mathsf{T}} \otimes \mathbb{I})\,)\,,$

and from right to left

$\mathfrak{T}_{21\to 12} : (X \times Y) \times Z \longrightarrow X \times (Y \times Z)$
$\mathfrak{T}_{21\to 12} := (\,(\mathbb{I} \otimes \pi^{\mathsf{T}})\ \between \rho^{\mathsf{T}}\,;\rho'^{\mathsf{T}}) = (\pi_1\,;\,\pi_2 \between\ (\rho_2 \otimes \mathbb{I})\,)\,.$

These turn out to be bijective mappings. One can simply transpose the first in order to obtain the second. As some sort of a justification, we show that

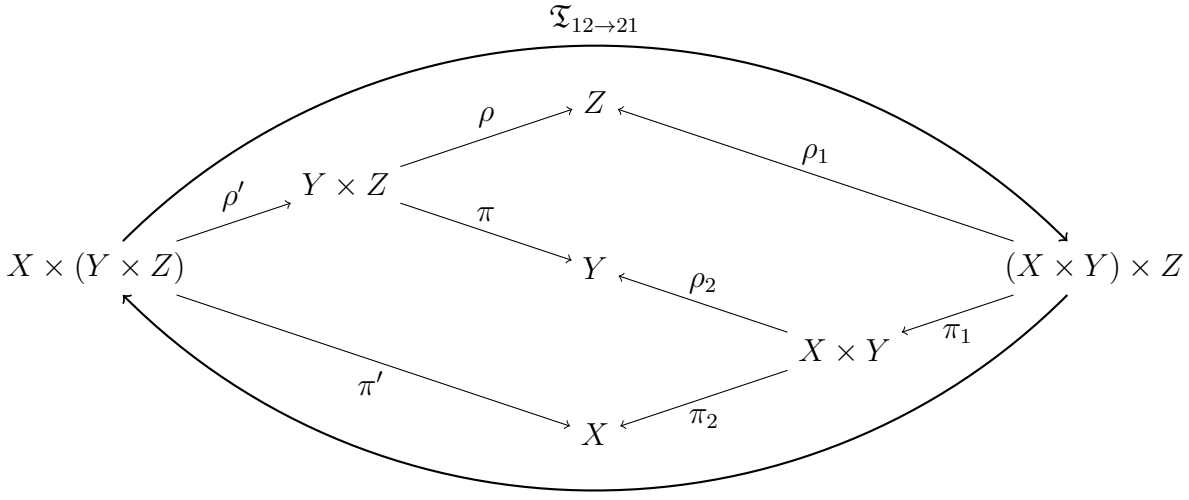
$\mathfrak{T}_{12\to 21}\,;\,\mathfrak{T}_{21\to 12} = (\,(\mathbb{I} \otimes \pi)\ \between \rho'\,;\rho)\ ;\ (\,(\mathbb{I} \otimes \pi^{\mathsf{T}})\ \between \rho^{\mathsf{T}}\,;\rho'^{\mathsf{T}})$
$= (\mathbb{I} \otimes \pi)\,;\,(\mathbb{I} \otimes \pi^{\mathsf{T}}) \cap \rho'\,;\rho\,;\rho^{\mathsf{T}}\,;\rho'^{\mathsf{T}}$
$= (\mathbb{I} \otimes \pi\,;\pi^{\mathsf{T}}) \cap \rho'\,;\rho\,;\rho^{\mathsf{T}}\,;\rho'^{\mathsf{T}}$
$= (\pi'\,;\pi'^{\mathsf{T}} \cap \rho'\,;\pi\,;\pi^{\mathsf{T}}\,;\rho'^{\mathsf{T}}) \cap\ \rho'\,;\rho\,;\rho^{\mathsf{T}}\,;\rho'^{\mathsf{T}} = \pi'\,;\pi'^{\mathsf{T}} \cap (\rho'\,;\pi\,;\pi^{\mathsf{T}}\,;\rho'^{\mathsf{T}} \cap\ \rho'\,;\rho\,;\rho^{\mathsf{T}}\,;\rho'^{\mathsf{T}})$
$= \pi'\,;\pi'^{\mathsf{T}} \cap \rho'\,;(\pi\,;\pi^{\mathsf{T}} \cap\ \rho\,;\rho^{\mathsf{T}})\,;\rho'^{\mathsf{T}} = \pi'\,;\pi'^{\mathsf{T}} \cap \rho'\,;\mathbb{I}\,;\rho'^{\mathsf{T}} = \mathbb{I}$

One will observe that the transition between the two versions of each brace change shuffle follows the scheme of a projection migration as in Prop. 3.1.2 — in case the Kronecker operator is replaced by join and fork. Since there are so many projections, those migrating are colored in green:

$\mathfrak{T}_{12\to 21} = (\,(\mathbb{I} \otimes \pi)\ \between \rho'\,;\rho)$
$= (\,(\pi_2^{\mathsf{T}} \between \pi\,;\rho_2^{\mathsf{T}})\ \between \rho'\,;\rho) = (\pi_2^{\mathsf{T}}\,;\pi_1^{\mathsf{T}} \between (\pi\,;\rho_2^{\mathsf{T}} \between \rho))$ projection migration Prop. 3.1.2.i

$= (\pi_2^\mathsf{T} ; \pi_1^\mathsf{T} \between (\rho_2^\mathsf{T} \otimes \mathbb{I}))$

$\mathfrak{T}_{21\to 12} := ((\mathbb{I} \otimes \pi^\mathsf{T}) \between \rho^\mathsf{T} ; \rho'^\mathsf{T})$
$= ((\pi_2 \between \rho_2 ; \pi^\mathsf{T}) \between \rho^\mathsf{T} ; \rho'^\mathsf{T}) = (\pi_1 ; \pi_2 \between (\rho_2 ; \pi^\mathsf{T} \between \rho^\mathsf{T}))$ projection migration Prop. 3.1.2.ii
$= (\pi_1 ; \pi_2 \between (\rho_2 \otimes \mathbb{I}))$



	((α,1),a)	((β,1),a)	((α,1),b)	((α,2),a)	((β,1),b)	((α,1),c)	((γ,1),a)	((α,2),b)	((β,1),c)	((β,2),a)	((γ,1),b)	((α,2),c)	((γ,2),a)	((β,2),b)	((γ,1),c)	((γ,2),b)	((β,2),c)	((γ,2),c)
(α,(1,a))	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(β,(1,a))	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(α,(2,a))	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(γ,(1,a))	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
(β,(2,a))	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
(α,(1,b))	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(γ,(2,a))	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
(β,(1,b))	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
(α,(2,b))	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
(γ,(1,b))	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
(β,(2,b))	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
(α,(1,c))	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
(γ,(2,b))	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
(β,(1,c))	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
(α,(2,c))	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
(γ,(1,c))	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
(β,(2,c))	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
(γ,(2,c))	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Fig. 3.1.4 Typing brace change $\mathfrak{T}_{12\to 21}$ and relation using mandrel representation for the product

---

A binary relational operation M — necessarily typed $M : X \times X \longrightarrow X$ as unification will reveal — is called **associative** in case

$(M \otimes \mathbb{I}) ; M = \mathfrak{T}_{21\to 12} ; (\mathbb{I} \otimes M) ; M.$

We also refer to [13, Fig. 8.2] or [15, Fig. 3.7].

The next conceivable effect is distributivity as known from the very beginning of mathematics, but also from scalar multiplication of vectors

$s \cdot (a + b) = s \cdot a + s \cdot b.$

Working with such constructs requires computer support. In the formulae below, we have denoted the projections explicitly as π, ρ. State of the art typing and unification mechanisms, not least of TituRel, are able to handle that correctly.

The following version is copied from [15, Fig. 3.9]:

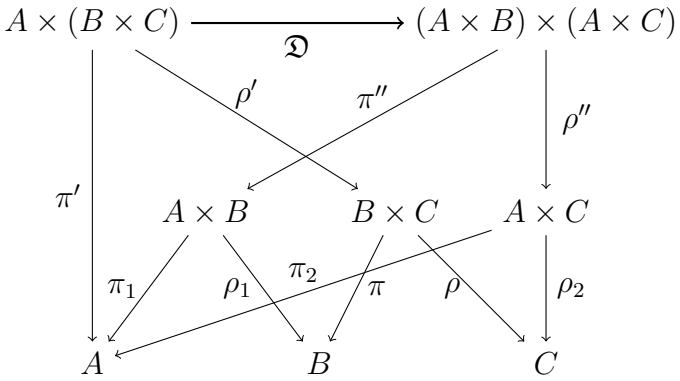


Fig. 3.1.5 Typing the distributivity rearrangement

3.1.3 Definition. For this situation we introduce the following notation:

i) $\mathfrak{D} := (\,(\mathbb{I} \otimes\!\!\!

$= (\mathbb{I} \otimes \pi) ; (\mathbb{I} \otimes \pi^\mathsf{T}) \cap (\mathbb{I} \otimes \rho) ; (\mathbb{I} \otimes \rho^\mathsf{T})$
$= (\mathbb{I} \otimes \pi ; \pi^\mathsf{T}) \cap (\mathbb{I} \otimes \rho ; \rho^\mathsf{T})$
$= \pi' ; \pi'^\mathsf{T} \cap \rho' ; \pi ; \pi^\mathsf{T} ; \rho'^\mathsf{T} \cap \pi' ; \pi'^\mathsf{T} \cap \rho' ; \rho ; \rho^\mathsf{T} ; \rho'^\mathsf{T}$
$= \pi' ; \pi'^\mathsf{T} \cap \rho' ; (\pi ; \pi^\mathsf{T} \cap \rho ; \rho^\mathsf{T}) ; \rho'^\mathsf{T}$
$= \pi' ; \pi'^\mathsf{T} \cap \rho' ; \rho'^\mathsf{T} = \mathbb{I}$

ii) We use the second variant of the definition of $\mathfrak{D}$ to have

$\pi'^\mathsf{T} ; \mathfrak{D} = (\pi_1^\mathsf{T} \oslash \pi_2^\mathsf{T})$ and $\rho'^\mathsf{T} ; \mathfrak{D} = (\rho_1^\mathsf{T} \otimes \rho_2^\mathsf{T})$

and continue for both statements simultaneously with

$(R \otimes (S \otimes T)) ; \mathfrak{D} = (R ; \pi'^\mathsf{T} \oslash (S \otimes T) ; \rho'^\mathsf{T}) ; \mathfrak{D}$
$= (R ; \pi'^\mathsf{T} ; \mathfrak{D} \oslash (S \otimes T) ; \rho'^\mathsf{T} ; \mathfrak{D})$ injectivity of $\mathfrak{D}$
$= (R ; (\pi_1^\mathsf{T} \oslash \pi_2^\mathsf{T}) \oslash (S \otimes T) ; (\rho_1^\mathsf{T} \otimes \rho_2^\mathsf{T}))$ see above
$= (R ; (\pi_1^\mathsf{T} \oslash \pi_2^\mathsf{T}) \oslash (S ; \rho_1^\mathsf{T} \otimes T ; \rho_2^\mathsf{T}))$
$= ((R ; \pi_1^\mathsf{T} \oslash R ; \pi_2^\mathsf{T}) \oslash (S ; \rho_1^\mathsf{T} \otimes T ; \rho_2^\mathsf{T}))$ if R is univalent, otherwise just "$\subseteq$"
$= \pi_0' ; (R ; \pi_1^\mathsf{T} \oslash R ; \pi_2^\mathsf{T}) \cap \rho_0' ; (S ; \rho_1^\mathsf{T} \otimes T ; \rho_2^\mathsf{T})$
$= \pi_0' ; R ; \pi_1^\mathsf{T} ; \pi''^\mathsf{T} \cap \pi_0' ; R ; \pi_2^\mathsf{T} ; \rho''^\mathsf{T} \cap \rho_0' ; \pi_0 ; S ; \rho_1^\mathsf{T} ; \pi''^\mathsf{T} \cap \rho_0' ; \rho_0 ; T ; \rho_2^\mathsf{T} ; \rho''^\mathsf{T}$
$= \pi_0' ; R ; \pi_1^\mathsf{T} ; \pi''^\mathsf{T} \cap \rho_0' ; \pi_0 ; S ; \rho_1^\mathsf{T} ; \pi''^\mathsf{T} \cap \pi_0' ; R ; \pi_2^\mathsf{T} ; \rho''^\mathsf{T} \cap \rho_0' ; \rho_0 ; T ; \rho_2^\mathsf{T} ; \rho''^\mathsf{T}$ rearranged
$= (\pi_0' ; R ; \pi_1^\mathsf{T} \cap \rho_0' ; \pi_0 ; S ; \rho_1^\mathsf{T} \oslash \pi_0' ; R ; \pi_2^\mathsf{T} \cap \rho_0' ; \rho_0 ; T ; \rho_2^\mathsf{T})$
$= ((R \otimes \pi_0 ; S) \oslash (R \otimes \rho_0 ; T))$
$= ((\mathbb{I} \otimes \pi_0) ; (R \otimes S) \oslash (\mathbb{I} \otimes \rho_0) ; (R \otimes T))$
$= ((\mathbb{I} \otimes \pi_0) \oslash (\mathbb{I} \otimes \rho_0)) ; ((R \otimes S) \otimes (R \otimes T))$
$= \mathfrak{D}_0 ; ((R \otimes S) \otimes (R \otimes T))$ □

Later we will need yet another result of this type, namely the Kronecker-fork shuffling. The typing scheme is as follows. One might say in view of the following result that $\otimes$ and $\oslash$ together are **bi-commutative**.

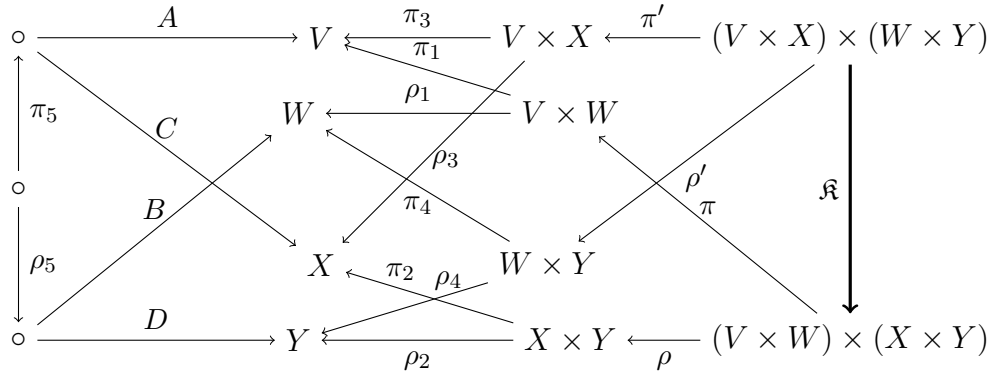


Fig. 3.1.6 Typing the Kronecker-fork shuffle

3.1.5 Proposition. We define **Kronecker-fork shuffling** as a relation

$\mathfrak{K} : (V \times X) \times (W \times Y) \longrightarrow (V \times W) \times (X \times Y)$

$\mathfrak{K} := \pi' ; \pi_3 ; \pi_1^\mathsf{T} ; \pi^\mathsf{T} \cap \rho' ; \pi_4 ; \rho_1^\mathsf{T} ; \pi^\mathsf{T} \cap \pi' ; \rho_3 ; \pi_2^\mathsf{T} ; \rho^\mathsf{T} \cap \rho' ; \rho_4 ; \rho_2^\mathsf{T} ; \rho^\mathsf{T}$

or, differently grouped, when so required:

$= \pi' ; (\pi_3 ; \pi_1^\intercal ; \pi^\intercal \cap \rho_3 ; \pi_2^\intercal ; \rho^\intercal) \cap \rho' ; (\pi_4 ; \rho_1^\intercal ; \pi^\intercal \cap \rho_4 ; \rho_2^\intercal ; \rho^\intercal)$
$= \big(\, (\pi_1 \otimes \pi_2) \;\oslash\; (\rho_1 \otimes \rho_2) \,\big)^\intercal$
$= (\pi' ; \pi_3 ; \pi_1^\intercal \cap \rho' ; \pi_4 ; \rho_1^\intercal) ; \pi^\intercal \cap (\pi' ; \rho_3 ; \pi_2^\intercal \cap \rho' ; \rho_4 ; \rho_2^\intercal) ; \rho^\intercal$
$= \big(\, (\pi_3 \otimes \pi_4) \;\oslash\; (\rho_3 \otimes \rho_4) \,\big)$.

Then the following holds

$\big(\, (A \oslash C) \;\otimes\; (B \oslash D) \,\big) ; \mathfrak{K} = \big(\, (A \otimes B) \;\oslash\; (C \otimes D) \,\big)$.

Proof: We omit the lengthy proof that $\mathfrak{K}$ is a bijective mapping and focus on proving that it satisfies useful formulae such as

$\mathfrak{K} ; (\pi_1 \otimes \pi_2) = \big(\, (\pi_3 \otimes \pi_4) \;\oslash\; (\rho_3 \otimes \rho_4) \,\big) ; (\pi_1 \otimes \pi_2)$ expanded
$= \big(\, (\pi_3 \otimes \pi_4) \;; \pi_1 \;\oslash\; (\rho_3 \otimes \rho_4) \;; \pi_2 \big)$
$= (\pi' ; \pi_3 \oslash \pi' ; \rho_3) = \pi' ; (\pi_3 \oslash \rho_3) = \pi' ; (\pi_3 ; \pi_3^\intercal \cap \rho_3 ; \rho_3^\intercal) = \pi' ; \mathbb{I} = \pi'$

$\mathfrak{K} ; (\rho_1 \otimes \rho_2) = \rho' \qquad \pi'^\intercal ; \mathfrak{K} = (\pi_1 \otimes \pi_2)^\intercal \qquad \rho'^\intercal ; \mathfrak{K} = (\rho_1 \otimes \rho_2)^\intercal$

Using this $\mathfrak{K}$, we prove

$\big(\, (A \oslash C) \;\otimes\; (B \oslash D) \,\big) ; \mathfrak{K} = (\pi_5 ; (A \oslash C) ; \pi'^\intercal \cap \rho_5 ; (B \oslash D) ; \rho'^\intercal) ; \mathfrak{K}$
$= \pi_5 ; (A \oslash C) ; (\pi_1 \otimes \pi_2)^\intercal \cap \rho_5 ; (B \oslash D) ; (\rho_1 \otimes \rho_2)^\intercal$ using the above
$= \pi_5 ; (A ; \pi_1^\intercal \oslash C ; \pi_2^\intercal) \cap \rho_5 ; (B ; \rho_1^\intercal \oslash D ; \rho_2^\intercal)$
$= \pi_5 ; A ; \pi_1^\intercal ; \pi^\intercal \cap \pi_5 ; C ; \pi_2^\intercal ; \rho^\intercal \cap \rho_5 ; B ; \rho_1^\intercal ; \pi^\intercal \cap \rho_5 ; D ; \rho_2^\intercal ; \rho^\intercal$
$= \big[\pi_5 ; A ; \pi_1^\intercal \cap \rho_5 ; B ; \rho_1^\intercal\big] ; \pi^\intercal \cap \big[\pi_5 ; C ; \pi_2^\intercal \cap \rho_5 ; D ; \rho_2^\intercal\big] ; \rho^\intercal$
$= \big(\, (A \otimes B) \;\oslash\; (C \otimes D) \,\big)$ □

In the last propositions, we have managed to lift several standard constructions such as commutativity, associativity, distributivity, e.g., that require quantifiers when being defined, to the algebraic level. These steps, we are traditionally used to execute element-wise, may now be handled on the upper level. One may complain that the proofs necessary to establish these results were sometimes fairly difficult, in particular the shuffling results. However, looking at the final resulting formulae, one will agree that these purely algebraic formulae are concise and open to intuitive comprehension.

We are now prepared to treat binary mappings algebraically.

3.1.6 Proposition. i) $PO := \pi ; \mathbb{S}^+ ; \pi^\intercal \cup (\mathbb{I} \otimes \mathbb{S}^*)$ is a linear ordering, the **pair order**.

ii) $PSO := \pi ; \mathbb{S}^+ ; \pi^\intercal \cup (\mathbb{I} \otimes \mathbb{S}^+)$ is a linear strictorder, the **pair strictorder**.

Proof: We mainly restrict to proving (ii).

$\mathbb{I} \cup PSO = (\mathbb{I} \otimes \mathbb{I}) \cup PSO = (\mathbb{I} \otimes \mathbb{I}) \cup \pi ; \mathbb{S}^+ ; \pi^\intercal \cup (\mathbb{I} \otimes \mathbb{S}^+)$
$= \pi ; \mathbb{S}^+ ; \pi^\intercal \cup (\mathbb{I} \otimes \mathbb{I} \cup \mathbb{S}^+) = \pi ; \mathbb{S}^+ ; \pi^\intercal \cup (\mathbb{I} \otimes \mathbb{S}^*) = PO$

$$
\begin{array}{l}
PSO;PSO = \left[\pi;\mathbb{S}^{+};\pi^{\intercal} \cup (\mathbb{I} \otimes \mathbb{S}^{+})\ \right];\left[\pi;\mathbb{S}^{+};\pi^{\intercal} \cup (\mathbb{I} \otimes \mathbb{S}^{+})\ \right] \\
= \pi;\mathbb{S}^{+};\pi^{\intercal};\pi;\mathbb{S}^{+};\pi^{\intercal} \cup \pi;\mathbb{S}^{+};\pi^{\intercal};(\mathbb{I} \otimes \mathbb{S}^{+}) \cup (\mathbb{I} \otimes \mathbb{S}^{+});\pi;\mathbb{S}^{+};\pi^{\intercal} \cup (\mathbb{I} \otimes \mathbb{S}^{+});(\mathbb{I} \otimes \mathbb{S}^{+}) \\
\subseteq \pi;\mathbb{S}^{+};\mathbb{S}^{+};\pi^{\intercal} \cup \pi;\mathbb{S}^{+};[\pi^{\intercal} \cap \mathbb{T};\mathbb{S}^{+};\rho^{\intercal}] \cup [\pi \cap \rho;\mathbb{S}^{+};\mathbb{T}];\mathbb{S}^{+};\pi^{\intercal} \cup (\mathbb{I} \otimes \mathbb{S}^{+};\mathbb{S}^{+}) \\
\subseteq \pi;\mathbb{S}^{+};\pi^{\intercal} \cup \pi;[\mathbb{S}^{+};\pi^{\intercal} \cap \mathbb{T};\mathbb{S}^{+};\rho^{\intercal}] \cup [\pi;\mathbb{S}^{+} \cap \rho;\mathbb{S}^{+};\mathbb{T}];\pi^{\intercal} \cup (\mathbb{I} \otimes \mathbb{S}^{+};\mathbb{S}^{+}) \\
\subseteq \pi;\mathbb{S}^{+};\pi^{\intercal} \cup (\mathbb{I} \otimes \mathbb{S}^{+};\mathbb{S}^{+}) = PSO
\end{array}
$$

□

PO is visualized in Fig. 3.1.7

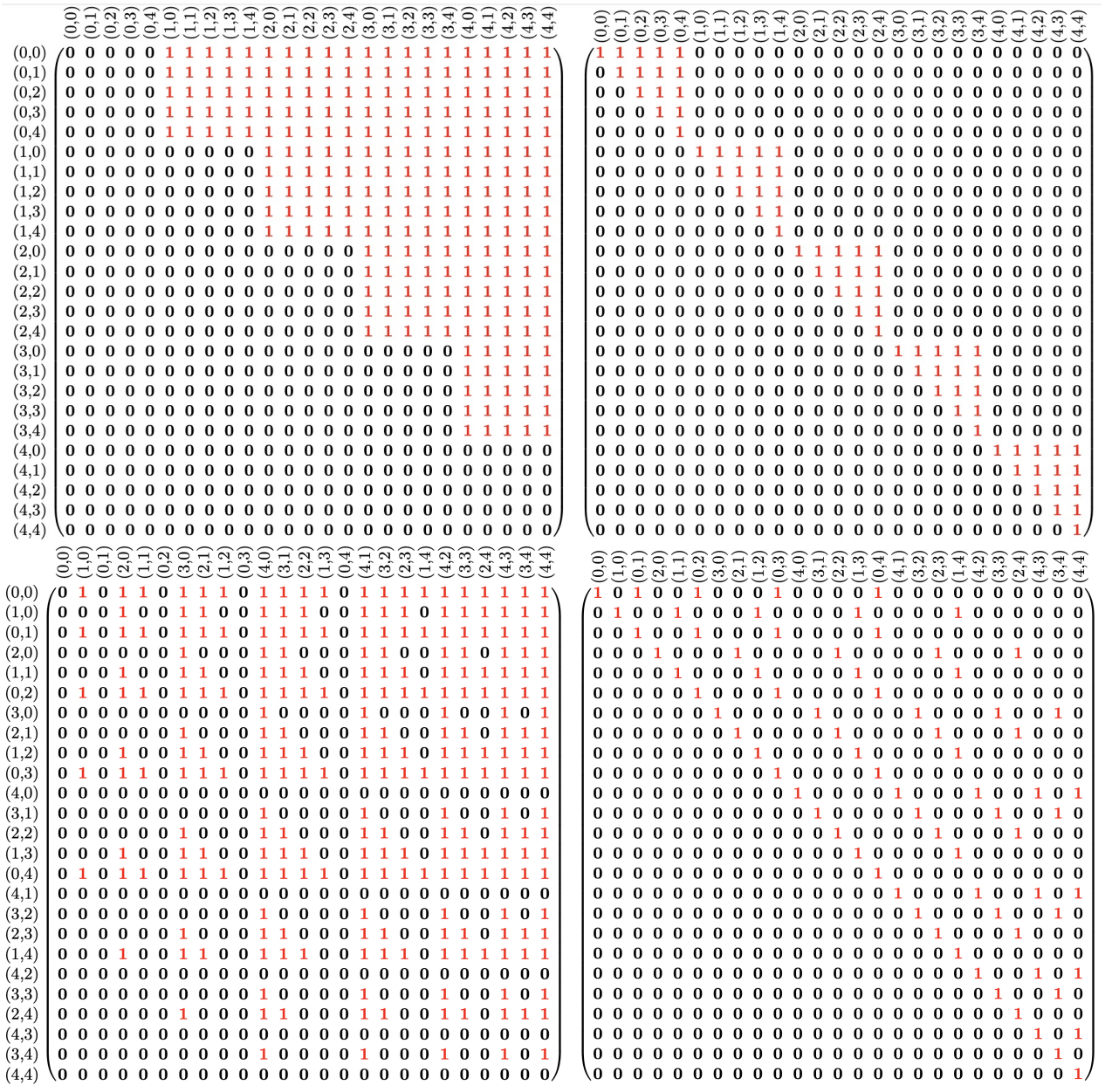


Fig. 3.1.7 The two constituents $\pi;\mathbb{S}^{+};\pi^{\intercal}$ and $(\mathbb{I} \otimes \mathbb{S}^{*})$ of the pair order PO; in the lower line also represented with Cantor enumeration

One should remember that $(\mathbb{S}^{*} \otimes \mathbb{S}^{*})$ cannot be used since it looks like this:

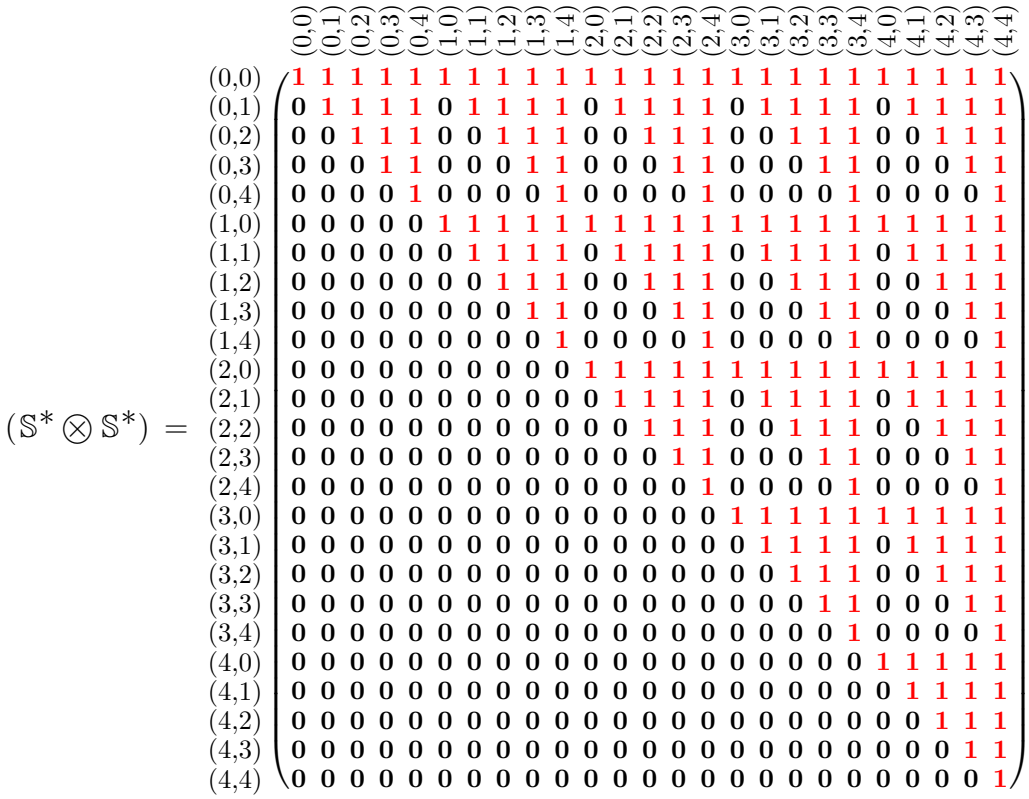


Fig. 3.1.8 Non-linear ordering of pairs $(\mathbb{S}^* \otimes \mathbb{S}^*)$

Also for binary relational operations, it is possible to ask for points and matrix entries and their interdependence.

Given the Point Axiom, a binary relational operation $R : X \times Y \longrightarrow Z$ has a representation as

$$R = \mathtt{sup}\left\{ (x \otimes y) ; z^\mathsf{T} \mid (x \otimes y) ; z^\mathsf{T} \subseteq R \mid x \in \mathtt{pts}(X), y \in \mathtt{pts}(Y), z \in \mathtt{pts}(Z) \right\}.$$

This simplifies, in case R is univalent and delivers $R^\mathsf{T} ; (x \otimes y)$ for the argument $(x \otimes y)$, to

$$\begin{array}{l} \mathtt{sup}\left\{ (x \otimes y) ; (x \otimes y)^\mathsf{T} ; R \mid (x \otimes y) \subseteq R ; \mathbb{T} \mid x \in \mathtt{pts}(X), y \in \mathtt{pts}(Y) \right\} \\ = \mathtt{sup}\left\{ (x \otimes y) ; (x \otimes y)^\mathsf{T} \mid (x \otimes y) \subseteq R ; \mathbb{T} \mid x \in \mathtt{pts}(X), y \in \mathtt{pts}(Y) \right\} ; R \\ = \mathtt{sup}\left\{ (x ; x^\mathsf{T} \otimes y ; y^\mathsf{T}) \mid (x \otimes y) \subseteq R ; \mathbb{T} \mid x \in \mathtt{pts}(X), y \in \mathtt{pts}(Y) \right\} ; R \quad \text{Prop. 2.1.1.ii} \\ = (\mathbb{I} \cap R ; \mathbb{T}) ; R = \mathbb{I} ; R \cap R ; \mathbb{T} = R. \end{array}$$

3.2 Inverses

The basic idea of right-inversion is easily captured when considering the right side of Fig. 3.2.1, with R the addition $\mathbb{A}$ and $R^{\leftarrowtail}$ the corresponding subtraction $\mathfrak{S}$.

3.2.1 Definition. Given any binary relational operation $R : X \times Y \longrightarrow Z$ with projections $\pi : X \times Y \longrightarrow X$ and $\rho : X \times Y \longrightarrow Y$ and, correspondingly π', ρ', we define its **right-** resp. **left-inverse** as

$$R^{\leftarrowtail} := (R^\mathsf{T} \otimes \rho^\mathsf{T}) ; \pi \qquad {}^{\rightarrowtail}R := (\pi^\mathsf{T} \otimes R^\mathsf{T}) ; \rho. \qquad \square$$

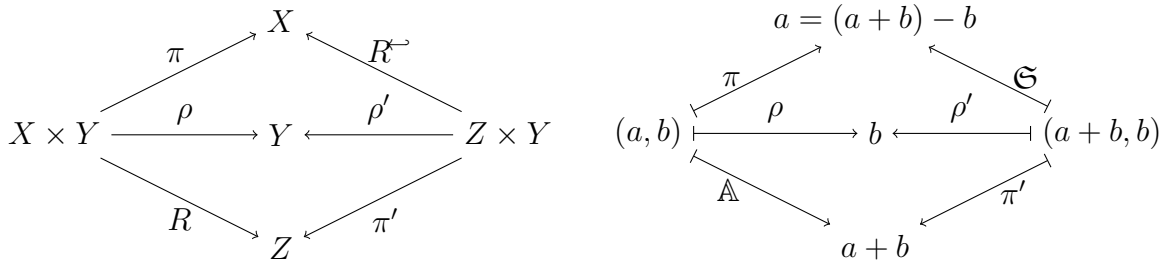


Fig. 3.2.1 Right-inverting a binary relational operation

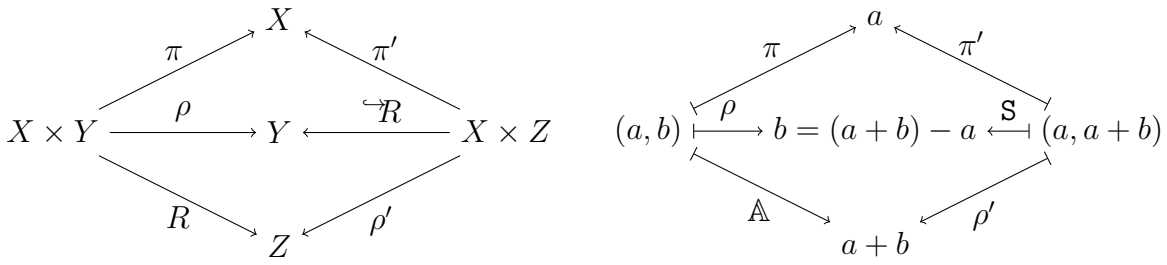


Fig. 3.2.2 Left-inverting a binary relational operation

Note, that no conditions whatsoever are postulated for R in the preceding definition as, e.g., being univalent or being a mapping. All the more surprising is the following result. It establishes — in the first case — a bijection between the relations of type $X \times Y \longrightarrow Z$ and $Z \times Y \longrightarrow X$. It is at least clear that both relation sets have — when finite — the same number of elements, namely $2^{|X|\cdot|Y|\cdot|Z|}$. For illustrating examples see the forthcoming Figs. 8.1.6 and 8.1.7.

3.2.2 Proposition. The inversions $R \mapsto R^{\leftarrowtail}$ and $R \mapsto {}^{\hookrightarrow}R$ of binary relational operations are involutions,

$$R = [R^{\leftarrowtail}]^{\leftarrowtail} \qquad R = {}^{\hookrightarrow}[{}^{\hookrightarrow}R].$$

Proof: We start the proof with an abbreviation after having expanded the right-hand side

$$[R^{\leftarrowtail}]^{\leftarrowtail} = [\,(R^\mathsf{T} \circledcirc \rho^\mathsf{T}) ; \pi]^{\leftarrowtail} = ([\,(R^\mathsf{T} \circledcirc \rho^\mathsf{T}) ; \pi]^\mathsf{T} \circledcirc \rho'^\mathsf{T}) ; \pi' = (\pi^\mathsf{T} ; (R \circledcirc \rho) \circledcirc \rho'^\mathsf{T}) ; \pi'$$
$$= \big(\pi ; \pi^\mathsf{T} ; (R \circledcirc \rho) \cap \rho ; \rho'^\mathsf{T}\big) ; \pi' = \big(\pi ; \pi^\mathsf{T} ; (R ; \pi'^\mathsf{T} \cap \rho ; \rho'^\mathsf{T}) \cap \rho ; \rho'^\mathsf{T}\big) ; \pi' =: A ; \pi'.$$

Direction "$\supseteq$":

$$A ; \pi' \subseteq R \quad \Longleftrightarrow \quad A \subseteq R ; \pi'^\mathsf{T} \quad \text{shunting}$$

To A on the left side, we apply the Dedekind rule in

$$A \subseteq \big(\pi ; \pi^\mathsf{T} \cap \rho ; \rho'^\mathsf{T} ; (R ; \pi'^\mathsf{T} \cap \rho ; \rho'^\mathsf{T})^\mathsf{T}\big) ; \big((R ; \pi'^\mathsf{T} \cap \rho ; \rho'^\mathsf{T}) \cap \pi ; \pi^\mathsf{T} ; \rho ; \rho'^\mathsf{T}\big)$$
$$\subseteq (\pi ; \pi^\mathsf{T} \cap \rho ; \rho'^\mathsf{T} ; \rho' ; \rho^\mathsf{T}) ; R ; \pi'^\mathsf{T}$$
$$\subseteq (\pi ; \pi^\mathsf{T} \cap \rho ; \rho^\mathsf{T}) ; R ; \pi'^\mathsf{T} \quad \text{since } \rho' \text{ is univalent}$$
$$= \mathbb{I} ; R ; \pi'^\mathsf{T} \quad \text{direct product property of } \pi, \rho$$

Direction "$\subseteq$":

$$R = R \cap \top = R \cap \rho ; \rho'^\mathsf{T} ; \pi' = (R ; \pi'^\mathsf{T} \cap \rho ; \rho'^\mathsf{T}) ; \pi' = \big((R ; \pi'^\mathsf{T} \cap \rho ; \rho'^\mathsf{T}) \cap \rho ; \rho'^\mathsf{T}\big) ; \pi'$$
$$\subseteq (\pi ; \pi^\mathsf{T} ; (R ; \pi'^\mathsf{T} \cap \rho ; \rho'^\mathsf{T}) \cap \rho ; \rho'^\mathsf{T}) ; \pi' = A ; \pi' \quad \text{here } \pi ; \pi^\mathsf{T} \text{ is obviously an equivalence}$$ □

One will recognize that this is a generalization of the bijective mechanism of Prop. 2.2.5

$$v \mapsto R := \mathtt{rel}(v) = \pi^\mathsf{T} ; (\rho \cap v ; \mathbb{T}) \quad \Longleftrightarrow \quad R \mapsto v := \mathtt{vec}(R) = (\pi ; R \cap \rho) ; \mathbb{T} = (R \oslash \mathbb{I}) ; \mathbb{T}$$

in as far as

$$v : X \times Y \longrightarrow \mathbb{1} \quad \text{and} \quad v^{\hookleftarrow} = R^\mathsf{T} : \mathbb{1} \times Y \longrightarrow X \quad \text{where } \mathbb{1} \times Y \approx Y \quad \text{so that}$$
$$R^\mathsf{T} : \mathbb{1} \times Y \longrightarrow X \quad \text{and} \quad (R^\mathsf{T})^{\hookleftarrow} = v : X \times Y \longrightarrow \mathbb{1} \quad \text{where again } \mathbb{1} \times Y \approx Y$$

We have quite often developed a relational counterpart out of a predicate-logic formulation in some semi-formal way. For this purpose, Prop. 3.2.2 is best suited: It allows — preserving the meaning — to switch from

$$\forall (x, y) \in X \times Y : \forall z \in Z : R_{(x,y),z}$$

to

$$\forall (z, y) \in Z \times Y : \forall x \in X : R^{\hookleftarrow}_{(z,y),x}.$$

In the following, we find us again limited by the Point Axiom when investigating how far inverses respect the usual operations. Only one of the symmetric variants is proved in detail.

3.2.3 Proposition. Let be given relations $R, S : X \times Y \longrightarrow Z$. Then

i) $\quad {\hookrightarrow}R = \mathfrak{P}' ; (\mathfrak{P} ; R)^{\hookleftarrow} \qquad R^{\hookleftarrow} = \mathfrak{P}'' ; {\hookrightarrow}(\mathfrak{P} ; R)$

ii) $\quad {\hookrightarrow}(\overline{R}) = \overline{{\hookrightarrow}R} \qquad (\overline{R})^{\hookleftarrow} = \overline{R^{\hookleftarrow}}$

iii) $\quad {\hookrightarrow}(R \cup S) = {\hookrightarrow}R \cup {\hookrightarrow}S \qquad (R \cup S)^{\hookleftarrow} = R^{\hookleftarrow} \cup S^{\hookleftarrow}$

$\quad {\hookrightarrow}(R \cap S) = {\hookrightarrow}R \cap {\hookrightarrow}S \qquad (R \cap S)^{\hookleftarrow} = R^{\hookleftarrow} \cap S^{\hookleftarrow}$

$\quad {\hookrightarrow}(R ; S) \supseteq (\mathbb{I} \otimes S^\mathsf{T}) ; {\hookrightarrow}R \qquad (R ; S)^{\hookleftarrow} \supseteq (S^\mathsf{T} \otimes \mathbb{I}) ; R^{\hookleftarrow}$ holds in general

$\quad {\hookrightarrow}(R ; S) = (\mathbb{I} \otimes S^\mathsf{T}) ; {\hookrightarrow}R \qquad (R ; S)^{\hookleftarrow} = (S^\mathsf{T} \otimes \mathbb{I}) ; R^{\hookleftarrow}$ if S injective or Point Axiom holds

iv) $\quad {\hookrightarrow}R \subseteq {\hookrightarrow}S \iff R \subseteq S \iff R^{\hookleftarrow} \subseteq S^{\hookleftarrow}$

v) $\quad (A^\mathsf{T} \oslash \rho^\mathsf{T}) = (A^{\hookleftarrow} \oslash \rho') \qquad (\pi^\mathsf{T} \oslash A^\mathsf{T}) = (\pi' \oslash {\hookrightarrow}A)$

The following are versions that do not explicitly refer to differently typed projections:

$$(A^\mathsf{T} \oslash (\mathbb{T} \oslash \mathbb{I})) = (A^{\hookleftarrow} \oslash (\mathbb{T} \oslash \mathbb{I})) \qquad ((\mathbb{I} \oslash \mathbb{T}) \oslash A^\mathsf{T}) = ((\mathbb{I} \oslash \mathbb{T}) \oslash {\hookrightarrow}A)$$

vi) $\quad (R \oslash S)^{\hookleftarrow} = \pi ; R^\mathsf{T} \cap \mathtt{vec}(S^\mathsf{T}) \qquad (v ; \mathbb{T} \oslash S)^{\hookleftarrow} = \mathtt{vec}(S^\mathsf{T}) ; \mathbb{T} ; v^\mathsf{T}$

$\quad {\hookrightarrow}(R \oslash S) = \mathtt{vec}(R^\mathsf{T}) \cap \rho ; S^\mathsf{T} \qquad {\hookrightarrow}(R \oslash w ; \mathbb{T}) = \mathtt{vec}(R^\mathsf{T}) ; \mathbb{T} ; w^\mathsf{T}$

vii) $\quad \pi'^\mathsf{T} ; {\hookrightarrow}R = \mathtt{rel}(R ; \mathbb{T}) \qquad \rho^\mathsf{T} ; R^{\hookleftarrow} = \big[\mathtt{rel}(R ; \mathbb{T})\big]^\mathsf{T}$

$\quad \rho'^\mathsf{T} ; {\hookrightarrow}R = R^\mathsf{T} ; \rho \qquad \pi'^\mathsf{T} ; R^{\hookleftarrow} = R^\mathsf{T} ; \pi$

viii) $\quad [{\hookrightarrow}R]^{\hookleftarrow} = \mathfrak{P} ; R^{\hookleftarrow} \qquad \mathtt{vec}({\hookrightarrow}R) = \mathfrak{T}_{21 \to 12} ; (\mathbb{I} \otimes \mathfrak{P}) ; \mathfrak{T}_{12 \to 21} ; \mathtt{vec}(R)$

$\quad {\hookrightarrow}[R^{\hookleftarrow}] = \mathfrak{P} ; {\hookrightarrow}R \qquad {\hookrightarrow}R = \mathtt{rel}(\mathfrak{T}_{21 \to 12} ; (\mathbb{I} \otimes \mathfrak{P}) ; \mathfrak{T}_{12 \to 21} ; \mathtt{vec}(R))$

If A is commutative, it will satisfy ${\hookrightarrow}A = \mathfrak{P} ; A^{\hookleftarrow}$.

ix) $\hookrightarrow([\hookrightarrow R]^{\hookleftarrow}) = \mathfrak{P} ; R = (\hookrightarrow[R^{\hookleftarrow}])^{\hookleftarrow}$

Proof: i) We are careful to distinguish the permutation operators as

$\mathfrak{P} : Y \times X \longrightarrow X \times Y \qquad \mathfrak{P}' : X \times Z \longrightarrow Z \times X \qquad \mathfrak{P}'' : Z \times Y \longrightarrow Y \times Z$

and prove the first of the two formulae as follows:

$\mathfrak{P}' ; (\mathfrak{P} ; R)^{\hookleftarrow} = \mathfrak{P}' ; ((\mathfrak{P} ; R)^{\mathsf{T}} \circledcirc \rho^{\mathsf{T}}) ; \pi = \mathfrak{P}' ; (R^{\mathsf{T}} ; \mathfrak{P}^{\mathsf{T}} \circledcirc \rho^{\mathsf{T}}) ; \pi = (\rho^{\mathsf{T}} \circledcirc R^{\mathsf{T}} ; \mathfrak{P}^{\mathsf{T}}) ; \pi$
$= (\rho^{\mathsf{T}} ; \mathfrak{P} ; \mathfrak{P}^{\mathsf{T}} \circledcirc R^{\mathsf{T}} ; \mathfrak{P}^{\mathsf{T}}) ; \pi = (\rho^{\mathsf{T}} ; \mathfrak{P} \circledcirc R^{\mathsf{T}}) ; \mathfrak{P}^{\mathsf{T}} ; \pi = (\pi^{\mathsf{T}} \circledcirc R^{\mathsf{T}}) ; \rho = {}^{\hookrightarrow}R$

---

ii) This proof is still far from elegant!!!

$\mathbb{T} = \rho' ; \mathbb{T} = \rho' ; \rho^{\mathsf{T}} ; \pi$ since ρ' is total and π, ρ a direct product
$= \left[\pi' ; \mathbb{T} \cap \rho' ; \rho^{\mathsf{T}}\right] ; \pi$ because π' is total
$= \left[(\pi' ; R^{\mathsf{T}} \cup \pi' ; \overline{R}^{\mathsf{T}}) \cap \rho' ; \rho^{\mathsf{T}}\right] ; \pi$
$= \left[(\pi' ; R^{\mathsf{T}} \cap \rho' ; \rho^{\mathsf{T}}) \cup (\pi' ; \overline{R}^{\mathsf{T}} \cap \rho' ; \rho^{\mathsf{T}})\right] ; \pi$
$= (\pi' ; R^{\mathsf{T}} \cap \rho' ; \rho^{\mathsf{T}}) ; \pi \cup (\pi' ; \overline{R}^{\mathsf{T}} \cap \rho' ; \rho^{\mathsf{T}}) ; \pi$
$= (R^{\mathsf{T}} \circledcirc \rho^{\mathsf{T}}) ; \pi \cup (\overline{R}^{\mathsf{T}} \circledcirc \rho^{\mathsf{T}}) ; \pi = R^{\hookleftarrow} \cup (\overline{R})^{\hookleftarrow}$

so that $\overline{R^{\hookleftarrow}} \subseteq (\overline{R})^{\hookleftarrow}$. It remains to show $\overline{R^{\hookleftarrow}} \supseteq (\overline{R})^{\hookleftarrow}$, i.e.:

$(\overline{R}^{\mathsf{T}} \circledcirc \rho^{\mathsf{T}}) ; \pi \subseteq \overline{(R^{\mathsf{T}} \circledcirc \rho^{\mathsf{T}}) ; \pi}$
$\iff \quad (\overline{R}^{\mathsf{T}} \circledcirc \rho^{\mathsf{T}}) \subseteq \overline{(R^{\mathsf{T}} \circledcirc \rho^{\mathsf{T}}) ; \pi ; \pi^{\mathsf{T}}}$ shunting the map π that slips below negation
$\iff \quad (R^{\mathsf{T}} \circledcirc \rho^{\mathsf{T}}) ; \pi ; \pi^{\mathsf{T}} \subseteq \overline{(\overline{R}^{\mathsf{T}} \circledcirc \rho^{\mathsf{T}})} = \overline{\pi' ; \overline{R}^{\mathsf{T}} \cap \rho' ; \rho^{\mathsf{T}}} = \pi' ; R^{\mathsf{T}} \cup \overline{\rho' ; \rho^{\mathsf{T}}}$ negated
$\iff \quad (\pi' ; R^{\mathsf{T}} \cap \rho' ; \rho^{\mathsf{T}}) ; \pi ; \pi^{\mathsf{T}} \cap \rho' ; \rho^{\mathsf{T}} \subseteq \pi' ; R^{\mathsf{T}}$

This can indeed be proved employing the Dedekind rule and properties of a direct product:

$(\pi' ; R^{\mathsf{T}} \cap \rho' ; \rho^{\mathsf{T}}) ; \pi ; \pi^{\mathsf{T}} \cap \rho' ; \rho^{\mathsf{T}} \subseteq ((\pi' ; R^{\mathsf{T}} \cap \rho' ; \rho^{\mathsf{T}}) \cap \rho' ; \rho^{\mathsf{T}} ; \pi ; \pi^{\mathsf{T}}) ; (\pi ; \pi^{\mathsf{T}} \cap (R ; \pi'^{\mathsf{T}} \cap \rho ; \rho'^{\mathsf{T}}) ; \rho' ; \rho^{\mathsf{T}})$
$= ((\pi' ; R^{\mathsf{T}} \cap \rho' ; \rho^{\mathsf{T}}) \cap \mathbb{T}) ; (\pi ; \pi^{\mathsf{T}} \cap (R ; \pi'^{\mathsf{T}} ; \rho' \cap \rho) ; \rho^{\mathsf{T}})$ since $\rho^{\mathsf{T}} ; \pi = \mathbb{T}$; destroy and append
$\subseteq (\pi' ; R^{\mathsf{T}} \cap \rho' ; \rho^{\mathsf{T}}) ; (\pi ; \pi^{\mathsf{T}} \cap \rho ; \rho^{\mathsf{T}}) = (\pi' ; R^{\mathsf{T}} \cap \rho' ; \rho^{\mathsf{T}}) ; \mathbb{I} \subseteq \pi' ; R^{\mathsf{T}}$

---

iii) $(R \cup S)^{\hookleftarrow} = (R^{\mathsf{T}} \cup S^{\mathsf{T}} \circledcirc \rho^{\mathsf{T}}) ; \pi$ by definiton
$= \big((R^{\mathsf{T}} \circledcirc \rho^{\mathsf{T}}) \cup (S^{\mathsf{T}} \circledcirc \rho^{\mathsf{T}}) \big) ; \pi$
$= (R^{\mathsf{T}} \circledcirc \rho^{\mathsf{T}}) ; \pi \cup (S^{\mathsf{T}} \circledcirc \rho^{\mathsf{T}}) ; \pi = R^{\hookleftarrow} \cup S^{\hookleftarrow}$

Left inversion for "$\cup$" is proved analogously.

The case "$\cap$" is handled by a Boolean manipulation:

$(R \cap S)^{\hookleftarrow} = \left[\overline{\overline{R} \cup \overline{S}}\right]^{\hookleftarrow} = \overline{(\overline{R} \cup \overline{S})^{\hookleftarrow}} = \overline{(\overline{R})^{\hookleftarrow} \cup (\overline{S})^{\hookleftarrow}} = \overline{\overline{R^{\hookleftarrow}} \cup \overline{S^{\hookleftarrow}}} = R^{\hookleftarrow} \cap S^{\hookleftarrow}$

Right inversion of a product doesn't allow an easy formula. Again, we encounter that the result depends on the Point Axiom:

$(R ; S)^{\hookleftarrow} = (S^{\mathsf{T}} ; R^{\mathsf{T}} \circledcirc \rho^{\mathsf{T}}) ; \pi$
$\supseteq (S^{\mathsf{T}} \otimes \mathbb{I}) ; (R^{\mathsf{T}} \circledcirc \rho^{\mathsf{T}}) ; \pi$ see [9, Prop. 7.5.ii]
$= (S^{\mathsf{T}} \otimes \mathbb{I}) ; R^{\hookleftarrow}$

with equality in case S should be injective — or the Point Axiom is assumed.

---

iv) $R \subseteq S \implies (R^{\mathsf{T}} \circledcirc \rho^{\mathsf{T}}) ; \pi \subseteq (S^{\mathsf{T}} \circledcirc \rho^{\mathsf{T}}) ; \pi$, i.e. $R^{\hookleftarrow} \subseteq S^{\hookleftarrow}$

The reverse implication follows via Prop. 3.2.2.

---

v) Carefully obeying the typing, we expand both sides:

$(A \circledcirc \rho)^\intercal = (A^\intercal \circledcirc \rho^\intercal) = \pi' ; A^\intercal \cap \rho' ; \rho^\intercal$

$(A^{\hookleftarrow} \circledcirc \rho') = ((A^\intercal \circledcirc \rho^\intercal) ; \pi \circledcirc \rho') = (\pi' ; A^\intercal \cap \rho' ; \rho^\intercal) ; \pi ; \pi^\intercal \cap \rho' ; \rho^\intercal$

Now, trivially, the first term is contained in the second. It takes more effort to convince us from the reverse containment via the Dedekind rule:

$(\pi' ; A^\intercal \cap \rho' ; \rho^\intercal) ; \pi ; \pi^\intercal \cap \rho' ; \rho^\intercal \subseteq \big((\pi' ; A^\intercal \cap \rho' ; \rho^\intercal) \cap \rho' ; \rho^\intercal ; \pi ; \pi^\intercal\big) ; \big(\pi ; \pi^\intercal \cap (\pi' ; A^\intercal \cap \rho' ; \rho^\intercal)^\intercal ; \rho' ; \rho^\intercal\big)$

$\subseteq (\pi' ; A^\intercal \cap \rho' ; \rho^\intercal) ; \big(\pi ; \pi^\intercal \cap \rho ; \rho'^\intercal ; \rho' ; \rho^\intercal\big)$ since $\rho^\intercal ; \pi = \top$

$= (\pi' ; A^\intercal \cap \rho' ; \rho^\intercal) ; \big(\pi ; \pi^\intercal \cap \rho ; \rho^\intercal\big)$ since $\rho'^\intercal ; \rho' = \mathbb{I}$

$= \pi' ; A^\intercal \cap \rho' ; \rho^\intercal$ since π, ρ form a direct product

---

vi) $(R \circledcirc S)^{\hookleftarrow} = ((R \circledcirc S)^\intercal \circledcirc \rho^\intercal) ; \pi = ((R^\intercal \circledcirc S^\intercal) \circledcirc \rho^\intercal) ; \pi$

$= (R^\intercal ; \pi^\intercal \cap S^\intercal ; \rho^\intercal \circledcirc \rho^\intercal) ; \pi$

$= \big[\pi' ; (R^\intercal ; \pi^\intercal \cap S^\intercal ; \rho^\intercal) \cap \rho' ; \rho^\intercal\big] ; \pi = \big[\pi' ; R^\intercal ; \pi^\intercal \cap (\pi' ; S^\intercal \cap \rho') ; \rho^\intercal\big] ; \pi$

$= \pi' ; R^\intercal \cap (\pi' ; S^\intercal \cap \rho') ; \top = \pi' ; R^\intercal \cap (S^\intercal \circledcirc \mathbb{I}) ; \top$ destroy and append

$= \pi' ; R^\intercal \cap \mathtt{vec}(S^\intercal) = (\pi' \cap \mathtt{vec}(S^\intercal)) ; R^\intercal$

Proof variant with projection migration:

$(R \circledcirc S)^{\hookleftarrow} = ((R \circledcirc S)^\intercal \circledcirc \rho^\intercal) ; \pi = ((R^\intercal \circledcirc S^\intercal) \circledcirc \rho^\intercal) ; \pi$

$= (\pi' ; R^\intercal \circledcirc (S^\intercal \circledcirc \mathbb{I})) ; \pi$ projection migration according to Prop. 3.1.2.ii

$= \pi' ; R^\intercal \cap (S^\intercal \circledcirc \mathbb{I}) ; \top = \pi' ; R^\intercal \cap \mathtt{vec}(S^\intercal)$ destroy and append

$= (\pi' \cap \mathtt{vec}(S^\intercal)) ; R^\intercal$

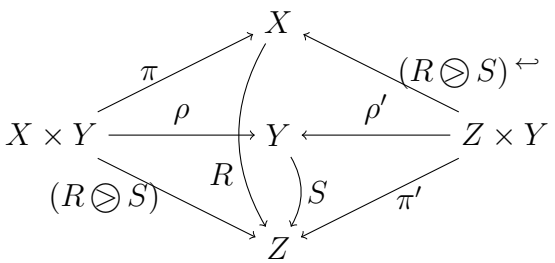


Fig. 3.2.3 Specialization of Fig. 3.2.1, helping to understand the typing of (vi)

$(v ; \mathbb{T} \circledcirc S)^{\hookleftarrow} = ((v ; \mathbb{T} \circledcirc S)^\intercal \circledcirc \rho^\intercal) ; \pi = ((\mathbb{T} ; v^\intercal \circledcirc S^\intercal) \circledcirc \rho^\intercal) ; \pi$

$= (\mathbb{T} ; v^\intercal ; \pi^\intercal \cap S^\intercal ; \rho^\intercal \circledcirc \rho^\intercal) ; \pi$

$= \big[\pi' ; (\mathbb{T} ; v^\intercal ; \pi^\intercal \cap S^\intercal ; \rho^\intercal) \cap \rho' ; \rho^\intercal\big] ; \pi = \big[\mathbb{T} ; v^\intercal ; \pi^\intercal \cap (\pi' ; S^\intercal \cap \rho') ; \rho^\intercal\big] ; \pi$

$= \mathbb{T} ; v^\intercal \cap (\pi' ; S^\intercal \cap \rho') ; \top$ destroy and append

$= \mathbb{T} ; v^\intercal \cap \mathtt{vec}(S^\intercal) ; \top = \mathtt{vec}(S^\intercal) ; \mathbb{T} ; v^\intercal$

Proof variant with projection migration:

$(v ; \mathbb{T} \circledcirc S)^{\hookleftarrow} = ((v ; \mathbb{T} \circledcirc S)^\intercal \circledcirc \rho^\intercal) ; \pi = ((\mathbb{T} ; v^\intercal \circledcirc S^\intercal) \circledcirc \rho^\intercal) ; \pi$

$= (\mathbb{T} ; v^\intercal \circledcirc (S^\intercal \circledcirc \mathbb{I})) ; \pi$ projection migration with π total

$= \mathbb{T} ; v^\intercal \cap (S^\intercal \circledcirc \mathbb{I}) ; \top$ destroy and append

$= \mathbb{T} ; v^\intercal \cap \mathtt{vec}(S^\intercal) ; \top = \mathtt{vec}(S^\intercal) ; \mathbb{T} ; v^\intercal$

---

vii) $\pi'^{\mathsf{T}};\overset{\hookrightarrow}{R} = \pi'^{\mathsf{T}};(\pi^{\mathsf{T}} \circledS R^{\mathsf{T}});\rho = \pi'^{\mathsf{T}};(\pi';\pi^{\mathsf{T}} \cap \rho';R^{\mathsf{T}});\rho$ by definition
$= (\pi^{\mathsf{T}} \cap \pi'^{\mathsf{T}};\rho';R^{\mathsf{T}});\rho = (\pi^{\mathsf{T}} \cap \mathbb{T};R^{\mathsf{T}});\rho$ destroy and append
$= \pi^{\mathsf{T}};(\rho \cap R;\mathbb{T})$ mask shifting
$= \mathtt{rel}(R;\mathbb{T})$

$\rho'^{\mathsf{T}};\overset{\hookleftarrow}{R} = \rho'^{\mathsf{T}};(R^{\mathsf{T}} \circledS \rho^{\mathsf{T}});\pi = \rho'^{\mathsf{T}};(\pi';R^{\mathsf{T}} \cap \rho';\rho^{\mathsf{T}});\pi$ by definition
$= (\mathbb{T};R^{\mathsf{T}} \cap \rho^{\mathsf{T}});\pi = \big[\pi^{\mathsf{T}};(\rho \cap R;\mathbb{T})\big]^{\mathsf{T}} = \big[\mathtt{rel}(R;\mathbb{T})\big]^{\mathsf{T}}$

$\rho'^{\mathsf{T}};\overset{\hookrightarrow}{R} = \rho'^{\mathsf{T}};(\pi^{\mathsf{T}} \circledS R^{\mathsf{T}});\rho = \rho'^{\mathsf{T}};(\pi';\pi^{\mathsf{T}} \cap \rho';R^{\mathsf{T}});\rho = (\mathbb{T} \cap R^{\mathsf{T}});\rho = R^{\mathsf{T}};\rho$

$\pi'^{\mathsf{T}};\overset{\hookleftarrow}{R} = \pi'^{\mathsf{T}};(R^{\mathsf{T}} \circledS \rho^{\mathsf{T}});\pi = \pi'^{\mathsf{T}};(\pi';R^{\mathsf{T}} \cap \rho';\rho^{\mathsf{T}});\pi = (R^{\mathsf{T}} \cap \mathbb{T});\pi = R^{\mathsf{T}};\pi$

---

viii) In order to prove $\big[\overset{\hookrightarrow}{R}\big]^{\hookleftarrow} = \mathfrak{P};\overset{\hookleftarrow}{R}$, we observe which permutations happen. Assuming as original typing $R : X \times Y \longrightarrow Z$, we follow construction of the typing of the left term stepwise

$\overset{\hookrightarrow}{R} : X \times Z \longrightarrow Y \qquad \big[\overset{\hookrightarrow}{R}\big]^{\hookleftarrow} : Y \times Z \longrightarrow X$

and correspondingly the right one

$\overset{\hookleftarrow}{R} : Z \times Y \longrightarrow X \qquad \mathfrak{P};\overset{\hookleftarrow}{R} : Y \times Z \longrightarrow X.$

This is remodelled with brace shuffling, applied to the respective vectorizations — during which process nothing happens to the relation R in question:

$$\begin{aligned}
\mathtt{vec}(\overset{\hookrightarrow}{R}) &: (X \times Z) \times Y \longrightarrow \mathbb{1} \\
\\
\mathtt{vec}(R) &: (X \times Y) \times Z \longrightarrow \mathbb{1} \\
\mathfrak{T}_{12\to 21};\mathtt{vec}(R) &: X \times (Y \times Z) \longrightarrow \mathbb{1} \\
(\mathbb{I} \otimes \mathfrak{P});\mathfrak{T}_{12\to 21};\mathtt{vec}(R) &: X \times (Z \times Y) \longrightarrow \mathbb{1} \\
\mathfrak{T}_{21\to 12};(\mathbb{I} \otimes \mathfrak{P});\mathfrak{T}_{12\to 21};\mathtt{vec}(R) &: (X \times Z) \times Y \longrightarrow \mathbb{1}
\end{aligned}$$

In total therefore

$\mathtt{vec}(\overset{\hookrightarrow}{R}) = \mathfrak{T}_{21\to 12};(\mathbb{I} \otimes \mathfrak{P});\mathfrak{T}_{12\to 21};\mathtt{vec}(R)$

which to prove in our usual style seems forbiddingly complicated and is for now postponed. (It will probably require the Point Axiom.)

$\mathfrak{P};\overset{\hookrightarrow}{R} = \mathfrak{P};(\pi^{\mathsf{T}} \circledS R^{\mathsf{T}});\rho = (R^{\mathsf{T}} \circledS \pi^{\mathsf{T}});\rho$
$\overset{\hookrightarrow}{[R^{\hookleftarrow}]} = (\pi'^{\mathsf{T}} \circledS (R^{\hookleftarrow})^{\mathsf{T}});\rho'$

Due to a former version, we present here the proof of equality for the transposed terms. For reasons of symmetry in view of Prop. 3.2.2, we prove just one direction:

$\rho'^{\mathsf{T}};(\pi' \circledS R^{\hookleftarrow}) \subseteq \rho^{\mathsf{T}};(R \circledS \pi) \qquad \Longleftrightarrow \qquad (\pi' \circledS R^{\hookleftarrow}) \subseteq \rho';\rho^{\mathsf{T}};(R \circledS \pi)$ shunting

Now we apply the Dedekind rule to the left side

$(\pi' \circledS R^{\hookleftarrow}) = (\pi' \circledS (R^{\mathsf{T}} \circledS \rho^{\mathsf{T}});\pi) = \pi';\pi''^{\mathsf{T}} \cap (R^{\mathsf{T}} \circledS \rho^{\mathsf{T}});\pi;\rho''^{\mathsf{T}}$
$= \pi';\pi''^{\mathsf{T}} \cap (\pi';R^{\mathsf{T}} \cap \rho';\rho^{\mathsf{T}});\ \pi;\rho''^{\mathsf{T}}$
$\subseteq \big[(\pi';R^{\mathsf{T}} \cap \rho';\rho^{\mathsf{T}}) \cap \pi';\pi''^{\mathsf{T}};\ \rho'';\pi^{\mathsf{T}}\big];\big[\pi;\rho''^{\mathsf{T}} \cap (R;\pi'^{\mathsf{T}} \cap \rho;\rho'^{\mathsf{T}});\pi';\pi''^{\mathsf{T}}\big]$ Dedekind rule
$\subseteq \rho';\rho^{\mathsf{T}};\big[\pi;\rho''^{\mathsf{T}} \cap (R;\pi'^{\mathsf{T}} \cap \rho;\rho'^{\mathsf{T}});\ \pi';\pi''^{\mathsf{T}}\big]$
$= \rho';\rho^{\mathsf{T}};\big[\pi;\rho''^{\mathsf{T}} \cap (R \cap \rho;\rho'^{\mathsf{T}};\pi');\pi''^{\mathsf{T}}\big]$ destroy and append
$= \rho';\rho^{\mathsf{T}};\big[\pi;\rho''^{\mathsf{T}} \cap R;\pi''^{\mathsf{T}}\big]$
$= \rho';\rho^{\mathsf{T}};(R \circledS \pi)$

If A is commutative:

$$\overset{\hookrightarrow}{A} = (\pi^{\intercal} \circledcirc A^{\intercal}) ; \rho = (\pi^{\intercal} \circledcirc A^{\intercal}) ; \mathfrak{P} ; \pi = (\pi^{\intercal} ; \mathfrak{P} \circledcirc A^{\intercal} ; \mathfrak{P}) ; \pi$$
$$= (\rho^{\intercal} \circledcirc A^{\intercal}) ; \pi = \mathfrak{P} ; (A^{\intercal} \circledcirc \rho^{\intercal}) ; \pi = \mathfrak{P} ; \overset{\hookleftarrow}{A}$$

---

ix) Using (viii), we have

$$\overset{\hookrightarrow}{([\overset{\hookrightarrow}{R}]^{\hookleftarrow})} = \mathfrak{P} ; \overset{\hookrightarrow}{[\overset{\hookrightarrow}{R}]} = \mathfrak{P} ; R = \mathfrak{P} ; [\overset{\hookleftarrow}{R}]^{\hookleftarrow} = (\overset{\hookrightarrow}{[\overset{\hookleftarrow}{R}]})^{\hookleftarrow}$$ □

There might easily arise the question as to a commutation rule for operations "$.^{\intercal}$" and "$.^{\hookleftarrow}$". In general $R^{\intercal}$ is not a binary relational operation when R is; however, it may be. In this case, indeed by definition $(R^{\intercal})^{\hookleftarrow} = (R \circledcirc \rho^{\intercal}) ; \pi = ((R^{\intercal})^{\hookleftarrow} \circledcirc \rho') ; \pi$ using (v); and this holds indeed via destroy and append.

In general, there exist just a few possibilities to modify a binary relational operation iterating the three operators

$$R \mapsto \overset{\hookrightarrow}{R} \qquad R \mapsto \overset{\hookleftarrow}{R} \qquad R \mapsto \mathfrak{P} ; R$$

corresponding to the permutations

$$(X, Y, Z) \mapsto (X, Z, Y) \qquad (X, Y, Z) \mapsto (Z, Y, X) \qquad (X, Y, Z) \mapsto (Y, X, Z),$$

that are sufficient to generate all the permutations on three elements. Right inversion leaves the right argument fixed and exchanges the left one with the target. Left inversion leaves the left argument fixed and exchanges the right one with the target. This is purely combinatorial; the relation proper is unaffected:

$$R : X \times Y \longrightarrow Z$$

$$\overset{\hookrightarrow}{R} : X \times Z \longrightarrow Y \qquad\qquad \overset{\hookleftarrow}{R} : Z \times Y \longrightarrow X$$

$$[\overset{\hookrightarrow}{R}]^{\hookleftarrow} : Y \times Z \longrightarrow X \qquad\qquad \overset{\hookrightarrow}{[\overset{\hookleftarrow}{R}]} : Z \times X \longrightarrow Y$$

$$\overset{\hookrightarrow}{([\overset{\hookrightarrow}{R}]^{\hookleftarrow})} : Y \times X \longrightarrow Z \quad \approx \quad (\mathfrak{P} ; R) : Y \times X \longrightarrow Z \quad \approx \quad (\overset{\hookrightarrow}{[\overset{\hookleftarrow}{R}]})^{\hookleftarrow} : Y \times X \longrightarrow Z$$

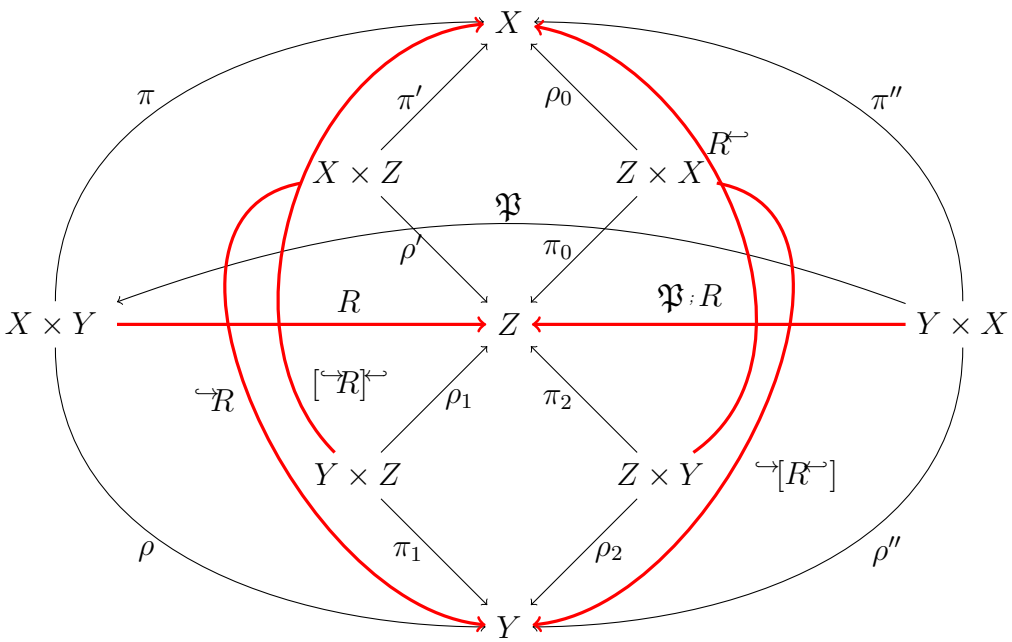


Fig. 3.2.4 Illustrating all 6 iterated applications of inversion

All these purely combinatorial effects are now visualized as matrices:

$$R = \begin{array}{c} \\ (\mathrm{W},\spadesuit)\\(\mathrm{W},\heartsuit)\\(\mathrm{W},\diamondsuit)\\(\mathrm{W},\clubsuit)\\(\mathrm{D},\spadesuit)\\(\mathrm{D},\heartsuit)\\(\mathrm{D},\diamondsuit)\\(\mathrm{D},\clubsuit)\\(\mathrm{L},\spadesuit)\\(\mathrm{L},\heartsuit)\\(\mathrm{L},\diamondsuit)\\(\mathrm{L},\clubsuit) \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \end{array} \quad \mathfrak{P}\,;\, R = \begin{array}{c} \\ (\spadesuit,\mathrm{W})\\(\spadesuit,\mathrm{D})\\(\spadesuit,\mathrm{L})\\(\heartsuit,\mathrm{W})\\(\heartsuit,\mathrm{D})\\(\heartsuit,\mathrm{L})\\(\diamondsuit,\mathrm{W})\\(\diamondsuit,\mathrm{D})\\(\diamondsuit,\mathrm{L})\\(\clubsuit,\mathrm{W})\\(\clubsuit,\mathrm{D})\\(\clubsuit,\mathrm{L}) \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \end{array} \quad \mathtt{vec}(R) = \begin{array}{c} \\ ((\mathrm{W},\spadesuit),1)\\((\mathrm{W},\spadesuit),2)\\((\mathrm{W},\heartsuit),1)\\((\mathrm{W},\heartsuit),2)\\((\mathrm{W},\diamondsuit),1)\\((\mathrm{W},\diamondsuit),2)\\((\mathrm{W},\clubsuit),1)\\((\mathrm{W},\clubsuit),2)\\((\mathrm{D},\spadesuit),1)\\((\mathrm{D},\spadesuit),2)\\((\mathrm{D},\heartsuit),1)\\((\mathrm{D},\heartsuit),2)\\((\mathrm{D},\diamondsuit),1)\\((\mathrm{D},\diamondsuit),2)\\((\mathrm{D},\clubsuit),1)\\((\mathrm{D},\clubsuit),2)\\((\mathrm{L},\spadesuit),1)\\((\mathrm{L},\spadesuit),2)\\((\mathrm{L},\heartsuit),1)\\((\mathrm{L},\heartsuit),2)\\((\mathrm{L},\diamondsuit),1)\\((\mathrm{L},\diamondsuit),2)\\((\mathrm{L},\clubsuit),1)\\((\mathrm{L},\clubsuit),2) \end{array} \begin{array}{c} 1 \\ \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{pmatrix} \end{array}$$

$$\hookrightarrow R = \begin{array}{c} \\ (\mathrm{W},1)\\(\mathrm{W},2)\\(\mathrm{D},1)\\(\mathrm{D},2)\\(\mathrm{L},1)\\(\mathrm{L},2) \end{array} \begin{array}{c} \begin{array}{cccc} \spadesuit & \heartsuit & \diamondsuit & \clubsuit \end{array} \\ \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix} \end{array} \quad \mathtt{vec}(\hookrightarrow R) = \begin{array}{c} \\ ((\mathrm{W},1),\spadesuit)\\((\mathrm{W},1),\heartsuit)\\((\mathrm{W},1),\diamondsuit)\\((\mathrm{W},1),\clubsuit)\\((\mathrm{W},2),\spadesuit)\\((\mathrm{W},2),\heartsuit)\\((\mathrm{W},2),\diamondsuit)\\((\mathrm{W},2),\clubsuit)\\((\mathrm{D},1),\spadesuit)\\((\mathrm{D},1),\heartsuit)\\((\mathrm{D},1),\diamondsuit)\\((\mathrm{D},1),\clubsuit)\\((\mathrm{D},2),\spadesuit)\\((\mathrm{D},2),\heartsuit)\\((\mathrm{D},2),\diamondsuit)\\((\mathrm{D},2),\clubsuit)\\((\mathrm{L},1),\spadesuit)\\((\mathrm{L},1),\heartsuit)\\((\mathrm{L},1),\diamondsuit)\\((\mathrm{L},1),\clubsuit)\\((\mathrm{L},2),\spadesuit)\\((\mathrm{L},2),\heartsuit)\\((\mathrm{L},2),\diamondsuit)\\((\mathrm{L},2),\clubsuit) \end{array} \begin{array}{c} 1 \\ \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{pmatrix} \end{array}$$

$$R\hookleftarrow = \begin{array}{c} \\ (1,\spadesuit)\\(1,\heartsuit)\\(1,\diamondsuit)\\(1,\clubsuit)\\(2,\spadesuit)\\(2,\heartsuit)\\(2,\diamondsuit)\\(2,\clubsuit) \end{array} \begin{array}{c} \begin{array}{ccc} \mathrm{W} & \mathrm{D} & \mathrm{L} \end{array} \\ \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix} \end{array} \quad \mathtt{vec}(R\hookleftarrow) = \begin{array}{c} \\ ((1,\spadesuit),\mathrm{W})\\((1,\spadesuit),\mathrm{D})\\((1,\spadesuit),\mathrm{L})\\((1,\heartsuit),\mathrm{W})\\((1,\heartsuit),\mathrm{D})\\((1,\heartsuit),\mathrm{L})\\((1,\diamondsuit),\mathrm{W})\\((1,\diamondsuit),\mathrm{D})\\((1,\diamondsuit),\mathrm{L})\\((1,\clubsuit),\mathrm{W})\\((1,\clubsuit),\mathrm{D})\\((1,\clubsuit),\mathrm{L})\\((2,\spadesuit),\mathrm{W})\\((2,\spadesuit),\mathrm{D})\\((2,\spadesuit),\mathrm{L})\\((2,\heartsuit),\mathrm{W})\\((2,\heartsuit),\mathrm{D})\\((2,\heartsuit),\mathrm{L})\\((2,\diamondsuit),\mathrm{W})\\((2,\diamondsuit),\mathrm{D})\\((2,\diamondsuit),\mathrm{L})\\((2,\clubsuit),\mathrm{W})\\((2,\clubsuit),\mathrm{D})\\((2,\clubsuit),\mathrm{L}) \end{array} \begin{array}{c} 1 \\ \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{pmatrix} \end{array}$$

$$
\begin{array}{c|cccc}
 & \spadesuit & \heartsuit & \diamondsuit & \clubsuit \\
(1,\mathrm{W}) & \mathbf{1} & 0 & 0 & 0 \\
(1,\mathrm{D}) & 0 & 0 & 0 & \mathbf{1} \\
(1,\mathrm{L}) & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 \\
(2,\mathrm{W}) & 0 & 0 & 0 & 0 \\
(2,\mathrm{D}) & 0 & 0 & 0 & 0 \\
(2,\mathrm{L}) & 0 & 0 & 0 & \mathbf{1}
\end{array}
\qquad
\begin{array}{c|c}
 & 1 \\
((1,\mathrm{W}),\spadesuit) & \mathbf{1} \\
((1,\mathrm{W}),\heartsuit) & 0 \\
((1,\mathrm{W}),\diamondsuit) & 0 \\
((1,\mathrm{W}),\clubsuit) & 0 \\
((1,\mathrm{D}),\spadesuit) & 0 \\
((1,\mathrm{D}),\heartsuit) & 0 \\
((1,\mathrm{D}),\diamondsuit) & 0 \\
((1,\mathrm{D}),\clubsuit) & \mathbf{1} \\
((1,\mathrm{L}),\spadesuit) & \mathbf{1} \\
((1,\mathrm{L}),\heartsuit) & \mathbf{1} \\
((1,\mathrm{L}),\diamondsuit) & \mathbf{1} \\
((1,\mathrm{L}),\clubsuit) & 0 \\
((2,\mathrm{W}),\spadesuit) & 0 \\
((2,\mathrm{W}),\heartsuit) & 0 \\
((2,\mathrm{W}),\diamondsuit) & 0 \\
((2,\mathrm{W}),\clubsuit) & 0 \\
((2,\mathrm{D}),\spadesuit) & 0 \\
((2,\mathrm{D}),\heartsuit) & 0 \\
((2,\mathrm{D}),\diamondsuit) & 0 \\
((2,\mathrm{D}),\clubsuit) & 0 \\
((2,\mathrm{L}),\spadesuit) & 0 \\
((2,\mathrm{L}),\heartsuit) & 0 \\
((2,\mathrm{L}),\diamondsuit) & 0 \\
((2,\mathrm{L}),\clubsuit) & \mathbf{1}
\end{array}
$$

$\hookrightarrow[R^{\hookleftarrow}]$ $\quad$ $\mathtt{vec}(\hookrightarrow[R^{\hookleftarrow}])$

$$
\begin{array}{c|ccc}
 & \mathrm{W} & \mathrm{D} & \mathrm{L} \\
(\spadesuit,1) & \mathbf{1} & 0 & \mathbf{1} \\
(\spadesuit,2) & 0 & 0 & 0 \\
(\heartsuit,1) & 0 & 0 & \mathbf{1} \\
(\heartsuit,2) & 0 & 0 & 0 \\
(\diamondsuit,1) & 0 & 0 & \mathbf{1} \\
(\diamondsuit,2) & 0 & 0 & 0 \\
(\clubsuit,1) & 0 & \mathbf{1} & 0 \\
(\clubsuit,2) & 0 & 0 & \mathbf{1}
\end{array}
\qquad
\begin{array}{c|c}
 & 1 \\
((\spadesuit,1),\mathrm{W}) & \mathbf{1} \\
((\spadesuit,1),\mathrm{D}) & 0 \\
((\spadesuit,1),\mathrm{L}) & \mathbf{1} \\
((\spadesuit,2),\mathrm{W}) & 0 \\
((\spadesuit,2),\mathrm{D}) & 0 \\
((\spadesuit,2),\mathrm{L}) & 0 \\
((\heartsuit,1),\mathrm{W}) & 0 \\
((\heartsuit,1),\mathrm{D}) & 0 \\
((\heartsuit,1),\mathrm{L}) & \mathbf{1} \\
((\heartsuit,2),\mathrm{W}) & 0 \\
((\heartsuit,2),\mathrm{D}) & 0 \\
((\heartsuit,2),\mathrm{L}) & 0 \\
((\diamondsuit,1),\mathrm{W}) & 0 \\
((\diamondsuit,1),\mathrm{D}) & 0 \\
((\diamondsuit,1),\mathrm{L}) & \mathbf{1} \\
((\diamondsuit,2),\mathrm{W}) & 0 \\
((\diamondsuit,2),\mathrm{D}) & 0 \\
((\diamondsuit,2),\mathrm{L}) & 0 \\
((\clubsuit,1),\mathrm{W}) & 0 \\
((\clubsuit,1),\mathrm{D}) & \mathbf{1} \\
((\clubsuit,1),\mathrm{L}) & 0 \\
((\clubsuit,2),\mathrm{W}) & 0 \\
((\clubsuit,2),\mathrm{D}) & 0 \\
((\clubsuit,2),\mathrm{L}) & \mathbf{1}
\end{array}
$$

$[\hookrightarrow R]^{\hookleftarrow}$ $\quad$ $\mathtt{vec}([\hookrightarrow R]^{\hookleftarrow})$

$$
\begin{array}{c|cc}
 & 1 & 2 \\
(\spadesuit,\mathrm{W}) & \mathbf{1} & 0 \\
(\spadesuit,\mathrm{D}) & 0 & 0 \\
(\spadesuit,\mathrm{L}) & \mathbf{1} & 0 \\
(\heartsuit,\mathrm{W}) & 0 & 0 \\
(\heartsuit,\mathrm{D}) & 0 & 0 \\
(\heartsuit,\mathrm{L}) & \mathbf{1} & 0 \\
(\diamondsuit,\mathrm{W}) & 0 & 0 \\
(\diamondsuit,\mathrm{D}) & 0 & 0 \\
(\diamondsuit,\mathrm{L}) & \mathbf{1} & 0 \\
(\clubsuit,\mathrm{W}) & 0 & 0 \\
(\clubsuit,\mathrm{D}) & \mathbf{1} & 0 \\
(\clubsuit,\mathrm{L}) & 0 & \mathbf{1}
\end{array}
\qquad
\begin{array}{c|c}
 & 1 \\
((\spadesuit,\mathrm{W}),1) & \mathbf{1} \\
((\spadesuit,\mathrm{W}),2) & 0 \\
((\spadesuit,\mathrm{D}),1) & 0 \\
((\spadesuit,\mathrm{D}),2) & 0 \\
((\spadesuit,\mathrm{L}),1) & \mathbf{1} \\
((\spadesuit,\mathrm{L}),2) & 0 \\
((\heartsuit,\mathrm{W}),1) & 0 \\
((\heartsuit,\mathrm{W}),2) & 0 \\
((\heartsuit,\mathrm{D}),1) & 0 \\
((\heartsuit,\mathrm{D}),2) & 0 \\
((\heartsuit,\mathrm{L}),1) & \mathbf{1} \\
((\heartsuit,\mathrm{L}),2) & 0 \\
((\diamondsuit,\mathrm{W}),1) & 0 \\
((\diamondsuit,\mathrm{W}),2) & 0 \\
((\diamondsuit,\mathrm{D}),1) & 0 \\
((\diamondsuit,\mathrm{D}),2) & 0 \\
((\diamondsuit,\mathrm{L}),1) & \mathbf{1} \\
((\diamondsuit,\mathrm{L}),2) & 0 \\
((\clubsuit,\mathrm{W}),1) & 0 \\
((\clubsuit,\mathrm{W}),2) & 0 \\
((\clubsuit,\mathrm{D}),1) & \mathbf{1} \\
((\clubsuit,\mathrm{D}),2) & 0 \\
((\clubsuit,\mathrm{L}),1) & 0 \\
((\clubsuit,\mathrm{L}),2) & \mathbf{1}
\end{array}
$$

$\hookrightarrow([\hookrightarrow R]^{\hookleftarrow})$ $\quad$ $\mathtt{vec}(\hookrightarrow([\hookrightarrow R]^{\hookleftarrow}))$

$$
\begin{array}{c|cc}
 & 1 & 2 \\
(\spadesuit,\mathrm{W}) & \mathbf{1} & 0 \\
(\spadesuit,\mathrm{D}) & 0 & 0 \\
(\spadesuit,\mathrm{L}) & \mathbf{1} & 0 \\
(\heartsuit,\mathrm{W}) & 0 & 0 \\
(\heartsuit,\mathrm{D}) & 0 & 0 \\
(\heartsuit,\mathrm{L}) & \mathbf{1} & 0 \\
(\diamondsuit,\mathrm{W}) & 0 & 0 \\
(\diamondsuit,\mathrm{D}) & 0 & 0 \\
(\diamondsuit,\mathrm{L}) & \mathbf{1} & 0 \\
(\clubsuit,\mathrm{W}) & 0 & 0 \\
(\clubsuit,\mathrm{D}) & \mathbf{1} & 0 \\
(\clubsuit,\mathrm{L}) & 0 & \mathbf{1}
\end{array}
\qquad
\begin{array}{c|c}
 & 1 \\
((\spadesuit,\mathrm{W}),1) & \mathbf{1} \\
((\spadesuit,\mathrm{W}),2) & 0 \\
((\spadesuit,\mathrm{D}),1) & 0 \\
((\spadesuit,\mathrm{D}),2) & 0 \\
((\spadesuit,\mathrm{L}),1) & \mathbf{1} \\
((\spadesuit,\mathrm{L}),2) & 0 \\
((\heartsuit,\mathrm{W}),1) & 0 \\
((\heartsuit,\mathrm{W}),2) & 0 \\
((\heartsuit,\mathrm{D}),1) & 0 \\
((\heartsuit,\mathrm{D}),2) & 0 \\
((\heartsuit,\mathrm{L}),1) & \mathbf{1} \\
((\heartsuit,\mathrm{L}),2) & 0 \\
((\diamondsuit,\mathrm{W}),1) & 0 \\
((\diamondsuit,\mathrm{W}),2) & 0 \\
((\diamondsuit,\mathrm{D}),1) & 0 \\
((\diamondsuit,\mathrm{D}),2) & 0 \\
((\diamondsuit,\mathrm{L}),1) & \mathbf{1} \\
((\diamondsuit,\mathrm{L}),2) & 0 \\
((\clubsuit,\mathrm{W}),1) & 0 \\
((\clubsuit,\mathrm{W}),2) & 0 \\
((\clubsuit,\mathrm{D}),1) & \mathbf{1} \\
((\clubsuit,\mathrm{D}),2) & 0 \\
((\clubsuit,\mathrm{L}),1) & 0 \\
((\clubsuit,\mathrm{L}),2) & \mathbf{1}
\end{array}
$$

$(\hookrightarrow[R^{\hookleftarrow}])^{\hookleftarrow}$ $\quad$ $\mathtt{vec}((\hookrightarrow[R^{\hookleftarrow}])^{\hookleftarrow})$

	1
(W,(♠,1))	**1**
(W,(♠,2))	**0**
(W,(♡,1))	**0**
(W,(♡,2))	**0**
(W,(♢,1))	**0**
(W,(♢,2))	**0**
(W,(♣,1))	**0**
(W,(♣,2))	**0**
(D,(♠,1))	**0**
(D,(♠,2))	**0**
(D,(♡,1))	**0**
(D,(♡,2))	**0**
(D,(♢,1))	**0**
(D,(♢,2))	**0**
(D,(♣,1))	**1**
(D,(♣,2))	**0**
(L,(♠,1))	**1**
(L,(♠,2))	**0**
(L,(♡,1))	**1**
(L,(♡,2))	**0**
(L,(♢,1))	**1**
(L,(♢,2))	**0**
(L,(♣,1))	**0**
(L,(♣,2))	**1**

$\mathfrak{T}_{12\to21} \fatsemi \mathtt{vec}(R)$

	1
(W,(1,♠))	**1**
(W,(1,♡))	**0**
(W,(1,♢))	**0**
(W,(1,♣))	**0**
(W,(2,♠))	**0**
(W,(2,♡))	**0**
(W,(2,♢))	**0**
(W,(2,♣))	**0**
(D,(1,♠))	**0**
(D,(1,♡))	**0**
(D,(1,♢))	**0**
(D,(1,♣))	**1**
(D,(2,♠))	**0**
(D,(2,♡))	**0**
(D,(2,♢))	**0**
(D,(2,♣))	**0**
(L,(1,♠))	**1**
(L,(1,♡))	**1**
(L,(1,♢))	**1**
(L,(1,♣))	**0**
(L,(2,♠))	**0**
(L,(2,♡))	**0**
(L,(2,♢))	**0**
(L,(2,♣))	**1**

$(\mathbb{I} \otimes \mathfrak{P}) \fatsemi \mathfrak{T}_{12\to21} \fatsemi \mathtt{vec}(R)$

	1
((W,1),♠)	**1**
((W,1),♡)	**0**
((W,1),♢)	**0**
((W,1),♣)	**0**
((W,2),♠)	**0**
((W,2),♡)	**0**
((W,2),♢)	**0**
((W,2),♣)	**0**
((D,1),♠)	**0**
((D,1),♡)	**0**
((D,1),♢)	**0**
((D,1),♣)	**1**
((D,2),♠)	**0**
((D,2),♡)	**0**
((D,2),♢)	**0**
((D,2),♣)	**0**
((L,1),♠)	**1**
((L,1),♡)	**1**
((L,1),♢)	**1**
((L,1),♣)	**0**
((L,2),♠)	**0**
((L,2),♡)	**0**
((L,2),♢)	**0**
((L,2),♣)	**1**

$\mathfrak{T}_{21\to12} \fatsemi (\mathbb{I} \otimes \mathfrak{P}) \fatsemi \mathfrak{T}_{12\to21} \fatsemi \mathtt{vec}(R)$

4 Graph-theoretic aspects

We recall concepts of graph theory and from the very beginning of mathematics.

Given an arbitrary homogeneous relation R, the construct

$\mathtt{Nontrans}(R) = R \cap \overline{R ; R}$

is called its **non-transitive part**.[1] In more specific cases — as will occur here — we will have the situation that R is transitive and $R = \left[R \cap \overline{R ; R}\right]^+$; then this non-transitive part is called the **Hasse relation** of R

$\mathtt{Hasse}(R) := R \cap \overline{R ; R}$.

4.1 Regressive finiteness

Regressive finiteness is a relational concept related with being well-ordered. It needs some explanation and visualization which is provided with Fig. 4.1.1 and Fig. 4.1.2. We refer to [11, 12], where the definition reads

$R : X \longrightarrow X$ regressively finite $\quad :\Longleftrightarrow \quad \forall Y \subseteq X : \; Y \subseteq R^\mathsf{T} ; Y \;\rightarrow\; Y = \mathbb{L}$.

In our lifted version of relational mathematics, however, we must never quantify over subsets $Y \subseteq X$ — only over elements or points; here they will necessarily reside in the powerset obtained via the membership $\varepsilon : X \longrightarrow \mathbf{2}^X$. So, we should conceive of $y := \mathtt{syq}(\varepsilon, Y)$ and $Y = \varepsilon ; y$ to start formulating

$\forall y \in \mathbf{2}^X : \; \varepsilon ; y \subseteq R^\mathsf{T} ; \varepsilon ; y \;\rightarrow\; \varepsilon ; y = \mathbb{L}$.

This still predicate-logic style property shall now informally and step by step be brought to a quantifier-free version. Because y is a point and may thus be shunted, the subjunction may also be expressed as

$\forall y \in \mathbf{2}^X : \; \varepsilon ; y ; y^\mathsf{T} \subseteq R^\mathsf{T} ; \varepsilon \;\rightarrow\; \varepsilon ; y \subseteq \mathbb{L}$,
$\forall y \in \mathbf{2}^X : \; \varepsilon^\mathsf{T} ; \overline{R^\mathsf{T} ; \varepsilon} \subseteq \overline{y ; y^\mathsf{T}} \;\rightarrow\; \varepsilon^\mathsf{T} ; \mathbb{T} \subseteq \overline{y}$ Schröder equivalence twice
$\forall y \in \mathbf{2}^X : \; y ; y^\mathsf{T} \subseteq \overline{\varepsilon^\mathsf{T} ; \overline{R^\mathsf{T} ; \varepsilon}} \;\rightarrow\; y \subseteq \overline{\varepsilon^\mathsf{T} ; \mathbb{T}}$ negated
$\forall y \in \mathbf{2}^X : \; y ; y^\mathsf{T} \subseteq \overline{\varepsilon^\mathsf{T} ; \overline{R^\mathsf{T} ; \varepsilon}} \;\rightarrow\; y ; y^\mathsf{T} \subseteq \overline{\varepsilon^\mathsf{T} ; \mathbb{T}}$ because $y ; y^\mathsf{T} \subseteq A ; \mathbb{T} \iff y \subseteq A ; \mathbb{T} ; y = A ; \mathbb{T}$
$\mathbb{I} \cap \overline{\varepsilon^\mathsf{T} ; \overline{R^\mathsf{T} ; \varepsilon}} \subseteq \overline{\varepsilon^\mathsf{T} ; \mathbb{T}}$ since y is a point and all the $y ; y^\mathsf{T}$ united form the identity on $\mathbf{2}^X$,
$\mathbb{I} \cap \varepsilon^\mathsf{T} ; \mathbb{T} \subseteq \varepsilon^\mathsf{T} ; \overline{R^\mathsf{T} ; \varepsilon}$.

In the second last version, the right residual shows whether a column of ε is contained in the corresponding column $R^\mathsf{T} ; \varepsilon$ of its successors. This must never be the case — except for the empty "first" column of ε.

[1] See [1] where the theory of discreteness, non-transitive part, and Hasse-relations is developed in general. It is recorded in the textbooks [11, Def. 6.5.7] and also in [12, Def. 6.5.7].

4.1.1 Definition. A homogeneous relation R will be called **regressively finite** provided $\mathbb{I} \cap \varepsilon^{\mathsf{T}};\mathbb{T} \subseteq \varepsilon^{\mathsf{T}};\overline{R^{\mathsf{T}};\varepsilon}$. □

In the following two figures, R is regressively finite while R' is not.

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{array} = R$$

	{}	{0}	{1}	{0,1}	{2}	{0,2}	{1,2}	{0,1,2}	{3}	{0,3}	{1,3}	{0,1,3}	{2,3}	{0,2,3}	{1,2,3}	{0,1,2,3}	{4}	{0,4}	{1,4}	{0,1,4}	{2,4}	{0,2,4}	{1,2,4}	{0,1,2,4}	{3,4}	{0,3,4}	{1,3,4}	{0,1,3,4}	{2,3,4}	{0,2,3,4}	{1,2,3,4}	{0,1,2,3,4}
{}	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
{0}	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
{1}	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**
{0,1}	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
{2}	**1**	**1**	**0**	**0**	**1**	**1**	**0**	**0**	**1**	**1**	**0**	**0**	**1**	**1**	**0**	**0**	**1**	**1**	**0**	**0**	**1**	**1**	**0**	**0**	**1**	**1**	**0**	**0**	**1**	**1**	**0**	**0**
{0,2}	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
{1,2}	**1**	**1**	**1**	**0**	**1**	**1**	**1**	**0**	**1**	**1**	**1**	**0**	**1**	**1**	**1**	**0**	**1**	**1**	**1**	**0**	**1**	**1**	**1**	**0**	**1**	**1**	**1**	**0**	**1**	**1**	**1**	**0**
{0,1,2}	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
{3}	**1**	**1**	**1**	**1**	**0**	**0**	**0**	**0**	**1**	**1**	**1**	**1**	**0**	**0**	**0**	**0**	**1**	**1**	**1**	**1**	**0**	**0**	**0**	**0**	**1**	**1**	**1**	**1**	**0**	**0**	**0**	**0**
{0,3}	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
{1,3}	**1**	**1**	**1**	**1**	**1**	**0**	**1**	**0**	**1**	**1**	**1**	**1**	**1**	**0**	**1**	**0**	**1**	**1**	**1**	**1**	**1**	**0**	**1**	**0**	**1**	**1**	**1**	**1**	**1**	**0**	**1**	**0**
{0,1,3}	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
{2,3}	**1**	**1**	**1**	**1**	**1**	**1**	**0**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**0**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**0**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**0**	**0**
{0,2,3}	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
{1,2,3}	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**0**
{0,1,2,3}	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
{4}	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
{0,4}	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
{1,4}	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**
{0,1,4}	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
{2,4}	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**0**	**0**	**1**	**1**	**0**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**0**	**0**	**1**	**1**	**0**	**0**	**1**	**1**
{0,2,4}	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
{1,2,4}	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**0**	**1**	**1**	**1**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**0**	**1**	**1**	**1**	**0**	**1**	**1**
{0,1,2,4}	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
{3,4}	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**0**	**0**	**0**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**0**	**0**	**0**	**0**
{0,3,4}	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
{1,3,4}	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**0**	**1**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**0**	**1**	**0**	**1**
{0,1,3,4}	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
{2,3,4}	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**0**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**0**	**0**
{0,2,3,4}	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
{1,2,3,4}	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**0**
{0,1,2,3,4}	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**

Fig. 4.1.1 Regressively finite: $\varepsilon^{\mathsf{T}};\overline{R^{\mathsf{T}};\varepsilon}$ contains the diagonal except for the upper left point

In the next case, we have a very simple loop so that the relation R' is not regressively finite. We might run $4, 3, 4, 3, \ldots$ infinitely often. We may also think of points $\ldots -4, -3, -2, -1$ added that are related via S. Then an infinite descent via S^{T} would have a similar effect.

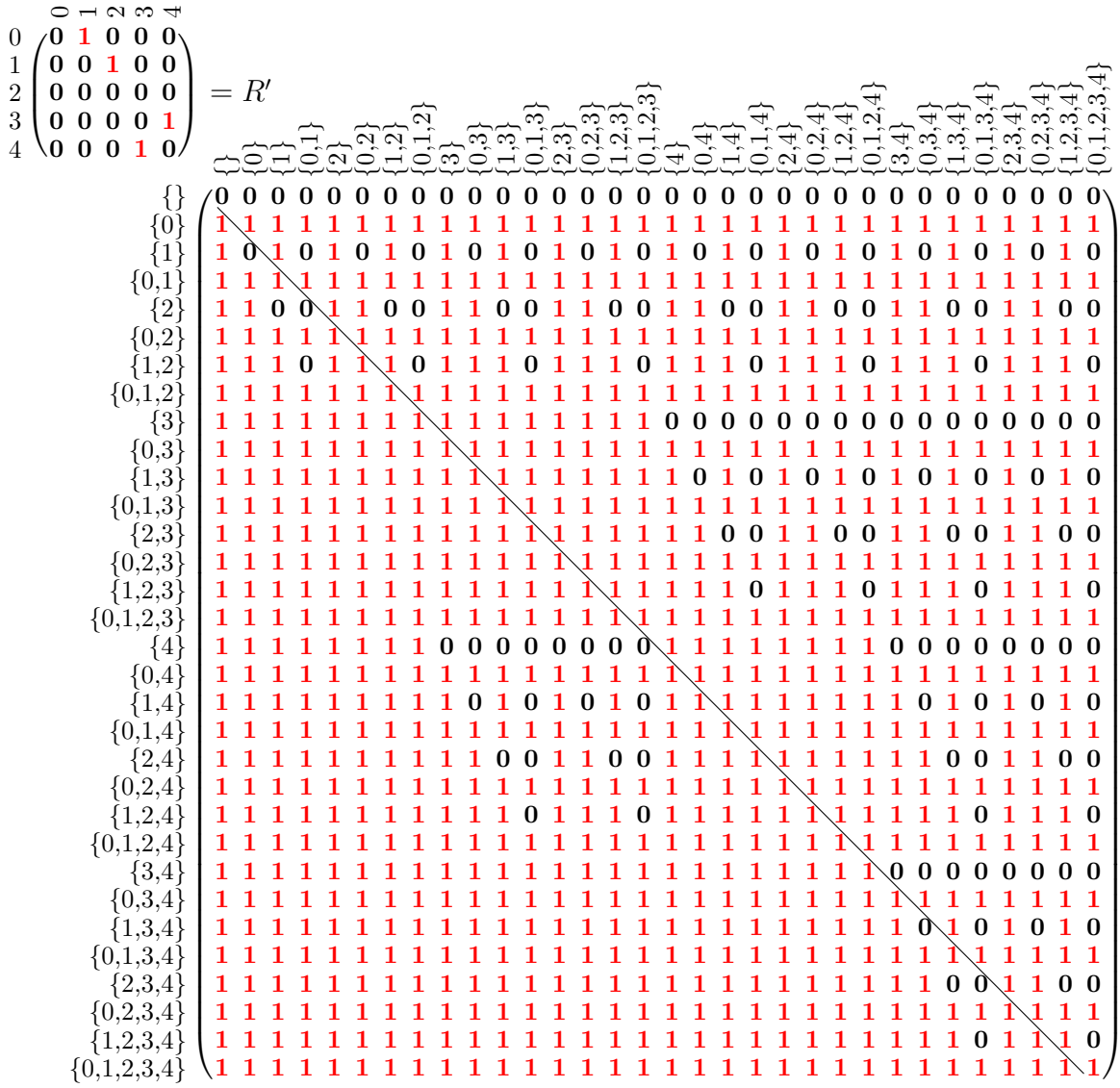


Fig. 4.1.2 The R' from Fig. 5.2.3 is not regressively finite: see diagonal of $\varepsilon^{\mathsf{T}} ; \overline{R'^{\mathsf{T}} ; \varepsilon}$ at row $\{3,4\}$.

4.1.2 Proposition. Whenever one has a regressively finite relation $R : X \longrightarrow X$ and some subset F of X satisfying $\overline{F} \subseteq R^{\mathsf{T}} ; \overline{F}$, this set will be the full set, $F = \top\!\!\top$.

Proof: Let us define the point $a := \mathtt{syq}(\varepsilon, \overline{F})$, so that $\varepsilon ; a = \overline{F}$. This allows us to make use of the algebraic condition for regressive finiteness. We have, namely, from Def. 4.1.1

$$\mathbb{I} \cap \varepsilon^{\mathsf{T}} ; \top\!\!\top \subseteq \varepsilon^{\mathsf{T}} ; \overline{R^{\mathsf{T}} ; \varepsilon}$$

which implies

$$\begin{aligned} a \cap \varepsilon^{\mathsf{T}} ; \top\!\!\top &= (\mathbb{I} \cap \varepsilon^{\mathsf{T}} ; \top\!\!\top) ; a \subseteq \varepsilon^{\mathsf{T}} ; \overline{R^{\mathsf{T}} ; \varepsilon} ; a = \varepsilon^{\mathsf{T}} ; \overline{R^{\mathsf{T}} ; \varepsilon ; a} = \varepsilon^{\mathsf{T}} ; \overline{R^{\mathsf{T}} ; \overline{F}} \subseteq \varepsilon^{\mathsf{T}} ; \overline{\overline{F}} \\ &= \varepsilon^{\mathsf{T}} ; F = \varepsilon^{\mathsf{T}} ; \overline{\varepsilon ; a} = \varepsilon^{\mathsf{T}} ; \overline{\varepsilon} ; a = \overline{\Omega} ; a = \overline{\Omega ; a} \subseteq \overline{\mathbb{I} ; a} = \overline{a} \quad \text{with } \Omega = \overline{\varepsilon^{\mathsf{T}} ; \overline{\varepsilon}} \text{ the powerset order} \\ \iff \quad & \top\!\!\top = \overline{a} \cup \overline{\varepsilon^{\mathsf{T}} ; \top\!\!\top} \cup \overline{a} \quad \iff \quad a \subseteq \overline{\varepsilon^{\mathsf{T}} ; \top\!\!\top} \end{aligned}$$

Finally, $\overline{F} = \varepsilon ; a \subseteq \varepsilon ; \overline{\varepsilon^{\mathsf{T}} ; \top\!\!\top} = \bot\!\!\bot$, or else, $F = \top\!\!\top$. □

A relation R is regressively finite if and only if its transitive closure R^+ is:

4.1.3 Proposition. R regressively finite $\quad\Longleftrightarrow\quad$ R^+ regressively finite

Proof: Following Def. 4.1.1, we have to convince us that

$\mathbb{I} \cap \varepsilon^\intercal ; \mathbb{T} \subseteq \varepsilon^\intercal ; \overline{R^\intercal ; \varepsilon} \quad\Longleftrightarrow\quad \mathbb{I} \cap \varepsilon^\intercal ; \mathbb{T} \subseteq \varepsilon^\intercal ; \overline{R^{\intercal+} ; \varepsilon}$,

of which direction "$\Longleftarrow$" is trivial because $R \subseteq R^+$.

For the other direction, we assume that R is regressively finite, i.e. that

$\forall Y \subseteq X : \; Y \subseteq R^\intercal ; Y \;\rightarrow\; Y = \mathbb{L}$.

The question is, whether then also R^+ is regressively finite, i.e.

$\forall Z \subseteq X : \; Z \subseteq R^{\intercal+} ; Z \;\rightarrow\; Z = \mathbb{L}$.

Assume some subset Z satisfying $Z \subseteq R^{\intercal+} ; Z$ to be given. Then we form the set $W := R^{\intercal*} ; Z$ and show that it satisfies

$W = Z \cup R^\intercal ; R^{\intercal*} ; Z \subseteq R^{\intercal+} ; Z \cup R^\intercal ; W = R^\intercal ; R^{\intercal*} ; Z \cup R^\intercal ; W = R^\intercal ; W \cup R^\intercal ; W = R^\intercal ; W$.

Therefore, we have $W = \mathbb{L}$ and in consequence also $Z = \mathbb{L}$. □

Another result concerning the propagation of regressive finiteness:

4.1.4 Proposition. Assume a mapping $\varphi : X \longrightarrow X'$ that is a homomorphism from the relation $B : X \longrightarrow X$ into $B' : X' \longrightarrow X'$, i.e. that satisfies $B ; \varphi \subseteq \varphi ; B'$. If B' is regressively finite, so will be B.

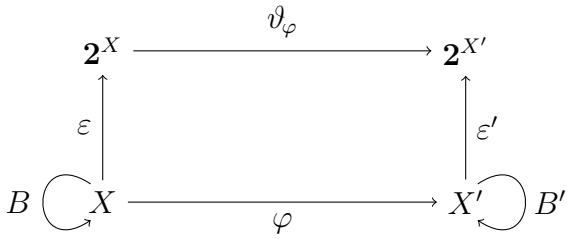


Fig. 4.1.3 Typing for a homomorphism between relations

Proof: By assumption B' is regressively finite, i.e. $\mathbb{I}_{\mathbf{2}^{X'}} \cap \varepsilon'^\intercal ; \mathbb{T} \subseteq \varepsilon'^\intercal ; \overline{B'^\intercal ; \varepsilon'}$. We recall — from, e.g. [14, 15], — the concept of an existential image of φ, defined as

$\vartheta_\varphi := \mathtt{syq}(\varphi^\intercal ; \varepsilon, \varepsilon')$

that satisfies this rule

$\varphi^\intercal ; \varepsilon = \varepsilon' ; \vartheta_\varphi^\intercal$

— among others — and estimate

$\varepsilon^\intercal ; \overline{B^\intercal ; \varepsilon} \supseteq \varepsilon^\intercal ; \overline{\overline{\varphi ; B'}^\intercal ; \varepsilon}$ because $B ; \varphi \subseteq \varphi ; B' \iff B \subseteq \overline{\varphi ; B'} ; \varphi^\intercal$

$= \varepsilon^\intercal ; \overline{\overline{\varphi ; \overline{B'^\intercal ; \varphi^\intercal}} ; \varepsilon} = \varepsilon^\intercal ; \varphi ; \overline{B'^\intercal ; \varphi^\intercal ; \varepsilon} = \varepsilon^\intercal ; \varphi ; \overline{B'^\intercal ; \varphi^\intercal ; \varepsilon} = \vartheta_\varphi ; \varepsilon'^\intercal ; \overline{B'^\intercal ; \varepsilon' ; \vartheta_\varphi^\intercal}$ since φ is a map

$= \vartheta_\varphi ; \varepsilon'^\intercal ; \overline{B'^\intercal ; \varepsilon'} ; \vartheta_\varphi^\intercal$ since ϑ_φ is a map

$\supseteq \vartheta_\varphi ; \left[\mathbb{I}_{\mathbf{2}^{X'}} \cap \varepsilon'^\intercal ; \mathbb{T}\right] ; \vartheta_\varphi^\intercal$ by assumption

$= \vartheta_\varphi ; \vartheta_\varphi^\intercal \cap \vartheta_\varphi ; \varepsilon'^\intercal ; \mathbb{T} ; \vartheta_\varphi^\intercal = \vartheta_\varphi ; \vartheta_\varphi^\intercal \cap \varepsilon^\intercal ; \varphi ; \mathbb{T} \supseteq \mathbb{I}_{\mathbf{2}^X} \cap \varepsilon^\intercal ; \mathbb{T}$ since $\varphi, \vartheta_\varphi$ are maps □

The essence of this proposition has been posed as Exercise 6.3.3 in the books [11, 12]. Here the proof is included since none seems as of yet to have been published.

4.2 A meta result

The next lemma is planned to be used fairly often in the various inductive definitions to come.

4.2.1 Lemma. Considering a functional $f(X) := F_0 \cup f_H(X)$ of which the functional f_H satisfies $f_H(\perp\!\!\!\perp) = \perp\!\!\!\perp$ and is

isotonic,	i.e. $A \subseteq B \implies f_H(A) \subseteq f_H(B)$
additive,	i.e. $f_H(X \cup Y) = f_H(X) \cup f_H(Y)$ and $f_H(\mathtt{sup}\{\, X \mid c(X) \,\}) = \mathtt{sup}\{\, f_H(X) \mid c(X) \,\}$ with $c(X)$ a criterion
$\cap$-sub-hereditary,	i.e. $f_H(X \cap Y) \subseteq f_H(X) \cap f_H(Y)$ and $f_H(\mathtt{inf}\{\, X \mid c(X) \,\}) \subseteq \mathtt{inf}\{\, f_H(X) \mid c(X) \,\}$ with $c(X)$ a criterion

the following will hold for the construct $F := \mathtt{inf}\,\{\, X \mid f(X) \subseteq X \,\}$:

i) $F = f(F)$

ii) $f^n(\perp\!\!\!\perp) \subseteq F$ for all $n \geq 0$

iii) $F = \mathtt{sup}_{n\geq 0}\, f^n(\perp\!\!\!\perp)$

iv) $f^n(\perp\!\!\!\perp) = \mathtt{sup}_{0\leq i<n}\, f_H^i(F_0)$ for all $n \geq 1$, so that $F = \mathtt{sup}_{n\geq 0}\, f_H^n(F_0)$

Proof: It is easy to show in advance that also f is isotonic, additive, and $\cap$-sub-hereditary:
$f(X \cup Y) = F_0 \cup f_H(X \cup Y) = F_0 \cup f_H(X) \cup f_H(Y) = f(X) \cup f(Y)$
$f(X \cap Y) = F_0 \cup f_H(X \cap Y) \subseteq F_0 \cup \big[f_H(X) \cap f_H(Y)\big] = f(X) \cap f(Y)$

---

i) $f(F) = f(\mathtt{inf}\,\{\, X \mid f(X) \subseteq X \,\})$ expanding F
$\subseteq \mathtt{inf}\,\{\, f(X) \mid f(X) \subseteq X \,\}$ $\cap$-sub-hereditary
$\subseteq \mathtt{inf}\,\{\, X \mid f(X) \subseteq X \,\}$ selection criterion
$= F$ by definition
$f(f(F)) \subseteq f(F)$ using the latter and isotonicity
By definition of F follows $F \subseteq f(F)$; this is then combined with the first observation.

---

ii) Proof by induction started with $f^0(\perp\!\!\!\perp) = \perp\!\!\!\perp$, $f^1(\perp\!\!\!\perp) = F_0 \subseteq F$, using (i), and induction hypothesis $f^n(\perp\!\!\!\perp) \subseteq F$ for $n \geq 1$, leading to
$f^{n+1}(\perp\!\!\!\perp) = f(f^n(\perp\!\!\!\perp)) \subseteq f(F) = F$.

---

iii) Due to additivity, the supremum in question satisfies the selection criterion,
$f(\mathtt{sup}_{n\geq 0}\, f^n(\perp\!\!\!\perp)) = \mathtt{sup}_{n\geq 0}\, f(f^n(\perp\!\!\!\perp)) = \mathtt{sup}_{k\geq 1}\, f^k(\perp\!\!\!\perp) \subseteq \mathtt{sup}_{n\geq 0}\, f^n(\perp\!\!\!\perp)$.
Therefore $F \subseteq \mathtt{sup}_{n\geq 0}\, f^n(\perp\!\!\!\perp)$. Equality follows immediately from (ii).

---

iv) We use $f(\perp\!\!\!\perp) = F_0$ as the rather obvious start of an induction for $n \geq 1$ with induction hypothesis $f^n(\perp\!\!\!\perp) = \mathtt{sup}_{0\leq i<n}\, f_H^i(F_0)$:
$f^{n+1}(\perp\!\!\!\perp) = f(f^n(\perp\!\!\!\perp)) = f(\mathtt{sup}_{0\leq i<n}\, f_H^i(F_0))$
$= F_0 \cup f_H(\mathtt{sup}_{0\leq i<n}\, f_H^i(F_0)) = \mathtt{sup}_{0\leq i<n+1}\, f_H^i(F_0)$.
Then we prove how F is formulated explicitly:
$F = \mathtt{sup}_{n\geq 0}\, f^n(\perp\!\!\!\perp) = \mathtt{sup}_{n\geq 0}\, \big[\mathtt{sup}_{0\leq i<n}\, f_H^i(F_0)\big] = \mathtt{sup}_{n\geq 0}\, f_H^n(F_0)$ □

Applications of this meta result include Props. 4.3.1, 5.2.1, 5.2.2, 5.3.1, 5.4.1, 6.2.1, 7.3.1, 8.2.3, 9.6.1, 10.1.7, 10.2.3, 10.3.7. It is, thus, indeed helpful to have abstracted it from any unnecessary details.

4.3 Transitive closures

Now we investigate which point of X is — in the graph-theoretical sense — *reachable* from some point via transitions according to some given homogeneous relation R. This means to recall several well-established relational as well as graph-theoretical concepts.

We instantiate Lemma 4.2.1 with functionals, now denoted t, t_H instead of f, f_H:

4.3.1 Proposition and Definition. Let be given any homogeneous relation R and consider the functional

$$t(X) := T_0 \cup t_H(X) \quad \text{with} \quad T_0 := R \quad \text{and} \quad t_H(X) := R ; X.$$

Defining the constructs

$$R^+ := \mathtt{inf}\ \{\, X \mid t(X) \subseteq X \,\} \quad \text{and} \quad R^* := \mathbb{I} \cup R^+,$$

the following will hold:

i) $R^+ = t(R) = R \cup R ; R^+ = \mathtt{sup}_{n\geq 1}\, R^n$

ii) $R^+ = R ; R^* = R^* ; R$

iii) $R^* = \mathtt{sup}_{n\geq 0}\, R^n$

Traditionally, one calls R^+ the **transitive closure** and R^* the **reflexive-transitive closure** of the relation R.

Proof: We observe in advance that Lemma 4.2.1 may be applied since the functional t_H is

isotonic, i.e.	$A \subseteq B \implies R ; A \subseteq R ; B$,	i.e. $t_H(A) \subseteq t_H(B)$,
additive, i.e.	$t_H(X \cup Y) = R ; (X \cup Y) = R ; X \cup R ; Y = t_H(X) \cup t_H(Y)$,	
$\cap$-sub-hereditary, i.e.	$t_H(X \cap Y) = R ; (X \cap Y) \subseteq R ; X \cap R ; Y = t_H(X) \cap t_H(Y)$.	

i) This follows from Lemma 4.2.1.i,iv, observing that

$$R^+ = \mathtt{sup}_{n\geq 0}\, t_H^n(T_0) = \mathtt{sup}_{n\geq 0}\, R^n ; R = \mathtt{sup}_{n\geq 1}\, R^n$$

ii) $R ; R^* = R ; (\mathbb{I} \cup R^+) = R \cup R ; R^+ = R^+$

iii) is trivial recalling that $R^0 = \mathbb{I}$. □

Altogether, with Lemma 4.2.1 and Prop. 4.3.1, we have proved anew, how to define the transitive closure in both ways, as a supremum and as an infimum.

5 Introducing numbers

We address a very basic problem of mathematics: the stepwise introduction of numbers, traditionally obtained in this sequence

$$\begin{array}{lcl} \text{natural} & \longrightarrow & \text{integer} \\ \mathbb{N} & & \mathbb{Z} \end{array} \longrightarrow \begin{array}{l} \text{rational} \\ \mathbb{Q} \end{array} \longrightarrow \begin{array}{l} \text{real} \\ \mathbb{R} \end{array} \longrightarrow \begin{array}{l} \text{complex} \\ \mathbb{C} \end{array} \longrightarrow \begin{array}{l} \text{quaternion} \\ \mathbb{H}. \end{array}$$

In the well-known way, we have a successor relation and in the first two transitions a product-forming accompanied by extending the operations to classes according to an equivalence. Then occurs a cut completion. Finally, we have two times simply an extension of the operations to pairs. We focus on natural numbers $\mathbb{N}$ first.

A remark is necessary concerning constructs such as $\mathtt{sup}_{0\leq i<n}$: We are going to introduce natural numbers via recursion applied to functionals, formulated in terms of relations. This requires to apply these functionals once, two times or more often, and already while still in the process of defining naturals, we are thus triggered to use them. It is, however, not so and just a notational convention letting us avoid an un-intuitive way of presentation.

5.1 Natural numbers

For the very start we have to introduce natural numbers. Algebraic semantics in its standard form assumes a totally defined injective "successor" mapping $S : X \longrightarrow X$, in order to resemble $n \mapsto n+1$, and considers the initial model of the term algebra as constituting the natural numbers $\mathbb{N}$. Thinking puristically in terms of relation algebra, this concept is not sufficiently immediate, because it works with a characterization of the initial model comparing it with *all* the other models — an amorphous multitude.

We are going to rephrase the introduction of natural numbers relationally, starting again from $\mathbb{S} : \mathbb{N} \longrightarrow \mathbb{N}$, with the zero point $z := \overline{\mathbb{S}^{\mathsf{T}}; \mathbb{T}} : \mathbb{N} \longrightarrow \mathbb{1}$ derived therefrom, and using the membership $\varepsilon : \mathbb{N} \longrightarrow \mathbf{2}^{\mathbb{N}}$ — generically defined uniquely up to isomorphism — with which a more equational definition becomes possible.

Notational convention With the following investigations we aim at the set $\mathbb{N}$ of natural numbers, but will often give examples where these are limited to some initial finite section thereof. The axiomatization will be chosen so that solely the request that $\mathbb{S}$ be totally defined makes the difference. Notationally we will use $\mathbb{S} : \mathbb{N} \longrightarrow \mathbb{N}$, be these still limited or not. Whether $\mathbb{S}$ is total will always be made clear from the context. □

5.1.1 Definition. The homogeneous relation $\mathbb{S} : \mathbb{N} \longrightarrow \mathbb{N}$ will be called the **successor relation** on (a possibly limited initial section of) the naturals provided

i) $\mathbb{S}$ is injective, univalent and non-empty i.e., satisfies $\mathbb{S} ; \mathbb{S}^\intercal \subseteq \mathbb{I}$, $\mathbb{S}^\intercal ; \mathbb{S} \subseteq \mathbb{I}$ $\mathbb{S} \neq \mathbb{\bot}$,

ii) $z := \overline{\mathbb{S}^\intercal ; \mathbb{T}}$ is a point, i.e., satisfies $z ; z^\intercal \subseteq \mathbb{I}$, $z = z ; \mathbb{T}$, $\mathbb{T} ; z = \mathbb{T}$,

iii) $\mathbb{S}$ is regressively finite[1], i.e., satisfies $\mathbb{I} \cap \varepsilon^\intercal ; \mathbb{T} \subseteq \varepsilon^\intercal ; \overline{\mathbb{S}^\intercal ; \varepsilon}$.

Should $\mathbb{S}$ in addition be total, i.e., satisfy $\mathbb{S} ; \mathbb{S}^\intercal \supseteq \mathbb{I}$, thus turning out to be an injective map, we will speak of "the" **successor relation on natural numbers** $\mathbb{N}$. □

The case $\mathbb{S} = \mathbb{\bot} : \mathbb{1} \longrightarrow \mathbb{1}$, that might occur in case $\mathbb{S}$ is not yet total, is explicitly excluded with (i). Otherwise, we might encounter paradoxical effects, since all the other requirements of Def. 5.1.1 are satisfied, recalling

$$\varepsilon = 1 \begin{pmatrix} \overset{\mathbb{1}}{\mathbf{0}} & \overset{\mathbb{1}}{\mathbf{1}} \end{pmatrix}.$$

So, there will besides $z ; \mathbb{T}$, the "number 0", always exist $e := \mathbb{S}^\intercal ; z ; \mathbb{T}$, the "number 1".

When not demanding (iii), unpleasant situations as with the S in Fig. 5.2.3, Fig. 5.2.6 and Fig. 5.2.7 are still possible.

The above definition allows as a very first immediate result a disjoint partition of the identity — irrespective of $\mathbb{S}$ being total or regressively finite:

5.1.2 Proposition. $z ; z^\intercal \cup \mathbb{S}^\intercal ; \mathbb{S} = \mathbb{I}$ $\mathbb{S} \cap \mathbb{T} ; z^\intercal = \mathbb{\bot}$

Proof: "$\subseteq$" follows from $\mathbb{S}$ univalent and z injective. To obtain "$\supseteq$", we start decomposing

$$\mathbb{I} = \mathbb{I} \cap \mathbb{T} = (\mathbb{I} \cap \mathbb{S}^\intercal ; \mathbb{T}) \cup (\mathbb{I} \cap \overline{\mathbb{S}^\intercal ; \mathbb{T}}) = (\mathbb{I} \cap \mathbb{S}^\intercal ; \mathbb{T}) \cup (\mathbb{I} \cap z ; \mathbb{T})$$

and proceed with the Dedekind rule:

$$\mathbb{S}^\intercal ; \mathbb{T} \cap \mathbb{I} \subseteq (\mathbb{S}^\intercal \cap \mathbb{I} ; \mathbb{T}^\intercal) ; (\mathbb{T} \cap \mathbb{S} ; \mathbb{I}) = \mathbb{S}^\intercal ; \mathbb{S}$$
$$\overline{\mathbb{S}^\intercal ; \mathbb{T}} \cap \mathbb{I} = z ; \mathbb{T} \cap \mathbb{I} \subseteq (z \cap \mathbb{I} ; \mathbb{T}^\intercal) ; (\mathbb{T} \cap z^\intercal ; \mathbb{I}) = z ; z^\intercal$$

The second claim follows from (ii):

$$\mathbb{S} \cap \mathbb{T} ; z^\intercal = \mathbb{\bot} \iff \mathbb{T} = \overline{\mathbb{S}} \cup \overline{\mathbb{T} ; z^\intercal} \iff \mathbb{T} ; z^\intercal \subseteq \overline{\mathbb{S}} \iff \mathbb{S} ; z \subseteq \mathbb{\bot} \iff \mathbb{S}^\intercal ; \mathbb{T} \subseteq \overline{z}$$ □

The following propositions will all proceed gradually. Of course, we ultimately aim at the naturals in orange as shown in Fig. 5.1.1. Some results in number theory may, however, be obtained already for initial finite segments of these, i.e. in the yellow region. The white area has just one task: It shall be used for examples showing what may happen if regressive finiteness is not yet demanded. (The grey area is nothing other than the union of yellow and orange introduced for the comment.)

[1] According to [11, 12, Prop. 6.3.5] $\mathbb{S}$ is then due to its finite degree, also regressively bounded. Therefore, it satisfies $\mathbb{T} = \mathtt{sup}_{h \geq 0} \overline{\mathbb{S}^{\intercal h} ; \mathbb{T}}$ — following [11, 12, Prop. 6.3.3.iii].

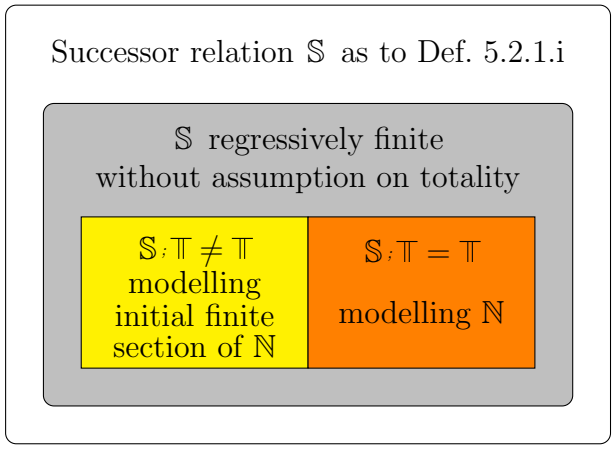


Fig. 5.1.1 Introducing properties of successor relations step by step

Such a decomposition seems reasonable. A proof of the Goldbach conjecture, e.g., might be possible already in the yellow area.

5.2 Iterations over successor relations

Later, we consider in particular the transitive closure of the successor relation $\mathbb{S}$ hoping for the linear strictorder on the possibly finite initial section of naturals. However, there has to be made use of regressive finiteness in a sophisticated way.

With the following iterative concept we aim at restricting to what is reachable from z. To this end, we now instantiate Lemma 4.2.1 with a new functional, denoted v.

5.2.1 Definition and Proposition. Given the successor relation $\mathbb{S}$ as discussed before, we consider the functional

$$v(X) := V_0 \cup v_H(X) \quad \text{with} \quad V_0 := z;\mathbb{T} \quad \text{and} \quad v_H(X) := \mathbb{S}^{\mathsf{T}};X$$

and define

$$V := V_{\mathbb{S}} := \mathtt{inf}\,\{\, X \mid v(X) \subseteq X\},$$

what shall afterwards be called the **vector of $\mathbb{S}$-reachable points**. In addition, we define even and odd numbers as partial diagonal or as a vector:

$$\begin{aligned}
\mathtt{eve} &:= \mathtt{sup}_{i\geq 0}\ \mathbb{I} \cap \quad (\mathbb{S}^{\mathsf{T}2})^i;z;\mathbb{T} = \mathbb{I} \cap \quad (\mathbb{S}^{\mathsf{T}2})^*;z;\mathbb{T}\\
\mathtt{odd} &:= \mathtt{sup}_{i\geq 0}\ \mathbb{I} \cap \mathbb{S}^{\mathsf{T}};(\mathbb{S}^{\mathsf{T}2})^i;z;\mathbb{T} = \mathbb{I} \cap \mathbb{S}^{\mathsf{T}};(\mathbb{S}^{\mathsf{T}2})^*;z;\mathbb{T}\\
\mathtt{eve};\mathbb{T} &:= \mathtt{sup}_{i\geq 0}\ \quad (\mathbb{S}^{\mathsf{T}2})^i;z;\mathbb{T} = \quad (\mathbb{S}^{\mathsf{T}2})^*;z;\mathbb{T}\\
\mathtt{odd};\mathbb{T} &:= \mathtt{sup}_{i\geq 0}\ \mathbb{S}^{\mathsf{T}};(\mathbb{S}^{\mathsf{T}2})^i;z;\mathbb{T} = \mathbb{S}^{\mathsf{T}};(\mathbb{S}^{\mathsf{T}2})^*;z;\mathbb{T}
\end{aligned}$$

i) Then in general:

a) $V = v(V) = z;\mathbb{T} \cup \mathbb{S}^{\mathsf{T}};V = \mathtt{sup}_{n\geq 0}\, v_H^n(V_0) = \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\mathsf{T}n};z;\mathbb{T} = \mathbb{S}^{\mathsf{T}*};z;\mathbb{T}$

b) $\mathbb{S}^k;\mathbb{S}^{\mathsf{T}n} = \mathbb{S}^{\mathsf{T}n-k} \cap \mathbb{S}^k;\mathbb{T} \qquad \mathbb{S}^{\mathsf{T}k};\mathbb{S}^n = \mathbb{S}^{n-k} \cap \mathbb{S}^{\mathsf{T}k};\mathbb{T} \qquad$ in case $n \geq k$

$\mathbb{S}^k;\mathbb{S}^{\mathsf{T}n} = \mathbb{S}^{k-n} \cap \mathbb{T};\mathbb{S}^{\mathsf{T}n} \qquad \mathbb{S}^{\mathsf{T}k};\mathbb{S}^n = \mathbb{S}^{\mathsf{T}k-n} \cap \mathbb{T};\mathbb{S}^n \qquad$ in case $k \geq n$

$\mathbb{S}^{\intercal k};\mathbb{S}^{n} \cap \mathbb{S}^{n};\mathbb{T} = \mathbb{S}^{n};\mathbb{S}^{\intercal k} \cap \mathbb{S}^{\intercal k};\mathbb{T}$ for arbitrary k, n

$\mathbb{S};\mathbb{S}^{\intercal n} \cap \mathbb{T};\mathbb{S} = \mathbb{S}^{\intercal n};\mathbb{S} \cap \mathbb{S};\mathbb{T}$ for all $n \geq 0$

c) $\mathbb{I} \supseteq \mathbb{S};\mathbb{S}^{\intercal} \supseteq \mathbb{S}^{2};\mathbb{S}^{\intercal 2} \supseteq \ldots \supseteq \mathbb{S}^{n};\mathbb{S}^{\intercal n} \supseteq \ldots$

d) For arbitrary $n \geq 0$ the construct $\mathbb{S}^{\intercal n};z;\mathbb{T}$ is either a point or it is equal to $\perp\!\!\!\perp$ and this implication holds: $\mathbb{S}^{\intercal n};z;\mathbb{T} = \perp\!\!\!\perp \implies \mathbb{S}^{\intercal n+1};z;\mathbb{T} = \perp\!\!\!\perp$

In any case, $n \neq k$ implies $\mathbb{S}^{\intercal n};z;\mathbb{T} \cap \mathbb{S}^{\intercal k};z;\mathbb{T} = \perp\!\!\!\perp$.

e) $\mathbb{T};z^{\intercal};\mathbb{S}^{n};\mathbb{S}^{\intercal k};z;\mathbb{T} = \begin{cases} \perp\!\!\!\perp & \text{in case } n \neq k \\ \perp\!\!\!\perp & \text{in case } n = k \text{ and } \mathbb{S}^{\intercal n};z;\mathbb{T} = \perp\!\!\!\perp \\ \mathbb{T} & \text{in case } n = k \text{ and } \mathbb{S}^{\intercal n};z;\mathbb{T} \neq \perp\!\!\!\perp, \text{ so that } \mathbb{S}^{\intercal n};z;\mathbb{T} \text{ is a point} \end{cases}$

f) If $\mathbb{S};\mathbb{S} \neq \perp\!\!\!\perp$, then $\mathbb{S}^{\intercal};z;\mathbb{T} \subseteq \mathbb{S};\mathbb{T}$.

g) $\mathbb{S}^{\intercal i};z;\mathbb{T} \cap \mathbb{S}^{m} = \mathbb{S}^{m} \cap \mathbb{T};z^{\intercal};\mathbb{S}^{i+m} = \mathbb{S}^{\intercal i};z;\mathbb{T};z^{\intercal};\mathbb{S}^{i+m}$ for all $i, m \geq 0$

$\mathbb{T};z^{\intercal};\mathbb{S}^{j} \cap \mathbb{S}^{\intercal n} = \mathbb{S}^{\intercal n} \cap \mathbb{S}^{\intercal j+n};z;\mathbb{T} = \mathbb{S}^{\intercal j+n};z;\mathbb{T};z^{\intercal};\mathbb{S}^{j}$ for all $j, n \geq 0$

h) $(\mathbb{T};z^{\intercal} \oslash z;\mathbb{T});(\overline{e;\mathbb{T}} \oslash \overline{e;\mathbb{T}}) = z;\mathbb{T}$ [2])

i) $\mathbb{S}^{\intercal n};\mathbb{S}^{n} = \mathbb{I} \cap \mathbb{S}^{\intercal n};\mathbb{T} = \mathbb{I} \cap \mathbb{T};\mathbb{S}^{n}$ for $n \geq 0$

j) $(\mathbb{I} \oslash \mathbb{I});z;\mathbb{T} = (\mathbb{I} \oslash z;\mathbb{T};z^{\intercal});z;\mathbb{T} = (z;\mathbb{T};z^{\intercal} \oslash \mathbb{I});z;\mathbb{T}$

k) $\mathbb{S}^{\intercal n+1};z;\mathbb{T} \neq \perp\!\!\!\perp \implies \mathbb{S}^{*};\mathbb{S}^{\intercal n+1};z;\mathbb{T} = \mathbb{S}^{*};\mathbb{S}^{\intercal n};z;\mathbb{T} \cup \mathbb{S}^{\intercal n+1};z;\mathbb{T}$

ii) In case $\mathbb{S}$ is regressively finite, there hold in addition

a) $V = \mathbb{T} = \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\intercal n};z;\mathbb{T} = \mathbb{S}^{\intercal *};z;\mathbb{T}$ $\quad \overline{z;\mathbb{T}} = \mathbb{S}^{\intercal +};z;\mathbb{T}$ $\quad \mathbb{S}^{+} \subseteq \overline{\mathbb{I}}$

b) $z;\mathbb{T} \subseteq \mathbb{S}^{*}$ and $z;\mathbb{T} \subseteq \mathbb{S};\mathbb{T}$

For $n \geq 0$ always $\mathbb{S}^{\intercal n};z;\mathbb{T} \cap \overline{\mathtt{sup}_{0\leq i<n}\, \mathbb{T};z^{\intercal};\mathbb{S}^{i}} \subseteq \mathbb{S}^{*}$

c) For $n \geq 0$ holds $\overline{\mathbb{S}^{\intercal n+1};\mathbb{T}} = \mathtt{sup}_{0\leq i\leq n}\, \mathbb{S}^{\intercal i};z;\mathbb{T}$ $\quad \mathtt{inf}_{n\geq 1}\, \mathbb{S}^{+n} = \perp\!\!\!\perp$

$\mathtt{sup}_{n\geq 0}\, \overline{\mathbb{S}^{\intercal n};\mathbb{T}} = \mathbb{T}$

d) $\mathbb{S}^{\intercal n};z;\mathbb{T} = \perp\!\!\!\perp \implies \mathbb{S}^{n} = \perp\!\!\!\perp$

e) $\mathbb{S};\mathbb{S}^{\intercal};z;\mathbb{T} = z;\mathbb{T}$

f) $\mathbb{S}^{i} \cap \mathbb{S}^{k} = \begin{cases} \perp\!\!\!\perp & i \neq k \\ \mathbb{S}^{i} & i = k \end{cases}$

g) $\mathtt{eve} \cap \mathtt{odd} = \perp\!\!\!\perp$ $\quad \mathtt{eve} \cup \mathtt{odd} = \mathbb{I}$

$\mathtt{eve};\mathbb{T} \cap \mathtt{odd};\mathbb{T} = \perp\!\!\!\perp$ $\quad \mathtt{eve};\mathbb{T} \cup \mathtt{odd};\mathbb{T} = \mathbb{T}$

[2] Later, we will recognize that $M_0 = (\mathbb{T};z^{\intercal} \oslash z;\mathbb{T})^{\intercal}$ is the start of the multiplication iteration.

$\mathtt{odd} ; \mathbb{T} = \mathbb{S}^{\mathsf{T}} ; \mathtt{eve} ; \mathbb{T} \qquad \mathtt{eve} ; \mathbb{T} = z ; \mathbb{T} \cup \mathbb{S}^{\mathsf{T}} ; \mathtt{odd} ; \mathbb{T}$

$\mathbb{S} ; \mathtt{odd} ; \mathbb{T} \subseteq \mathtt{eve} ; \mathbb{T} \qquad \mathbb{S} ; \mathtt{eve} ; \mathbb{T} \subseteq \mathtt{odd} ; \mathbb{T}$

$\mathbb{S}^{\mathsf{T}} ; \mathtt{eve} ; \mathbb{S} = \mathtt{odd} \qquad \mathbb{S}^{\mathsf{T}} ; \mathtt{odd} ; \mathbb{S} \subseteq \mathtt{eve} \qquad \mathbb{S} ; \mathtt{odd} ; \mathbb{S}^{\mathsf{T}} \subseteq \mathtt{eve} \qquad \mathbb{S} ; \mathtt{eve} ; \mathbb{S}^{\mathsf{T}} \subseteq \mathtt{odd}$

h) Given the two points p, q, all these tiny formulae are equivalent:

$p ; q^{\mathsf{T}} \subseteq \mathbb{S} \iff p \subseteq \mathbb{S} ; q \iff p = \mathbb{S} ; q \iff q \subseteq \mathbb{S}^{\mathsf{T}} ; p \iff q = \mathbb{S}^{\mathsf{T}} ; p$

$\iff \mathbb{S} \cap p ; \mathbb{T} = p ; q^{\mathsf{T}} \iff \mathbb{S}^{\mathsf{T}} \cap q ; \mathbb{T} = q ; p^{\mathsf{T}}$

i) $\forall n \geq 0 : \ \mathtt{sup}_{0 \leq i \leq n} \, \mathbb{S}^{\mathsf{T}i} ; z ; \mathbb{T} \cup \mathtt{sup}_{0 \leq j \leq n} \, \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{j} \subseteq \mathbb{S}^{*} \cup \mathbb{S}^{\mathsf{T}*}$

$(\mathbb{S}^{\mathsf{T}} ; \mathbb{S}^{*} \cap \mathbb{S}^{*}) ; \mathbb{T} = \mathbb{S}^{\mathsf{T}} ; \mathbb{T}$

j) For $i \geq 0$ always $\mathbb{S}^{\mathsf{T}*} ; \mathbb{S}^{\mathsf{T}i} ; z ; \mathbb{T} = \mathbb{S}^{\mathsf{T}i} ; \mathbb{T}$

k) $\mathbb{S}^{+}$ is regressively finite.

iii) If $\mathbb{S}$ happens to be regressively finite but not total, i.e. $\mathbb{S} ; \mathbb{T} \neq \mathbb{T}$,

a) $g_{\mathbb{S}} := \overline{\mathbb{S} ; \mathbb{T}}$ is a point

b) $g_{\mathbb{S}} ; g_{\mathbb{S}}^{\mathsf{T}} \cup \mathbb{S} ; \mathbb{S}^{\mathsf{T}} = \mathbb{I}$ is a disjoint decomposition of the identity, $\mathbb{S} ; \mathbb{S}^{\mathsf{T}} = \mathbb{I} \cap \overline{g_{\mathbb{S}} ; \mathbb{T}}$

c) $\mathbb{S}^{\mathsf{T}n_{\mathbb{S}}} ; z ; \mathbb{T} = g_{\mathbb{S}}$ for some $n_{\mathbb{S}}$ and $\mathbb{S}^{\mathsf{T}n} ; z ; \mathbb{T} = \mathbb{\perp}$ as well as $\mathbb{S}^{n} = \mathbb{\perp}$ if $n > n_{\mathbb{S}}$

d) $\mathbb{S}^{n_{\mathbb{S}}} = z ; \mathbb{T} ; g_{\mathbb{S}}^{\mathsf{T}} \qquad \mathbb{S}^{n_{\mathbb{S}} - n} ; g_{\mathbb{S}} = \mathbb{S}^{\mathsf{T}n} ; z$ for $0 \leq n \leq n_{\mathbb{S}}$

e) $V = \mathbb{T} = \mathtt{sup}_{n_{\mathbb{S}} \geq n \geq 0} \, \mathbb{S}^{\mathsf{T}n} ; z ; \mathbb{T}$

f) $\mathtt{FlipTD} :== \mathtt{sup}_{n_{\mathbb{S}} \geq n \geq 0} \, \mathbb{S}^{\mathsf{T}n} ; z ; \mathbb{T} ; g_{\mathbb{S}}^{\mathsf{T}}$ is an involutory bijective map

$\mathtt{FlipPairsTD} := (\mathtt{FlipTD} \otimes \mathtt{FlipTD})$

g) $\mathtt{FlipTD} ; z = g_{\mathbb{S}} \qquad \mathtt{FlipTD} ; g_{\mathbb{S}} = z \qquad \mathtt{FlipTD} \cap \mathbb{T} ; g_{\mathbb{S}}^{\mathsf{T}} ; \mathbb{S}^{\mathsf{T}k} = \mathbb{S}^{\mathsf{T}k} ; z ; \mathbb{T} ; g_{\mathbb{S}}^{\mathsf{T}} ; \mathbb{S}^{\mathsf{T}k}$

h) $\mathtt{FlipTD}^{\mathsf{T}} = \mathtt{FlipTD} \qquad \mathtt{FlipTD} ; \mathtt{FlipTD} = \mathbb{I}$

i) $\mathtt{FlipTD} ; \mathbb{S}^{\mathsf{T}n} = \mathbb{S}^{n} ; \mathtt{FlipTD}$ as well as $\mathtt{FlipTD} ; \mathbb{S}^{n} = \mathbb{S}^{\mathsf{T}n} ; \mathtt{FlipTD}$ for $n \geq 0$

j) $\mathtt{FlipPairsTD} ; \mathtt{FlipPairsTD} = \mathbb{I} \qquad \mathfrak{P} ; \mathtt{FlipPairsTD} = \mathtt{FlipPairsTD} ; \mathfrak{P}$

k) $(g_{\mathbb{S}} ; \mathbb{T} \circledcirc \mathbb{I}) ; (\mathtt{FlipTD} \cap \mathbb{T} ; g_{\mathbb{S}}^{\mathsf{T}} ; \mathbb{S}^{\mathsf{T}n}) \subseteq (\mathbb{S}^{\mathsf{T}n} \circledcirc \mathbb{S}^{\mathsf{T}n} ; z ; \mathbb{T})$

l) $\mathbb{S}^{\mathsf{T}n} ; z ; \mathbb{T} \subseteq \mathbb{S} ; \mathbb{T}$ for $0 \leq n < n_{\mathbb{S}}$

m) $\mathbb{S}^{\mathsf{T}i} ; z ; \mathbb{T} \cap \mathbb{S}^{m} = \mathbb{\perp} \iff i + m > n_{\mathbb{S}}$

n) $\mathbb{S} ; \mathbb{S}^{\mathsf{T}} = \mathbb{I} \cap \mathbb{S} ; \mathbb{T} = \mathbb{I} \cap \overline{g_{\mathbb{S}} ; \mathbb{T}} = \mathbb{I} \cap \overline{g_{\mathbb{S}} ; \mathbb{T} ; g_{\mathbb{S}}^{\mathsf{T}}}$

o) $g_{\mathbb{S}} ; \mathbb{T} \cup \mathbb{S} ; \mathtt{eve} ; \mathbb{T} = \overline{\mathbb{S} ; \mathtt{odd} ; \mathbb{T}} \qquad g_{\mathbb{S}} ; \mathbb{T} \cup \mathbb{S} ; \mathtt{odd} ; \mathbb{T} = \overline{\mathbb{S} ; \mathtt{eve} ; \mathbb{T}}$

iv) In case $\mathbb{S}$ is regressively finite and also total, i.e. $\mathbb{S};\mathbb{T} = \mathbb{T}$,

a) $\mathbb{S}$ is an injective map, $\mathbb{S};\mathbb{S}^\intercal = \mathbb{I}$, and $g_{\mathbb{S}} = \mathbb{L}$.

b) $\mathbb{S}^k;\mathbb{S}^{\intercal n} = \mathbb{S}^{\intercal n-k}$ if $n \geq k$

$\mathbb{S}^k;\mathbb{S}^{\intercal n} = \mathbb{S}^{k-n}$ if $k \geq n$

$\mathbb{S}^{\intercal k};\mathbb{S}^n = \mathbb{S}^n;\mathbb{S}^{\intercal k} \cap \mathbb{S}^{\intercal k};\mathbb{T}$ for arbitrary k, n

c) $\mathbb{S};\mathtt{odd};\mathbb{T} = \mathtt{eve};\mathbb{T}$ $\mathbb{S};\mathtt{eve};\mathbb{T} = \mathtt{odd};\mathbb{T}$

Proof: We observe in advance that Lemma 4.2.1 may be applied since the functional v_H is

isotonic, i.e. $A \subseteq B \implies \mathbb{S}^\intercal;A \subseteq \mathbb{S}^\intercal;B$, i.e. $v_H(A) \subseteq v_H(B)$,
additive, i.e. $v_H(X \cup Y) = \mathbb{S}^\intercal;(X \cup Y) = \mathbb{S}^\intercal;X \cup \mathbb{S}^\intercal;Y = v_H(X) \cup v_H(Y)$,
$\cap$-hereditary, i.e. $v_H(X \cap Y) = \mathbb{S}^\intercal;(X \cap Y) = \mathbb{S}^\intercal;X \cap \mathbb{S}^\intercal;Y = v_H(X) \cap v_H(Y)$,
for the latter using that $\mathbb{S}$ is injective.

---

i.a) This follows from Lemma 4.2.1.i,iv. The last equality follows from Prop. 4.3.1.iii.

---

i.b) In the first four cases, "$\subseteq$" follows from injectivity and univalency. Direction "$\supseteq$" uses the Dedekind formula — only explicated for the very first equation:

$\mathbb{S}^k;\mathbb{T} \cap \mathbb{S}^{\intercal n-k} \subseteq (\mathbb{S}^k \cap \mathbb{S}^{\intercal n-k};\mathbb{T}^\intercal);(\mathbb{T} \cap \mathbb{S}^{\intercal k};\mathbb{S}^{\intercal n-k}) \subseteq \mathbb{S}^k;\mathbb{S}^{\intercal n}$

The final part is proved to be a cyclic containment:

$\mathbb{S}^n;\mathbb{T} \cap \mathbb{S}^{\intercal k};\mathbb{S}^n \subseteq (\mathbb{S}^n \cap \mathbb{S}^{\intercal k};\mathbb{S}^n;\mathbb{T});(\mathbb{T} \cap \mathbb{S}^{\intercal n};\mathbb{S}^{\intercal k};\mathbb{S}^n)$ Dedekind
$= (\mathbb{S}^n \cap \mathbb{S}^{\intercal k};\mathbb{S}^n;\mathbb{T});\mathbb{S}^{\intercal n+k};\mathbb{S}^n$
$= \mathbb{S}^n;\mathbb{S}^{\intercal k};\mathbb{S}^{\intercal n};\mathbb{S}^n \cap \mathbb{S}^{\intercal k};\mathbb{S}^n;\mathbb{T}$ masking
$\subseteq \mathbb{S}^n;\mathbb{S}^{\intercal k} \cap \mathbb{S}^{\intercal k};\mathbb{T}$ since $\mathbb{S}$ is univalent
$= \mathbb{S}^{\intercal k};\mathbb{T} \cap \mathbb{S}^n;\mathbb{S}^{\intercal k}$
$\subseteq (\mathbb{S}^{\intercal k} \cap \mathbb{S}^n;\mathbb{S}^{\intercal k};\mathbb{T});(\mathbb{T} \cap \mathbb{S}^k;\mathbb{S}^n;\mathbb{S}^{\intercal k})$ Dedekind
$= (\mathbb{S}^{\intercal k} \cap \mathbb{S}^n;\mathbb{S}^{\intercal k};\mathbb{T});\mathbb{S}^{k+n};\mathbb{S}^{\intercal k}$
$= \mathbb{S}^{\intercal k};\mathbb{S}^n;\mathbb{S}^k;\mathbb{S}^{\intercal k} \cap \mathbb{S}^n;\mathbb{S}^{\intercal k};\mathbb{T}$ masking
$\subseteq \mathbb{S}^{\intercal k};\mathbb{S}^n \cap \mathbb{S}^n;\mathbb{T}$ since $\mathbb{S}$ is injective

---

The claim will certainly hold for $n = 0$. The proof — not an induction! — of the case $n > 0$ consists of the two non-trivial mutual containments and of two estimations:

$\mathbb{S};\mathbb{S}^{\intercal n} \cap \mathbb{T};\mathbb{S} \subseteq \mathbb{S}^{\intercal n-1} \cap \mathbb{T};\mathbb{S}$ since $\mathbb{S}$ is injective
$\subseteq \big(\mathbb{T} \cap \mathbb{S}^{\intercal n-1};\mathbb{S}^\intercal\big);\big(\mathbb{S} \cap \mathbb{T};\mathbb{S}^{\intercal n-1}\big) \subseteq \mathbb{S}^{\intercal n};\mathbb{S}$ Dedekind formula
$\mathbb{S}^{\intercal n};\mathbb{S} \cap \mathbb{S};\mathbb{T} \subseteq \mathbb{S}^{\intercal n-1} \cap \mathbb{S};\mathbb{T}$ because $\mathbb{S}$ is univalent
$\subseteq \big(\mathbb{S} \cap \mathbb{S}^{\intercal n-1};\mathbb{T}\big);\big(\mathbb{T} \cap \mathbb{S}^\intercal;\mathbb{S}^{\intercal n-1}\big) \subseteq \mathbb{S};\mathbb{S}^{\intercal n}$ Dedekind formula

---

i.c) by induction: $\mathbb{I} \supseteq \mathbb{S}^n;\mathbb{S}^{\intercal n}$ implies $\mathbb{I} \supseteq \mathbb{S}^n;\mathbb{I};\mathbb{S}^{\intercal n} \supseteq \mathbb{S}^n;\mathbb{S};\mathbb{S}^\intercal;\mathbb{S}^{\intercal n} = \mathbb{S}^{n+1};\mathbb{S}^{\intercal n+1}$

---

i.d) $p := \mathbb{S}^{\intercal n};z;\mathbb{T}$ is certainly row-constant. We have injectivity of p,

$p;p^\intercal = \mathbb{S}^{\intercal n};z;\mathbb{T};z^\intercal;\mathbb{S}^n \subseteq \mathbb{S}^{\intercal n};\mathbb{S}^n \subseteq \mathbb{I}$,

since z is injective and $\mathbb{S}$ is univalent. Now either $p = \mathbb{L}$ or $p \neq \mathbb{L}$, in which case p is a point. The final implication is then trivial.

Without loss of generality, we assume $n < k$ and apply the Schröder equivalence for

$\mathbb{S}^n;\mathbb{S}^{\mathsf{T}k};z;\mathbb{T} = (\mathbb{S}^{\mathsf{T}k-n} \cap \mathbb{S}^n;\mathbb{T});z;\mathbb{T} \subseteq \mathbb{S}^{\mathsf{T}};\mathbb{T} = \overline{z;\mathbb{T}}$ using (i.b) with k, n exchanged

$\iff \quad \mathbb{S}^{\mathsf{T}n};z;\mathbb{T} \subseteq \overline{\mathbb{S}^{\mathsf{T}k};z;\mathbb{T}}$

$\iff \quad \mathbb{S}^{\mathsf{T}n};z;\mathbb{T} \cap \mathbb{S}^{\mathsf{T}k};z;\mathbb{T} = \bot\!\!\!\bot$

---

i.e) The first case is obtained reducing via injectivity of $\mathbb{S}$ and finally applying $\mathbb{S};z = \bot\!\!\!\bot$. The second case and the third are then trivial.

---

i.f) We prove in advance that $z;\mathbb{T};\mathbb{T} \cap \mathbb{S}^2 = z;\mathbb{T};z^{\mathsf{T}};\mathbb{S}^2$. Direction "$\supseteq$" is obvious since z is injective. The other direction follows with the Dedekind rule:

$z;\mathbb{T};\mathbb{T} \cap \mathbb{S}^2 \subseteq (z;\mathbb{T} \cap \mathbb{S}^2;\mathbb{T}^{\mathsf{T}});(\mathbb{T} \cap \mathbb{T};z^{\mathsf{T}};\mathbb{S}^2) \subseteq z;\mathbb{T};z^{\mathsf{T}};\mathbb{S}^2$

Assuming $\mathbb{S}^{\mathsf{T}};z;\mathbb{T} \subseteq \mathbb{S};\mathbb{T}$ were *not* satisfied, we certainly have $\mathbb{S}^{\mathsf{T}};z;\mathbb{T} \neq \bot\!\!\!\bot$, and it is according to (i.d) a point, so that necessarily $\mathbb{S}^{\mathsf{T}};z;\mathbb{T} \subseteq \overline{\mathbb{S};\mathbb{T}}$. This in turn is equivalent with $\mathbb{S};\mathbb{S};\mathbb{T} \subseteq \overline{z;\mathbb{T}}$ and implies the contradiction

$(z;\mathbb{T};\mathbb{T} \cap \mathbb{S}^2);\mathbb{T} \subseteq \mathbb{S};\mathbb{S};\mathbb{T} \subseteq \overline{z;\mathbb{T}}$

unless $\mathbb{S};\mathbb{S} = \bot\!\!\!\bot$.

An example with $\mathbb{S};\mathbb{S} = \bot\!\!\!\bot$ is provided by the limitation of $\mathbb{N}$ to just $\{0,1\}$:

$$\mathbb{S} = \begin{array}{c} \\ 0 \\ 1 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \end{array} \qquad \mathbb{S}^{\mathsf{T}};z;\mathbb{T} = \begin{array}{c} \\ 0 \\ 1 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} \end{pmatrix} \end{array} \not\subseteq \begin{array}{c} \\ 0 \\ 1 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \end{array} = \mathbb{S};\mathbb{T}$$

---

i.g) It has to be shown that $\mathbb{S}^{\mathsf{T}i};z;\mathbb{T} \cap \mathbb{S}^m = \mathbb{S}^m \cap \mathbb{T};z^{\mathsf{T}};\mathbb{S}^{i+m}$. The left side is contained in the right one, since with the Dedekind formula

$\mathbb{S}^{\mathsf{T}i};z;\mathbb{T} \cap \mathbb{S}^m \subseteq (\mathbb{S}^{\mathsf{T}i} \cap \mathbb{S}^m;\mathbb{T};z^{\mathsf{T}});(z;\mathbb{T} \cap \mathbb{S}^i;\mathbb{S}^m) \subseteq (\mathbb{T};z^{\mathsf{T}});(\mathbb{S}^i;\mathbb{S}^m) = \mathbb{T};z^{\mathsf{T}};\mathbb{S}^{i+m}$,

and also for the opposite direction

$\mathbb{T};z^{\mathsf{T}};\mathbb{S}^{i+m} \cap \mathbb{S}^m \subseteq (\mathbb{T};z^{\mathsf{T}} \cap \mathbb{S}^m;\mathbb{S}^{\mathsf{T}i+m});(\mathbb{S}^{i+m} \cap z;\mathbb{T};\mathbb{S}^m) \subseteq \mathbb{S}^m;\mathbb{S}^{\mathsf{T}i+m};z;\mathbb{T} \subseteq \mathbb{S}^{\mathsf{T}i};z;\mathbb{T}$

For the third version certainly

$\mathbb{S}^{\mathsf{T}i};z;\mathbb{T} \cap \mathbb{S}^m = \mathbb{S}^m \cap \mathbb{T};z^{\mathsf{T}};\mathbb{S}^{i+m} \supseteq \mathbb{S}^{\mathsf{T}i};z;\mathbb{T};z^{\mathsf{T}};\mathbb{S}^{i+m}$,

for which univalency of $\mathbb{S}$ and injectivity of z has been used. Containment holds also in the other direction since with the Dedekind formula:

$\mathbb{S}^{\mathsf{T}i};z;\mathbb{T} \cap \mathbb{S}^m \subseteq (\mathbb{S}^{\mathsf{T}i} \cap \mathbb{S}^m;\mathbb{T};z^{\mathsf{T}});(z;\mathbb{T} \cap \mathbb{S}^i;\mathbb{S}^m)$

$\subseteq \mathbb{S}^{\mathsf{T}i};z;\mathbb{T} \cap \mathbb{T};z^{\mathsf{T}};\mathbb{S}^{i+m} = \mathbb{S}^{\mathsf{T}i};z;\mathbb{T};z^{\mathsf{T}};\mathbb{S}^{i+m}$ masking

---

The second version has been obtained by transposition together with renaming $(i, m) \mapsto (j, n)$.

---

i.h) $(\mathbb{T};z^{\mathsf{T}} \circledcirc z;\mathbb{T});(\overline{e;\mathbb{T}} \circledcirc \overline{e;\mathbb{T}}) = (\mathbb{T};z^{\mathsf{T}};\pi^{\mathsf{T}} \cap z;\mathbb{T};\rho^{\mathsf{T}});(\pi;\overline{e;\mathbb{T}} \cap \rho;\overline{e;\mathbb{T}})$

$= (\mathbb{T};z^{\mathsf{T}};\pi^{\mathsf{T}} \cap z;\mathbb{T});(\pi;\overline{e;\mathbb{T}} \cap \rho;\overline{e;\mathbb{T}})$ since ρ is total

$= \mathbb{T};z^{\mathsf{T}};\pi^{\mathsf{T}};(\pi;\overline{e;\mathbb{T}} \cap \rho;\overline{e;\mathbb{T}}) \cap z;\mathbb{T}$ masking

$= \mathbb{T};z^{\mathsf{T}};(\overline{e;\mathbb{T}} \cap \pi^{\mathsf{T}};\rho;\overline{e;\mathbb{T}}) \cap z;\mathbb{T}$ destroy and append

$= \mathbb{T};z^{\mathsf{T}};(\overline{e;\mathbb{T}} \cap \mathbb{T};\overline{e;\mathbb{T}}) \cap z;\mathbb{T}$

$= \mathbb{T};z^{\mathsf{T}};\overline{e;\mathbb{T}} \cap z;\mathbb{T}$

$= \mathbb{T};z^{\mathsf{T}};\overline{\mathbb{S}^{\mathsf{T}};z;\mathbb{T}} \cap z;\mathbb{T}$ by definition of e

$= \mathbb{T} ; z^{\mathsf{T}} ; \left[\mathbb{S}^{\mathsf{T}} ; \overline{z ; \mathbb{T}} \cup \overline{\mathbb{S}^{\mathsf{T}} ; \mathbb{T}} \right] \cap z ; \mathbb{T}$ since $\mathbb{S}$ is injective
$= \left[\mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\mathsf{T}} ; \overline{z ; \mathbb{T}} \cup \mathbb{T} ; z^{\mathsf{T}} ; \overline{\mathbb{S}^{\mathsf{T}} ; \mathbb{T}} \right] \cap z ; \mathbb{T}$
$= \left[\mathbb{L} \cup \mathbb{T} ; z^{\mathsf{T}} ; z ; \mathbb{T} \right] \cap z ; \mathbb{T}$ by definition of z and because $\mathbb{S} ; z = \mathbb{L}$
$= \mathbb{T} \cap z ; \mathbb{T}$
$= z ; \mathbb{T}$

---

i.i) The $\mathbb{S}^n$ are obviously all injective. We prove the relevant direction for an arbitrary injective relation R:

$R^{\mathsf{T}} ; \mathbb{T} \cap \mathbb{I} \subseteq (R^{\mathsf{T}} \cap \mathbb{I} ; \mathbb{T}) ; (\mathbb{T} \cap R ; \mathbb{I}) = R^{\mathsf{T}} ; R$

---

i.j) $(\mathbb{I} \oslash \mathbb{I}) ; z ; \mathbb{T} = (z ; \mathbb{T} \oslash z ; \mathbb{T})$ since $z ; \mathbb{T}$ is injective
$= \pi ; z ; \mathbb{T} \cap \rho ; z ; \mathbb{T}$
$\subseteq (\pi \cap \rho ; z ; \mathbb{T} ; z^{\mathsf{T}}) ; (z ; \mathbb{T} \cap \pi^{\mathsf{T}} ; \rho ; z ; \mathbb{T})$ Dedekind
$= (\mathbb{I} \oslash z ; \mathbb{T} ; z^{\mathsf{T}}) ; z ; \mathbb{T}$ because with z a point $\pi^{\mathsf{T}} ; \rho ; z ; \mathbb{T} = \mathbb{T} ; z ; \mathbb{T} = \mathbb{T}$
$\subseteq (\mathbb{I} \oslash \mathbb{I}) ; z ; \mathbb{T}$ since $z ; \mathbb{T}$ is injective

Closing this cycle, all are equal.

---

i.k) We have to prove

$\mathbb{S}^* ; \mathbb{S}^{\mathsf{T}} ; \mathbb{S}^{\mathsf{T}^n} ; z ; \mathbb{T} = (\mathbb{S}^* \cup \mathbb{S}^{\mathsf{T}}) ; \mathbb{S}^{\mathsf{T}^n} ; z ; \mathbb{T}$

and show first that direction „$\subseteq$“ is equivalent via shunting a point with

$\mathbb{S}^* ; \mathbb{S}^{\mathsf{T}} ; \mathbb{S}^{\mathsf{T}^n} ; z ; \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^n \subseteq \mathbb{S}^* ; \mathbb{S}^{\mathsf{T}} ; \mathbb{I} = (\mathbb{S}^+ \cup \mathbb{I}) ; \mathbb{S}^{\mathsf{T}} \subseteq \mathbb{S}^+ ; \mathbb{S}^{\mathsf{T}} \cup \mathbb{S}^{\mathsf{T}} \subseteq \mathbb{S}^* \cup \mathbb{S}^{\mathsf{T}}$.

For „$\supseteq$“ the second part is trivially satisfied. The first is obvious via shunting a point:

$\mathbb{S}^* ; \mathbb{S}^{\mathsf{T}^n} ; z ; \mathbb{T} \subseteq \mathbb{S}^* ; \mathbb{S}^{\mathsf{T}^{n+1}} ; z ; \mathbb{T} \quad \Longleftrightarrow \quad \mathbb{S}^* ; \mathbb{S}^{\mathsf{T}^n} ; z ; \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{n+1} \subseteq \mathbb{S}^* ; \mathbb{S} \subseteq \mathbb{S}^*$

---

ii.a) We prove $V = \mathbb{T}$, starting with the result (i.a) negated

$\overline{V} = \overline{z ; \mathbb{T}} \cap \overline{\mathbb{S}^{\mathsf{T}} ; V} = \mathbb{S}^{\mathsf{T}} ; \mathbb{T} \cap [\mathbb{S}^{\mathsf{T}} ; \overline{V} \cup \overline{\mathbb{S}^{\mathsf{T}} ; \mathbb{T}}]$ since $\mathbb{S}$ is injective
$= [\mathbb{S}^{\mathsf{T}} ; \mathbb{T} \cap \mathbb{S}^{\mathsf{T}} ; \overline{V}] \cup [\mathbb{S}^{\mathsf{T}} ; \mathbb{T} \cap \overline{\mathbb{S}^{\mathsf{T}} ; \mathbb{T}}]$
$= \mathbb{S}^{\mathsf{T}} ; \overline{V} \cup \mathbb{L} = \mathbb{S}^{\mathsf{T}} ; \overline{V}$,

enabling us to make use of the condition for regressive finiteness. We have, namely, from Def. 5.1.1.iii that $\overline{V} = \mathbb{L}$, or else, $V = \mathbb{T}$, according to the introductory remark of Sect. 4.1 or Prop. 4.1.2. Furthermore

$\mathbb{S}^{\mathsf{T}^+} ; z ; \mathbb{T} = \mathbb{S}^{\mathsf{T}} ; \mathbb{S}^{\mathsf{T}^*} ; z ; \mathbb{T} = \mathbb{S}^{\mathsf{T}} ; \mathbb{T} = \overline{z ; \mathbb{T}}$.

To show that $\mathbb{S}^+$ is irreflexive, we assume there might exist a point x with $x ; x^{\mathsf{T}} \subseteq \mathbb{S}^+ \cap \mathbb{I}$. Shunting, we get $x \subseteq (\mathbb{S}^+ \cap \mathbb{I}) ; x = \mathbb{S}^+ ; x \cap x$, so that $x \subseteq \mathbb{S}^+ ; x$ and, thus, also that $x \subseteq \mathbb{S}^{+\mathsf{T}} ; x$. Considering the set $G := \mathbb{S}^{*\mathsf{T}} ; x$, we find out that

$G = \mathbb{S}^{*\mathsf{T}} ; x \subseteq \mathbb{S}^{*\mathsf{T}} ; \mathbb{S}^{+\mathsf{T}} ; x = \mathbb{S}^{+\mathsf{T}} ; \mathbb{S}^{*\mathsf{T}} ; x = \mathbb{S}^{\mathsf{T}} ; \mathbb{S}^{*\mathsf{T}} ; \mathbb{S}^{*\mathsf{T}} ; x = \mathbb{S}^{\mathsf{T}} ; \mathbb{S}^{*\mathsf{T}} ; x = \mathbb{S}^{\mathsf{T}} ; G$.

This gives the opportunity to apply Prop. 4.1.2 and find out that $G = \mathbb{L}$; a contradiction.

---

ii.b) From (ii.a), we have

$\mathbb{T} \subseteq \mathbb{S}^{\mathsf{T}^*} ; z ; \mathbb{T} \quad \Longleftrightarrow \quad \mathbb{T} ; z^{\mathsf{T}} \subseteq \mathbb{S}^{\mathsf{T}^*} \quad \Longleftrightarrow \quad z ; \mathbb{T} \subseteq \mathbb{S}^*$ shunting and transposing

For the second property, we assume that it does not hold, i.e. $z ; \mathbb{T} \not\subseteq \mathbb{S} ; \mathbb{T}$. Then $z ; \mathbb{T} \subseteq \overline{\mathbb{S} ; \mathbb{T}}$, since z is a point. By equivalence $\mathbb{S} ; \mathbb{T} \subseteq \overline{z ; \mathbb{T}}$ and with the Schröder equivalence $\mathbb{S}^{\mathsf{T}} ; z ; \mathbb{T} \subseteq \mathbb{\bot}$, which leads to a contradiction to Def. 5.1.1.i when looking at (ii.a):

$\mathbb{T} = \mathbb{S}^{\mathsf{T}*} ; z ; \mathbb{T} = z ; \mathbb{T} \cup \mathtt{sup}_{n \geq 1} \, \mathbb{S}^{\mathsf{T}n} ; z ; \mathbb{T} = z ; \mathbb{T} \cup \mathbb{\bot} = z ; \mathbb{T} = \overline{\mathbb{S}^{\mathsf{T}} ; \mathbb{T}}$

$\Longrightarrow \quad \mathbb{S}^{\mathsf{T}} ; \mathbb{T} = \mathbb{\bot} \quad \Longrightarrow \quad \mathbb{S} = \mathbb{\bot}$

The last statement will be proved by induction. The case $n = 0$ has just been shown. We prove an equivalent statement, bringing the negated part on right side, i.e.

$\mathbb{S}^{\mathsf{T}n} ; z ; \mathbb{T} \subseteq \mathbb{S}^* \cup \mathtt{sup}_{0 \leq i < n} \, \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^i$

and start the induction as follows:

$\mathbb{S}^{\mathsf{T}n+1} ; z ; \mathbb{T} = \mathbb{S}^{\mathsf{T}} ; \big[\mathbb{S}^{\mathsf{T}n} ; z ; \mathbb{T} \cap \mathbb{S}^{\mathsf{T}n} ; z ; \mathbb{T} \big]$

$= \mathbb{S}^{\mathsf{T}} ; \big[\mathbb{S}^{\mathsf{T}n} ; z ; \mathbb{T} \cap \big\{ \mathbb{S}^* \cup \mathtt{sup}_{0 \leq i < n} \, \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^i \big\} \big]$ hypothesis; maintaining the point $\mathbb{S}^{\mathsf{T}n} ; z ; \mathbb{T}$

$= \mathbb{S}^{\mathsf{T}} ; \big[\mathbb{S}^{\mathsf{T}n} ; z ; \mathbb{T} \cap \big\{ (\mathbb{I} \cup \mathbb{S}^+) \cup \mathtt{sup}_{0 \leq i < n} \, \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^i \big\} \big]$

$= \mathbb{S}^{\mathsf{T}} ; \big[(\mathbb{S}^{\mathsf{T}n} ; z ; \mathbb{T} \cap \mathbb{I}) \cup (\mathbb{S}^{\mathsf{T}n} ; z ; \mathbb{T} \cap \mathbb{S}^+) \cup (\mathbb{S}^{\mathsf{T}n} ; z ; \mathbb{T} \cap \mathtt{sup}_{0 \leq i < n} \, \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^i) \big]$

$\subseteq \mathbb{S}^{\mathsf{T}} ; \big[(\mathbb{S}^{\mathsf{T}n} ; z ; \mathbb{T} \cap \mathbb{I}) \cup \mathbb{S} ; \mathbb{S}^* \cup \mathtt{sup}_{0 \leq i < n} \, \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^i \big]$

$= \mathbb{S}^{\mathsf{T}} ; (\mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^n \cap \mathbb{I}) \cup \mathbb{S}^{\mathsf{T}} ; \mathbb{S} ; \mathbb{S}^* \cup \mathbb{S}^{\mathsf{T}} ; \mathtt{sup}_{0 \leq i < n} \, \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^i$

$\subseteq \mathbb{S}^{\mathsf{T}} ; \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^n \cup \mathbb{S}^* \cup \mathtt{sup}_{0 \leq i < n} \, \mathbb{S}^{\mathsf{T}} ; \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^i$

$\subseteq \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^n \cup \mathbb{S}^* \cup \mathtt{sup}_{0 \leq i < n} \, \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^i = \mathbb{S}^* \cup \mathtt{sup}_{0 \leq i < n+1} \, \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^i$

---

ii.c) In case $n = 0$, we have $\overline{\mathbb{S}^{\mathsf{T}} ; \mathbb{T}} = z ; \mathbb{T} = \mathbb{S}^{\mathsf{T}0} ; z ; \mathbb{T}$ by definition. This starts an induction:

$\overline{\mathbb{S}^{\mathsf{T}n+1} ; \mathbb{T}} = \overline{\mathbb{S}^{\mathsf{T}} ; \mathbb{S}^{\mathsf{T}n} ; \mathbb{T}} = \mathbb{S}^{\mathsf{T}} ; \overline{\mathbb{S}^{\mathsf{T}n} ; \mathbb{T}} \cup \overline{\mathbb{S}^{\mathsf{T}} ; \mathbb{T}}$ standard rule for univalent $\mathbb{S}^{\mathsf{T}}$

$= \mathbb{S}^{\mathsf{T}} ; \big[\mathtt{sup}_{0 \leq i \leq n} \, \mathbb{S}^{\mathsf{T}i} ; z ; \mathbb{T} \big] \cup z ; \mathbb{T}$ induction hypothesis and definition of $z = z ; \mathbb{T}$

$= \mathtt{sup}_{0 \leq i \leq n+1} \, \mathbb{S}^{\mathsf{T}i} ; z ; \mathbb{T}$

In view of the immediately preceding result, this means

$\mathtt{sup}_{k \geq 1} \, \overline{\mathbb{S}^{\mathsf{T}k} ; \mathbb{T}} = \mathtt{sup}_{k \geq 1} \, \mathtt{sup}_{0 \leq i < k} \, \mathbb{S}^{\mathsf{T}i} ; z ; \mathbb{T} = \mathtt{sup}_{0 \leq i} \, \mathbb{S}^{\mathsf{T}i} ; z ; \mathbb{T} = \mathbb{T}$ due to (ii.a).

Reformulation via negation and the obvious $\mathbb{S}^{\mathsf{T}k} ; \mathbb{T} = \mathbb{S}^{+\mathsf{T}k} ; \mathbb{T}$ for $k \geq 1$ leads to

$\mathbb{\bot} = \mathtt{inf}_{k \geq 1} \, \mathbb{S}^{\mathsf{T}k} ; \mathbb{T} = \mathtt{inf}_{k \geq 1} \, \mathbb{S}^{+\mathsf{T}k} ; \mathbb{T} \supseteq \mathtt{inf}_{k \geq 1} \, \mathbb{S}^{+\mathsf{T}k} ; \mathbb{I} = \mathtt{inf}_{k \geq 1} \, \mathbb{S}^{+\mathsf{T}k}$.

Now the last result:

$\mathtt{sup}_{n \geq 0} \, \overline{\mathbb{S}^{\mathsf{T}n} ; \mathbb{T}} = z ; \mathbb{T} \cup \mathtt{sup}_{k \geq 0} \, \overline{\mathbb{S}^{\mathsf{T}k+1} \mathbb{T}} = z ; \mathbb{T} \cup \mathtt{sup}_{k \geq 0} \, \mathtt{sup}_{0 \leq i \leq k} \, \mathbb{S}^{\mathsf{T}i} ; z ; \mathbb{T}$ see before

$= \mathtt{sup}_{0 \leq i} \, \mathbb{S}^{\mathsf{T}i} ; z ; \mathbb{T} = \mathbb{T}$ see (ii.a)

---

ii.d) $\mathbb{T} = \mathtt{sup}_{k \geq 0} \, \mathbb{S}^{\mathsf{T}k} ; z ; \mathbb{T} = \mathtt{sup}_{n > k \geq 0} \, \mathbb{S}^{\mathsf{T}k} ; z ; \mathbb{T} \cup \mathtt{sup}_{k \geq n} \, \mathbb{S}^{\mathsf{T}k} ; z ; \mathbb{T}$ partitioning the result (ii.a)

$= \mathtt{sup}_{n > k \geq 0} \, \mathbb{S}^{\mathsf{T}k} ; z ; \mathbb{T} \cup \mathbb{\bot}$ by assumption $\mathbb{S}^{\mathsf{T}n} ; z ; \mathbb{T} = \mathbb{\bot}$

$= \overline{\mathbb{S}^{\mathsf{T}n} ; \mathbb{T}}$ following (ii.c),

asserting $\mathbb{S}^{\mathsf{T}n} ; \mathbb{T} = \mathbb{\bot}$ and, thus, $\mathbb{S}^n = \mathbb{\bot}$.

---

ii.e) $\mathbb{S} ; \mathbb{S}^{\mathsf{T}} ; z ; \mathbb{T} = (\mathbb{S}^0 \cap \mathbb{S} ; \mathbb{T}) ; z ; \mathbb{T}$ (i.b)

$= (\mathbb{I} \cap \mathbb{S} ; \mathbb{T}) ; z ; \mathbb{T}$

$= \mathbb{I} ; z ; \mathbb{T} \cap \mathbb{S} ; \mathbb{T}$ masking

$= z ; \mathbb{T}$ due to (ii.b)

---

ii.f) The variant with $i = k$ is certainly true and it remains to treat $i \neq k$. We prove this point-wise, starting from $\mathbb{T} = \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\intercal n} ; z ; \mathbb{T}$ in (ii.a), and assuming two points $x = \mathbb{S}^{\intercal p} ; z ; \mathbb{T}$ and $y = \mathbb{S}^{\intercal q} ; z ; \mathbb{T}$ satisfying

$x ; y^{\intercal} \subseteq \mathbb{S}^i \cap \mathbb{S}^k$.
$\iff$ $y \subseteq \mathbb{S}^{\intercal i} ; x$ and $y \subseteq \mathbb{S}^{\intercal k} ; x$ shunting the point x and transposing
$\iff$ $\mathbb{S}^{\intercal q} ; z ; \mathbb{T} \subseteq \mathbb{S}^{\intercal i} ; \mathbb{S}^{\intercal p} ; z ; \mathbb{T}$ and $\mathbb{S}^{\intercal q} ; z ; \mathbb{T} \subseteq \mathbb{S}^{\intercal k} ; \mathbb{S}^{\intercal p} ; z ; \mathbb{T}$

which is only possible if $q = i + p$ and $q = k + p$, contradicting the assumption $i \neq k$.

---

ii.g) This set of statements will be proved in a slightly sloppy form: The first two equations $\mathtt{eve} \cap \mathtt{odd} = \mathbb{\bot}$ and $\mathtt{eve} \cup \mathtt{odd} = \mathbb{I}$ are obvious from the definition together with (i.d). Therefore also $\mathtt{eve} ; \mathbb{T} \cap \mathtt{odd} ; \mathbb{T} = \mathbb{\bot}$ and $\mathtt{eve} ; \mathbb{T} \cup \mathtt{odd} ; \mathbb{T} = \mathbb{T}$.

The results $\mathtt{odd} ; \mathbb{T} = \mathbb{S}^{\intercal} ; \mathtt{eve} ; \mathbb{T}$ and $\mathtt{eve} ; \mathbb{T} = z ; \mathbb{T} \cup \mathbb{S}^{\intercal} ; \mathtt{odd} ; \mathbb{T}$ take into account that $\mathbb{S}^{\intercal} ; \mathtt{odd} ; \mathbb{T}$ cannot produce z.

The next two, $\mathbb{S} ; \mathtt{odd} ; \mathbb{T} \subseteq \mathtt{eve} ; \mathbb{T}$ and $\mathbb{S} ; \mathtt{eve} ; \mathbb{T} \subseteq \mathtt{odd} ; \mathbb{T}$, hold because $\mathbb{S}$ is injective and may not yet be total.

Going up and down or down and up with $\mathbb{S}$ is represented by

$\mathbb{S}^{\intercal} ; \mathtt{eve} ; \mathbb{S} = \mathtt{odd}$ $\quad$ $\mathbb{S}^{\intercal} ; \mathtt{odd} ; \mathbb{S} \subseteq \mathtt{eve}$ $\quad$ $\mathbb{S} ; \mathtt{odd} ; \mathbb{S}^{\intercal} \subseteq \mathtt{eve}$ $\quad$ $\mathbb{S} ; \mathtt{eve} ; \mathbb{S}^{\intercal} \subseteq \mathtt{odd}$

where, looking at $z ; \mathbb{T}$ just one equation will hold in general.

---

ii.h) $p ; q^{\intercal} \subseteq \mathbb{S} \iff p \subseteq \mathbb{S} ; q$ follows from shunting the point q
$\iff$ $p = \mathbb{S} ; q$ since also $\mathbb{S} ; q$ is a point in Prop. 2.2.2.i
$\iff$ $q \subseteq \mathbb{S}^{\intercal} ; p$ correspondingly when starting from the transposed version $q ; p^{\intercal} \subseteq \mathbb{S}^{\intercal}$
$\iff$ $q = \mathbb{S}^{\intercal} ; p$ again with Prop. 2.2.2.i, since also $\mathbb{S}^{\intercal} ; p$ is a point
$\iff$ $\mathbb{S} \cap p ; \mathbb{T} = p ; q^{\intercal}$ the non-trivial direction follows with the Dedekind rule:
$p ; \mathbb{T} \cap \mathbb{S} \subseteq (p \cap \mathbb{S} ; \mathbb{T}) ; (\mathbb{T} \cap p^{\intercal} ; \mathbb{S}) \subseteq p ; p^{\intercal} ; \mathbb{S} = p ; q^{\intercal}$ see above

---

ii.i) Proof by induction with the trivial start $z ; \mathbb{T} \subseteq \mathbb{S}^*$ according to (ii.b) above. The we reason for $n \longrightarrow n + 1$ as follows:

$\mathtt{sup}_{0\leq i\leq n+1}\, \mathbb{S}^{\intercal i} ; z ; \mathbb{T} \cup \mathtt{sup}_{0\leq j\leq n+1}\, \mathbb{T} ; z^{\intercal} ; \mathbb{S}^j$
$\mathbb{S}^{\intercal n+1} ; z ; \mathbb{T} \cup \mathtt{sup}_{0\leq i\leq n}\, \mathbb{S}^{\intercal i} ; z ; \mathbb{T} \cup \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n+1} \cup \mathtt{sup}_{0\leq j\leq n}\, \mathbb{T} ; z^{\intercal} ; \mathbb{S}^j$
$= \left[\mathbb{S}^{\intercal n+1} ; z ; \mathbb{T} \cap \overline{\mathtt{sup}_{0\leq j\leq n}\, \mathbb{T} ; z^{\intercal} ; \mathbb{S}^j} \right] \cup \left[\mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n+1} \cap \overline{\mathtt{sup}_{0\leq j\leq n}\, \mathbb{S}^{\intercal j} ; z ; \mathbb{T}} \right]$
$\quad \cup \mathtt{sup}_{0\leq i\leq n}\, \mathbb{S}^{\intercal i} ; z ; \mathbb{T} \cup \mathtt{sup}_{0\leq j\leq n}\, \mathbb{T} ; z^{\intercal} ; \mathbb{S}^j$
$\subseteq \mathbb{S}^* \cup \mathbb{S}^{\intercal *}$ via hypothesis and (ii.b)

The direction "$\subseteq$" is trivial. For the other we assume the Point Axiom and take some point

$p ; \mathbb{T} \subseteq \mathbb{S}^{\intercal} ; \mathbb{T} \iff \mathbb{T} \subseteq \mathbb{T} ; p^{\intercal} ; \mathbb{S}^{\intercal} ; \mathbb{T}$.

In order to prove the following, we shunt the point

$p ; \mathbb{T} \subseteq (\mathbb{S}^{\intercal} ; \mathbb{S}^* \cap \mathbb{S}^*) ; \mathbb{T} \iff \mathbb{T} \subseteq \mathbb{T} ; p^{\intercal} ; (\mathbb{S}^{\intercal} ; \mathbb{S}^* \cap \mathbb{S}^*) ; \mathbb{T}$

and prove

$\mathbb{T} ; p^{\intercal} ; (\mathbb{S}^{\intercal} ; \mathbb{S}^* \cap \mathbb{S}^*) ; \mathbb{T} = (\mathbb{T} ; p^{\intercal} ; \mathbb{S}^{\intercal} ; \mathbb{S}^* \cap \mathbb{T} ; p^{\intercal} ; \mathbb{S}^*) ; \mathbb{T}$
$= \mathbb{T} ; p^{\intercal} ; \mathbb{S}^{\intercal} ; \mathbb{S}^* ; (\mathbb{T} \cap \mathbb{S}^{\intercal *} ; p ; \mathbb{T})$ mask shifting

$= \mathbb{T} ; p^\intercal ; \mathbb{S}^\intercal ; \mathbb{S}^* ; \mathbb{S}^{\intercal *} ; p ; \mathbb{T} = \mathbb{T} ; p^\intercal ; \mathbb{S}^\intercal ; \mathbb{T} ; p ; \mathbb{T} = \mathbb{T} ; p^\intercal ; \mathbb{S}^\intercal ; \mathbb{T}$ since $p ; \mathbb{T}$ is a point
$\supseteq \mathbb{T}$ see the initial condition

---

ii.j) $\mathbb{S}^{\intercal *} ; \mathbb{S}^{\intercal i} ; z ; \mathbb{T} = [\mathtt{sup}_{k \geq 0} \mathbb{S}^{\intercal k}] ; \mathbb{S}^{\intercal i} ; z ; \mathbb{T}$
$= \mathtt{sup}_{k \geq 0} \mathbb{S}^{\intercal k} ; \mathbb{S}^{\intercal i} ; z ; \mathbb{T} = \mathtt{sup}_{k \geq 0} \mathbb{S}^{\intercal i} ; \mathbb{S}^{\intercal k} ; z ; \mathbb{T} = \mathbb{S}^{\intercal i} ; [\mathtt{sup}_{k \geq 0} \mathbb{S}^{\intercal k} ; z ; \mathbb{T}]$
$= \mathbb{S}^{\intercal i} ; \mathbb{T}$ via (ii.a)

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ii.k) This follows directly from Prop. 4.1.3.

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iii.a) If $\mathbb{S}$ does not satisfy $\mathbb{S} ; \mathbb{T} = \mathbb{T}$, this means $g_{\mathbb{S}} \neq \mathbb{\bot}$ and consequently $\mathbb{T} ; g_{\mathbb{S}} = \mathbb{T}$, since $g_{\mathbb{S}} = g_{\mathbb{S}} ; \mathbb{T}$. The problem is how to establish injectivity of $g_{\mathbb{S}}$.

For a first attempt let us assume points $x \subseteq g_{\mathbb{S}}$ and $y \subseteq g_{\mathbb{S}}$ with $x \neq y$, thus violating injectivity. Due to reachability in (ii.a), it follows that $x \subseteq \mathbb{S}^{\intercal *} ; z ; \mathbb{T}$ and $y \subseteq \mathbb{S}^{\intercal *} ; z ; \mathbb{T}$. Since $z ; \mathbb{T}$ is by definition a point, it may be shunted so that the latter is equivalent with $z ; \mathbb{T} \subseteq \mathbb{S}^* ; y$. In total we have, thus,

$x \subseteq \mathbb{S}^{\intercal *} ; \mathbb{S}^* ; y = (\mathtt{sup}_{n \geq 0} \mathbb{S}^{\intercal n}) ; (\mathtt{sup}_{k \geq 0} \mathbb{S}^k) ; y.$

Since x is a point, it is using Prop. 2.2.2 contained in at least one of these, say, in

$x \subseteq \mathbb{S}^{\intercal n} ; \mathbb{S}^k ; y.$

canceling by univalence $\mathbb{S}^\intercal ; \mathbb{S} \subseteq \mathbb{I}$ as often as possible, we obtain one of these situations

— for $n = k$ that $x \subseteq y$, i.e. $x = y$ since both are points, a contradiction

— for $n < k$ that $x \subseteq \mathbb{S}^{k-n} ; y \subseteq \mathbb{S} ; \mathbb{T} = \overline{g}$, a contradiction

— for $n > k$ that $x \subseteq \mathbb{S}^{\intercal n-k} ; y \iff y \subseteq \mathbb{S}^{n-k} ; x \subseteq \mathbb{S} ; \mathbb{T} = \overline{g}$, a contradiction

---

iii.b) "$\subseteq$" follows from $\mathbb{S}$ and $g_{\mathbb{S}}$ injective. To obtain "$\supseteq$", we decompose

$\mathbb{I} = (\mathbb{I} \cap \mathbb{S} ; \mathbb{T}) \cup (\mathbb{I} \cap \overline{\mathbb{S} ; \mathbb{T}}) = \mathbb{S} ; \mathbb{S}^\intercal \cup (\mathbb{I} \cap g_{\mathbb{S}} ; \mathbb{T})$ by definition of $g_{\mathbb{S}}$

and proceed with the Dedekind formula

$g_{\mathbb{S}} ; \mathbb{T} \cap \mathbb{I} \subseteq (g_{\mathbb{S}} \cap \mathbb{I} ; \mathbb{T}^\intercal) ; (\mathbb{T} \cap g_{\mathbb{S}}^\intercal ; \mathbb{I}) = g_{\mathbb{S}} ; g_{\mathbb{S}}^\intercal.$

---

iii.c) From (ii.a), we now get $g_{\mathbb{S}} \subseteq \mathbb{T} = \mathtt{sup}_{n \geq 0} \mathbb{S}^{\intercal n} ; z ; \mathbb{T}$, and since $g_{\mathbb{S}}$ is a point, also that it is fully contained and thus equal to one of these $g_{\mathbb{S}} = \mathbb{S}^{\intercal n_{\mathbb{S}}} ; z ; \mathbb{T}$, the exponent of which we will henceforth denote as $n_{\mathbb{S}}$.

For $n > n_{\mathbb{S}}$, always $\mathbb{S}^{\intercal n} ; z ; \mathbb{T} = \mathbb{\bot}$, because already $\mathbb{S}^\intercal ; \mathbb{S}^{\intercal n_{\mathbb{S}}} ; z ; \mathbb{T} = \mathbb{S}^\intercal ; g_{\mathbb{S}} ; \mathbb{T} = \mathbb{S}^\intercal ; \overline{\mathbb{S} ; \mathbb{T}} \subseteq \mathbb{\bot}$. The other result uses (ii.d).

---

iii.d) From $\mathbb{S}^{\intercal n_{\mathbb{S}}} ; z ; \mathbb{T} = g_{\mathbb{S}}$ immediately follows $\mathbb{S}^{\intercal n_{\mathbb{S}}} ; z ; \mathbb{T} ; z^\intercal = g_{\mathbb{S}} ; \mathbb{T} ; z^\intercal$ and with injectivity of $z ; \mathbb{T}$ and transposing $\mathbb{S}^{n_{\mathbb{S}}} \subseteq z ; \mathbb{T} ; g_{\mathbb{S}}^\intercal$, i.e. the first half of the claim. The second half uses the definition of $g_{\mathbb{S}}$ and injectivity of $z ; \mathbb{T}$:

$z ; \mathbb{T} ; g_{\mathbb{S}}^\intercal = z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{n_{\mathbb{S}}} \subseteq \mathbb{I} ; \mathbb{S}^{n_{\mathbb{S}}} = \mathbb{S}^{n_{\mathbb{S}}}$

Since $\mathbb{S}^{n_\mathbb{S}} = z;\mathbb{T};g_\mathbb{S}^\intercal \implies \mathbb{S}^{n_\mathbb{S}-0};g_\mathbb{S} = z;\mathbb{T};g_\mathbb{S}^\intercal;g_\mathbb{S} = z;\mathbb{T} = \mathbb{S}^{\intercal 0};z;\mathbb{T}$, the other claim holds for $n = 0$. Furthermore by induction as long as $n+1 \leq n_\mathbb{S}$

$\mathbb{S}^{\intercal n+1};z;\mathbb{T} = \mathbb{S}^\intercal;\mathbb{S}^{\intercal n};z;\mathbb{T} = \mathbb{S}^\intercal;\mathbb{S}^{n_\mathbb{S}-n};g_\mathbb{S} \subseteq \mathbb{S}^{n_\mathbb{S}-(n+1)};g_\mathbb{S}$ with $\mathbb{S}$ univalent.

This implies equality because both sides are points.

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iii.e) follows from (ii.a) together with (iii.c).

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iii.f) `FlipTD` is total:

$\texttt{FlipTD};\mathbb{T} = \texttt{sup}_{n_\mathbb{S}\geq n\geq 0}\,\mathbb{S}^{\intercal n};z;\mathbb{T};g_\mathbb{S}^\intercal;\mathbb{S}^{\intercal n};\mathbb{T} = \texttt{sup}_{n_\mathbb{S}\geq n\geq 0}\,\mathbb{S}^{\intercal n};z;\mathbb{T} = \mathbb{T}$, due to (iii.e)

since $\mathbb{S}^n;g_\mathbb{S};\mathbb{T}$ is always a point, so that $\mathbb{T};g_\mathbb{S}^\intercal;\mathbb{S}^{\intercal n};\mathbb{T} = \mathbb{T}$.

$\texttt{FlipTD}^\intercal;\texttt{FlipTD}$
$= \left[\texttt{sup}_{n_\mathbb{S}\geq n\geq 0}\,\mathbb{S}^{\intercal n};z;\mathbb{T};g_\mathbb{S}^\intercal;\mathbb{S}^{\intercal n}\right]^\intercal;\left[\texttt{sup}_{n_\mathbb{S}\geq k\geq 0}\,\mathbb{S}^{\intercal k};z;\mathbb{T};g_\mathbb{S}^\intercal;\mathbb{S}^{\intercal k}\right]$
$= \texttt{sup}_{n_\mathbb{S}\geq n\geq 0}\,\texttt{sup}_{n_\mathbb{S}\geq k\geq 0}\,\mathbb{S}^n;g_\mathbb{S};\mathbb{T};\mathbb{T};z^\intercal;\mathbb{S}^n;\mathbb{S}^{\intercal k};z;\mathbb{T};g_\mathbb{S}^\intercal;\mathbb{S}^{\intercal k}$
$= \texttt{sup}_{n_\mathbb{S}\geq n\geq 0}\,\mathbb{S}^n;g_\mathbb{S};\mathbb{T};z^\intercal;\mathbb{S}^n;\mathbb{S}^{\intercal n};z;\mathbb{T};g_\mathbb{S}^\intercal;\mathbb{S}^{\intercal n}$ avoiding $\mathbb{S};z = \mathbb{1}$ implies $k := n$
$= \texttt{sup}_{n_\mathbb{S}\geq n\geq 0}\,\mathbb{S}^n;g_\mathbb{S};\mathbb{T};g_\mathbb{S}^\intercal;\mathbb{S}^{\intercal n}$ since $\mathbb{S}^{\intercal n};z;\mathbb{T}$ is a point
$\subseteq \mathbb{I}$ since $g_\mathbb{S}$ is a point and $\mathbb{S}$ is injective

$$\texttt{FlipTD} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{array}$$

Fig. 5.2.1 `FlipTD` in case $\mathbb{S}$ is not total for $n_\mathbb{S} = 7$

$\texttt{FlipTD};\texttt{FlipTD}$
$= \left[\texttt{sup}_{n_\mathbb{S}\geq n\geq 0}\,\mathbb{S}^{\intercal n};z;\mathbb{T};g_\mathbb{S}^\intercal;\mathbb{S}^{\intercal n}\right];\left[\texttt{sup}_{n_\mathbb{S}\geq k\geq 0}\,\mathbb{S}^{\intercal k};z;\mathbb{T};g_\mathbb{S}^\intercal;\mathbb{S}^{\intercal k}\right]$
$= \texttt{sup}_{n_\mathbb{S}\geq n\geq 0}\,\texttt{sup}_{n_\mathbb{S}\geq k\geq 0}\,\mathbb{S}^{\intercal n};z;\mathbb{T};g_\mathbb{S}^\intercal;\mathbb{S}^{\intercal n};\mathbb{S}^{\intercal k};z;\mathbb{T};g_\mathbb{S}^\intercal;\mathbb{S}^{\intercal k}$
$= \texttt{sup}_{n_\mathbb{S}\geq n\geq 0}\,\texttt{sup}_{n_\mathbb{S}\geq k\geq 0}\,\mathbb{S}^{\intercal n};z;\mathbb{T};z^\intercal;\mathbb{S}^{n_\mathbb{S}-n};\mathbb{S}^{\intercal k};z;\mathbb{T};g_\mathbb{S}^\intercal;\mathbb{S}^{\intercal k}$ following (iii.d)
$= \texttt{sup}_{n_\mathbb{S}\geq k\geq 0}\,\mathbb{S}^{\intercal n_\mathbb{S}-k};z;\mathbb{T};z^\intercal;\mathbb{S}^k;\mathbb{S}^{\intercal k};z;\mathbb{T};g_\mathbb{S}^\intercal;\mathbb{S}^{\intercal k}$ since $n_\mathbb{S} - n \neq k$ implies $\mathbb{S};z = \mathbb{1}$
$\subseteq \texttt{sup}_{n_\mathbb{S}\geq k\geq 0}\,\mathbb{S}^{\intercal n_\mathbb{S}-k};z;\mathbb{T};g_\mathbb{S}^\intercal;\mathbb{S}^{\intercal k}$
$= \texttt{sup}_{n_\mathbb{S}\geq k\geq 0}\,\mathbb{S}^{\intercal n_\mathbb{S}-k};\mathbb{S}^{n_\mathbb{S}};\mathbb{S}^{\intercal k}$ due to (iii.d)
$\subseteq \mathbb{I}$ since $\mathbb{S}$ is injective and univalent

This implies equality because `FlipTD` is a map.

---

iii.g) $\texttt{FlipTD};z = \texttt{sup}_{n_\mathbb{S}\geq n\geq 0}\,\mathbb{S}^{\intercal n};z;\mathbb{T};g_\mathbb{S}^\intercal;\mathbb{S}^{\intercal n};z = \texttt{sup}_{n_\mathbb{S}\geq n\geq 0}\,\mathbb{S}^{\intercal n};z;\mathbb{T};z^\intercal;\mathbb{S}^{n_\mathbb{S}-n};z$ (iii.d)
$= \mathbb{S}^{\intercal n_\mathbb{S}};z;\mathbb{T};z^\intercal;\mathbb{S}^0;z$ since $n_\mathbb{S} - n > 0$ would imply $\mathbb{S};z = \mathbb{1}$
$= \mathbb{S}^{\intercal n_\mathbb{S}};z;\mathbb{T} = g_\mathbb{S};\mathbb{T}$ by definition

The second result is proved simultaneously. The third result:

$\texttt{FlipTD} \cap \mathbb{T};g_\mathbb{S}^\intercal;\mathbb{S}^{\intercal k} = \left[\texttt{sup}_{n_\mathbb{S}\geq n\geq 0}\,\mathbb{S}^{\intercal n};z;\mathbb{T};g_\mathbb{S}^\intercal;\mathbb{S}^{\intercal n}\right] \cap \mathbb{T};g_\mathbb{S}^\intercal;\mathbb{S}^{\intercal k}$
$= \texttt{sup}_{n_\mathbb{S}\geq n\geq 0}\left[\mathbb{S}^{\intercal n};z;\mathbb{T};g_\mathbb{S}^\intercal;\mathbb{S}^{\intercal n} \cap \mathbb{T};g_\mathbb{S}^\intercal;\mathbb{S}^{\intercal k}\right]$
$= \texttt{sup}_{n_\mathbb{S}\geq n\geq 0}\,\mathbb{S}^{\intercal n};z;\left[\mathbb{T};g_\mathbb{S}^\intercal;\mathbb{S}^{\intercal n} \cap \mathbb{T};g_\mathbb{S}^\intercal;\mathbb{S}^{\intercal k}\right]$ masking

$= \mathbb{S}^{\tau k} ; z ; \mathbb{T} ; g_{\mathbb{S}}^{\intercal} ; \mathbb{S}^{\tau k}$ since $k \neq n$ implies different points, i.e. $\mathbb{S}^{n} ; g_{\mathbb{S}} \cap \mathbb{S}^{k} ; g_{\mathbb{S}} = \mathbb{\perp}$

---

iii.h) $\mathtt{FlipTD}^{\intercal} = \left[\mathtt{sup}_{n_{\mathbb{S}} \geq n \geq 0} \, \mathbb{S}^{\tau n} ; z ; \mathbb{T} ; g_{\mathbb{S}}^{\intercal} ; \mathbb{S}^{\tau n} \right]^{\intercal} = \mathtt{sup}_{n_{\mathbb{S}} \geq n \geq 0} \, \mathbb{S}^{n} ; g_{\mathbb{S}} ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n}$
$= \mathtt{sup}_{n_{\mathbb{S}} \geq k \geq 0} \, \mathbb{S}^{n_{\mathbb{S}} - k} ; g_{\mathbb{S}} ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n_{\mathbb{S}} - k}$ sequence reversed
$= \mathtt{sup}_{n_{\mathbb{S}} \geq k \geq 0} \, \mathbb{S}^{\tau k} ; z ; \mathbb{T} ; g_{\mathbb{S}}^{\intercal} ; \mathbb{S}^{\tau k} = \mathtt{FlipTD}$ (iii.d)

$\mathtt{FlipTD} ; \mathtt{FlipTD} = \mathbb{I}$ follows directly from being involutory, i.e. from (iii.f)

---

iii.i) The case $n = 0$ is trivial. We only need to show this for $n = 1$; the rest follows by induction.

$\mathtt{FlipTD} ; \mathbb{S}^{\intercal} = \mathtt{sup}_{n_{\mathbb{S}} \geq n \geq 0} \, \mathbb{S}^{\tau n} ; z ; \mathbb{T} ; g_{\mathbb{S}}^{\intercal} ; \mathbb{S}^{\tau n} ; \mathbb{S}^{\intercal}$
$= \mathtt{sup}_{n_{\mathbb{S}} > n \geq 0} \, \mathbb{S}^{\tau n} ; z ; \mathbb{T} ; g_{\mathbb{S}}^{\intercal} ; \mathbb{S}^{\tau n+1}$ since $\mathbb{S}^{n_{\mathbb{S}}+1} ; g_{\mathbb{S}} = \mathbb{\perp}$
$= \mathtt{sup}_{n_{\mathbb{S}} > n \geq 0} \, \mathbb{T} ; g_{\mathbb{S}}^{\intercal} ; \mathbb{S}^{\tau n+1} \cap \mathbb{S}^{\tau n} ; z ; \mathbb{T}$ masking
$= \mathtt{sup}_{n_{\mathbb{S}} > n \geq 0} \, \mathbb{T} ; g_{\mathbb{S}}^{\intercal} ; \mathbb{S}^{\tau n+1} \cap \mathbb{S}^{n_{\mathbb{S}} - n} ; g_{\mathbb{S}} ; \mathbb{T}$ (iii.d)
$= \mathtt{sup}_{n_{\mathbb{S}} > n \geq 0} \, \mathbb{T} ; g_{\mathbb{S}}^{\intercal} ; \mathbb{S}^{\tau n+1} \cap \mathbb{S} ; \mathbb{S}^{n_{\mathbb{S}} - (n+1)} ; g_{\mathbb{S}} ; \mathbb{T}$
$= \mathtt{sup}_{n_{\mathbb{S}} > n \geq 0} \, \mathbb{S} ; \left[\mathbb{T} ; g_{\mathbb{S}}^{\intercal} ; \mathbb{S}^{\tau n+1} \cap \mathbb{S}^{n_{\mathbb{S}} - (n+1)} ; g_{\mathbb{S}} ; \mathbb{T} \right]$ masking
$= \mathbb{S} ; \left[\mathtt{sup}_{n_{\mathbb{S}} > n \geq 0} \, \mathbb{T} ; g_{\mathbb{S}}^{\intercal} ; \mathbb{S}^{\tau n+1} \cap \mathbb{S}^{\tau n+1} ; z ; \mathbb{T} \right]$ again (iii.d)
$= \mathbb{S} ; \left[\mathtt{sup}_{n_{\mathbb{S}} \geq k \geq 0} \, \mathbb{S}^{\tau k} ; z ; \mathbb{T} ; g_{\mathbb{S}}^{\intercal} ; \mathbb{S}^{\tau k} \right]$ with $k := n + 1$ because $\mathbb{S} ; z = \mathbb{\perp}$ when $k = 0$
$= \mathbb{S} ; \mathtt{FlipTD}$

This proves the second claim (trivial for $n = 0$) for $n = 1$ with two starting points:

$\mathtt{FlipTD} ; \mathbb{S} = \mathtt{FlipTD}^{\intercal} ; \mathbb{S} = \mathtt{sup}_{n_{\mathbb{S}} \geq k \geq 0} \, \mathbb{S}^{k} ; g_{\mathbb{S}} ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{k} ; \mathbb{S}$
$= \mathtt{sup}_{n_{\mathbb{S}} > k \geq 0} \, \mathbb{S}^{k} ; g_{\mathbb{S}} ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{k+1}$ since $\mathbb{S}^{\tau m} ; z = \mathbb{\perp}$ if $m > n_{\mathbb{S}}$, following (iii.c)

$\mathbb{S}^{\intercal} ; \mathtt{FlipTD} = \mathtt{sup}_{n_{\mathbb{S}} \geq l \geq 0} \, \mathbb{S}^{\intercal} ; \mathbb{S}^{\tau l} ; z ; \mathbb{T} ; g_{\mathbb{S}}^{\intercal} ; \mathbb{S}^{\tau l}$
$= \mathtt{sup}_{n_{\mathbb{S}} > l \geq 0} \, \mathbb{S}^{\tau l+1} ; z ; \mathbb{T} ; g_{\mathbb{S}}^{\intercal} ; \mathbb{S}^{\tau l}$ since $\mathbb{S}^{\tau m} ; z = \mathbb{\perp}$ when $m > n_{\mathbb{S}}$, again (iii.c)

Now we invert the indexing direction, shift indices by 1 and use Prop. 5.2.1.iii.d, i.e., that $\mathbb{S}^{n_{\mathbb{S}} - n} ; g_{\mathbb{S}} = \mathbb{S}^{\tau n} ; z$. The induction for higher n is immediate.

---

iii.j) $(\mathtt{FlipTD} \otimes \mathtt{FlipTD}) ; (\mathtt{FlipTD} \otimes \mathtt{FlipTD}) = (\mathtt{FlipTD} ; \mathtt{FlipTD} \otimes \mathtt{FlipTD} ; \mathtt{FlipTD})$
$= (\mathbb{I} \otimes \mathbb{I}) = \mathbb{I}$ following (iii.h)

$\mathfrak{P} ; \mathtt{FlipPairsTD} = (\pi \leftthreetimes \rho) ; (\mathtt{FlipTD} \otimes \mathtt{FlipTD})$
$= ((\pi \leftthreetimes \rho) ; \mathtt{FlipTD} \otimes (\pi \leftthreetimes \rho) ; \mathtt{FlipTD}) = (\rho ; \mathtt{FlipTD} \otimes \pi ; \mathtt{FlipTD})$

Afterwards correspondingly for the right side.

---

iii.k) $(g_{\mathbb{S}} ; \mathbb{T} \rightthreetimes \mathbb{I}) ; (\mathtt{FlipTD} \cap \mathbb{T} ; g_{\mathbb{S}}^{\intercal} ; \mathbb{S}^{\tau n}) = (g_{\mathbb{S}} ; \mathbb{T} \rightthreetimes \mathbb{I}) ; \mathbb{S}^{\tau n} ; z ; \mathbb{T} ; g_{\mathbb{S}}^{\intercal} ; \mathbb{S}^{\tau n}$ (iii.g)
$= (g_{\mathbb{S}} ; \mathbb{T} ; \mathbb{S}^{\tau n} ; z ; \mathbb{T} ; g_{\mathbb{S}}^{\intercal} ; \mathbb{S}^{\tau n} \rightthreetimes \mathbb{S}^{\tau n} ; z ; \mathbb{T} ; g_{\mathbb{S}}^{\intercal} ; \mathbb{S}^{\tau n})$ the factor is injective
$\subseteq (\mathbb{S}^{\tau n} \rightthreetimes \mathbb{S}^{\tau n} ; z ; \mathbb{T})$ since $g_{\mathbb{S}}$ is injective

---

iii.l) When $0 \leq n < n_{\mathbb{S}}$, we have $n_{\mathbb{S}} - n \geq 1$, and thus
$\mathbb{S}^{\tau n} ; z ; \mathbb{T} = \mathbb{S}^{n_{\mathbb{S}} - n} ; g_{\mathbb{S}} ; \mathbb{T} \subseteq \mathbb{S} ; \mathbb{T}$ due to (iii.d).

---

iii.m) $\mathbb{S}^{\tau i} ; z ; \mathbb{T} \cap \mathbb{S}^{m} = \mathbb{\perp} \iff \mathbb{S}^{\tau i} ; z ; \mathbb{T} \subseteq \overline{\mathbb{S}^{m}} \iff \mathbb{S}^{i} ; \mathbb{S}^{m} \subseteq \overline{z ; \mathbb{T}}$

$\iff \quad z;\mathbb{T} = \mathbb{S}^{n_{\mathbb{S}}};g_{\mathbb{S}};\mathbb{T} \subseteq \overline{\mathbb{S}^{i+m}} \quad \iff \quad g_{\mathbb{S}}^{\intercal};\mathbb{S}^{\intercal n_{\mathbb{S}}};\mathbb{S}^{i+m} \subseteq \mathbb{\perp}$
$\iff \quad \mathbb{S}^{\intercal i+m};\mathbb{S}^{n_{\mathbb{S}}};g_{\mathbb{S}} \subseteq \mathbb{\perp}$

This finally means that $i+m > n_{\mathbb{S}}$.

---

iii.n) The first equality recalls (i.i). In addition, one has to observe the definition of $g_{\mathbb{S}}$. Finally with masking and transposition

$\mathbb{I} \cap \overline{g_{\mathbb{S}};\mathbb{T};g_{\mathbb{S}}^{\intercal}} = \mathbb{I} \cap \overline{g_{\mathbb{S}};\mathbb{T} \cap \mathbb{T};g_{\mathbb{S}}^{\intercal}} = \mathbb{I} \cap [\overline{g_{\mathbb{S}};\mathbb{T}} \cup \overline{\mathbb{T};g_{\mathbb{S}}^{\intercal}}]$.

---

iii.o) We prove just one:

$\overline{\mathbb{S};\mathtt{eve};\mathbb{T}} = \mathbb{S};\overline{\mathtt{eve};\mathbb{T}} \cup \overline{\mathbb{S};\mathbb{T}}$ since $\mathbb{S}$ is univalent
$= \mathbb{S};\mathtt{odd};\mathbb{T} \cup g_{\mathbb{S}};\mathbb{T}$

---

iv.a) $\mathbb{S}$ is univalent and injective by definition; now comes totality in addition. Looking at the definition $g_{\mathbb{S}} := \overline{\mathbb{S};\mathbb{T}}$ in (iii), it is immediate that $g_{\mathbb{S}} = \mathbb{\perp}$.

---

iv.b) is a trivial consequence of (i.b) when $\mathbb{S}$ is total.

---

iv.c) Now, $\mathbb{S}$ is a map and, e.g. $\mathtt{odd};\mathbb{T} \subseteq \mathbb{S};\mathtt{eve};\mathbb{T} \quad \iff \quad \mathbb{S}^{\intercal};\mathtt{odd};\mathbb{T} \subseteq \mathtt{eve};\mathbb{T}$, which is true. □

In (ii.g) and (iv.c), borderline effects may occur. While it is easy to correct with $z;\mathbb{T}$, the corresponding correction at the upper border is possible, but here avoided, since we will hardly need it.

When we do not demand regressive finiteness, this leaves us still with situations as those of Fig. 5.2.3 and Fig. 5.2.6, where S is indeed univalent and injective and $z := \overline{S^{\intercal};\mathbb{T}}$ is a point, but not yet $V = \mathbb{T}$. In Prop. 5.2.2 we will also investigate when $D = \mathbb{I}$ and in Def. 5.3.1 when $\mathbb{V}$ is a linear order in similar iterations.

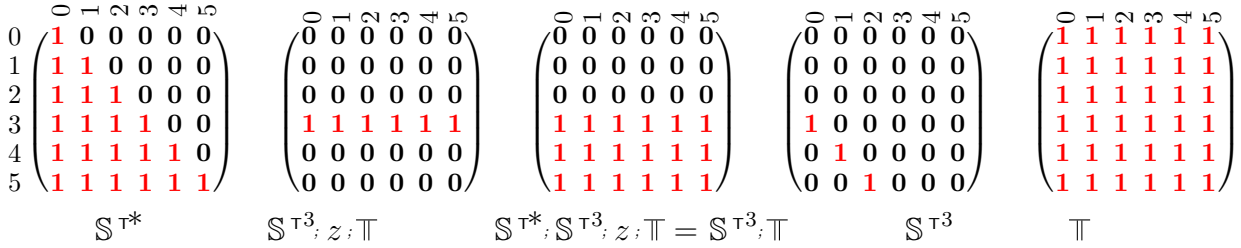


Fig. 5.2.2 Illustrating Prop. 5.2.1.ii.j

Remark Obviously, we might also have introduced odd and even numbers in the standard style followed here several times: This would mean the functional $\omega_H(X) := \mathbb{S}^{\intercal 2};X;\mathbb{S}^2$ together with either the start $z;\mathbb{T};z^{\intercal}$ for evens or $e;\mathbb{T};e^{\intercal}$ for the odds. □

In Fig. 5.2.3 one will observe that $S^3 \neq \perp\!\!\!\perp$ in spite of $S^3;z = \perp\!\!\!\perp$; thus illustrating Prop. 5.2.1.ii.d. Adding A and multiplying M are here used for demonstration purposes — anticipating later definitions.

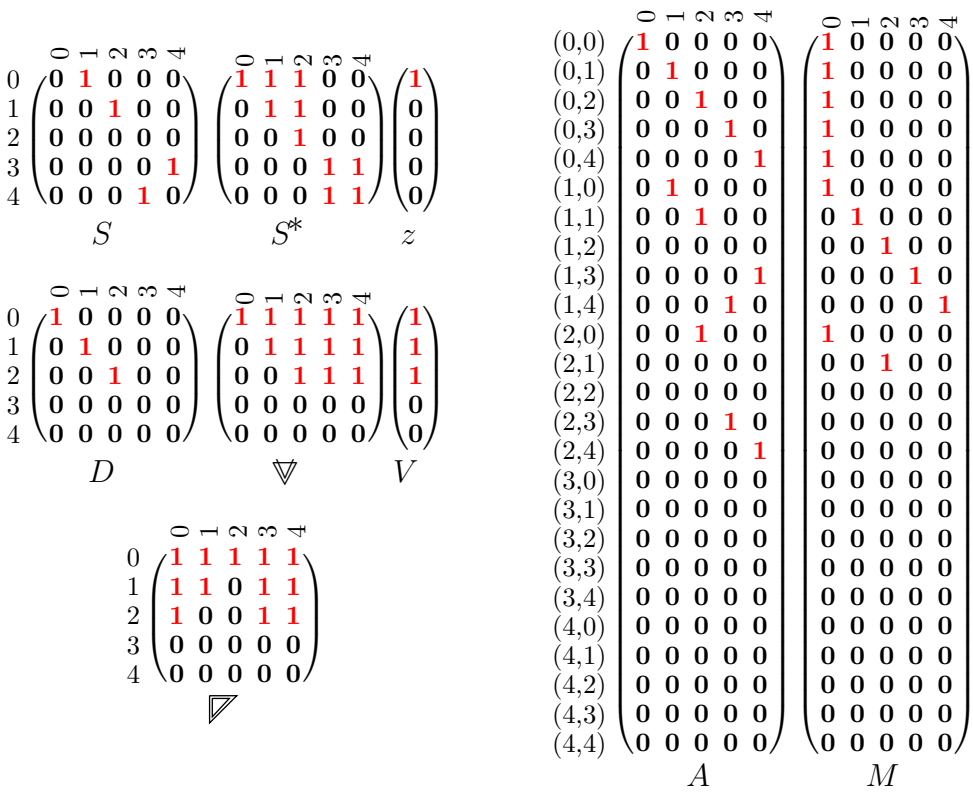


Fig. 5.2.3 Univalent and injective but not regressively finite

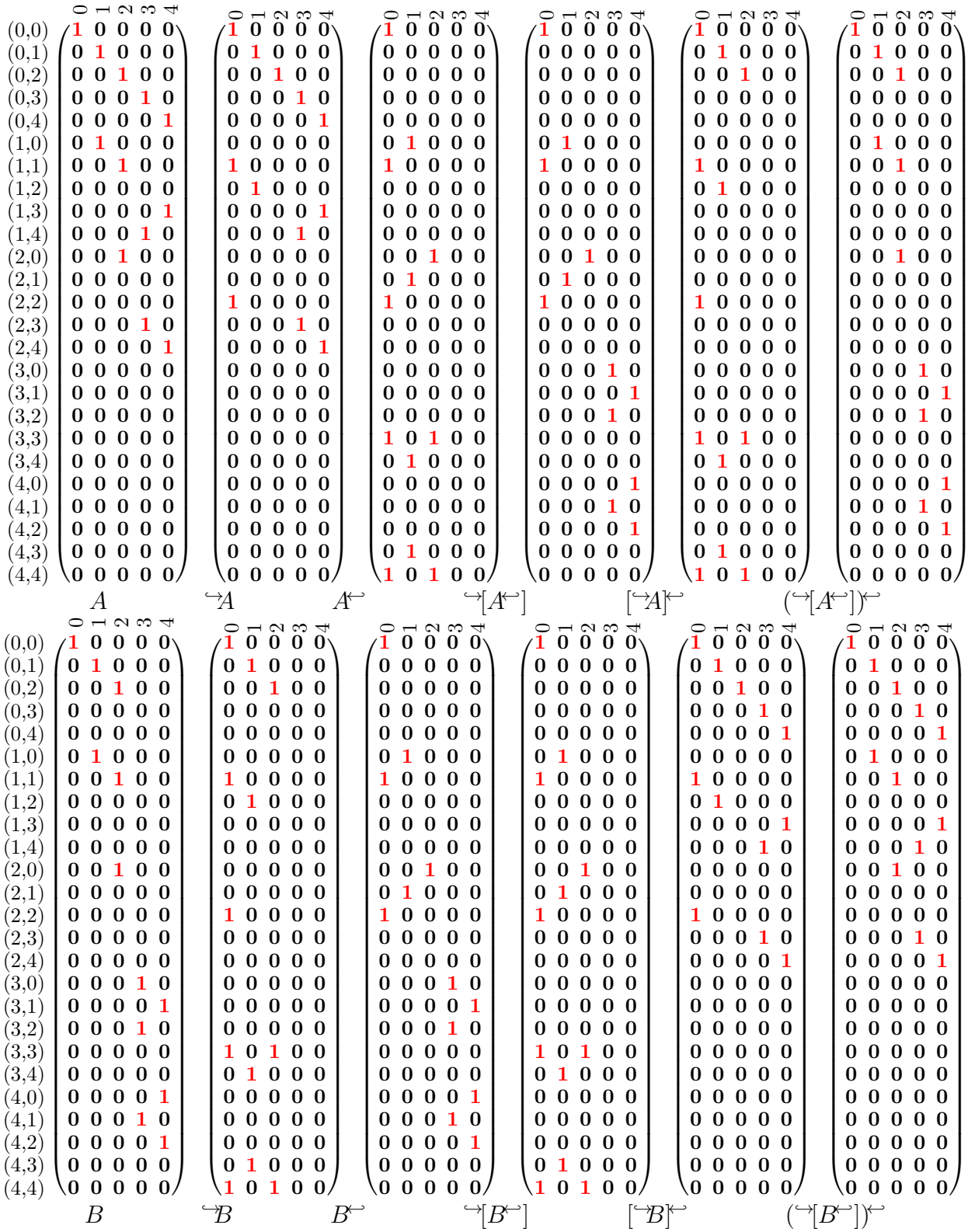


Fig. 5.2.4 Examples of iterated inversions

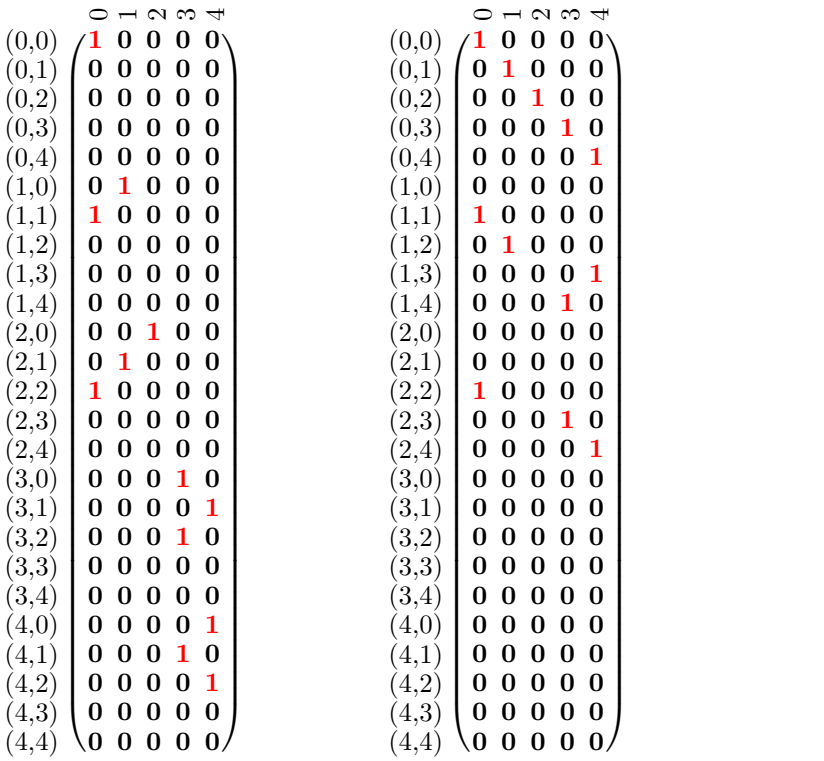


Fig. 5.2.5 $= \mathfrak{S}_{\text{by iteration}} = {}^{\hookrightarrow}[A^{\hookleftarrow}] = B^{\hookleftarrow}$ $= {}^{\hookrightarrow}A = [{}^{\hookrightarrow}B]^{\hookleftarrow} = \mathfrak{s}_{\text{by iteration}}$

The example of Fig. 5.2.3 shows that inverting and iterating to obtain subtraction bring different results!

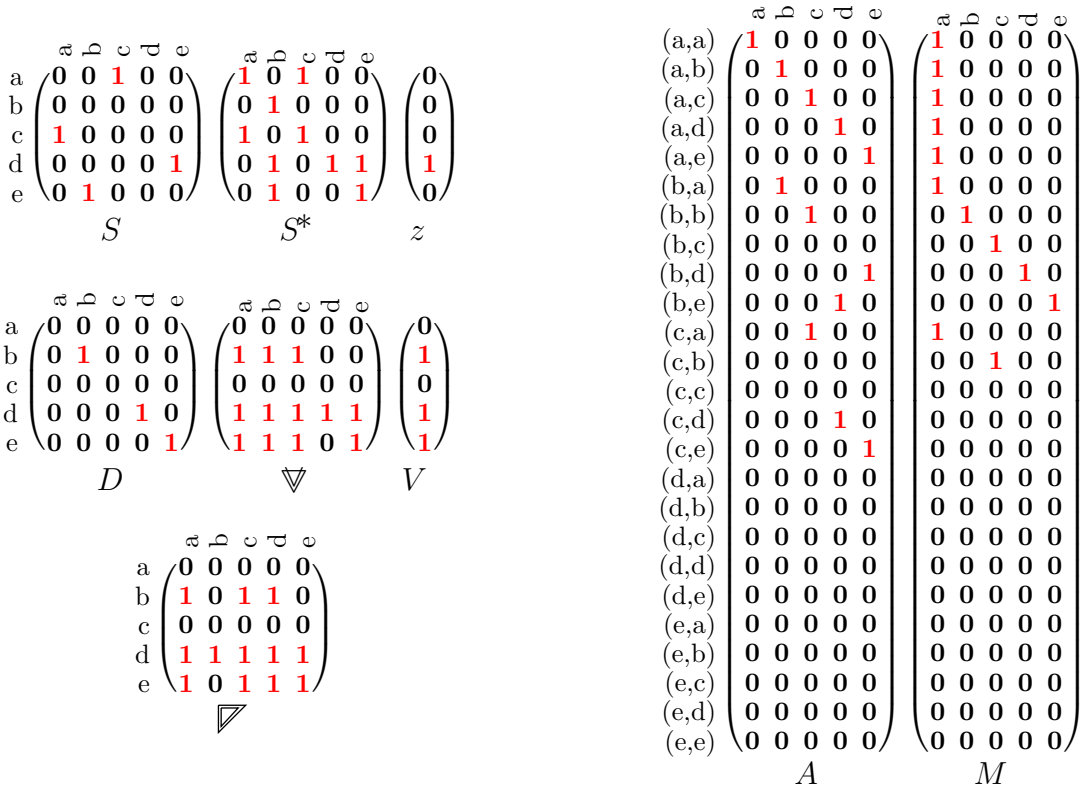


Fig. 5.2.6 Univalent and injective but not regressively finite — same as before, less nicely arranged

Even worse is this:

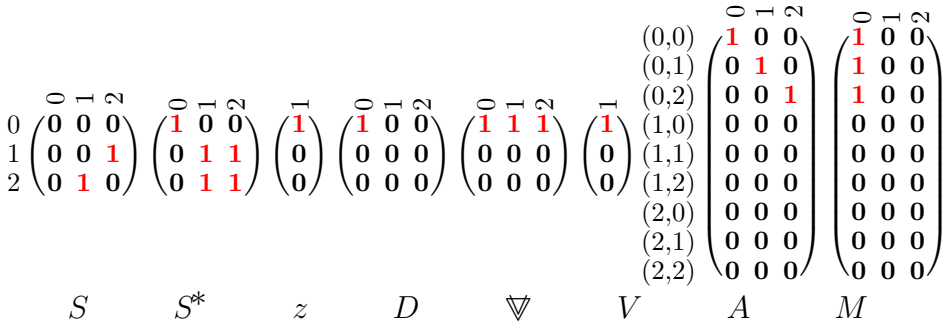


Fig. 5.2.7 Univalent and injective but not regressively finite

Looking at Fig. 5.2.3 or Fig. 5.2.6 or Fig. 5.2.7, we do not expect any relationship between S^* and $\mathbb{V}$. To reveal it, we now include regressive finiteness in the chain of proofs:

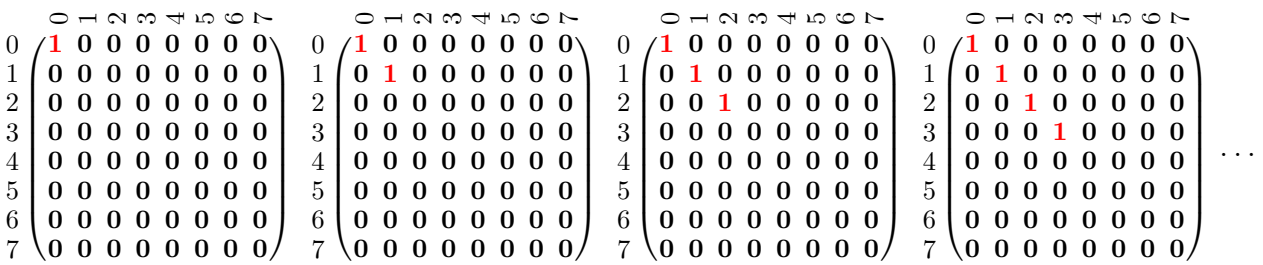


Fig. 5.2.8 Progress of the functional $d(\perp\!\!\!\perp), d(d(\perp\!\!\!\perp)), \ldots$ to exhaust the diagonal

The construct $\mathbb{S}^{\intercal n}{;}\, z {;}\, \mathbb{T}$ may be equal to $\mathbb{\perp}$; however, if it is not, it constitutes a point resembling the number n and all the $\mathbb{S}^{\intercal k}{;}\, z {;}\, \mathbb{T}$ with $k \leq n$ will also be points. The set $\mathtt{pts}(\mathbb{N})$ may be obtained as

$$\mathtt{pts}(\mathbb{N}) = \{\, \mathbb{S}^{\intercal 0}{;}\, z {;}\, \mathbb{T}, \mathbb{S}^{\intercal 1}{;}\, z {;}\, \mathbb{T}, \mathbb{S}^{\intercal 2}{;}\, z {;}\, \mathbb{T}, \mathbb{S}^{\intercal 3}{;}\, z {;}\, \mathbb{T}, \ldots \},$$

which will end with $\mathbb{S}^{\intercal n_{\mathbb{S}}}{;}\, z {;}\, \mathbb{T}$ when $\mathbb{S}$ is finite and otherwise run infinitely. Without regressive finiteness, one may find the situation of the D in Fig. 5.2.3 or Fig. 5.2.6 that do not succeed in exhausting.

5.2.2 Definition and Proposition. Given the successor relation $\mathbb{S}$ as discussed before, we consider the functional

$$d(X) := D_0 \cup d_H(X) \quad \text{with} \quad D_0 := z {;}\, z^{\intercal} \quad \text{and} \quad d_H(X) := \mathbb{S}^{\intercal} {;}\, X {;}\, \mathbb{S}$$

and define what shall afterwards be called the **diagonal of $\mathbb{S}$-reachable points** as

$$D := D_{\mathbb{S}} := \mathtt{inf}\, \{\, X \mid d(X) \subseteq X \},$$

for which the following will hold

i) in general:

a) $D = d(D) = z {;}\, z^{\intercal} \cup \mathbb{S}^{\intercal} {;}\, D {;}\, \mathbb{S} = \mathtt{sup}_{n \geq 0}\, d_H^n(D_0) = \mathtt{sup}_{n \geq 0}\, \mathbb{S}^{\intercal n} {;}\, z {;}\, \mathbb{T} {;}\, z^{\intercal} {;}\, \mathbb{S}^n \subseteq \mathbb{I}$

b) $D \subseteq \overline{\mathbb{S}^n}$ for $n \geq 1$

c) $D {;}\, \mathbb{T} = V$

ii) In case $\mathbb{S}$ is regressively finite, there hold in addition

a) $D = \mathbb{I} = \mathtt{sup}_{n \geq 0}\, \mathbb{S}^{\intercal n} {;}\, z {;}\, \mathbb{T} {;}\, z^{\intercal} {;}\, \mathbb{S}^n$

b) $\mathbb{I} \subseteq \overline{\mathbb{S}^n}$ for $n \geq 1$

c) $\mathtt{sup}_{n \geq 0}\, \mathbb{S}^{\intercal n} {;}\, z {;}\, \mathbb{T} {;}\, z^{\intercal} {;}\, (\mathbb{S}^i)^* {;}\, \mathbb{S}^n = \mathtt{sup}_{k \geq 0}\, \mathbb{S}^{k * i} = (\mathbb{S}^i)^*$ for arbitrary $i \geq 0$

d) For arbitrary $i \geq 0$, $(\mathbb{S}^i)^* \cup (\mathbb{S}^{\intercal i})^*$ is an equivalence.

iii) If $\mathbb{S}$ happens to be regressively finite but not total, i.e. $\mathbb{S} {;}\, \mathbb{T} \neq \mathbb{T}$,

a) $\mathbb{S}$ is not total $\implies$ $\mathbb{I} = \mathtt{sup}_{n_{\mathbb{S}} \geq n \geq 0}\, \mathbb{S}^{\intercal n} {;}\, z {;}\, \mathbb{T} {;}\, z^{\intercal} {;}\, \mathbb{S}^n$

iv) In case $\mathbb{S}$ is regressively finite and also total, i.e. $\mathbb{S} {;}\, \mathbb{T} = \mathbb{T}$,

a)

Proof: We observe in advance that Lemma 4.2.1 may be applied since the functional d_H is

isotonic, i.e. $A \subseteq B \implies \mathbb{S}^\intercal ; A ; \mathbb{S} \subseteq \mathbb{S}^\intercal ; B ; \mathbb{S}$, i.e. $d_H(A) \subseteq d_H(B)$,
additive, i.e. $d_H(X \cup Y) = \mathbb{S}^\intercal ; (X \cup Y) ; \mathbb{S} = \mathbb{S}^\intercal ; X ; \mathbb{S} \cup \mathbb{S}^\intercal ; Y ; \mathbb{S} = d_H(X) \cup d_H(Y)$,
$\cap$-hereditary, i.e. $d_H(X \cap Y) = \mathbb{S}^\intercal ; (X \cap Y) ; \mathbb{S} = \mathbb{S}^\intercal ; X ; \mathbb{S} \cap \mathbb{S}^\intercal ; Y ; \mathbb{S} = d_H(X) \cap d_H(Y)$.

---

i.a) This follows from Lemma 4.2.1.i,iv. Obviously $D \subseteq \mathbb{I}$, because the selection criterion is satisfied in the following form

$d(\mathbb{I}) = z ; z^\intercal \cup \mathbb{S}^\intercal ; \mathbb{I} ; \mathbb{S} \subseteq \mathbb{I}$.

---

i.b) We show that $\overline{\mathbb{S}^n}$ satisfies the selection criterion $d(\overline{\mathbb{S}^n}) = z ; z^\intercal \cap \mathbb{S}^\intercal ; \overline{\mathbb{S}^n} ; \mathbb{S} \subseteq \overline{\mathbb{S}^n}$. Firstly,

$z ; z^\intercal \subseteq \overline{\mathbb{S}^n} \iff \mathbb{S}^n ; z \subseteq \overline{z}$,

where the latter indeed holds for $n \geq 1$ since $\mathbb{S} ; z = \mathbb{L}$. Using injectivity and $\mathbb{S}^\intercal ; \overline{\mathbb{S}^n} \subseteq \overline{\mathbb{S}^{n-1}}$

$\mathbb{S}^n ; \mathbb{S}^\intercal = \mathbb{S}^{n-1} ; \mathbb{S} ; \mathbb{S}^\intercal \subseteq \mathbb{S}^{n-1} \implies \overline{\mathbb{S}^{n-1}} ; \mathbb{S} \subseteq \overline{\mathbb{S}^n} \implies \mathbb{S}^\intercal ; \overline{\mathbb{S}^n} ; \mathbb{S} \subseteq \overline{\mathbb{S}^n}$.

---

i.c) $D ; \mathbb{T} = \left[\mathtt{sup}_{n \geq 0} \, \mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^n \right] ; \mathbb{T} = \mathtt{sup}_{n \geq 0} \, \mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^n ; \mathbb{T} = \mathtt{sup}_{n \geq 0} \, \mathbb{S}^{\intercal n} ; z ; \mathbb{T} = V$

One has to observe the behaviour of $\mathbb{T} ; z^\intercal ; \mathbb{S}^n$: Either $\mathbb{T} ; z^\intercal ; \mathbb{S}^n = \mathbb{L}$ or not. This implies, however, that $\mathbb{T} ; z^\intercal ; \mathbb{S}^n ; \mathbb{T} = \mathbb{L}$ or $\mathbb{T} ; z^\intercal ; \mathbb{S}^n ; \mathbb{T} = \mathbb{T}$, justifying the second last "=".

---

ii.a) One half, $D \subseteq \mathbb{I}$, of this claim is already established with (i.a). The other is intricate because the vector/subset description as a vector or as a diagonal relation pops up — in principle trivial, but tedious when to be proved in a fully formal way.

The intention is to show that $\overline{D ; \mathbb{T}} \subseteq \mathbb{S}^\intercal ; \overline{D ; \mathbb{T}}$, leading to $D ; \mathbb{T} = \mathbb{T}$ by regressive finiteness according to Prop. 4.1.2 and, thus, to $D = \mathbb{I}$.

A purely technical step is it to find out with $D \subseteq \mathbb{I}$ that $\overline{D ; \mathbb{T}} = (\mathbb{I} \cap \overline{D}) ; \mathbb{T}$ (see Prop. 2.1.1.vi). We have, thus, to show $(\mathbb{I} \cap \overline{D}) ; \mathbb{T} \subseteq \mathbb{S}^\intercal ; (\mathbb{I} \cap \overline{D}) ; \mathbb{T}$. This will certainly hold if

$\mathbb{I} \cap \overline{D} \subseteq \mathbb{S}^\intercal ; (\mathbb{I} \cap \overline{D}) ; \mathbb{S}$.

For the left side, we have

$\mathbb{I} \cap \overline{D} = \mathbb{I} \cap \overline{z ; z^\intercal \cup \mathbb{S}^\intercal ; D ; \mathbb{S}}$ due to (i)
$= \mathbb{I} \cap \overline{z ; z^\intercal} \cap \overline{\mathbb{S}^\intercal ; D ; \mathbb{S}}$
$= \mathbb{S}^\intercal ; \mathbb{S} \cap \overline{\mathbb{S}^\intercal ; D ; \mathbb{S}}$ Prop. 5.1.2
$= \mathbb{S}^\intercal ; \mathbb{S} \cap \mathbb{S}^\intercal ; \mathbb{T} \cap \mathbb{T} ; \mathbb{S} \cap \overline{\mathbb{S}^\intercal ; D ; \mathbb{S}}$
$= \mathbb{S}^\intercal ; \mathbb{S} \cap \mathbb{S}^\intercal ; \overline{D} ; \mathbb{S}$ applying two times the rule $Q ; \overline{R} = \overline{Q ; R} \cap Q ; \mathbb{T}$ for univalent Q
$= \mathbb{S}^\intercal ; (\mathbb{I} \cap \overline{D}) ; \mathbb{S}$

Now regressive finiteness may indeed be applied as already announced.

---

ii.b) is trivial with (i.b) and (ii.a).

---

ii.c) Assuming some $i \geq 0$, we proceed as follows:

$\mathtt{sup}_{n \geq 0} \, \mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; z^\intercal ; (\mathbb{S}^i)^* ; \mathbb{S}^n = \mathtt{sup}_{n \geq 0} \, \mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^n ; (\mathbb{S}^i)^*$ arranged differently
$= \left[\mathtt{sup}_{n \geq 0} \, \mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^n \right] ; (\mathbb{S}^i)^* = \mathbb{I} ; (\mathbb{S}^i)^*$ due to (ii.a)

---

ii.d) Symmetry and reflexivity are provided by construction. In order to establish transitivity, it suffices to convince ourselves that due to injectivity and univalency, e.g.

$(\mathbb{S}^{\,i})^*;(\mathbb{S}^{\,\intercal i})^* = \big[\,\mathtt{sup}_{n\geq 0}(\mathbb{S}^{\,i})^n\big];\big[\,\mathtt{sup}_{k\geq 0}(\mathbb{S}^{\,\intercal i})^k\big]$
$= \mathtt{sup}_{n\geq 0}\,\mathtt{sup}_{k\geq 0}\,\mathbb{S}^{\,i*n};\mathbb{S}^{\,\intercal i*k} \subseteq (\mathbb{S}^{\,i})^* \cup (\mathbb{S}^{\,\intercal i})^*$

---

iii) This combines (ii.a) with Prop. 5.2.1.iii.c.

---

iv) □

There is a well-known connection between vector V and diagonal D, namely

$D = \mathbb{I} \cap V;\mathbb{T}$ and $V = D;\mathbb{T}$,

via which even more details may be transferred from one side to the other.

5.3 Order and function properties of the successor

While the preceding result was for the identity, or the diagonal, we now work for the "upper right or left triangle", i.e. the order on the naturals, again with an iteration. Concerning (ii.c), we should remind us of the fact that the strictorder C on reals has no Hasse relation, since $\mathtt{Nontrans}(C) = \mathbb{L}$.

5.3.1 Definition and Proposition. We define as **ordering** for $\mathbb{S}$ the construct

$\mathbb{V} := \mathbb{V}_{\mathbb{S}} := \mathtt{inf}\,\{\, X \mid g(X) \subseteq X \,\}$,

where we have in view of Lemma 4.2.1

$g(X) := \mathbb{V}_0 \cup g_H(X)$ with $\mathbb{V}_0 := z;\mathbb{T}$ and $g_H(X) := \mathbb{S}^{\,\intercal};X;\mathbb{S}$.

Respecting possible further properties of $\mathbb{S}$, the relation $\mathbb{V}$ satisfies

i) in general:

a) $\mathbb{V} = g(\mathbb{V}) = z;\mathbb{T} \cup \mathbb{S}^{\,\intercal};\mathbb{V};\mathbb{S} = \mathtt{sup}_{n\geq 0}\, g_H^n(\mathbb{V}_0) = \mathtt{sup}_{n\geq 0}\,\mathbb{S}^{\,\intercal n};z;\mathbb{T};\mathbb{S}^{\,n}$

b) $\mathbb{V};\mathbb{T} = V$

c) $\mathbb{V}_{n+1} := g(\mathbb{V}_n) = \mathbb{S}^{\,\intercal n};z;\mathbb{T};\mathbb{S}^{\,n}$ form a monotonic sequence $\mathbb{V}_0 \subseteq \mathbb{V}_1 \subseteq \ldots$

ii) additionally, in case $\mathbb{S}$ is regressively finite:

a) $\mathbb{V} = \mathbb{S}^* = z;\mathbb{T} \cup \mathbb{S}^{\,\intercal};\mathbb{S}^*;\mathbb{S} = z;\mathbb{T} \cup \mathbb{S}^{\,\intercal};\mathbb{S}^+ = \mathtt{sup}_{n\geq 0}\,\mathbb{S}^{\,\intercal n};z;\mathbb{T};\mathbb{S}^{\,n}$.

b) $\mathbb{S}^+ = \mathbb{S};\mathbb{S}^* = \mathtt{sup}_{n\geq 1}\,\mathbb{S}^{\,n}$ is a linear strictorder $\mathbb{S}^+ = \mathtt{sup}_{n\geq 0}\,\mathbb{S}^{\,\intercal n};z;\mathbb{T};\overline{\mathbb{T};z^{\intercal}};\mathbb{S}^{\,n}$

c) $\mathbb{S}$ is the Hasse relation of $\mathbb{S}^+$.

$\mathbb{S} = \mathbb{S}^+ \cap \overline{\mathbb{S}^+;\mathbb{S}^+} = \mathtt{Hasse}(\mathbb{S}^+)$

$\mathbb{S}^+$ is path-length bounded and discrete.

d) $\overline{(\mathbb{S}^+ \circledcirc \mathbb{I})\,;\mathbb{T}} = (\mathbb{I} \circledcirc \mathbb{S}^*)\,;\mathbb{T}$

iii) If $\mathbb{S}$ happens to be regressively finite but not total, i.e. $\mathbb{S}\,;\mathbb{T} \neq \mathbb{T}$,

a) $\mathbb{S}^* = \mathtt{sup}_{n_{\mathbb{S}} \geq n \geq 0}\, \mathbb{S}^{\intercal n}\,;z\,;\mathbb{T}\,;\mathbb{S}^n$

b) $\mathbb{S}^{\intercal *} = \mathtt{sup}_{n_{\mathbb{S}} \geq n \geq 0}\, \mathbb{S}^n\,;g_{\mathbb{S}}\,;\mathbb{T}\,;\mathbb{S}^{\intercal n}$

c) $g_{\mathbb{S}}$ is the greatest element of the strictorder.

iv) additionally, if $\mathbb{S}$ is regressively finite and total:

a)

Proof: We observe in advance that Lemma 4.2.1 may be applied since the functional g_H is

isotonic, i.e. $A \subseteq B \implies \mathbb{S}^{\intercal}\,;A\,;\mathbb{S} \subseteq \mathbb{S}^{\intercal}\,;B\,;\mathbb{S}$, i.e. $g_H(A) \subseteq g_H(B)$,
additive, i.e. $g_H(X \cup Y) = \mathbb{S}^{\intercal}\,;(X \cup Y)\,;\mathbb{S} = \mathbb{S}^{\intercal}\,;X\,;\mathbb{S} \cup \mathbb{S}^{\intercal}\,;Y\,;\mathbb{S} = g_H(X) \cup g_H(Y)$,
$\cap$-hereditary, i.e. $g_H(X \cap Y) = \mathbb{S}^{\intercal}\,;(X \cap Y)\,;\mathbb{S} = \mathbb{S}^{\intercal}\,;X\,;\mathbb{S} \cap \mathbb{S}^{\intercal}\,;Y\,;\mathbb{S} = g_H(X) \cap g_H(Y)$.

---

i.a) This follows from Lemma 4.2.1.i,iv.

---

i.b) $\mathbb{V}\,;\mathbb{T} = \left[\mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\intercal n}\,;z\,;\mathbb{T}\,;\mathbb{S}^n\right]\,;\mathbb{T} = \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\intercal n}\,;z\,;\mathbb{T}\,;\mathbb{S}^n\,;\mathbb{T} = \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\intercal n}\,;z\,;\mathbb{T} = V$
One has to observe the behaviour of $\mathbb{S}^n$: Either $\mathbb{S}^n = \mathbb{L}$ or not. This implies, however, that $\mathbb{T}\,;\mathbb{S}^n\,;\mathbb{T} = \mathbb{L}$ or $\mathbb{T}\,;\mathbb{S}^n\,;\mathbb{T} = \mathbb{T}$, justifying the second last "=".

---

i.c) is trivial.

---

ii.a) In order to finally obtain $\mathbb{V} \subseteq \mathbb{S}^*$, we recall Prop. 5.2.1.ii.b saying $z\,;\mathbb{T} \subseteq \mathbb{S}^*$. Using that, we are able to show that $\mathbb{S}^*$ satisfies the present selection criterion:

$g(\mathbb{S}^*) = z\,;\mathbb{T} \cup \mathbb{S}^{\intercal}\,;\mathbb{S}^*\,;\mathbb{S}$
$\subseteq \mathbb{S}^* \cup \mathbb{S}^{\intercal}\,;\mathbb{S}\,;\mathbb{S}^*$ due to Prop. 4.3.1.ii
$\subseteq \mathbb{S}^* \cup \mathbb{S}^* = \mathbb{S}^*$, since $\mathbb{S}$ is univalent

$\mathbb{V} \supseteq \mathbb{S}^*$ remains to be proved proceeding by induction with hypothesis $\mathbb{S}^{\intercal n}\,;z\,;\mathbb{T} \cap \mathbb{S}^* \subseteq \mathbb{V}$:

$z\,;\mathbb{T} \cap \mathbb{S}^* \subseteq z\,;\mathbb{T} \subseteq \mathbb{V}$ for $n = 0$

and

$\mathbb{S}^{\intercal n+1}\,;z\,;\mathbb{T} \cap \mathbb{S}^* = \mathbb{S}^{\intercal}\,;\mathbb{S}^{\intercal n}\,;z\,;\mathbb{T} \cap \mathbb{S}^*$ for $n \geq 0$
$\subseteq (\mathbb{S}^{\intercal} \cap \mathbb{S}^*\,;[\mathbb{S}^{\intercal n}\,;z\,;\mathbb{T}]^{\intercal})\,;(\mathbb{S}^{\intercal n}\,;z\,;\mathbb{T} \cap \mathbb{S}\,;\mathbb{S}^*)$ Dedekind rule
$\subseteq \mathbb{S}^{\intercal}\,;(\mathbb{S}^{\intercal n}\,;z\,;\mathbb{T} \cap \mathbb{S}^*\,;\mathbb{S})$
$= \mathbb{S}^{\intercal}\,;(\mathbb{S}^{\intercal n}\,;z\,;\mathbb{T} \cap \mathbb{S}^*)\,;\mathbb{S}$ masking
$\subseteq \mathbb{S}^{\intercal}\,;\mathbb{V}\,;\mathbb{S}$ induction hypothesis
$\subseteq \mathbb{V}$ (i.a)

Now with Prop. 5.2.1.ii.a we have in total
$\mathbb{S}^* = \mathbb{T} \cap \mathbb{S}^* = \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\intercal^n} ; z ; \mathbb{T} \cap \mathbb{S}^* \subseteq \mathbb{V}$.

---

ii.b) $\mathbb{S}^+$ is asymmetric: Assume the contrary, i.e. points x, y with $x ; y^\intercal \subseteq \mathbb{S}^+ \cap \mathbb{S}^{+\intercal}$. Then with shunting points

$x ; y^\intercal \subseteq \mathbb{S}^+ \iff x \subseteq \mathbb{S}^+ ; y$
$x ; y^\intercal \subseteq \mathbb{S}^{+\intercal} \iff y ; x^\intercal \subseteq \mathbb{S}^+ \iff y \subseteq \mathbb{S}^+ ; x,$

so that $x \subseteq \mathbb{S}^+ ; \mathbb{S}^+ ; x$, expressing that a circuit with two or more steps leads from x to x. More formally, $x \subseteq \mathtt{sup}_{n\geq 2}\, \mathbb{S}^n ; x$. Since x is a point, there exists an index $n \geq 2$ with $x \subseteq \mathbb{S}^n ; x$. The construct $\mathbb{S}^n ; x$ is obviously again a point and we get a finite circuit

$x,\ \mathbb{S} ; x,\ \mathbb{S}^2 ; x,\ \ldots\ \mathbb{S}^n ; x = x.$

Every of these points has a predecessor, so that z cannot be among them, because $\mathbb{S} ; z = \mathbb{\perp}$.

We consider the set $W := \mathbb{S}^+ ; x$ of points over which this circuit runs and aim at finding out that $W \subseteq \mathbb{S}^\intercal ; W$. From regressive finiteness then, namely, follows $W = \mathbb{\perp}$, contradicting $W \supseteq y$ with y a point.

$W = \mathbb{S}^+ ; x = \mathbb{T} \cap \mathbb{S}^+ ; x = \mathbb{S}^{\intercal *} ; z ; \mathbb{T} \cap \mathbb{S}^+ ; x$ Prop. 5.2.1.ii.a
$= (z ; \mathbb{T} \cap \mathbb{S}^+ ; x) \cup (\mathbb{S}^\intercal ; \mathbb{S}^{\intercal *} ; z ; \mathbb{T} \cap \mathbb{S}^+ ; x)$
$= \mathbb{S}^\intercal ; \mathbb{S}^{\intercal *} ; z ; \mathbb{T} \cap \mathbb{S}^+ ; x$ since z cannot have a predecessor as all points in $\mathbb{S}^+ ; x$ have
$\subseteq (\mathbb{S}^\intercal \cap \ldots) ; (\mathbb{S}^{\intercal *} ; z ; \mathbb{T} \cap \mathbb{S} ; \mathbb{S}^+ ; x)$ Dedekind rule
$\subseteq \mathbb{S}^\intercal ; \mathbb{S} ; \mathbb{S}^+ ; x$
$\subseteq \mathbb{S}^\intercal ; \mathbb{S}^+ ; x = \mathbb{S}^\intercal ; W$

The strictorder $\mathbb{S}^+$ is linear, i.e. $\overline{\mathbb{I}} \subseteq \mathbb{S}^+ \cup \mathbb{S}^{+\intercal}$: Assume to the contrary, that there exist points $x \neq y$ with $x ; y^\intercal \subseteq \overline{\mathbb{S}^+} \cap \overline{\mathbb{S}^+}^\intercal$, i.e. both, $x ; y^\intercal \subseteq \overline{\mathbb{S}^+}$ and $x ; y^\intercal \subseteq \overline{\mathbb{S}^+}^\intercal$. From Prop. 5.2.1.ii.a, we know that $x \subseteq \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\intercal^n} ; z ; \mathbb{T} = \mathbb{T}$ and also $y \subseteq \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\intercal^n} ; z ; \mathbb{T} = \mathbb{T}$. Since both are points, there exist exponents i, k such that $x = \mathbb{S}^{\intercal^i} ; z ; \mathbb{T}$ and $y = \mathbb{S}^{\intercal^k} ; z ; \mathbb{T}$. This, however, leads to a contradiction:

$x ; y^\intercal = \mathbb{S}^{\intercal^i} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^k,$

because maximal canceling in the middle will — having in mind $x \neq y$ — end with a power of $\mathbb{S}^\intercal$ on the left or a power of $\mathbb{S}$ on the right.

For the other point, it suffices to prove by induction that

$\mathbb{S}^+ \cap \mathbb{S}^{\intercal^n} ; z ; \mathbb{T} = \mathbb{S}^{\intercal^n} ; z ; \mathbb{T} ; \overline{\mathbb{T} ; z^\intercal} ; \mathbb{S}^n,$

starting with $n = 0$:

$\mathbb{S}^+ \cap z ; \mathbb{T} = (\overline{\mathbb{I}} \cap \mathbb{S}^*) \cap z ; \mathbb{T} = \overline{\mathbb{I}} \cap z ; \mathbb{T} ; \mathbb{T}$ because $z ; \mathbb{T} \subseteq \mathbb{S}^*$ due to Prop. 5.2.1.ii.b
$\subseteq (z ; \mathbb{T} \cap \overline{\mathbb{I}} ; \mathbb{T}^\intercal) ; (\mathbb{T} \cap \mathbb{T} ; z^\intercal ; \overline{\mathbb{I}})$ Dedekind rule
$\subseteq z ; \mathbb{T} ; \overline{\mathbb{T} ; z^\intercal} ; \overline{\mathbb{I}} = z ; \mathbb{T} ; \overline{\mathbb{T} ; z^\intercal}$ masking

But also $z ; \mathbb{T} ; \overline{\mathbb{T} ; z^\intercal} \subseteq \overline{\mathbb{I}}$, since equivalent via the Schröder rule with $\mathbb{T} ; z^\intercal ; \mathbb{I} \subseteq \mathbb{T} ; z^\intercal$.

Induction step $n \mapsto n + 1$:

$\mathbb{S}^{\intercal^{n+1}} ; z ; \mathbb{T} ; \overline{\mathbb{T} ; z^\intercal} ; \mathbb{S}^{n+1} = \mathbb{S}^\intercal ; (\mathbb{S}^{\intercal^n} ; z ; \mathbb{T} ; \overline{\mathbb{T} ; z^\intercal} ; \mathbb{S}^n) ; \mathbb{S}$
$= \mathbb{S}^\intercal ; (\mathbb{S}^+ \cap \mathbb{S}^{\intercal^n} ; z ; \mathbb{T}) ; \mathbb{S}$ hypothesis
$= \mathbb{S}^\intercal ; (\mathbb{S}^+ ; \mathbb{S} \cap \mathbb{S}^{\intercal^n} ; z ; \mathbb{T}) = \mathbb{S}^\intercal ; (\mathbb{S} ; \mathbb{S}^+ \cap \mathbb{S}^{\intercal^n} ; z ; \mathbb{T})$ masking
$= \mathbb{S}^\intercal ; \mathbb{S} ; \mathbb{S}^+ \cap \mathbb{S}^{\intercal^{n+1}} ; z ; \mathbb{T}$ since $\mathbb{S}$ is injective

$\subseteq \mathbb{S}^+ \cap \mathbb{S}^{\intercal n+1}; z; \mathbb{T}$

but also

$\mathbb{S}^+ \cap \mathbb{S}^{\intercal n+1}; z; \mathbb{T} = \mathbb{S}^+ \cap \mathbb{S}^{\intercal}; \mathbb{S}^{\intercal n}; z; \mathbb{T}$
$\subseteq (\mathbb{S}^{\intercal} \cap \mathbb{S}^+; \mathbb{T}; z^{\intercal}; \mathbb{S}^n); (\mathbb{S}^{\intercal n}; z; \mathbb{T} \cap \mathbb{S}; \mathbb{S}^+)$ Dedekind rule
$\subseteq \mathbb{S}^{\intercal}; (\mathbb{S}^{\intercal n}; z; \mathbb{T} \cap \mathbb{S}^+; \mathbb{S}) = \mathbb{S}^{\intercal}; (\mathbb{S}^{\intercal n}; z; \mathbb{T} \cap \mathbb{S}^+); \mathbb{S}$ masking
$= \mathbb{S}^{\intercal}; (\mathbb{S}^{\intercal n}; z; \mathbb{T}; \overline{\mathbb{T}; z^{\intercal}}; \mathbb{S}^n); \mathbb{S}$ hypothesis
$= \mathbb{S}^{\intercal n+1}; z; \mathbb{T}; \overline{\mathbb{T}; z^{\intercal}}; \mathbb{S}^{n+1}$

---

ii.c) The non-transitive part of R is defined as $\mathtt{Nontrans}(R) := R \cap \overline{R; R}$, and indeed

$\mathtt{Nontrans}(\mathbb{S}^+) = \mathbb{S}^+ \cap \overline{\mathbb{S}^+; \mathbb{S}^+} = \mathtt{sup}_{n\geq 1}\, \mathbb{S}^n \cap \overline{\mathtt{sup}_{n\geq 1}\, \mathbb{S}^n; \mathtt{sup}_{m\geq 1}\, \mathbb{S}^m} = \mathbb{S}\,,$

implying $\big[\mathtt{Nontrans}(\mathbb{S}^+)\big]^+ = \mathbb{S}^+$ and qualifying it to be called the Hasse relation $\mathtt{Hasse}(\mathbb{S}^+) := \mathtt{Nontrans}(\mathbb{S}^+)$ of the strictorder $\mathbb{S}^+$.

As a transitive relation $\mathbb{S}^+$ is — according to a remark from [11, page 138], resp. [12, page 136] — path-length bounded by definition if

$\mathtt{sup}_{n\geq 1}\, \overline{(\mathbb{S}^+)^n} = \mathbb{T}$, or else if $\mathtt{inf}_{n\geq 1}(\mathbb{S}^+)^n = \mathbb{\bot}$.

Assuming that this does not hold, i.e. that there exist points $p = \mathbb{S}^{\intercal a}; z; \mathbb{T}, q = \mathbb{S}^{\intercal b}; z; \mathbb{T}$ with

$\forall n \geq 1 : p; q^{\intercal} \subseteq (\mathbb{S}^+)^n$,

we get a contradiction as soon as n is chosen above b, and thus annihilating the right side:

$\iff \quad p \subseteq (\mathbb{S}^+)^n; q \quad \iff \quad p \subseteq (\mathbb{S}^+)^n; \mathbb{S}^{\intercal b}; z; \mathbb{T}$

Due to [11, Prop. 6.5.5.ii], resp. [12, Prop. 6.5.5.ii], every path-length bounded relation is also discrete. Further, due to [11, Prop. 6.5.10], resp. [12, Prop. 6.5.10], the discrete strictorder $\mathbb{S}^+$ has a Hasse-relation, i.e. $\mathbb{S}^+ = \big[\mathtt{Hasse}(\mathbb{S}^+)\big]^+$ — a second argument for this fact.

---

ii.d) $(\mathbb{I} \oslash \mathbb{S}^*); \mathbb{T} = ((\mathbb{S}^{\intercal})^* \oslash \mathbb{I}); \mathbb{T}$ Prop. 2.1.1.iii
$= (\overline{\mathbb{S}^+} \oslash \mathbb{I}); \mathbb{T}$ since $\mathbb{S}$ is a linear strictorder
$= \overline{(\mathbb{S}^+ \oslash \mathbb{I}); \mathbb{T}}$ Prop. 2.1.1.v

---

iii.a) This follows from (ii.a) and Prop. 5.2.1.iii.c.

---

iii.b) Now we employ in addition Prop. 5.2.1.iii.h and Prop. 5.2.1.iii.i to obtain

$\mathbb{S}^{\intercal *} = \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\intercal n} = \mathtt{sup}_{n\geq 0}\, \mathtt{FlipTD}\,; \mathtt{FlipTD}\,; \mathbb{S}^{\intercal n} = \mathtt{FlipTD}\,; \mathtt{sup}_{n\geq 0}\, \mathbb{S}^n; \mathtt{FlipTD}$
$= \mathtt{FlipTD}\,; \mathbb{S}^*; \mathtt{FlipTD}$
$= \mathtt{FlipTD}\,; \big[\mathtt{sup}_{n_{\mathbb{S}}\geq n\geq 0}\, \mathbb{S}^{\intercal n}; z; \mathbb{T}; \mathbb{S}^n\big]; \mathtt{FlipTD}$ due to (iii.a)
$= \mathtt{sup}_{n_{\mathbb{S}}\geq n\geq 0}\, \mathtt{FlipTD}\,; \mathbb{S}^{\intercal n}; z; \mathbb{T}; \mathbb{S}^n; \mathtt{FlipTD}$
$= \mathtt{sup}_{n_{\mathbb{S}}\geq n\geq 0}\, \mathbb{S}^n; \mathtt{FlipTD}; z; \mathbb{T}; \mathtt{FlipTD}\,; \mathbb{S}^{\intercal n}$
$= \mathtt{sup}_{n_{\mathbb{S}}\geq n\geq 0}\, \mathbb{S}^n; g_{\mathbb{S}}\,; \mathbb{T}; \mathbb{S}^{\intercal n}$ since $\mathtt{FlipTD}$ is surjective and Prop. 5.2.1.iii.g

---

iii.c) The greatest point wrt. the ordering is $\mathtt{gre}_{\mathbb{S}^*}(\mathbb{T})$; the claim, thus, $\overline{\mathbb{S}; \mathbb{T}} = g_{\mathbb{S}} = \overline{\overline{\mathbb{S}^{*\intercal}}; \mathbb{T}}$. This is easily verified in an example, but non-trivial in a formal proof. For the direction "$\supseteq$" we start with irreflexivity and asymmetry of the linear strictorder $\mathbb{S}^+$:

$$\mathbb{S}^+ \subseteq \overline{\mathbb{I}}, \quad \mathbb{S}^{+\intercal} \subseteq \overline{\mathbb{S}^+} \quad \Longrightarrow \quad \mathbb{S}^+ \subseteq \overline{\mathbb{I} \cup \mathbb{S}^{+\intercal}} = \overline{\mathbb{S}^{*\intercal}}$$
$$\Longrightarrow \quad \mathbb{S}^+ ; \mathbb{T} \subseteq \overline{\mathbb{S}^{*\intercal}} ; \mathbb{T} \quad \Longrightarrow \quad \overline{\overline{\mathbb{S}^{*\intercal}} ; \mathbb{T}} \subseteq \overline{\mathbb{S}^+ ; \mathbb{T}} = \overline{\mathbb{S} ; \mathbb{T}},$$

i.e. one half of the claim, since trivially $\mathbb{S}^+ ; \mathbb{T} = \mathbb{S} ; \mathbb{T}$.

In the other direction, we use $z ; \mathbb{T} \subseteq \mathbb{S}^*$ from Prop. 5.2.1.ii.b:

$$\overline{\mathbb{S}^\intercal ; \mathbb{T}} = z ; \mathbb{T} \subseteq \mathbb{S}^* \quad \Longrightarrow \quad \mathtt{FlipTD} ; \overline{\mathbb{S}^\intercal ; \mathbb{T}} ; \mathtt{FlipTD} \subseteq \mathtt{FlipTD} ; \mathbb{S}^* ; \mathtt{FlipTD}$$
$$\Longrightarrow \quad \overline{\mathbb{S} ; \mathtt{FlipTD} ; \mathbb{T} ; \mathtt{FlipTD}} \subseteq \mathbb{S}^{\intercal *} ; \mathtt{FlipTD} ; \mathtt{FlipTD}, \text{ i.e. } \quad \overline{\mathbb{S} ; \mathbb{T}} \subseteq \mathbb{S}^{\intercal *},$$

so that indeed

$$\overline{\mathbb{S} ; \mathbb{T}} = \overline{\mathbb{S} ; \mathbb{T}} ; \mathbb{T} \subseteq \mathbb{S}^{\intercal *} \quad \Longleftrightarrow \quad \overline{\mathbb{S}^{\intercal *}} ; \mathbb{T}^\intercal \subseteq \mathbb{S} ; \mathbb{T} \quad \text{Schröder rule}$$
$$\overline{\mathbb{S} ; \mathbb{T}} \subseteq \overline{\mathbb{S}^{*\intercal}} ; \mathbb{T} = g_{\mathbb{S}}. \qquad \square$$

Verbal interpretation of (ii.d): If it is not the case that the first component of a pair has to be strictly increased in order to catch up with the second, the second will be less or equal to the first.

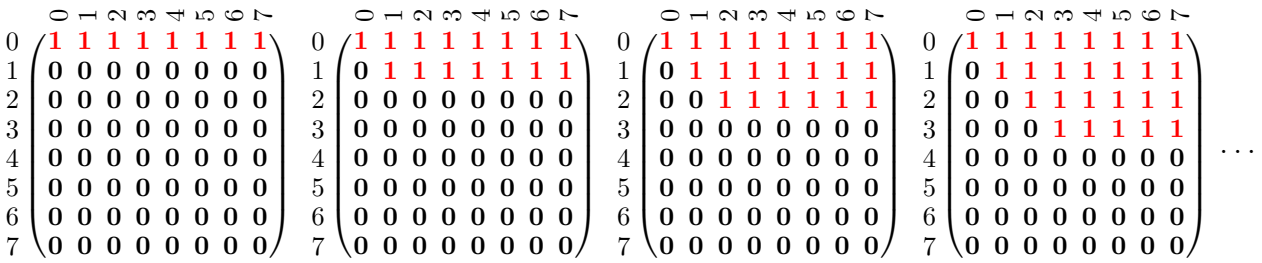


Fig. 5.3.1 Progress of the functional $g(\mathbb{\perp}), g(g(\mathbb{\perp})), \ldots$ to exhaust the upper right triangle

The item (iii) of the next proposition offers systematically a path that is more or less hidden by normal number situations: With $\mathbb{S}$ regressively finite and total, the result will always be $\mathbb{T}$ and, thus, uninteresting. Nevertheless, we start here our general procedure.

5.3.2 Definition and Proposition. We define as **left upper triangle** for $\mathbb{S}$ the construct

$$\mathbb{V} := \mathbb{V}_{\mathbb{S}} := \mathtt{inf} \{ X \mid u(X) \subseteq X \},$$

where we have in view of Lemma 4.2.1

$$u(X) := \mathbb{V}_0 \cup u_H(X) \quad \text{with} \quad \mathbb{V}_0 := z ; \mathbb{T} \quad \text{and} \quad u_H(X) := \mathbb{S}^\intercal ; X ; \mathbb{S}^\intercal.$$

Respecting possible further properties of $\mathbb{S}$, the relation $\mathbb{V}$ satisfies

i) in general:

a) $\mathbb{V} = z ; \mathbb{T} \cup \mathbb{S}^\intercal ; \mathbb{V} ; \mathbb{S}^\intercal = \mathtt{sup}_{n \geq 0} u_H^n(\mathbb{V}_0) = \mathtt{sup}_{n \geq 0} \mathbb{V}_n$ with $\mathbb{V}_n := \mathtt{sup}_{0 \leq i \leq n} \mathbb{S}^{\intercal i} ; z ; \mathbb{T} ; \mathbb{S}^{\intercal i}$

b) The supremum on the right is a disjoint union.

ii) additionally, in case $\mathbb{S}$ is regressively finite:

a) $\mathbb{V}$ is total.

iii) If $\mathbb{S}$ happens to be regressively finite but not total, i.e. $\mathbb{S}\fatsemi\mathbb{T} \neq \mathbb{T}$,

a) $\fatbslash = \mathbb{S}^*\fatsemi \texttt{FlipTD}$

iv) additionally, if $\mathbb{S}$ is regressively finite and total:

a) $\fatbslash = \mathbb{T}$

Proof: We observe in advance that Lemma 4.2.1 may be applied since the functional u_H is

isotonic, i.e. $A \subseteq B \implies \mathbb{S}^\intercal\fatsemi A\fatsemi\mathbb{S}^\intercal \subseteq \mathbb{S}^\intercal\fatsemi B\fatsemi\mathbb{S}^\intercal$, i.e. $u_H(A) \subseteq u_H(B)$,
additive, i.e. $u_H(X \cup Y) = \mathbb{S}^\intercal\fatsemi(X \cup Y)\fatsemi\mathbb{S}^\intercal = \mathbb{S}^\intercal\fatsemi X\fatsemi\mathbb{S}^\intercal \cup \mathbb{S}^\intercal\fatsemi Y\fatsemi\mathbb{S}^\intercal = u_H(X) \cup u_H(Y)$,
$\cap$-hereditary, i.e. $u_H(X \cap Y) = \mathbb{S}^\intercal\fatsemi(X \cap Y)\fatsemi\mathbb{S}^\intercal = \mathbb{S}^\intercal\fatsemi X\fatsemi\mathbb{S}^\intercal \cap \mathbb{S}^\intercal\fatsemi Y\fatsemi\mathbb{S}^\intercal = u_H(X) \cap u_H(Y)$.

---

i.a) This follows from Lemma 4.2.1.i,iv.

---

i.b) follows from $\mathbb{S}^{\intercal i}\fatsemi z\fatsemi\mathbb{T} \neq \mathbb{S}^{\intercal j}\fatsemi z\fatsemi\mathbb{T}$ for $i \neq j$; see Prop. 5.2.1.i.d

---

ii.a) From Prop. 5.2.1.ii.a, we know that $\mathbb{T} = \texttt{sup}_{n\geq 0}\, \mathbb{S}^{\intercal n}\fatsemi z\fatsemi\mathbb{T}$. Either $\mathbb{S}^{\intercal n}\fatsemi z\fatsemi\mathbb{T} = \perp\!\!\!\perp$ and the respective summand will not contribute anything. Or else $\mathbb{S}^{\intercal n}\fatsemi z\fatsemi\mathbb{T} \neq \perp\!\!\!\perp$, in which case necessarily $\mathbb{S}^{\intercal n} \neq \perp\!\!\!\perp$ due to Prop. 5.2.1.ii.d; then, however also $\mathbb{T}\fatsemi\mathbb{S}^{\intercal n}\fatsemi\mathbb{T} \neq \mathbb{T}$, implying

$\fatbslash\fatsemi\mathbb{T} = \big[\texttt{sup}_{n\geq 0}\, \mathbb{S}^{\intercal n}\fatsemi z\fatsemi\mathbb{T}\fatsemi\mathbb{S}^{\intercal n}\big]\fatsemi\mathbb{T} = \texttt{sup}_{n\geq 0}\, \mathbb{S}^{\intercal n}\fatsemi z\fatsemi\mathbb{T}\fatsemi\mathbb{S}^{\intercal n}\fatsemi\mathbb{T} = \texttt{sup}_{n\geq 0}\, \mathbb{S}^{\intercal n}\fatsemi z\fatsemi\mathbb{T} = \mathbb{T}$.

---

iii.a) The sequence $\fatbslash_0 := z\fatsemi\mathbb{T}$, $\fatbslash_{n+1} := u(\fatbslash_n) = z\fatsemi\mathbb{T} \cup \mathbb{S}^\intercal\fatsemi\fatbslash_n\fatsemi\mathbb{S}^\intercal$ satisfies

$\fatbslash_n = \mathbb{S}^*\fatsemi \texttt{FlipTD} \cap \texttt{sup}_{0\leq i\leq n}\, \mathbb{S}^{\intercal i}\fatsemi z\fatsemi\mathbb{T}$,

which is shown via induction as follows:

$\fatbslash_0 = z\fatsemi\mathbb{T} = z\fatsemi\mathbb{T}\fatsemi \texttt{FlipTD}$ since FlipTD is an involutory bijective map; Prop. 5.2.1
$= (\mathbb{S}^* \cap z\fatsemi\mathbb{T})\fatsemi \texttt{FlipTD}$ since $z\fatsemi\mathbb{T} \subseteq \mathbb{S}^*$ due to Prop. 5.2.1.ii.b
$= \mathbb{S}^*\fatsemi \texttt{FlipTD} \cap z\fatsemi\mathbb{T}$ masking
$= \mathbb{S}^*\fatsemi \texttt{FlipTD} \cap \mathbb{S}^{\intercal 0}\fatsemi z\fatsemi\mathbb{T}$

$\fatbslash_{n+1} = z\fatsemi\mathbb{T} \cup \mathbb{S}^\intercal\fatsemi\fatbslash_n\fatsemi\mathbb{S}^\intercal$
$= z\fatsemi\mathbb{T} \cup \mathbb{S}^\intercal\fatsemi\big[\mathbb{S}^*\fatsemi \texttt{FlipTD} \cap \texttt{sup}_{0\leq i\leq n}\, \mathbb{S}^{\intercal i}\fatsemi z\fatsemi\mathbb{T}\big]\fatsemi\mathbb{S}^\intercal$ hypothesis
$= z\fatsemi\mathbb{T} \cup \big[\mathbb{S}^\intercal\fatsemi\mathbb{S}^*\fatsemi \texttt{FlipTD} \cap \texttt{sup}_{0\leq i\leq n}\, \mathbb{S}^{\intercal i+1}\fatsemi z\fatsemi\mathbb{T}\big]\fatsemi\mathbb{S}^\intercal$ with $\mathbb{S}$ injective
$= z\fatsemi\mathbb{T} \cup \big[\mathbb{S}^\intercal\fatsemi\mathbb{S}^*\fatsemi \texttt{FlipTD}\fatsemi\mathbb{S}^\intercal \cap \texttt{sup}_{0\leq i\leq n}\, \mathbb{S}^{\intercal i+1}\fatsemi z\fatsemi\mathbb{T}\big]$ masking
$= z\fatsemi\mathbb{T} \cup \big[\mathbb{S}^\intercal\fatsemi\mathbb{S}^*\fatsemi\mathbb{S}\fatsemi \texttt{FlipTD} \cap \texttt{sup}_{0\leq i\leq n}\, \mathbb{S}^{\intercal i+1}\fatsemi z\fatsemi\mathbb{T}\big]$ due to Prop. 5.2.1.iii.i
$= z\fatsemi\mathbb{T} \cup \big[\mathbb{S}^\intercal\fatsemi\mathbb{S}\fatsemi\mathbb{S}^*\fatsemi \texttt{FlipTD} \cap \texttt{sup}_{0\leq i\leq n}\, \mathbb{S}^{\intercal i+1}\fatsemi z\fatsemi\mathbb{T}\big]$
$= z\fatsemi\mathbb{T} \cup \big[(\mathbb{I} \cap \overline{z\fatsemi\mathbb{T}})\fatsemi\mathbb{S}^*\fatsemi \texttt{FlipTD} \cap \texttt{sup}_{0\leq i\leq n}\, \mathbb{S}^{\intercal i+1}\fatsemi z\fatsemi\mathbb{T}\big]$
$= z\fatsemi\mathbb{T} \cup \big[\mathbb{S}^*\fatsemi \texttt{FlipTD} \cap \overline{z\fatsemi\mathbb{T}} \cap \texttt{sup}_{0\leq i\leq n}\, \mathbb{S}^{\intercal i+1}\fatsemi z\fatsemi\mathbb{T}\big]$ masking
$= \big\{z\fatsemi\mathbb{T} \cup \mathbb{S}^*\fatsemi \texttt{FlipTD}\big\} \cap \mathbb{T} \cap \big\{z\fatsemi\mathbb{T} \cup \texttt{sup}_{0\leq i\leq n}\, \mathbb{S}^{\intercal i+1}\fatsemi z\fatsemi\mathbb{T}\big\}$
$= \big\{(z\fatsemi\mathbb{T} \cup \mathbb{S}^*)\fatsemi \texttt{FlipTD}\big\} \cap \texttt{sup}_{0\leq k\leq n+1}\, \mathbb{S}^{\intercal k}\fatsemi z\fatsemi\mathbb{T}$ masking
$= \mathbb{S}^*\fatsemi \texttt{FlipTD} \cap \texttt{sup}_{0\leq k\leq n+1}\, \mathbb{S}^{\intercal k}\fatsemi z\fatsemi\mathbb{T}$ again since $z\fatsemi\mathbb{T} \subseteq \mathbb{S}^*$ due to Prop. 5.2.1.ii.b

This is now to be united over all n, making the united right terms to $\mathbb{T}$.

---

iv.a) $\fatbslash = \texttt{sup}_{n\geq 0}\, \mathbb{S}^{\intercal n}\fatsemi z\fatsemi\mathbb{T}\fatsemi\mathbb{S}^{\intercal n} = \texttt{sup}_{n\geq 0}\, \mathbb{S}^{\intercal n}\fatsemi z\fatsemi\mathbb{T}$ since $\mathbb{S}$ is total
$= \mathbb{T}$ due to Prop. 5.2.1.ii.a □

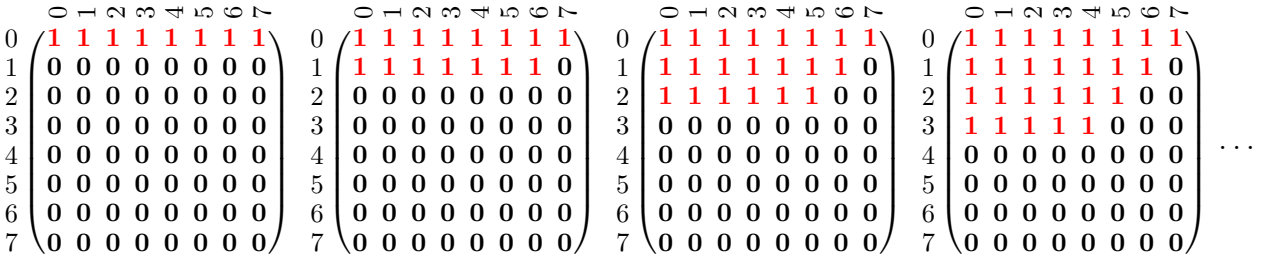


Fig. 5.3.2 Progress of $\mathbb{V}_0, \mathbb{V}_1, \ldots$ to exhaust the upper left triangle

We now consider an issue that seems pictorially related with $\mathbb{S}^*$ and $\mathbb{S}$, however mirrored along the middle vertical line so as to obtain

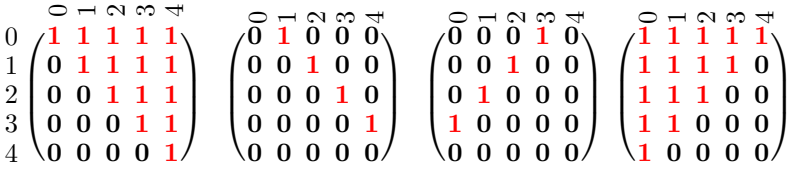


Fig. 5.3.3 $\mathbb{S}^*$ and $\mathbb{S}$ mirrored to $\not{/}_3$ and $\mathbb{V}$

The relational definitions differ deeply in their structure. Not least the concept of "the middle vertical line" restricts the idea of mirroring to the finite case. This may be seen in (iii) of the following

5.3.3 Definition and Proposition. We define

$$\not{/}_0 := z ; \mathbb{T} ; z^\mathsf{T} \qquad \not{/}_{n+1} := \mathbb{S}^\mathsf{T} ; \not{/}_n \cup \not{/}_n ; \mathbb{S}$$

Respecting possible further properties of $\mathbb{S}$, for all $n \geq 0$ the construct

i) in general:

a) is symmetric and satisfies $\mathbb{S} ; \not{/}_n ; \mathbb{S} \subseteq \not{/}_n$ for all $n \geq 0$

b) satisfies for all $n \geq 0$ the following: $\not{/}_n ; \not{/}_n \subseteq \mathbb{I}$

c) $\not{/}_n$ is univalent and injective for all $n \geq 0$.

d) $\not{/}_n ; \mathbb{S} ; \not{/}_n \subseteq \mathbb{S}^\mathsf{T}$ for all $n \geq 0$

e) $\not{/}_{n+1} ; \mathbb{S}^\mathsf{T} \subseteq \not{/}_n$ for all $n \geq 0$

f) $\mathbb{S} ; \not{/}_{n+1} \subseteq \not{/}_n$ for all $n \geq 0$

g) $\mathbb{S}^\mathsf{T} ; \not{/}_n \supseteq \mathbb{S}^{\mathsf{T}n+1} ; z ; \mathbb{T} ; z^\mathsf{T}$ for all $n \geq 0$

h) $\not{/}_n ; \mathbb{S} \supseteq z ; \mathbb{T} ; z^\mathsf{T} ; \mathbb{S}^{n+1}$ for all $n \geq 0$

i) $\not{/}_n = \mathtt{sup}_{0 \leq i \leq n} \, \mathbb{S}^{\mathsf{T}i} ; z ; \mathbb{T} ; z^\mathsf{T} ; \mathbb{S}^{n-i}$ for all $n \geq 0$

j) $\mathbb{S}^\mathsf{T} ; \not{/}_n = \not{/}_{n+1} \cap \mathbb{S}^\mathsf{T} ; \mathbb{T}$ for all $n \geq 0$

ii) additionally, in case $\mathbb{S}$ is regressively finite:

a)

iii) If $\mathbb{S}$ happens to be regressively finite but not total, i.e. $\mathbb{S};\mathbb{T} \neq \mathbb{T}$, there will exist the greatest number $n_{\mathbb{S}}$ and

a) $\not{/}_k = \mathbb{S}^{n_{\mathbb{S}}-k};\texttt{FlipTD} \quad$ for all $0 \leq k \leq n_{\mathbb{S}}$

iv) additionally, if $\mathbb{S}$ is regressively finite and total:

a)

Proof: i.a) Symmetry is proved by induction, starting with the obvious case $n = 0$:
$\not{/}^{\mathsf{T}}_{n+1} = \left[\mathbb{S}^{\mathsf{T}};\not{/}_n \cup \not{/}_n;\mathbb{S}\right]^{\mathsf{T}} = \not{/}^{\mathsf{T}}_n;\mathbb{S} \cup \mathbb{S}^{\mathsf{T}};\not{/}^{\mathsf{T}}_n$
$= \not{/}_n;\mathbb{S} \cup \mathbb{S}^{\mathsf{T}};\not{/}_n \quad$ hypothesis
$= \not{/}_{n+1}$

For $n = 0$ we see that
$\mathbb{S};\not{/}_0;\mathbb{S} = \mathbb{S};z;\mathbb{T};z^{\mathsf{T}};\mathbb{S} = \perp\!\!\!\perp \subseteq \not{/}_0$
Now the induction:
$\mathbb{S};\not{/}_{n+1};\mathbb{S} = \mathbb{S};\left[\mathbb{S}^{\mathsf{T}};\not{/}_n \cup \not{/}_n;\mathbb{S}\right];\mathbb{S} = \mathbb{S};\mathbb{S}^{\mathsf{T}};\not{/}_n;\mathbb{S} \cup \mathbb{S};\not{/}_n;\mathbb{S};\mathbb{S}$
$\subseteq \not{/}_n;\mathbb{S} \cup \not{/}_n;\mathbb{S} \quad$ since $\mathbb{S}$ is injective and by hypothesis
$= \not{/}_n;\mathbb{S} \subseteq \not{/}_{n+1}$

---

i.b) The proof is by induction started trivially $\not{/}_0;\not{/}_0 = z;\mathbb{T};z^{\mathsf{T}};z;\mathbb{T};z^{\mathsf{T}} \subseteq z;\mathbb{T};z^{\mathsf{T}} = \not{/}_0 \subseteq \mathbb{I}$:
$\not{/}_{n+1};\not{/}_{n+1} = \left[\mathbb{S}^{\mathsf{T}};\not{/}_n \cup \not{/}_n;\mathbb{S}\right];\left[\mathbb{S}^{\mathsf{T}};\not{/}_n \cup \not{/}_n;\mathbb{S}\right]$
$= \mathbb{S}^{\mathsf{T}};\not{/}_n;\mathbb{S}^{\mathsf{T}};\not{/}_n \cup \mathbb{S}^{\mathsf{T}};\not{/}_n;\not{/}_n;\mathbb{S} \cup \not{/}_n;\mathbb{S};\mathbb{S}^{\mathsf{T}};\not{/}_n \cup \not{/}_n;\mathbb{S};\not{/}_n;\mathbb{S}$
$\subseteq \not{/}_n;\not{/}_n \cup \mathbb{S}^{\mathsf{T}};\mathbb{S} \cup \not{/}_n;\not{/}_n \cup \not{/}_n;\not{/}_n \quad$ due to (i.a), hypothesis, and $\mathbb{S}$ injective
$\subseteq \mathbb{I} \quad$ hypothesis and $\mathbb{S}$ univalent

---

i.c) Symmetry of (i.a) together with (i.b) show that $\not{/}_n$ is univalent and injective for all $n \geq 0$.

---

i.d) Induction again with start $\not{/}_0;\mathbb{S};\not{/}_0 = z;\mathbb{T};z^{\mathsf{T}};\mathbb{S};z;\mathbb{T};z^{\mathsf{T}} = \perp\!\!\!\perp \subseteq \mathbb{S}^{\mathsf{T}}$ and induction step
$\not{/}_{n+1};\mathbb{S};\not{/}_{n+1} = \left[\mathbb{S}^{\mathsf{T}};\not{/}_n \cup \not{/}_n;\mathbb{S}\right];\mathbb{S};\left[\mathbb{S}^{\mathsf{T}};\not{/}_n \cup \not{/}_n;\mathbb{S}\right]$
$= \mathbb{S}^{\mathsf{T}};\not{/}_n;\mathbb{S};\mathbb{S}^{\mathsf{T}};\not{/}_n \cup \mathbb{S}^{\mathsf{T}};\not{/}_n;\mathbb{S};\not{/}_n;\mathbb{S} \cup \not{/}_n;\mathbb{S};\mathbb{S};\mathbb{S}^{\mathsf{T}};\not{/}_n \cup \not{/}_n;\mathbb{S};\mathbb{S};\not{/}_n;\mathbb{S}$
$\subseteq \mathbb{S}^{\mathsf{T}};\not{/}_n;\not{/}_n \cup \mathbb{S}^{\mathsf{T}};\not{/}_n;\not{/}_n \cup \not{/}_n;\mathbb{S};\not{/}_n \cup \not{/}_n;\mathbb{S};\not{/}_n \quad$ since $\mathbb{S}$ is injective, (i.a)
$\subseteq \mathbb{S}^{\mathsf{T}} \cup \mathbb{S}^{\mathsf{T}} \cup \mathbb{S}^{\mathsf{T}} \cup \mathbb{S}^{\mathsf{T}} \quad$ using (i.b) and hypothesis
$= \mathbb{S}^{\mathsf{T}}$

---

i.e) For $n = 0$ we have to show
$\left[\mathbb{S}^{\mathsf{T}};\not{/}_0 \cup \not{/}_0;\mathbb{S}\right];\mathbb{S}^{\mathsf{T}} \subseteq \not{/}_0$
which is obvious in view of $\mathbb{S};z;\mathbb{T} = \perp\!\!\!\perp$ and $\mathbb{S}$ injective in the expanded version

$\big[\mathbb{S}^\mathsf{T};z;\mathbb{T};z^\mathsf{T}\cup z;\mathbb{T};z^\mathsf{T};\mathbb{S}\big];\mathbb{S}^\mathsf{T}\subseteq z;\mathbb{T};z^\mathsf{T}$.

Furthermore

$\not\swarrow_{n+1};\mathbb{S}^\mathsf{T}=\big[\mathbb{S}^\mathsf{T};\not\swarrow_n\cup\not\swarrow_n;\mathbb{S}\big];\mathbb{S}^\mathsf{T}=\mathbb{S}^\mathsf{T};\not\swarrow_n;\mathbb{S}^\mathsf{T}\cup\not\swarrow_n;\mathbb{S};\mathbb{S}^\mathsf{T}$
$\subseteq\not\swarrow_n\cup\not\swarrow_n=\not\swarrow_n$ via (i.a) and injectivity of $\mathbb{S}$

---

i.f) is simply (i.e) in transposed form.

---

i.g) The case $n=0$ is obvious.

$\mathbb{S}^\mathsf{T};\not\swarrow_{n+1}\supseteq\mathbb{S}^\mathsf{T};(\mathbb{S}^\mathsf{T};\not\swarrow_n)\supseteq\mathbb{S}^\mathsf{T};\mathbb{S}^{\mathsf{T}n+1};z;\mathbb{T};z^\mathsf{T}=\mathbb{S}^{\mathsf{T}n+2};z;\mathbb{T};z^\mathsf{T}$ using the hypothesis

---

i.h) Again, (i.g) in transposed form.

---

i.i) The case $n=0$ is obvious. Induction now means

$\not\swarrow_{n+1}=\mathbb{S}^\mathsf{T};\not\swarrow_n\cup\not\swarrow_n;\mathbb{S}$
$=\mathbb{S}^\mathsf{T};[\mathtt{sup}_{0\le i\le n}\,\mathbb{S}^{\mathsf{T}i};z;\mathbb{T};z^\mathsf{T};\mathbb{S}^{n-i}]\cup[\mathtt{sup}_{0\le j\le n}\,\mathbb{S}^{\mathsf{T}j};z;\mathbb{T};z^\mathsf{T};\mathbb{S}^{n-j}];\mathbb{S}$ hypothesis
$=\big[\underbrace{\sup_{0\le i<n}\,\mathbb{S}^{\mathsf{T}i+1};z;\mathbb{T};z^\mathsf{T};\mathbb{S}^{n-i}}_{\text{equal with } j=i+1}\cup\mathbb{S}^{\mathsf{T}n+1};z;\mathbb{T};z^\mathsf{T}\big]$
$\cup\big[z;\mathbb{T};z^\mathsf{T};\mathbb{S}^n;\mathbb{S}\cup\underbrace{\sup_{0<j\le n}\,\mathbb{S}^{\mathsf{T}j};z;\mathbb{T};z^\mathsf{T};\mathbb{S}^{n-j};\mathbb{S}}\big]$
$=\mathtt{sup}_{0\le k\le n+1}\,\mathbb{S}^{\mathsf{T}k};z;\mathbb{T};z^\mathsf{T};\mathbb{S}^{(n+1)-k}$

---

i.j) Proof by induction starting with the case $n=0$:

$\not\swarrow_1\cap\mathbb{S}^\mathsf{T};\mathbb{T}=[\mathbb{S}^\mathsf{T};\not\swarrow_0\cup\not\swarrow_0;\mathbb{S}]\cap\mathbb{S}^\mathsf{T};\mathbb{T}$
$=[\mathbb{S}^\mathsf{T};z;\mathbb{T};z^\mathsf{T}\cup z;\mathbb{T};z^\mathsf{T};\mathbb{S}]\cap\mathbb{S}^\mathsf{T};\mathbb{T}$
$=[\mathbb{S}^\mathsf{T};z;\mathbb{T};z^\mathsf{T}\cap\mathbb{S}^\mathsf{T};\mathbb{T}]\cup[z;\mathbb{T};z^\mathsf{T};\mathbb{S}\cap\mathbb{S}^\mathsf{T};\mathbb{T}]=[\mathbb{S}^\mathsf{T};z;\mathbb{T};z^\mathsf{T}\cap\mathbb{S}^\mathsf{T};\mathbb{T}]\cup\perp\!\!\!\perp$
$=\mathbb{S}^\mathsf{T};\not\swarrow_0\cap\mathbb{S}^\mathsf{T};\mathbb{T}=\mathbb{S}^\mathsf{T};\not\swarrow_0$
$\not\swarrow_{n+1}\cap\mathbb{S}^\mathsf{T};\mathbb{T}=[\mathbb{S}^\mathsf{T};\not\swarrow_n\cup\not\swarrow_n;\mathbb{S}]\cap\mathbb{S}^\mathsf{T};\mathbb{T}=[\mathbb{S}^\mathsf{T};\not\swarrow_n\cap\mathbb{S}^\mathsf{T};\mathbb{T}]\cup[\not\swarrow_n;\mathbb{S}\cap\mathbb{S}^\mathsf{T};\mathbb{T}]$
$=\mathbb{S}^\mathsf{T};\not\swarrow_n\cup[\not\swarrow_n;\mathbb{S}\cap\mathbb{S}^\mathsf{T};\mathbb{T}]$

Now we just need an application of the Dedekind rule together with (i.a):

$\not\swarrow_n;\mathbb{S}\cap\mathbb{S}^\mathsf{T};\mathbb{T}\subseteq(\mathbb{S}^\mathsf{T}\cap\not\swarrow_n;\mathbb{S};\mathbb{T}^\mathsf{T});(\mathbb{T}\cap\mathbb{S};\not\swarrow_n;\mathbb{S})\subseteq\mathbb{S}^\mathsf{T};\mathbb{S};\not\swarrow_n;\mathbb{S}\subseteq\mathbb{S}^\mathsf{T};\not\swarrow_n$

---

ii)

---

iii) For $k=0$ we have

$\not\swarrow_0=z;\mathbb{T};z^\mathsf{T}=z;\mathbb{T};g^\mathsf{T}_\mathbb{S};\mathtt{FlipTD}$ Def. 5.2.1.iii.g,h
$=\mathbb{S}^{n_\mathbb{S}-0};\mathtt{FlipTD}$ Def. 5.2.1.iii.d

Now the induction step:

$\not\swarrow_{k+1}=\mathbb{S}^\mathsf{T};\not\swarrow_k\cup\not\swarrow_k;\mathbb{S}$
$=\mathbb{S}^\mathsf{T};\mathbb{S}^{n_\mathbb{S}-k};\mathtt{FlipTD}\cup\mathbb{S}^{n_\mathbb{S}-k};\mathtt{FlipTD};\mathbb{S}$ hypothesis

$= \mathbb{S}^{\intercal};\mathbb{S}^{n_\mathbb{S}-k};\mathtt{FlipTD} \cup \mathbb{S}^{n_\mathbb{S}-k};\mathbb{S}^{\intercal};\mathtt{FlipTD}$ Def. 5.2.1.iii.i
$= \big[\mathbb{S}^{\intercal};\mathbb{S}^{n_\mathbb{S}-k} \cup \mathbb{S}^{n_\mathbb{S}-k};\mathbb{S}^{\intercal}\big];\mathtt{FlipTD}$
$= \mathbb{S}^{n_\mathbb{S}-(k+1)};\mathtt{FlipTD}$

To justify the last step requires to show that always
$\mathbb{S}^{\intercal};\mathbb{S}^{n_\mathbb{S}-k} \cup \mathbb{S}^{n_\mathbb{S}-k};\mathbb{S}^{\intercal} = \mathbb{S}^{n_\mathbb{S}-(k+1)}$.

Applying Prop. 5.2.1.i.b twice,[3] we have indeed
$\mathbb{S}^{\intercal};\mathbb{S}^{n_\mathbb{S}-k} \cup \mathbb{S}^{n_\mathbb{S}-k};\mathbb{S}^{\intercal} = \big[\mathbb{S}^{(n_\mathbb{S}-k)-1} \cap \mathbb{S}^{\intercal};\mathbb{T}\big] \cup \big[\mathbb{S}^{(n_\mathbb{S}-k)-1} \cap \mathbb{T};\mathbb{S}^{\intercal}\big]$
$= \mathbb{S}^{(n_\mathbb{S}-k)-1} \cap \big[\mathbb{S}^{\intercal};\mathbb{T} \cup \mathbb{T};\mathbb{S}^{\intercal}\big]$
$= \mathbb{S}^{n_\mathbb{S}-(k+1)} \cap \big[\overline{z;\mathbb{T}} \cup \overline{g_\mathbb{S};\mathbb{T}}^{\intercal}\big] = \mathbb{S}^{n_\mathbb{S}-(k+1)} \cap \overline{z;\mathbb{T} \cap \mathbb{T};g_\mathbb{S}^{\intercal}} = \mathbb{S}^{n_\mathbb{S}-(k+1)} \cap \overline{z;\mathbb{T};g_\mathbb{S}^{\intercal}}$
$= \mathbb{S}^{n_\mathbb{S}-(k+1)} \cap \overline{\mathbb{S}^{n_\mathbb{S}}}$
$= \mathbb{S}^{n_\mathbb{S}-(k+1)}$

The latter will certainly hold since $n_\mathbb{S} - (k+1) < n_\mathbb{S}$.

---

iv) □

The following construct is determined to model what Fig. 5.3.4 shows.

5.3.4 Definition and Proposition. We define

$\mathscr{V}_0 := z;\mathbb{T};z^{\intercal} \qquad \mathscr{V}_{n+1} := \mathscr{V}_0 \cup \mathbb{S}^{\intercal};\mathscr{V}_n \cup \mathscr{V}_n;\mathbb{S}$

as well as

$\mathscr{V} := \mathtt{sup}_{n\geq 0}\, \mathscr{V}_n$.

Respecting possible further properties of $\mathbb{S}$, for all $n \geq 0$ the construct

i) in general:

a) is symmetric for all $n \geq 0$ and forms a monotone sequence, i.e. $\mathscr{V}_0 \subseteq \mathscr{V}_1 \subseteq \mathscr{V}_2 \subseteq \ldots$

$\mathscr{V}_n = \mathtt{sup}_{0\leq i\leq n}\, \mathtt{sup}_{0\leq j\leq n-i}\, \mathbb{S}^{\intercal i};z;\mathbb{T};z^{\intercal};\mathbb{S}^j$

$\mathscr{V} \supseteq z;\mathbb{T} \cap \overline{\mathtt{inf}_{k\geq 1}\, \mathbb{T};\mathbb{S}^k}$

b) satisfies $\not{\mathscr{V}}_n \subseteq \mathscr{V}_n$ for all $n \geq 0$ and also $\mathtt{sup}_{0\leq i\leq n}\, \not{\mathscr{V}}_i = \mathscr{V}_n$

c) For $n \geq 0$ always $\mathscr{V}_n \subseteq \overline{\mathbb{S}^{\intercal *};\mathbb{S}^{\intercal n+1};\mathbb{T}} \cap \overline{\mathbb{T};\mathbb{S}^{n+1};\mathbb{S}^*}$.

d) For $n \geq 0$ always $\mathscr{V}_n \cap \mathbb{S}^{\intercal n};z;\mathbb{T} = \mathbb{S}^{\intercal n};z;\mathbb{T} \cap \mathbb{T};z^{\intercal}$.

e) For $0 \leq n \leq n_\mathbb{S}$ always $\mathscr{V}_n \subseteq \mathbb{V}_n$.

ii) additionally, in case $\mathbb{S}$ is regressively finite:

a) $\mathscr{V} = \mathbb{T}$

[3] with different substitutions!

iii) If $\mathbb{S}$ happens to be regressively finite but not total,

a) $\not\mathbb{F}_{2*n_{\mathbb{S}}} = \mathbb{T}$

iv) additionally, if $\mathbb{S}$ is regressively finite and total:

a)

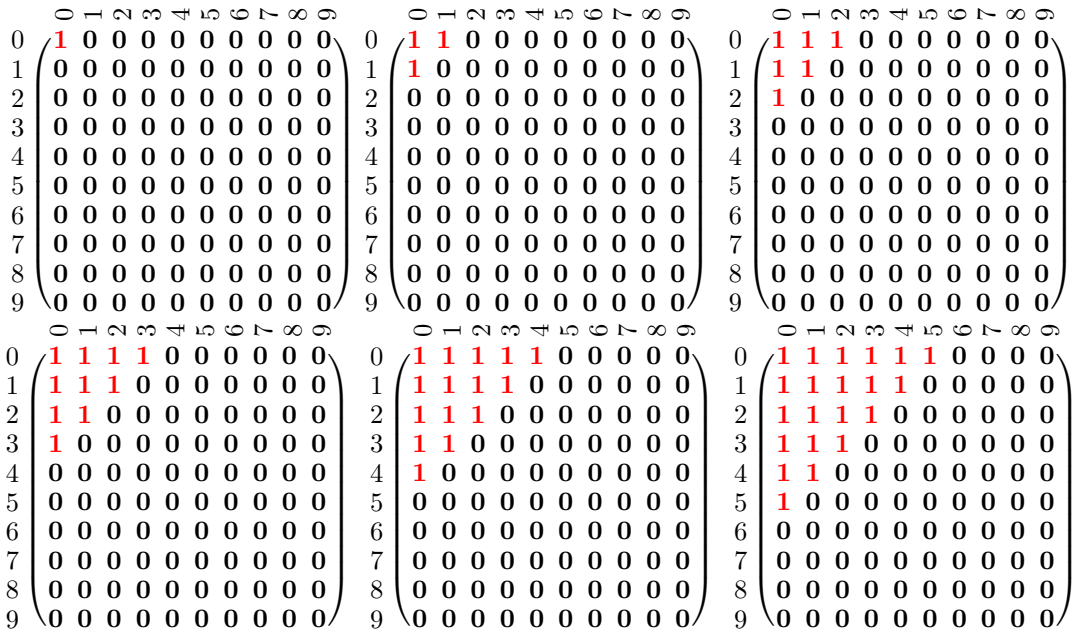


Fig. 5.3.4 $\not\mathbb{F}_i$ for $i = 0, 1, ..5$

Proof: i.a) Symmetry is proved by induction starting with the trivially satisfied case $n = 0$:

$\not\mathbb{F}^{\mathsf{T}}_{n+1} = \left[\not\mathbb{F}_0 \cup \mathbb{S}^{\mathsf{T}};\not\mathbb{F}_n \cup \not\mathbb{F}_n;\mathbb{S}\right]^{\mathsf{T}} = \not\mathbb{F}^{\mathsf{T}}_0 \cup \not\mathbb{F}^{\mathsf{T}}_n;\mathbb{S} \cup \mathbb{S}^{\mathsf{T}};\not\mathbb{F}^{\mathsf{T}}_n$

$= \not\mathbb{F}_0 \cup \not\mathbb{F}_n;\mathbb{S} \cup \mathbb{S}^{\mathsf{T}};\not\mathbb{F}_n = \not\mathbb{F}_{n+1}$ hypothesis

---

Also for monotony we use induction starting with the trivial $\not\mathbb{F}_0 \subseteq \not\mathbb{F}_1$ and continued by

$\not\mathbb{F}_{n+1} = \not\mathbb{F}_0 \cup \mathbb{S}^{\mathsf{T}};\not\mathbb{F}_n \cup \not\mathbb{F}_n;\mathbb{S} \subseteq \not\mathbb{F}_0 \cup \mathbb{S}^{\mathsf{T}};\not\mathbb{F}_{n+1} \cup \not\mathbb{F}_{n+1};\mathbb{S} \subseteq \not\mathbb{F}_{n+2}$

Now the inductive proof of the explicit formula, the start of which is immediate:

$\not\mathbb{F}_{n+1} = \not\mathbb{F}_0 \cup \mathbb{S}^{\mathsf{T}};\not\mathbb{F}_n \cup \not\mathbb{F}_n;\mathbb{S}$

$= z;\mathbb{T};z^{\mathsf{T}} \cup \mathbb{S}^{\mathsf{T}};\mathtt{sup}_{0\le i\le n}\,\mathtt{sup}_{0\le j\le n-i}\,\mathbb{S}^{\mathsf{T}i};z;\mathbb{T};z^{\mathsf{T}};\mathbb{S}^{j} \quad \cup$

$\mathtt{sup}_{0\le i\le n}\,\mathtt{sup}_{0\le j\le n-i}\,\mathbb{S}^{\mathsf{T}i};z;\mathbb{T};z^{\mathsf{T}};\mathbb{S}^{j};\mathbb{S}$ hypothesis

$= z;\mathbb{T};z^{\mathsf{T}} \cup \mathtt{sup}_{1\le k\le n+1}\,\mathtt{sup}_{0\le j\le (n+1)-k}\,\mathbb{S}^{\mathsf{T}k};z;\mathbb{T};z^{\mathsf{T}};\mathbb{S}^{j} \quad \cup$ using that $k := i+1$

$\mathtt{sup}_{0\le i\le n}\,\mathtt{sup}_{1\le l\le (n+1)-i}\,\mathbb{S}^{\mathsf{T}i};z;\mathbb{T};z^{\mathsf{T}};\mathbb{S}^{l}$ using that $l := j+1$

$= z;\mathbb{T};z^{\mathsf{T}} \cup \mathtt{sup}_{0\le k\le n+1}\,\mathtt{sup}_{0\le l\le (n+1)-k}\,\mathbb{S}^{\mathsf{T}k};z;\mathbb{T};z^{\mathsf{T}};\mathbb{S}^{l}$

We prove by induction that $\not\mathbb{F}_n \supseteq z;\mathbb{T} \cap \overline{\mathbb{T};\mathbb{S}^{n+1}}$, start with

$\not\mathbb{F}_0 = z;\mathbb{T};z^{\mathsf{T}} = z;\mathbb{T} \cap \mathbb{T};z^{\mathsf{T}} = z;\mathbb{T} \cap \overline{\mathbb{T};\mathbb{S}^{1}}$

and proceed for $n \mapsto n+1$

$$\begin{array}{l} \mathbb{V}_{n+1} = \mathbb{V}_0 \cup \mathbb{S}^\intercal ; \mathbb{V}_n \cup \mathbb{V}_n ; \mathbb{S} \supseteq \mathbb{V}_0 \cup \mathbb{V}_n ; \mathbb{S} \\ \supseteq (z ; \mathbb{T} \cap \mathbb{T} ; z^\intercal) \cup (z ; \mathbb{T} \cap \overline{\mathbb{T} ; \mathbb{S}^{n+1}}) ; \mathbb{S} \quad \text{hypothesis} \\ = (z ; \mathbb{T} \cap \mathbb{T} ; z^\intercal) \cup (z ; \mathbb{T} \cap \overline{\mathbb{T} ; \mathbb{S}^{n+1}} ; \mathbb{S}) = z ; \mathbb{T} \cap (\mathbb{T} ; z^\intercal \cup \overline{\mathbb{T} ; \mathbb{S}^{n+1}} ; \mathbb{S}) \\ = z ; \mathbb{T} \cap \overline{\mathbb{T} ; \mathbb{S}^{n+2}} \end{array}$$

since $\overline{\mathbb{T} ; \mathbb{S}^{n+2}} = \overline{\mathbb{T} ; \mathbb{S}^{n+1} ; \mathbb{S}} = \overline{\mathbb{T} ; \mathbb{S}^{n+1}} ; \mathbb{S} \cup \overline{\mathbb{T} ; \mathbb{S}} = \overline{\mathbb{T} ; \mathbb{S}^{n+1}} ; \mathbb{S} \cup \mathbb{T} ; z^\intercal$.

It remains to convince us that

$$\begin{array}{l} \mathbb{V} = \mathtt{sup}_{n \geq 0}\, \mathbb{V}_n \supseteq \mathtt{sup}_{n \geq 0}\, z ; \mathbb{T} \cap \overline{\mathbb{T} ; \mathbb{S}^{n+1}} = z ; \mathbb{T} \cap \mathtt{sup}_{n \geq 0}\, \overline{\mathbb{T} ; \mathbb{S}^{n+1}} \\ = z ; \mathbb{T} \cap \overline{\mathtt{inf}_{n \geq 0}\, \mathbb{T} ; \mathbb{S}^{n+1}} = z ; \mathbb{T} \cap \overline{\mathtt{inf}_{k \geq 1}\, \mathbb{T} ; \mathbb{S}^{k}} \end{array}$$

---

i.b) The induction starts for $\mathbb{W}$ as well as for $\mathbb{V}$ in case $n = 0$ with $z ; \mathbb{T} ; z^\intercal$. Induction step:

$$\mathbb{W}_{n+1} = \mathbb{S}^\intercal ; \mathbb{W}_n \cup \mathbb{W}_n ; \mathbb{S} \subseteq \mathbb{S}^\intercal ; \mathbb{V}_n \cup \mathbb{V}_n ; \mathbb{S} \subseteq \mathbb{V}_{n+1}$$

Using monotony of (i), we get

$$\mathtt{sup}_{0 \leq i \leq n}\, \mathbb{W}_i \subseteq \mathtt{sup}_{0 \leq i \leq n}\, \mathbb{V}_i = \mathbb{V}_n.$$

That equality holds is again proved by induction, starting with the trivial case $n = 0$:

$$\begin{array}{l} \mathbb{V}_{n+1} = \mathbb{V}_0 \cup \mathbb{S}^\intercal ; \mathbb{V}_n \cup \mathbb{V}_n ; \mathbb{S} = \mathbb{V}_0 \cup \mathbb{S}^\intercal ; \left[\mathtt{sup}_{0 \leq i \leq n}\, \mathbb{W}_i\right] \cup \left[\mathtt{sup}_{0 \leq i \leq n}\, \mathbb{W}_i\right] ; \mathbb{S} \quad \text{hypothesis} \\ = z ; \mathbb{T} ; z^\intercal \cup \mathtt{sup}_{0 \leq i \leq n} [\mathbb{S}^\intercal ; \mathbb{W}_i \cup \mathbb{W}_i ; \mathbb{S}] = \mathbb{W}_0 \cup \mathtt{sup}_{1 \leq j \leq n+1}\, \mathbb{W}_j = \mathtt{sup}_{0 \leq k \leq n+1}\, \mathbb{W}_k \end{array}$$

---

i.c) Looking at Fig. 5.3.4, we prove these horizontal and vertical limitations separately. The first induction starts with

$$\mathbb{V}_0 = z ; \mathbb{T} ; z^\intercal \subseteq z ; \mathbb{T} = \overline{\mathbb{S}^\intercal ; \overline{\mathbb{T}}} = \overline{\mathbb{S}^\intercal ; \mathbb{S}^{\intercal *} ; \overline{\mathbb{T}}} = \overline{\mathbb{S}^{\intercal *} ; \mathbb{S}^\intercal ; \mathbb{T}}$$

and is continued by

$$\begin{array}{l} \mathbb{V}_{n+1} = \mathbb{V}_0 \cup \mathbb{S}^\intercal ; \mathbb{V}_n \cup \mathbb{V}_n ; \mathbb{S} = z ; \mathbb{T} ; z^\intercal \cup \mathbb{S}^\intercal ; \mathbb{V}_n \cup \mathbb{V}_n ; \mathbb{S} \\ \subseteq z ; \mathbb{T} ; z^\intercal \cup \mathbb{S}^\intercal ; \overline{\mathbb{S}^{\intercal *} ; \mathbb{S}^{\intercal n+1} ; \overline{\mathbb{T}}} \cup \overline{\mathbb{S}^{\intercal *} ; \mathbb{S}^{\intercal n+1} ; \overline{\mathbb{T}}} ; \mathbb{S} \quad \text{hypothesis} \end{array}$$

It remains to prove these two items:

$$\begin{array}{l} \mathbb{S}^\intercal ; \overline{\mathbb{S}^{\intercal *} ; \mathbb{S}^{\intercal n+1} ; \overline{\mathbb{T}}} \subseteq \overline{\mathbb{S}^{\intercal *} ; \mathbb{S}^{\intercal n+2} ; \overline{\mathbb{T}}} \\ \quad \Longleftrightarrow \quad \mathbb{S} ; \mathbb{S}^{\intercal *} ; \mathbb{S}^{\intercal n+2} ; \mathbb{T} = \left(\mathbb{S} ; \mathbb{S}^\intercal\right) ; \mathbb{S}^{\intercal *} ; \mathbb{S}^{\intercal n+1} ; \mathbb{T} \subseteq \mathbb{S}^{\intercal *} ; \mathbb{S}^{\intercal n+1} ; \mathbb{T} \quad \text{Schröder rule} \end{array}$$

$$\begin{array}{l} \overline{\mathbb{S}^{\intercal *} ; \mathbb{S}^{\intercal n+1} ; \overline{\mathbb{T}}} ; \mathbb{S} \subseteq \overline{\mathbb{S}^{\intercal *} ; \mathbb{S}^{\intercal n+2} ; \overline{\mathbb{T}}} \\ \quad \Longleftrightarrow \quad \mathbb{S}^{\intercal *} ; \mathbb{S}^{\intercal n+2} ; \mathbb{T} ; \mathbb{S}^\intercal = \mathbb{S}^{\intercal *} ; \mathbb{S}^{\intercal n+1} ; \left(\mathbb{S}^\intercal ; \mathbb{T} ; \mathbb{S}^\intercal\right) \subseteq \mathbb{S}^{\intercal *} ; \mathbb{S}^{\intercal n+1} ; \mathbb{T} \quad \text{Schröder rule} \end{array}$$

A second induction starting with

$$\mathbb{V}_0 = z ; \mathbb{T} ; z^\intercal \subseteq \overline{\mathbb{T} ; \mathbb{S} ; \mathbb{S}^*} \quad \Longleftrightarrow \quad z ; \mathbb{T} \subseteq \overline{\mathbb{T} ; \mathbb{S} ; \mathbb{S}^*} ; z ; \mathbb{T} = \overline{\mathbb{T} ; \mathbb{S} ; \mathbb{S}^* ; z ; \mathbb{T}} = \overline{\mathbb{\bot}} = \mathbb{T}$$

and continued with

$$\begin{array}{l} \mathbb{V}_{n+1} = \mathbb{V}_0 \cup \mathbb{S}^\intercal ; \mathbb{V}_n \cup \mathbb{V}_n ; \mathbb{S} = z ; \mathbb{T} ; z^\intercal \cup \mathbb{S}^\intercal ; \mathbb{V}_n \cup \mathbb{V}_n ; \mathbb{S} \\ \subseteq z ; \mathbb{T} ; z^\intercal \cup \mathbb{S}^\intercal ; \overline{\mathbb{T} ; \mathbb{S}^{n+1} ; \mathbb{S}^*} \cup \overline{\mathbb{T} ; \mathbb{S}^{n+1} ; \mathbb{S}^*} ; \mathbb{S} \quad \text{hypothesis} \end{array}$$

It remains to prove these two items:

$$\begin{array}{l} \mathbb{S}^\intercal ; \overline{\mathbb{T} ; \mathbb{S}^{n+1} ; \mathbb{S}^*} \subseteq \overline{\mathbb{T} ; \mathbb{S}^{n+2} ; \mathbb{S}^*} \\ \quad \Longleftrightarrow \quad \mathbb{S} ; \mathbb{T} ; \mathbb{S}^{n+2} ; \mathbb{S}^* = \left(\mathbb{S} ; \mathbb{T} ; \mathbb{S}\right) ; \mathbb{S}^{n+1} ; \mathbb{S}^* \subseteq \mathbb{T} ; \mathbb{S}^{n+1} ; \mathbb{S}^* \quad \text{Schröder rule} \end{array}$$

$\overline{\mathbb{T};\mathbb{S}^{n+1};\mathbb{S}^*};\mathbb{S} \subseteq \overline{\mathbb{T};\mathbb{S}^{n+2};\mathbb{S}^*}$
$\iff \quad \mathbb{T};\mathbb{S}^{n+2};\mathbb{S}^*;\mathbb{S}^\intercal = \mathbb{T};\mathbb{S}^{n+1};\mathbb{S}^*;\big(\mathbb{S};\mathbb{S}^\intercal\big) \subseteq \mathbb{T};\mathbb{S}^{n+1};\mathbb{S}^*$ Schröder rule

---

i.d) Induction starting with

$\not{\mathbb{V}}_0 \cap z;\mathbb{T} = z;\mathbb{T};z^\intercal \cap z;\mathbb{T} = z;\mathbb{T};z^\intercal = z;\mathbb{T} \cap \mathbb{T};z^\intercal$

continued with

$\not{\mathbb{V}}_{n+1} \cap \mathbb{S}^{\intercal n+1};z;\mathbb{T} = \big[\not{\mathbb{V}}_0 \cup \mathbb{S}^\intercal;\not{\mathbb{V}}_n \cup \not{\mathbb{V}}_n;\mathbb{S}\big] \cap \mathbb{S}^{\intercal n+1};z;\mathbb{T}$
$= \big[z;\mathbb{T};z^\intercal \cap \mathbb{S}^{\intercal n+1};z;\mathbb{T}\big] \cup \big[\mathbb{S}^\intercal;\not{\mathbb{V}}_n \cap \mathbb{S}^{\intercal n+1};z;\mathbb{T}\big] \cup \big[\not{\mathbb{V}}_n;\mathbb{S} \cap \mathbb{S}^{\intercal n+1};z;\mathbb{T}\big]$
$= \mathbb{\perp} \cup \mathbb{S}^\intercal;\big[\not{\mathbb{V}}_n \cap \mathbb{S}^{\intercal n};z;\mathbb{T}\big] \cup \mathbb{\perp}$ due to (i.c)
$= \mathbb{S}^\intercal;\big[\mathbb{S}^{\intercal n};z;\mathbb{T} \cap \mathbb{T};z^\intercal\big]$ hypothesis
$= \mathbb{S}^{\intercal n+1};z;\mathbb{T} \cap \mathbb{T};z^\intercal$ masking

---

i.e) $\not{\mathbb{V}}_n \subseteq \not{\mathbb{V}}_n$ means in expanded form that
$\mathtt{sup}_{0\le i\le n}\,\mathtt{sup}_{0\le j\le n-i}\,\mathbb{S}^{\intercal i};z;\mathbb{T};z^\intercal;\mathbb{S}^j \subseteq \mathtt{sup}_{0\le k\le n}\,\mathbb{S}^{\intercal k};z;\mathbb{T};\mathbb{S}^{\intercal k}$.
The fact that $\mathbb{S}^{\intercal i};z;\mathbb{T};z^\intercal;\mathbb{S}^j$ should be contained in the right side implies $k := i$, i.e.
$\mathbb{S}^{\intercal i};z;\mathbb{T};z^\intercal;\mathbb{S}^j \subseteq \mathbb{S}^{\intercal i};z;\mathbb{T};\mathbb{S}^{\intercal i}$
$\iff \quad \mathbb{T};z^\intercal;\mathbb{S}^j \subseteq \mathbb{T};z^\intercal;\mathbb{S}^i;\mathbb{S}^{\intercal i};z;\mathbb{T};\mathbb{S}^{\intercal i} = \mathbb{T};\mathbb{S}^{\intercal i}$
$\iff \quad \mathbb{T} \subseteq \mathbb{T};\mathbb{S}^{\intercal i};\mathbb{S}^{\intercal j};z;\mathbb{T}$
$\iff \quad i + j \le n_{\mathbb{S}}$
with $n_{\mathbb{S}} = \overline{\mathbb{S};\mathbb{T}}$ the greatest existing number, which is indeed achieved with $n \le n_{\mathbb{S}}$.

---

ii.a) From (i.a) we know that $\not{\mathbb{V}}_n = \mathtt{sup}_{0\le i\le n}\,\mathtt{sup}_{0\le j\le n-i}\,\mathbb{S}^{\intercal i};z;\mathbb{T};z^\intercal;\mathbb{S}^j$, so that
$\not{\mathbb{V}} = \mathtt{sup}_{n\ge 0}\,\not{\mathbb{V}}_n = \mathtt{sup}_{n\ge 0}\,\mathtt{sup}_{i+j\le n}\,\mathbb{S}^{\intercal i};z;\mathbb{T};z^\intercal;\mathbb{S}^j = \mathtt{sup}_{i,j \text{ arbitrary}}\,\mathbb{S}^{\intercal i};z;\mathbb{T};z^\intercal;\mathbb{S}^j = \mathbb{T}$

---

iii)

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iv) □

5.4 Uniqueness up to isomorphism

We are now interested to which extent the definition given so far determines the naturals $\mathbb{N}$ uniquely up to isomorphism — given the situation with $\mathbb{S}$ total or not.

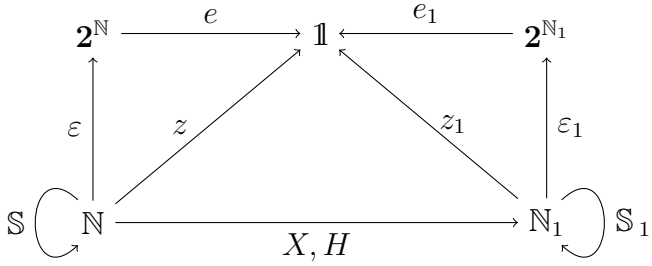


Fig. 5.4.1 Typing while checking unique definedness when introducing $\mathbb{N}$

5.4.1 Definition and Proposition. We define a **possibly partial** $(\mathbb{S},\mathbb{S}_1)$**-homomorphism**

$H := H_{\mathbb{S},\mathbb{S}_1} := \mathtt{inf}\ \{\, X \mid z ; z_1^\mathsf{T} \cup \mathbb{S}^\mathsf{T} ; X ; \mathbb{S}_1 \subseteq X \,\}$,

for successor relations $\mathbb{S}, \mathbb{S}_1$, where we have in view of Lemma 4.2.1

$h(X) := H_0 \cup h_H(X)$ with $H_0 = z ; z_1^\mathsf{T}$ and $h_H(X) := \mathbb{S}^\mathsf{T} ; X ; \mathbb{S}_1$.

Respecting possible further properties of $\mathbb{S}, \mathbb{S}_1$, the relation H satisfies the following:

i) In general, one has

a) $H = h(H) = z ; z_1^\mathsf{T} \cup \mathbb{S}^\mathsf{T} ; H ; \mathbb{S}_1 = \mathtt{sup}_{n\geq 0}\, h_H^n(H_0) = \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\mathsf{T}^n} ; z ; z_1^\mathsf{T} ; \mathbb{S}_1^n$

b) $\mathbb{S} ; H \subseteq H ; \mathbb{S}_1 \qquad \mathbb{S}_1 ; H^\mathsf{T} \subseteq H^\mathsf{T} ; \mathbb{S}$

c) H is univalent, $H^\mathsf{T} ; H \subseteq \mathbb{I}_1 \qquad H$ is injective, $H ; H^\mathsf{T} \subseteq \mathbb{I}$

ii) In case both, $\mathbb{S}, \mathbb{S}_1$, are regressively finite, there holds in addition

a) H is total **or** H is surjective — may be with both happening at the same time.

iii) If both, $\mathbb{S}, \mathbb{S}_1$, are regressively finite and $\mathbb{S}_1$ is total, there holds in addition

a) H is total

Proof: We observe in advance that Lemma 4.2.1 may be applied since the functional h_H is

isotonic, i.e. $A \subseteq B \implies \mathbb{S}^\mathsf{T} ; A ; \mathbb{S}_1 \subseteq \mathbb{S}^\mathsf{T} ; B ; \mathbb{S}_1$, i.e. $h_H(A) \subseteq h_H(B)$,

additive, i.e.
$$\begin{aligned} h_H(X \cup Y) &= \mathbb{S}^\mathsf{T} ; (X \cup Y) ; \mathbb{S}_1 \\ &= \mathbb{S}^\mathsf{T} ; X ; \mathbb{S}_1 \cup \mathbb{S}^\mathsf{T} ; Y ; \mathbb{S}_1 = h_H(X) \cup h_H(Y), \end{aligned}$$

$\cap$-hereditary, i.e.
$$\begin{aligned} h_H(X \cap Y) &= \mathbb{S}^\mathsf{T} ; (X \cap Y) ; \mathbb{S}_1 \\ &= \mathbb{S}^\mathsf{T} ; X ; \mathbb{S}_1 \cap \mathbb{S}^\mathsf{T} ; Y ; \mathbb{S}_1 = h_H(X) \cap h_H(Y). \end{aligned}$$

---

i.a) The claim collects the results of Lemma 4.2.1.i,iv. With $h_H^n(H_0) = \mathbb{S}^{\mathsf{T}^n} ; z ; z_1^\mathsf{T} ; \mathbb{S}_1^n$, we see, how H is formulated explicitly.

---

i.b) We prove only the first result; the second follows by symmetry.

$\mathbb{S} ; H = \mathbb{S} ; \big[z ; z_1^\mathsf{T} \cup \mathbb{S}^\mathsf{T} ; H ; \mathbb{S}_1 \big]$ due to (i.a)
$= \mathbb{S} ; z ; z_1^\mathsf{T} \cup \mathbb{S} ; \mathbb{S}^\mathsf{T} ; H ; \mathbb{S}_1$ since $\mathbb{S}$ is univalent
$\subseteq \mathbb{1} \cup \mathbb{S} ; \mathbb{S}^\mathsf{T} ; H ; \mathbb{S}_1 \subseteq H ; \mathbb{S}_1$ since $\mathbb{S} ; z = \mathbb{1}$ and $\mathbb{S}$ is injective

---

i.c) We observe $z^\mathsf{T} ; \mathbb{S}^n ; \mathbb{S}^{\mathsf{T}^l} ; z = \mathbb{1}$ whenever $n \neq l$ following from $\mathbb{S} ; \mathbb{S}^\mathsf{T} \subseteq \mathbb{I}$ and $\mathbb{S} ; z = \mathbb{1}$. Therefore with (i.a)

$H^\mathsf{T} ; H = \big[\mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\mathsf{T}^n} ; z ; z_1^\mathsf{T} ; \mathbb{S}_1^n \big]^\mathsf{T} ; \big[\mathtt{sup}_{l\geq 0}\, \mathbb{S}^{\mathsf{T}^l} ; z ; z_1^\mathsf{T} ; \mathbb{S}_1^l \big]$
$\subseteq \mathtt{sup}_{n\geq 0}\, \mathbb{S}_1^{\mathsf{T}^n} ; z_1 ; z^\mathsf{T} ; \mathbb{S}^n ; \mathbb{S}^{\mathsf{T}^n} ; z ; z_1^\mathsf{T} ; \mathbb{S}_1^n \subseteq \mathbb{I}_1$

The other result follows symmetrically.

---

ii.a) Assuming H neither to be total, nor to be surjective leads to a contradiction: There will then exist a point $x \subseteq \overline{H;\mathbb{T}}$ and a point $y \subseteq \overline{H^\intercal;\mathbb{T}}$. Regressive finiteness, Prop. 5.2.1.ii.a, implies for these points

$$x \subseteq \mathbb{S}^{*\intercal};z;\mathbb{T} \quad \text{and} \quad y \subseteq \mathbb{S}_1^{*\intercal};z_1;\mathbb{T}.$$

Since both, x and y, are points, they are fully contained in one of the disjoint sets over which the supremum of the reflexive-transitive closure is taken (see Prop. 2.2.2), i.e.

$$\exists i \geq 0 : x \subseteq \mathbb{S}^{\intercal i};z;\mathbb{T} \quad \text{and} \quad \exists k \geq 0 : y \subseteq \mathbb{S}_1^{\intercal k};z_1;\mathbb{T}.$$

Let for simplicity — and without restricting generality — be $i \leq k$. Then also $y_1 := \mathbb{S}_1^{\intercal i};z_1;\mathbb{T}$ is necessarily a point and, using (i.a), we obtain the contradiction

$$x;y_1^\intercal \subseteq \mathbb{S}^{\intercal i};z;\mathbb{T};z_1^\intercal;\mathbb{S}_1^i \subseteq H.$$

---

iii.a) We investigate the consequence of regressive finiteness:

$$\begin{aligned}
&\mathbb{S}^\intercal;\overline{H;\mathbb{T}} = \mathbb{S}^\intercal;\mathbb{T} \cap \overline{\mathbb{S}^\intercal;H;\mathbb{T}} \quad \text{since } \mathbb{S} \text{ is injective}\\
&= \mathbb{S}^\intercal;\mathbb{T} \cap \overline{\mathbb{S}^\intercal;H;\mathbb{S}_1;\mathbb{T}} \quad \text{since } \mathbb{S}_1 \text{ is total}\\
&= \overline{z;\mathbb{T}} \cap \overline{\mathbb{S}^\intercal;H;\mathbb{S}_1;\mathbb{T}} \quad \text{definition of } z\\
&= \overline{z;\mathbb{T} \cup \mathbb{S}^\intercal;H;\mathbb{S}_1;\mathbb{T}}\\
&= \overline{z;z_1^\intercal;\mathbb{T} \cup \mathbb{S}^\intercal;H;\mathbb{S}_1;\mathbb{T}} \quad \text{since } z_1^\intercal;\mathbb{T} = \mathbb{T}\\
&= \overline{(z;z_1^\intercal \cup \mathbb{S}^\intercal;H;\mathbb{S}_1);\mathbb{T}}\\
&= \overline{H;\mathbb{T}} \quad \text{(i.a)}
\end{aligned}$$

Now regressive finiteness of $\mathbb{S}$ may be applied via Prop. 4.1.2, resulting in $H;\mathbb{T} = \mathbb{T}$. □

Without loss of generality we assume H to be total (the case H surjective could be handled symmetrically). Then H will always be a map. A case distinction reveals

- $\mathbb{S}$ and $\mathbb{S}_1$ both non-finite and isomorphic, i.e. indeed the omnipresent naturals $\mathbb{N}$,
- $\mathbb{S}$ the initial section of length i of the naturals that is mapped into a longer initial section with $k \geq i$ of the infinite naturals,
- $\mathbb{S}$ the initial section of length i of the naturals that is mapped into the infinite naturals.

6 Additive structure

Thus having established $\mathbb{S}$ sufficiently precise, up to isomorphism, and leaving open only the question of $\mathbb{S}$ being total or not, we proceed to introducing addition and subtraction. Several different iterations are possible, and we will investigate their interdependence.

6.1 Binary relational operations on naturals

A construct $(\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash \mathbb{S}^{\intercal i} ; z ; \mathbb{T})$ may be equal to $\mathbb{L}$; however, if it is not, it constitutes a point in the product $\mathbb{N} \times \mathbb{N}$, meaning in the classical sense a pair of points resembling (n, i) and all the $(\mathbb{S}^{\intercal k} ; z ; \mathbb{T} \oslash \mathbb{S}^{\intercal j} ; z ; \mathbb{T})$ with $k \leq n$ and $j \leq i$ will also be pairs of points. When we later encounter, e.g., the addition represented as

$$\mathbb{A} = \mathtt{sup}_{n \geq 0}\, \mathtt{sup}_{i \geq 0}\, (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash \mathbb{S}^{\intercal i} ; z ; \mathbb{T}) ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n+i},$$

this means that to the point representing the pair (n, i) the $(n + i)$th successor is assigned — which may be a point or equal to $\mathbb{L}$ —, so as to provide an addition table. Therefore, a matrix entry is given: $pair ; point^{\intercal}$. This establishes a correspondence between our anthropomorphic sight on addition to the relation-algebraic one even when $\mathbb{S}$ should happen not to be total.

6.2 Addition

We usually call addition the mapping $\mathbb{A} : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ that satisfies

$(k + 1) + n = (k + n) + 1,$
$0 + n = n.$

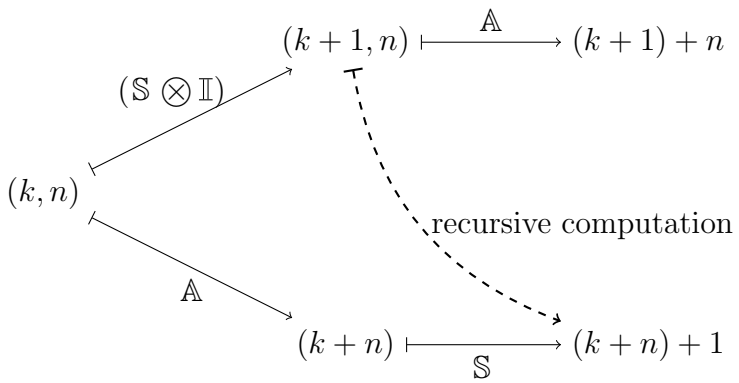


Fig. 6.2.1 Defining the addition relation on possibly finite naturals

The functional in the following lemma resembles this idea.

When introducing addition based on $\mathbb{S}$, we may proceed in two different ways: iterating over the first, giving $\mathbb{A}$, or over the second component resulting in B. To formally show that both iterations result in the same relation requires a technically challenging nested iterative proof. As long as $\mathbb{A} = B$ has not been established, we have to keep everything separate.

We instantiate Lemma 4.2.1 with two functionals, now denoted $a(X)$ and $b(X)$, respectively, instead of f.

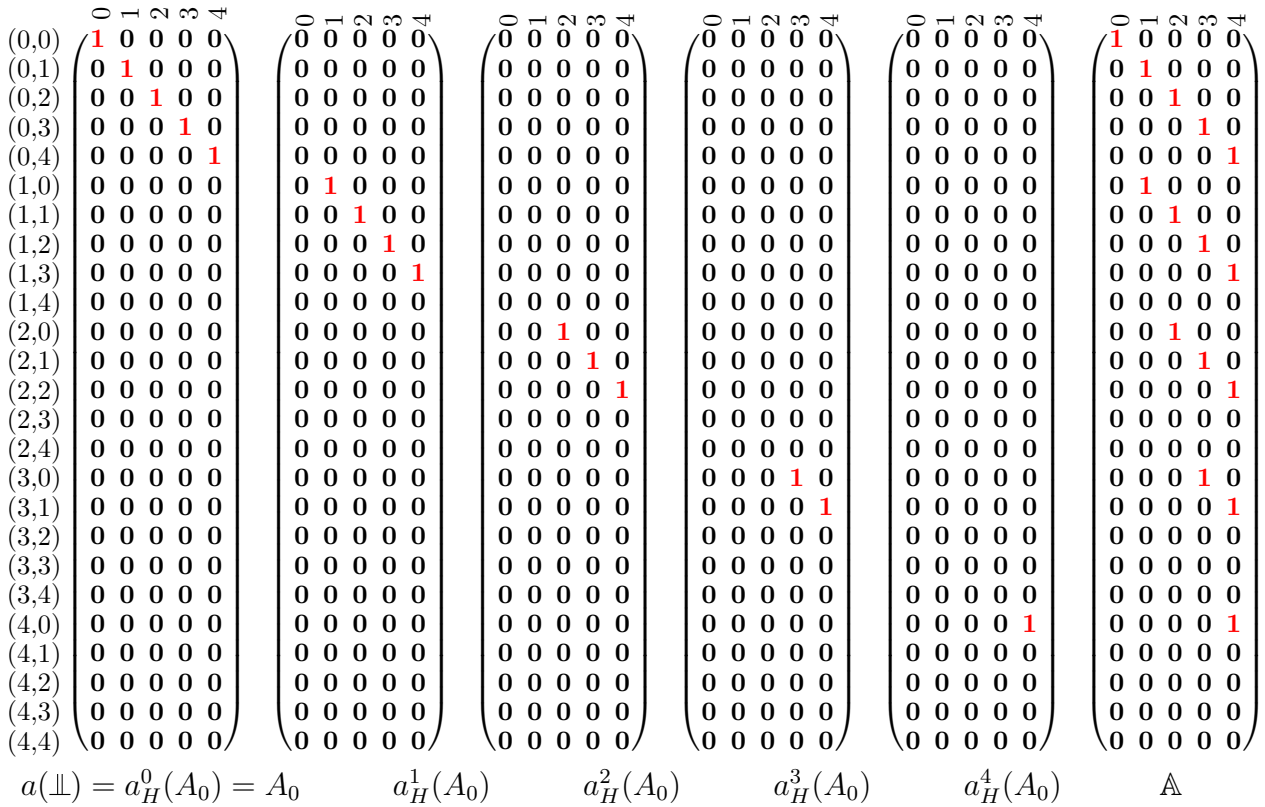


Fig. 6.2.2 Defining the addition relation iterating on first component

6.2.1 Definition and Proposition. We define as **addition** belonging to $\mathbb{S}$ the construct

$$\mathbb{A} := \mathbb{A}_{\mathbb{S}} := \mathtt{inf}\,\{\,X \mid a(X) \subseteq X\},$$
$$B := B_{\mathbb{S}} := \mathtt{inf}\,\{\,X \mid b(X) \subseteq X\},$$

where we have in view of Lemma 4.2.1

$$a(X) := \mathbb{A}_0 \cup a_H(X) \quad \text{with} \quad \mathbb{A}_0 = (z\,;\mathbb{T} \circledcirc \mathbb{I}) \quad \text{and} \quad a_H(X) = (\mathbb{S}^{\intercal} \otimes \mathbb{I})\,;X\,;\,\mathbb{S},$$
$$b(X) := B_0 \cup b_H(X) \quad \text{with} \quad B_0 = (\mathbb{I} \circledcirc z\,;\mathbb{T}) \quad \text{and} \quad b_H(X) = (\mathbb{I} \otimes \mathbb{S}^{\intercal})\,;X\,;\,\mathbb{S}.$$

Respecting possible further properties of $\mathbb{S}$, the versions $\mathbb{A}, B$ of addition satisfy

i) in general:

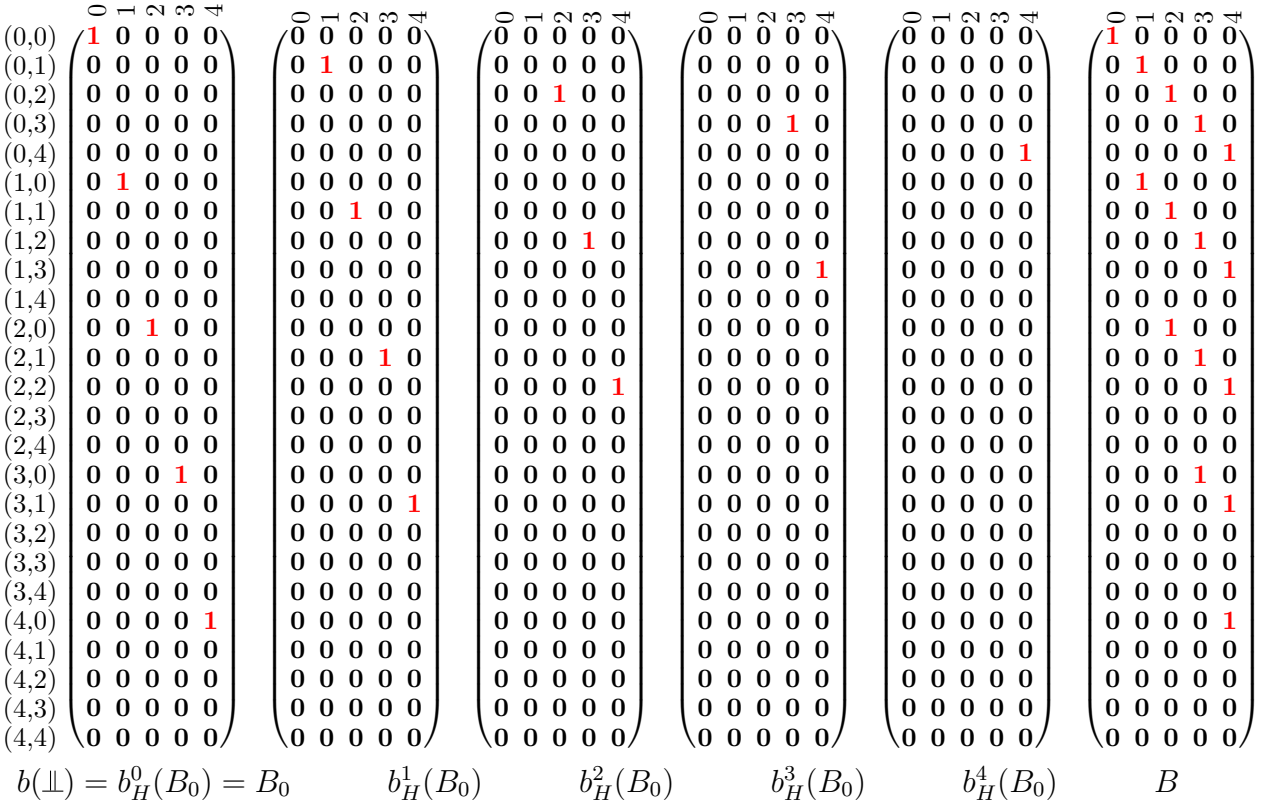


Fig. 6.2.3 Defining the addition relation iterating on second component

a) $\mathbb{A} = a(\mathbb{A}) = (z;\mathbb{T} \circledcirc \mathbb{I}) \cup (\mathbb{S}^\mathsf{T} \otimes \mathbb{I});\mathbb{A};\mathbb{S} = \mathtt{sup}_{n\geq 0}\, a_H^n(\mathbb{A}_0)$
$= \mathtt{sup}_{n\geq 0}\, (\mathbb{S}^{\mathsf{T}n};z;\mathbb{T} \circledcirc \mathbb{S}^n) \subseteq \pi;V$

$B = b(B) = (\mathbb{I} \circledcirc z;\mathbb{T}) \cup (\mathbb{I} \otimes \mathbb{S}^\mathsf{T});B;\mathbb{S} = \mathtt{sup}_{n\geq 0}\, b_H^n(B_0)$
$= \mathtt{sup}_{n\geq 0}\, (\mathbb{S}^n \circledcirc \mathbb{S}^{\mathsf{T}n};z;\mathbb{T}) \subseteq \rho;V$

b) $(\mathbb{S} \otimes \mathbb{I});\mathbb{A} \subseteq \mathbb{A};\mathbb{S}$ $(\mathbb{I} \otimes \mathbb{S});\mathbb{A} = \mathbb{A};\mathbb{S}$ $(\mathbb{S} \otimes \mathbb{S}^\mathsf{T});\mathbb{A} \subseteq \mathbb{A}$ $(\mathbb{S}^\mathsf{T} \otimes \mathbb{S});\mathbb{A} \subseteq \mathbb{A}$

$(\mathbb{I} \otimes \mathbb{S});B \subseteq B;\mathbb{S}$ $(\mathbb{S} \otimes \mathbb{I});B = B;\mathbb{S}$ $(\mathbb{S}^\mathsf{T} \otimes \mathbb{S});B \subseteq B$ $(\mathbb{S} \otimes \mathbb{S}^\mathsf{T});B \subseteq B$

$(\mathbb{I} \otimes \mathbb{S}^\mathsf{T});\mathbb{A} \cap \mathbb{T};\mathbb{S}^\mathsf{T} = \mathbb{A};\mathbb{S}^\mathsf{T} \cap \rho;\mathbb{S}^\mathsf{T};\mathbb{T}$ or as two estimations:

$(\mathbb{I} \otimes \mathbb{S}^\mathsf{T});\mathbb{A} \cap \mathbb{T};\mathbb{S}^\mathsf{T} \subseteq \mathbb{A};\mathbb{S}^\mathsf{T}$ $(\mathbb{I} \otimes \mathbb{S}^\mathsf{T});\mathbb{A} \supseteq \mathbb{A};\mathbb{S}^\mathsf{T} \cap \rho;\mathbb{S}^\mathsf{T};\mathbb{T}$

c) $\mathbb{A};z;\mathbb{T} = (z;\mathbb{T} \circledcirc z;\mathbb{T})$ $\mathbb{A}^\mathsf{T};(z;\mathbb{T} \circledcirc \mathbb{I}) = \mathbb{I}$ $\mathbb{A}^\mathsf{T};(\mathbb{I} \circledcirc z;\mathbb{T}) = D$

$B;z;\mathbb{T} = (z;\mathbb{T} \circledcirc z;\mathbb{T})$ $B^\mathsf{T};(\mathbb{I} \circledcirc z;\mathbb{T}) = \mathbb{I}$ $B^\mathsf{T};(z;\mathbb{T} \circledcirc \mathbb{I}) = D$

$\mathbb{A};e;\mathbb{T} = (z;\mathbb{T} \circledcirc e;\mathbb{T}) \cup (e;\mathbb{T} \circledcirc z;\mathbb{T})$ $\mathbb{A}^\mathsf{T};(z;\mathbb{T} \circledcirc z;\mathbb{T}) = z;\mathbb{T}$

$\mathbb{A} \cap \mathbb{T};z^\mathsf{T} = (z;\mathbb{T};z^\mathsf{T} \circledcirc \mathbb{I})$

d) $\mathfrak{P};B = \mathbb{A}$ $B = \mathfrak{P};\mathbb{A}$

e) $\mathbb{A}$ and B are both univalent

f) $\mathbb{A}$ and B need neither be commutative nor associative.

g) $\pi^{\mathsf{T}};\mathbb{A} = \mathbb{V} = \rho^{\mathsf{T}};B \qquad \rho^{\mathsf{T}};\mathbb{A} \cup \pi^{\mathsf{T}};B \subseteq \mathbb{S}^*$

h) $\mathbb{A}$ is left-cancellative, i.e. $\pi;\pi^{\mathsf{T}} \cap \mathbb{A};\mathbb{A}^{\mathsf{T}} \subseteq \mathbb{I}$.

B is right-cancellative, i.e. $B;B^{\mathsf{T}} \cap \rho;\rho^{\mathsf{T}} \subseteq \mathbb{I}$.

i) $(\mathbb{T};z^{\mathsf{T}};\mathbb{S}^n \oslash \mathbb{I});\mathbb{A} = \mathbb{S}^n \cap \mathbb{T};z^{\mathsf{T}};\mathbb{S}^n;\mathbb{S}^{\mathsf{T}n};z;\mathbb{T} = \begin{cases} \mathbb{S}^n & \text{when } \mathbb{S}^{\mathsf{T}n};z;\mathbb{T} \neq \mathbb{\bot} \\ \mathbb{\bot} & \text{otherwise} \end{cases}$

j) For $i,k \geq 0$ we have $\mathbb{A}^{\mathsf{T}};(\mathbb{S}^{\mathsf{T}i};z;\mathbb{T} \oslash \mathbb{S}^{\mathsf{T}k};z;\mathbb{T}) = \mathbb{S}^{\mathsf{T}i+k};z;\mathbb{T}$

k) For $l \geq 0$ we have $\mathtt{rel}(\mathbb{A};\mathbb{S}^{\mathsf{T}l};z;\mathbb{T}) = \mathtt{sup}_{0\leq n\leq l}\, \mathbb{S}^{\mathsf{T}n};z;\mathbb{T};z^{\mathsf{T}};\mathbb{S}^l;\mathbb{S}^{\mathsf{T}n} = \mathbb{/\!/}_l$

l) $\mathbb{A}^{\hookleftarrow} = \mathtt{sup}_{n\geq 0}\, (\mathbb{S}^{\mathsf{T}n} \oslash \mathbb{I});\mathbb{T};z^{\mathsf{T}};\mathbb{S}^n = \mathtt{sup}_{n\geq 0}\, (\mathbb{I} \oslash \mathbb{S}^n);\mathbb{T};z^{\mathsf{T}};\mathbb{S}^n$

$^{\hookrightarrow}\mathbb{A} = \mathtt{sup}_{n\geq 0}\, (\mathbb{S}^{\mathsf{T}n};z;\mathbb{T} \oslash \mathbb{S}^{\mathsf{T}n})$

$B^{\hookleftarrow} = \mathtt{sup}_{n\geq 0}\, (\mathbb{S}^{\mathsf{T}n} \oslash \mathbb{S}^{\mathsf{T}n};z;\mathbb{T})$

$^{\hookrightarrow}B = \mathtt{sup}_{n\geq 0}\, (\mathbb{I} \oslash \mathbb{S}^{\mathsf{T}n});\mathbb{T};z^{\mathsf{T}};\mathbb{S}^n = \mathtt{sup}_{n\geq 0}\, (\mathbb{S}^n \oslash \mathbb{I});\mathbb{T};z^{\mathsf{T}};\mathbb{S}^n$

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \quad \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \quad \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \quad \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \quad \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \quad \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Fig. 6.2.4 Illustrating (i.h) with left upper diagonals. Cf. also $\mathbb{A}_{table}$ in Fig. 6.3.10

ii) additionally, in case $\mathbb{S}$ is regressively finite:

a) $\mathbb{A} = B$, meaning not least that $\mathbb{A}$ and B are both commutative, i.e. $\mathbb{A} = \mathfrak{P};\mathbb{A}$

b) $\mathbb{A}^{\mathsf{T}};(\mathbb{I} \oslash z;\mathbb{T}) = \mathbb{I} \qquad \mathbb{A}^{\mathsf{T}};(z;\mathbb{T} \oslash \mathbb{I}) = \mathbb{I}$

c) $\pi^{\mathsf{T}};\mathbb{A} = \mathbb{S}^* = \rho^{\mathsf{T}};\mathbb{A}$

d) $(\mathbb{S} \otimes \mathbb{I});\mathbb{A} = \mathbb{A};\mathbb{S} = (\mathbb{I} \otimes \mathbb{S});\mathbb{A}$

e) $\mathbb{A} = \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\, (\mathbb{S}^{\mathsf{T}n};z;\mathbb{T} \oslash \mathbb{S}^{\mathsf{T}i};z;\mathbb{T});\mathbb{T};z^{\mathsf{T}};\mathbb{S}^{n+i}$

$\mathbb{A}^{\mathsf{T}};(\mathbb{S}^{\mathsf{T}n};z;\mathbb{T} \oslash \mathbb{S}^{\mathsf{T}i};z;\mathbb{T}) = \mathbb{S}^{\mathsf{T}n+i};z;\mathbb{T} \qquad \mathbb{A}^{\mathsf{T}};(\mathbb{S}^{\mathsf{T}n};z;\mathbb{T} \oslash \mathbb{I}) = \mathbb{S}^{\mathsf{T}n}$

$\mathbb{A}$ is associative, i.e. $\mathfrak{T};(\mathbb{I} \otimes \mathbb{A});\mathbb{A} = (\mathbb{A} \otimes \mathbb{I});\mathbb{A}$.

f) $\mathbb{A}^{\mathsf{T}};(\mathtt{eve};\mathbb{T} \oslash \mathtt{eve};\mathbb{T}) = \mathtt{eve};\mathbb{T} \qquad \mathbb{A}^{\mathsf{T}};(\mathtt{odd};\mathbb{T} \oslash \mathtt{odd};\mathbb{T}) \subseteq \mathtt{eve};\mathbb{T}$

$\mathbb{A}^{\mathsf{T}};(\mathtt{eve};\mathbb{T} \oslash \mathtt{odd};\mathbb{T}) = \mathtt{odd};\mathbb{T} \qquad \mathbb{A}^{\mathsf{T}};(\mathtt{odd};\mathbb{T} \oslash \mathtt{eve};\mathbb{T}) = \mathtt{odd};\mathbb{T}$

iii) If $\mathbb{S}$ happens to be regressively finite but not total, i.e. $\mathbb{S};\mathbb{T} \neq \mathbb{T}$,

a) $\mathbb{A} = \mathtt{sup}_{n_{\mathbb{S}} \geq n \geq 0} \, (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash \mathbb{S}^n)$

iv) In case $\mathbb{S}$ is regressively finite and also total, i.e. $\mathbb{S} ; \mathbb{T} = \mathbb{T}$,

a) $\mathbb{A}$ and B are both total

b) $(\mathbb{I} \otimes \mathbb{S}^\intercal) ; \mathbb{A} = \mathbb{A} ; \mathbb{S}^\intercal \cap \rho ; \mathbb{S}^\intercal ; \mathbb{T}$ $\qquad$ $(\mathbb{S}^\intercal \otimes \mathbb{I}) ; \mathbb{A} = \mathbb{A} ; \mathbb{S}^\intercal \cap \pi ; \mathbb{S}^\intercal ; \mathbb{T}$

Proof: We observe in advance that the functional a_H is

isotonic, i.e. $A \subseteq B \implies (\mathbb{S}^\intercal \otimes \mathbb{I}) ; A ; \mathbb{S} \subseteq (\mathbb{S}^\intercal \otimes \mathbb{I}) ; B ; \mathbb{S}$ i.e. $a_H(A) \subseteq a_H(B)$,
additive, i.e. $a_H(X \cup Y) = (\mathbb{S}^\intercal \otimes \mathbb{I}) ; (X \cup Y) ; \mathbb{S}$
$= (\mathbb{S}^\intercal \otimes \mathbb{I}) ; X ; \mathbb{S} \cup (\mathbb{S}^\intercal \otimes \mathbb{I}) ; Y ; \mathbb{S} = a_H(X) \cup a_H(Y)$,
$\cap$-hereditary, i.e. $a_H(X \cap Y) = (\mathbb{S}^\intercal \otimes \mathbb{I}) ; (X \cap Y) ; \mathbb{S}$ with $\mathbb{S}$ injective
$= (\mathbb{S}^\intercal \otimes \mathbb{I}) ; X ; \mathbb{S} \cap (\mathbb{S}^\intercal \otimes \mathbb{I}) ; Y ; \mathbb{S} = a_H(X) \cap a_H(Y)$,

so that Lemma 4.2.1 may indeed be applied. Also b_H satisfies these requirements.

---

i.a) The first equation is simply Lemma 4.2.1.i. Then we use Lemma 4.2.1.iv and prove how $\mathbb{A}$ is formulated explicitly:

$\mathbb{A} = \mathtt{sup}_{n \geq 0} \, a_H^n(\mathbb{A}_0) = \mathtt{sup}_{n \geq 0} \, (\mathbb{S}^{\intercal n} \otimes \mathbb{I}) ; (z ; \mathbb{T} \oslash \mathbb{I}) ; \mathbb{S}^n$
$= \mathtt{sup}_{n \geq 0} \, (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash \mathbb{I}) ; \mathbb{S}^n$
$= \mathtt{sup}_{n \geq 0} \, (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash \mathbb{S}^n)$ masking a join
$\subseteq \mathtt{sup}_{n \geq 0} \, \pi ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} = \pi ; \mathtt{sup}_{n \geq 0} \, \mathbb{S}^{\intercal n} ; z ; \mathbb{T} = \pi ; V$ Prop. 5.2.1.ii.a

The latter reminds us of the fact shown in Fig. 5.2.3, occurring when $\mathbb{S}$ is not yet assumed to be regressively finite.

The corresponding result for B follows completely analogously with components switched.

---

i.b) We proceed pairwise for the upper/lower variants of the first two lines, but handle only the upper:

$(\mathbb{S} \otimes \mathbb{I}) ; \mathbb{A} = (\mathbb{S} \otimes \mathbb{I}) ; \big[\, (z ; \mathbb{T} \oslash \mathbb{I}) \cup (\mathbb{S}^\intercal \otimes \mathbb{I}) ; \mathbb{A} ; \mathbb{S} \, \big]$ using (i.a)
$= (\mathbb{S} ; z ; \mathbb{T} \oslash \mathbb{I}) \cup (\mathbb{S} ; \mathbb{S}^\intercal \otimes \mathbb{I}) ; \mathbb{A} ; \mathbb{S}$ since $\mathbb{S}$ is univalent
$= \mathbb{L} \cup (\mathbb{S} ; \mathbb{S}^\intercal \otimes \mathbb{I}) ; \mathbb{A} ; \mathbb{S}$ since $\mathbb{S} ; z = \mathbb{L}$
$\subseteq (\mathbb{I} \otimes \mathbb{I}) ; \mathbb{A} ; \mathbb{S}$ since $\mathbb{S}$ is injective
$= \mathbb{I} ; \mathbb{A} ; \mathbb{S} = \mathbb{A} ; \mathbb{S}$

For an example with "$\neq$" see Fig. 5.2.7.

$(\mathbb{I} \otimes \mathbb{S}) ; \mathbb{A} = (\mathbb{I} \otimes \mathbb{S}) ; \big[\, \mathtt{sup}_{n \geq 0} \, (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash \mathbb{S}^n) \, \big]$ using (i.a)
$= \mathtt{sup}_{n \geq 0} \, (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash \mathbb{S} ; \mathbb{S}^n) = \mathtt{sup}_{n \geq 0} \, (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash \mathbb{S}^n ; \mathbb{S})$
$= \mathtt{sup}_{n \geq 0} \, (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash \mathbb{S}^n) ; \mathbb{S}$ masking a join
$= \big[\, \mathtt{sup}_{n \geq 0} \, (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash \mathbb{S}^n) \, \big] ; \mathbb{S} = \mathbb{A} ; \mathbb{S}$

$(\mathbb{S} \otimes \mathbb{S}^\intercal) ; \mathbb{A} = (\mathbb{S} \otimes \mathbb{S}^\intercal) ; \big[\, (z ; \mathbb{T} \oslash \mathbb{I}) \cup \mathtt{sup}_{n \geq 1} \, (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash \mathbb{S}^n) \, \big]$
$= \mathbb{L} \cup \mathtt{sup}_{n \geq 1} \, (\mathbb{S} ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash \mathbb{S}^\intercal ; \mathbb{S}^n)$ since $\mathbb{S} ; z = \mathbb{L}$
$\subseteq \mathtt{sup}_{n \geq 1} \, (\mathbb{S}^{\intercal n-1} ; z ; \mathbb{T} \oslash \mathbb{S}^{n-1})$ because $\mathbb{S}$ is injective and univalent
$= \mathtt{sup}_{k \geq 0} \, (\mathbb{S}^{\intercal k} ; z ; \mathbb{T} \oslash \mathbb{S}^k) = \mathbb{A}$ renumbered

---

$$
\begin{aligned}
&(\mathbb{S}^\intercal \otimes \mathbb{S})\fatsemi \mathbb{A} = (\mathbb{S}^\intercal \otimes \mathbb{S})\fatsemi \big[\, \mathtt{sup}_{n\geq 0}\ (\mathbb{S}^{\intercal n}\fatsemi z\fatsemi \mathbb{T} \oslash \mathbb{S}^n)\ \big]\\
&= \mathtt{sup}_{n\geq 0}\ (\mathbb{S}^{\intercal n+1}\fatsemi z\fatsemi \mathbb{T} \oslash \mathbb{S}^{n+1})\\
&= \mathtt{sup}_{k\geq 1}\ (\mathbb{S}^{\intercal k}\fatsemi z\fatsemi \mathbb{T} \oslash \mathbb{S}^k)\\
&\subseteq \mathbb{A}
\end{aligned}
$$

Analogously for the corresponding B-assertions.

$$
\begin{aligned}
&(\mathbb{I} \otimes \mathbb{S}^\intercal)\fatsemi \mathbb{A} \cap \mathbb{T}\fatsemi \mathbb{S}^\intercal = (\mathbb{I} \otimes \mathbb{S}^\intercal)\fatsemi \big[\, (z\fatsemi \mathbb{T} \oslash \mathbb{I}) \cup \mathtt{sup}_{n\geq 1}\ (\mathbb{S}^{\intercal n}\fatsemi z\fatsemi \mathbb{T} \oslash \mathbb{S}^n)\ \big] \cap \mathbb{T}\fatsemi \mathbb{S}^\intercal\\
&= \big[\, (\mathbb{I} \otimes \mathbb{S}^\intercal)\fatsemi (z\fatsemi \mathbb{T} \oslash \mathbb{I}) \cap \mathbb{T}\fatsemi \mathbb{S}^\intercal\big] \cup \big[\, (\mathbb{I} \otimes \mathbb{S}^\intercal)\fatsemi \mathtt{sup}_{n\geq 1}\ (\mathbb{S}^{\intercal n}\fatsemi z\fatsemi \mathbb{T} \oslash \mathbb{S}^n) \cap \mathbb{T}\fatsemi \mathbb{S}^\intercal\big]
\end{aligned}
$$

Since $z\fatsemi \mathbb{T}$ is not univalent, these two terms have to be treated separately

$$
\begin{aligned}
&(\mathbb{I} \otimes \mathbb{S}^\intercal)\fatsemi (z\fatsemi \mathbb{T} \oslash \mathbb{I}) \cap \mathbb{T}\fatsemi \mathbb{S}^\intercal = (z\fatsemi \mathbb{T} \oslash \mathbb{S}^\intercal) \cap \mathbb{T}\fatsemi \mathbb{S}^\intercal\\
&= \pi\fatsemi z\fatsemi \mathbb{T} \cap \rho\fatsemi \mathbb{S}^\intercal \cap \mathbb{T}\fatsemi \mathbb{S}^\intercal \cap \rho\fatsemi \mathbb{S}^\intercal\fatsemi \mathbb{T} \quad \text{with } \rho\fatsemi \mathbb{S}^\intercal \subseteq \rho\fatsemi \mathbb{S}^\intercal\fatsemi \mathbb{T}\\
&= \pi\fatsemi z\fatsemi \mathbb{T}\fatsemi \mathbb{S}^\intercal \cap \rho\fatsemi \mathbb{S}^\intercal \cap \rho\fatsemi \mathbb{S}^\intercal\fatsemi \mathbb{T} \quad \text{masking}\\
&= (z\fatsemi \mathbb{T} \oslash \mathbb{I})\fatsemi \mathbb{S}^\intercal \cap \rho\fatsemi \mathbb{S}^\intercal\fatsemi \mathbb{T},
\end{aligned}
$$

$$
\begin{aligned}
&(\mathbb{I} \otimes \mathbb{S}^\intercal)\fatsemi \mathtt{sup}_{n\geq 1}\ (\mathbb{S}^{\intercal n}\fatsemi z\fatsemi \mathbb{T} \oslash \mathbb{S}^n) \cap \mathbb{T}\fatsemi \mathbb{S}^\intercal\\
&= \mathtt{sup}_{n\geq 1}\ \pi\fatsemi \mathbb{S}^{\intercal n}\fatsemi z\fatsemi \mathbb{T} \cap \rho\fatsemi \mathbb{S}^\intercal\fatsemi \mathbb{S}^n \cap \mathbb{T}\fatsemi \mathbb{S}^\intercal\\
&= \mathtt{sup}_{n\geq 1}\ \pi\fatsemi \mathbb{S}^{\intercal n}\fatsemi z\fatsemi \mathbb{T} \cap \rho\fatsemi (\mathbb{S}^{n-1} \cap \mathbb{S}^\intercal\fatsemi \mathbb{T}) \cap \mathbb{T}\fatsemi \mathbb{S}^\intercal \quad \text{Prop. 5.2.1.i.b}\\
&= \mathtt{sup}_{n\geq 1}\ \pi\fatsemi \mathbb{S}^{\intercal n}\fatsemi z\fatsemi \mathbb{T}\fatsemi \mathbb{S}^\intercal \cap \rho\fatsemi \mathbb{S}^{n-1} \cap \rho\fatsemi \mathbb{T}\fatsemi \mathbb{S}^\intercal \cap \rho\fatsemi \mathbb{S}^\intercal\fatsemi \mathbb{T} \quad \text{masking and } \rho \text{ total}\\
&= \mathtt{sup}_{n\geq 1}\ (\mathbb{S}^{\intercal n}\fatsemi z\fatsemi \mathbb{T}\fatsemi \mathbb{S}^\intercal \oslash \mathbb{S}^{n-1} \cap \mathbb{T}\fatsemi \mathbb{S}^\intercal) \cap \rho\fatsemi \mathbb{S}^\intercal\fatsemi \mathbb{T}\\
&= \mathtt{sup}_{n\geq 1}\ (\mathbb{S}^{\intercal n}\fatsemi z\fatsemi \mathbb{T}\fatsemi \mathbb{S}^\intercal \oslash \mathbb{S}^n\fatsemi \mathbb{S}^\intercal) \cap \rho\fatsemi \mathbb{S}^\intercal\fatsemi \mathbb{T} \quad \text{Prop. 5.2.1.i.b}\\
&= \mathtt{sup}_{n\geq 1}\ (\mathbb{S}^{\intercal n}\fatsemi z\fatsemi \mathbb{T} \oslash \mathbb{S}^n)\fatsemi \mathbb{S}^\intercal \cap \rho\fatsemi \mathbb{S}^\intercal\fatsemi \mathbb{T}
\end{aligned}
$$

and then recombined to

$$
\big[\, (z\fatsemi \mathbb{T} \oslash \mathbb{I}) \cup \mathtt{sup}_{n\geq 1}\ (\mathbb{S}^{\intercal n}\fatsemi z\fatsemi \mathbb{T} \oslash \mathbb{S}^n)\ \big]\fatsemi \mathbb{S}^\intercal \cap \rho\fatsemi \mathbb{S}^\intercal\fatsemi \mathbb{T} \quad \text{univalence of } \mathbb{S}
$$
$$
= \mathbb{A}\fatsemi \mathbb{S}^\intercal \cap \rho\fatsemi \mathbb{S}^\intercal\fatsemi \mathbb{T}
$$

---

i.c) $\mathbb{A}\fatsemi z\fatsemi \mathbb{T} = \big[\, (z\fatsemi \mathbb{T} \oslash \mathbb{I}) \cup (\mathbb{S}^\intercal \otimes \mathbb{I})\fatsemi \mathbb{A}\fatsemi \mathbb{S}\big]\fatsemi z\fatsemi \mathbb{T}$ (i.a)
$= (z\fatsemi \mathbb{T} \oslash \mathbb{I})\fatsemi z\fatsemi \mathbb{T} \cup \mathbb{\bot}$ since $\mathbb{S}\fatsemi z = \mathbb{\bot}$
$= (z\fatsemi \mathbb{T} \oslash z\fatsemi \mathbb{T})$ masking a join

---

$$
\begin{aligned}
&\mathbb{A}^\intercal\fatsemi (z\fatsemi \mathbb{T} \oslash \mathbb{I}) = \big[(z\fatsemi \mathbb{T} \oslash \mathbb{I}) \cup (\mathbb{S}^\intercal \otimes \mathbb{I})\fatsemi \mathbb{A}\fatsemi \mathbb{S}\big]^\intercal\fatsemi (z\fatsemi \mathbb{T} \oslash \mathbb{I}) \quad \text{due to (i.a)}\\
&= (\mathbb{T}\fatsemi z^\intercal \oslash \mathbb{I})\fatsemi (z\fatsemi \mathbb{T} \oslash \mathbb{I}) \ \cup \mathbb{S}^\intercal\fatsemi \mathbb{A}^\intercal\fatsemi (\mathbb{S} \otimes \mathbb{I})\fatsemi (z\fatsemi \mathbb{T} \oslash \mathbb{I})\\
&= (\mathbb{T}\fatsemi z^\intercal\fatsemi z\fatsemi \mathbb{T} \cap \mathbb{I}) \ \cup \ \mathbb{S}^\intercal\fatsemi \mathbb{A}^\intercal\fatsemi (\mathbb{S}\fatsemi z\fatsemi \mathbb{T} \oslash \mathbb{I})\\
&= (\mathbb{T} \cap \mathbb{I}) \ \cup \mathbb{\bot} \quad \text{since } z \text{ is a point and } \mathbb{S}\fatsemi z = \mathbb{\bot}\\
&= \mathbb{I}
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{A}^\intercal\fatsemi (\mathbb{I} \oslash z\fatsemi \mathbb{T}) = \big[\, \mathtt{sup}_{n\geq 0}\ (\mathbb{S}^{\intercal n}\fatsemi z\fatsemi \mathbb{T} \oslash \mathbb{S}^n)\ \big]^\intercal\fatsemi (\mathbb{I} \oslash z\fatsemi \mathbb{T}) \quad \text{see (i.a)}\\
&= \mathtt{sup}_{n\geq 0}\ (\mathbb{T}\fatsemi z^\intercal\fatsemi \mathbb{S}^n \oslash \mathbb{S}^{\intercal n})\fatsemi (\mathbb{I} \oslash z\fatsemi \mathbb{T})\\
&= \mathtt{sup}_{n\geq 0}\ \mathbb{T}\fatsemi z^\intercal\fatsemi \mathbb{S}^n \cap \mathbb{S}^{\intercal n}\fatsemi z\fatsemi \mathbb{T}\\
&= \mathtt{sup}_{n\geq 0}\ \mathbb{S}^{\intercal n}\fatsemi z\fatsemi \mathbb{T}\fatsemi z^\intercal\fatsemi \mathbb{S}^n \quad \text{masking}\\
&= D \quad \text{due to Prop. 5.2.2.i.a}
\end{aligned}
$$

Again, the second line follows correspondingly.

$$
\begin{aligned}
&\mathbb{A}\fatsemi e\fatsemi \mathbb{T} = \mathtt{sup}_{n\geq 0}\ (\mathbb{S}^{\intercal n}\fatsemi z\fatsemi \mathbb{T} \oslash \mathbb{S}^n)\fatsemi e\fatsemi \mathbb{T}\\
&= (z\fatsemi \mathbb{T} \oslash \mathbb{I})\fatsemi e\fatsemi \mathbb{T} \cup (\mathbb{S}^\intercal\fatsemi z\fatsemi \mathbb{T} \oslash \mathbb{S})\fatsemi e\fatsemi \mathbb{T} \quad \text{since } \mathbb{S}^n\fatsemi e = \mathbb{\bot} \text{ for } n > 1
\end{aligned}
$$

$= (z ; \mathbb{T} ; e ; \mathbb{T} \oslash e ; \mathbb{T}) \cup (\mathbb{S}^{\intercal} ; z ; \mathbb{T} ; e ; \mathbb{T} \oslash \mathbb{S} ; e ; \mathbb{T})$
$= (z ; \mathbb{T} \oslash e ; \mathbb{T}) \cup (\mathbb{S}^{\intercal} ; z ; \mathbb{T} \oslash \mathbb{S} ; e ; \mathbb{T})$ since e is a point
$= (z ; \mathbb{T} \oslash e ; \mathbb{T}) \cup (e ; \mathbb{T} \oslash z ; \mathbb{T})$

$\mathbb{A}^{\intercal} ; (z ; \mathbb{T} \oslash z ; \mathbb{T}) = \big[(z ; \mathbb{T} \oslash \mathbb{I}) \cup (\mathbb{S}^{\intercal} \otimes \mathbb{I}) ; \mathbb{A} ; \mathbb{S} \big]^{\intercal} ; (z ; \mathbb{T} \oslash z ; \mathbb{T})$
$= \big[(\mathbb{T} ; z^{\intercal} \oslash \mathbb{I}) \cup \mathbb{S}^{\intercal} ; \mathbb{A}^{\intercal} ; (\mathbb{S} \otimes \mathbb{I}) \big] ; (z ; \mathbb{T} \oslash z ; \mathbb{T})$
$= (\mathbb{T} ; z^{\intercal} \oslash \mathbb{I}) ; (z ; \mathbb{T} \oslash z ; \mathbb{T}) \cup \mathbb{S}^{\intercal} ; \mathbb{A}^{\intercal} ; (\mathbb{S} \otimes \mathbb{I}) ; (z ; \mathbb{T} \oslash z ; \mathbb{T})$
$= (\mathbb{T} ; z^{\intercal} ; z ; \mathbb{T} \cap z ; \mathbb{T}) \cup \mathbb{\perp}$ because $\mathbb{S} ; z = \mathbb{\perp}$
$= \mathbb{T} \cap z ; \mathbb{T}$ since z is a point
$= z ; \mathbb{T}$

$\mathbb{A} \cap \mathbb{T} ; z^{\intercal} = \big[(z ; \mathbb{T} \oslash \mathbb{I}) \cup (\mathbb{S}^{\intercal} \otimes \mathbb{I}) ; \mathbb{A} ; \mathbb{S} \big] \cap \mathbb{T} ; z^{\intercal}$ due to (i.a)
$= \big[(z ; \mathbb{T} \oslash \mathbb{I}) \cap \mathbb{T} ; z^{\intercal} \big] \cup \big[(\mathbb{S}^{\intercal} \otimes \mathbb{I}) ; \mathbb{A} ; \mathbb{S} \cap \mathbb{T} ; z^{\intercal} \big]$
$= \big[\pi ; z ; \mathbb{T} \cap \rho \cap \mathbb{T} ; z^{\intercal} \big] \cup \big[(\mathbb{S}^{\intercal} \otimes \mathbb{I}) ; \mathbb{A} ; (\mathbb{S} \cap \mathbb{T} ; z^{\intercal}) \big]$ masking
$= \big[\pi ; z ; \mathbb{T} ; z^{\intercal} \cap \rho \big] \cup \mathbb{\perp}$ masking and Prop. 5.1.2
$= (z ; \mathbb{T} ; z^{\intercal} \oslash \mathbb{I})$

---

i.d) We apply $\mathfrak{P}$ to the $\mathtt{sup}$-version of (i):
$\mathfrak{P} ; \mathbb{A} = \mathfrak{P} ; \mathtt{sup}_{n \geq 0} (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash \mathbb{S}^{n}) = \mathtt{sup}_{n \geq 0} \mathfrak{P} ; (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash \mathbb{S}^{n})$
$= \mathtt{sup}_{n \geq 0} (\mathbb{S}^{n} \oslash \mathbb{S}^{\intercal n} ; z ; \mathbb{T}) = B$
and since $\mathfrak{P}$ is a bijective map satisfying $\mathfrak{P} ; \mathfrak{P} = \mathbb{I}$, also $\mathbb{A} = \mathfrak{P} ; B$.

---

i.e) $\mathbb{A}^{\intercal} ; \mathbb{A} = \big[\mathtt{sup}_{n \geq 0} (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash \mathbb{S}^{n}) \big]^{\intercal} ; \mathtt{sup}_{l \geq 0} (\mathbb{S}^{\intercal l} ; z ; \mathbb{T} \oslash \mathbb{S}^{l})$ due to (i.a)
$= \mathtt{sup}_{n \geq 0} \mathtt{sup}_{l \geq 0} (\mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n} \oslash \mathbb{S}^{\intercal n}) ; (\mathbb{S}^{\intercal l} ; z ; \mathbb{T} \oslash \mathbb{S}^{l})$
$= \mathtt{sup}_{n \geq 0} \mathtt{sup}_{l \geq 0} \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n} ; \mathbb{S}^{\intercal l} ; z ; \mathbb{T} \cap \mathbb{S}^{\intercal n} ; \mathbb{S}^{l}$
$\subseteq \mathtt{sup}_{n \geq 0} \mathbb{S}^{\intercal n} ; \mathbb{S}^{n}$ following Prop. 5.2.1.i.e
$= \mathbb{I}$ since $\mathbb{S}$ is univalent

---

i.f) Witness for the first — at least in case $\mathbb{A}$ — is the example of $(0, 3)$ as opposed to $(3, 0)$ in Fig. 5.2.3. Witness for the second — also just for case $\mathbb{A}$ — is the triple of $1, 2, 3$ in Fig. 5.2.3, where the row for $(1, 2)$ in A is empty and thus also $((1, 2), 3)$ not defined. In contrast, $(2, 3)$ results in 3 and $(1, 3)$ in 4.

---

i.g) We restrict to proving the left parts of the two statements, recall Prop. 5.3.1 and show
$g(\pi^{\intercal} ; \mathbb{A}) = z ; \mathbb{T} \cup \mathbb{S}^{\intercal} ; \pi^{\intercal} ; \mathbb{A} ; \mathbb{S}$
$= \pi^{\intercal} ; (z ; \mathbb{T} \oslash \mathbb{I}) \cup \pi^{\intercal} ; (\mathbb{S}^{\intercal} \otimes \mathbb{I}) ; \mathbb{A} ; \mathbb{S}$
$= \pi^{\intercal} ; \big[(z ; \mathbb{T} \oslash \mathbb{I}) \cup (\mathbb{S}^{\intercal} \otimes \mathbb{I}) ; \mathbb{A} ; \mathbb{S} \big]$
$= \pi^{\intercal} ; a(\mathbb{A})$
$= \pi^{\intercal} ; \mathbb{A}$

Thus, $\mathbb{V} \subseteq \pi^{\intercal} ; \mathbb{A}$. But also $\pi^{\intercal} ; \mathbb{A} \subseteq \mathbb{V}$, so that equality holds. We prove, namely, $\mathbb{A} \subseteq \pi ; \mathbb{V}$, what is equivalent by shunting, showing that $\pi ; \mathbb{V}$ satisfies the selection criterion for $\mathbb{A}$ in Prop. 6.2.1:
$a(\pi ; \mathbb{V}) = (z ; \mathbb{T} \oslash \mathbb{I}) \cup (\mathbb{S}^{\intercal} \otimes \mathbb{I}) ; \pi ; \mathbb{V} ; \mathbb{S}$
$\subseteq \pi ; z ; \mathbb{T} \cup \pi ; \mathbb{S}^{\intercal} ; \mathbb{V} ; \mathbb{S}$

$= \pi ; (z ; \mathbb{T} \cup \mathbb{S}^{\tau} ; \mathbb{V} ; \mathbb{S})$
$= \pi ; g(\mathbb{V}) = \pi ; \mathbb{V}$
Together, this establishes $\pi^{\tau} ; \mathbb{A} = \mathbb{V}$.

Now $\rho^{\tau} ; \mathbb{A}$ and — analogously — $\pi^{\tau} ; B$:
$\rho^{\tau} ; \mathbb{A} = \rho^{\tau} ; \mathtt{sup}_{n \geq 0} \, (\mathbb{S}^{\tau n} ; z ; \mathbb{T} \oslash \mathbb{S}^{n}) = \mathtt{sup}_{n \geq 0} \, \rho^{\tau} ; (\mathbb{S}^{\tau n} ; z ; \mathbb{T} \oslash \mathbb{S}^{n})$
$= \mathtt{sup}_{n \geq 0} \, \mathbb{T} ; \mathbb{S}^{\tau n} ; z ; \mathbb{T} \cap \mathbb{S}^{n}$
$\subseteq \mathtt{sup}_{n \geq 0} \, \mathbb{S}^{n} = \mathbb{S}^{*}$
One should have a look at Figs. 5.2.3 and 5.2.6 concerning the different behaviour.

---

i.h) We have to show $\pi ; \pi^{\tau} \cap \mathbb{A} ; \mathbb{A}^{\tau} \subseteq \mathbb{I}$:
$\pi ; \pi^{\tau} \cap \mathbb{A} ; \mathbb{A}^{\tau} = \pi ; \pi^{\tau} \cap \big[\, \mathtt{sup}_{n \geq 0} \, (\mathbb{S}^{\tau n} ; z ; \mathbb{T} \oslash \mathbb{S}^{n}) \, \big] ; \big[\, \mathtt{sup}_{k \geq 0} \, (\mathbb{T} ; z^{\tau} ; \mathbb{S}^{k} \oslash \mathbb{S}^{\tau k}) \, \big]$
$= \pi ; \pi^{\tau} \cap \mathtt{sup}_{n \geq 0} \, \mathtt{sup}_{k \geq 0} \, (\mathbb{S}^{\tau n} ; z ; \mathbb{T} \oslash \mathbb{S}^{n}) ; (\mathbb{T} ; z^{\tau} ; \mathbb{S}^{k} \oslash \mathbb{S}^{\tau k})$
$= \pi ; \pi^{\tau} \cap \mathtt{sup}_{n \geq 0} \, \mathtt{sup}_{k \geq 0} \, (\mathbb{S}^{\tau n} ; z ; \mathbb{T} ; z^{\tau} ; \mathbb{S}^{k} \otimes \mathbb{S}^{n} ; \mathbb{S}^{\tau k})$ Prop. 2.1.1.ii
$= \mathtt{sup}_{n \geq 0} \, \mathtt{sup}_{k \geq 0} \, (\mathbb{I} \cap \mathbb{S}^{\tau n} ; z ; \mathbb{T} ; z^{\tau} ; \mathbb{S}^{k} \otimes \mathbb{S}^{n} ; \mathbb{S}^{\tau k})$, which implies $n = k$ to be $\neq \mathbb{\bot}$
$= \mathtt{sup}_{n \geq 0} \, (\mathbb{I} \cap \mathbb{S}^{\tau n} ; z ; \mathbb{T} ; z^{\tau} ; \mathbb{S}^{n} \otimes \mathbb{S}^{n} ; \mathbb{S}^{\tau n})$
$\subseteq \mathtt{sup}_{n \geq 0} \, (\mathbb{I} \otimes \mathbb{I}) = \mathbb{I}$

---

i.i) $(\mathbb{T} ; z^{\tau} ; \mathbb{S}^{n} \oslash \mathbb{I}) ; \mathbb{A} = (\mathbb{T} ; z^{\tau} ; \mathbb{S}^{n} \oslash \mathbb{I}) ; \big[\, \mathtt{sup}_{j \geq 0} \, (\mathbb{S}^{\tau j} ; z ; \mathbb{T} \oslash \mathbb{S}^{j}) \, \big]$
$= \mathtt{sup}_{j \geq 0} \, \mathbb{T} ; z^{\tau} ; \mathbb{S}^{n} ; \mathbb{S}^{\tau j} ; z ; \mathbb{T} \cap \mathbb{S}^{j}$, implying $j := n$ to be $\neq \mathbb{\bot}$, since $\mathbb{S} ; z = \mathbb{\bot}$ otherwise
$= \mathbb{T} ; z^{\tau} ; \mathbb{S}^{n} ; \mathbb{S}^{\tau n} ; z ; \mathbb{T} \cap \mathbb{S}^{n}$
$= \begin{cases} \mathbb{S}^{n} & \text{when } \mathbb{S}^{\tau n} ; z ; \mathbb{T} \neq \mathbb{\bot} \\ \mathbb{\bot} & \text{otherwise} \end{cases}$
There is a point here to observe: The first term assumes as values $\mathbb{T}$ resp. $\mathbb{\bot}$ depending on $\mathbb{S}^{\tau n} ; z ; \mathbb{T}$ being $\neq \mathbb{\bot}$ or else $= \mathbb{\bot}$. In the first case we are done. In the second, we find out that also $(\mathbb{T} ; z^{\tau} ; \mathbb{S}^{n} \oslash \mathbb{I}) = \mathbb{\bot}$, which does without regressive finiteness, however, not imply that $\mathbb{S}^{n} = \mathbb{\bot}$; see Prop. 5.2.1.ii.d.

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i.j) $\mathbb{A}^{\tau} ; (\mathbb{S}^{\tau i} ; z ; \mathbb{T} \oslash \mathbb{S}^{\tau k} ; z ; \mathbb{T})$
$= \big[\, \mathtt{sup}_{n \geq 0} \, (\mathbb{S}^{\tau n} ; z ; \mathbb{T} \oslash \mathbb{S}^{n}) \, \big]^{\tau} ; (\mathbb{S}^{\tau i} ; z ; \mathbb{T} \oslash \mathbb{S}^{\tau k} ; z ; \mathbb{T})$
$= \mathtt{sup}_{n \geq 0} \, (\mathbb{T} ; z^{\tau} ; \mathbb{S}^{n} \oslash \mathbb{S}^{\tau n}) ; (\mathbb{S}^{\tau i} ; z ; \mathbb{T} \oslash \mathbb{S}^{\tau k} ; z ; \mathbb{T})$
$= \mathtt{sup}_{n \geq 0} \, \mathbb{T} ; z^{\tau} ; \mathbb{S}^{n} ; \mathbb{S}^{\tau i} ; z ; \mathbb{T} \cap \mathbb{S}^{\tau n} ; \mathbb{S}^{\tau k} ; z ; \mathbb{T}$ the first term implying $n := i$ to be $\neq \mathbb{\bot}$
$= \mathbb{T} ; z^{\tau} ; \mathbb{S}^{i} ; \mathbb{S}^{\tau i} ; z ; \mathbb{T} \cap \mathbb{S}^{\tau i} ; \mathbb{S}^{\tau k} ; z ; \mathbb{T}$
$= \mathbb{S}^{\tau i + k} ; z ; \mathbb{T}$
The final step has yet to be justified. Its first term assumes as values $\mathbb{T}$ resp. $\mathbb{\bot}$ depending on $\mathbb{S}^{\tau i} ; z ; \mathbb{T} \neq \mathbb{\bot}$ or else $= \mathbb{\bot}$. In the first case we are done. In the second, we find out that also $(\mathbb{S}^{\tau i} ; z ; \mathbb{T} \oslash \mathbb{S}^{\tau k} ; z ; \mathbb{T}) = \mathbb{\bot}$ and $\mathbb{S}^{\tau i + k} ; z ; \mathbb{T} = \mathbb{\bot}$, so that equality holds indeed.

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i.k) When $\mathbb{S}^{\tau l} ; z ; \mathbb{T} = \mathbb{\bot}$, this is obviously satisfied. In case $\neq \mathbb{\bot}$ we have a point $\mathbb{S}^{\tau l} ; z ; \mathbb{T}$:
$\mathtt{rel}(\, \mathbb{A} ; \mathbb{S}^{\tau l} ; z ; \mathbb{T}) = \mathtt{rel}(\big[\, \mathtt{sup}_{n \geq 0} \, (\mathbb{S}^{\tau n} ; z ; \mathbb{T} \oslash \mathbb{S}^{n}) \, \big] ; \mathbb{S}^{\tau l} ; z ; \mathbb{T})$
$= \mathtt{sup}_{n \geq 0} \, \mathtt{rel}(\, (\mathbb{S}^{\tau n} ; z ; \mathbb{T} \oslash \mathbb{S}^{n}) ; \mathbb{S}^{\tau l} ; z ; \mathbb{T})$ since $\mathtt{rel}$ is additive; see page 15
$= \mathtt{sup}_{n \geq 0} \, \mathtt{rel}(\, (\mathbb{S}^{\tau n} ; z ; \mathbb{T} ; \mathbb{S}^{\tau l} ; z ; \mathbb{T} \oslash \mathbb{S}^{n} ; \mathbb{S}^{\tau l} ; z ; \mathbb{T}) \,)$ since $\mathbb{S}^{\tau l} ; z ; \mathbb{T}$ is a point
$= \mathtt{sup}_{n \geq 0} \, \mathtt{rel}(\, (\mathbb{S}^{\tau n} ; z ; \mathbb{T} \oslash \mathbb{S}^{n} ; \mathbb{S}^{\tau l} ; z ; \mathbb{T}) \,)$

$= \mathtt{sup}_{n\geq 0}\, \pi^{\intercal} ; \big[\ (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \between \mathbb{S}^{n} ; \mathbb{S}^{\intercal l} ; z ; \mathbb{T}) \cap \rho \big]$ definition of `rel` in Prop. 2.2.5
$= \mathtt{sup}_{n\geq 0}\, \pi^{\intercal} ; \big[\pi ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \rho ; \mathbb{S}^{n} ; \mathbb{S}^{\intercal l} ; z ; \mathbb{T} \cap \rho \big]$
$= \mathtt{sup}_{n\geq 0}\, \pi^{\intercal} ; \big[\pi ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \rho ; (\mathbb{S}^{n} ; \mathbb{S}^{\intercal l} ; z ; \mathbb{T} \cap \mathbb{I})\big]$
$= \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \mathbb{T} ; (\mathbb{S}^{n} ; \mathbb{S}^{\intercal l} ; z ; \mathbb{T} \cap \mathbb{I})$ destroy and append
$= \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap (\mathbb{T} \cap \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{l} ; \mathbb{S}^{\intercal n}) ; \mathbb{I}$ mask shifting
$= \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{l} ; \mathbb{S}^{\intercal n}$
$= \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{l} ; \mathbb{S}^{\intercal n}$ masking; this implies $n \leq l$
$= \not{/\!/}_{l}$ Prop. 5.3.3.i.i

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i.l) $\mathbb{A}^{\hookleftarrow} = (\,\mathbb{A}^{\intercal} \between \rho^{\intercal}) ; \pi = (\big[\,\mathtt{sup}_{n\geq 0}\, (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \between \mathbb{S}^{n})\,\big]^{\intercal} \between \rho^{\intercal}) ; \pi$ Prop. 6.2.1.i.a
$= (\big[\,\mathtt{sup}_{n\geq 0}\, (\mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n} \between \mathbb{S}^{\intercal n})\,\big] \between \rho^{\intercal}) ; \pi$
$= \mathtt{sup}_{n\geq 0}\, (\,(\mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n} \between \mathbb{S}^{\intercal n}) \between \rho^{\intercal}) ; \pi$
$= \mathtt{sup}_{n\geq 0}\, \big[\pi ; (\mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n} \between \mathbb{S}^{\intercal n}) \cap \rho ; \rho^{\intercal}\big] ; \pi$
$= \mathtt{sup}_{n\geq 0}\, \big[\pi ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n} ; \pi^{\intercal} \cap (\pi ; \mathbb{S}^{\intercal n} \cap \rho) ; \rho^{\intercal}\big] ; \pi$
$= \mathtt{sup}_{n\geq 0}\, \big[\mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n} \cap (\pi ; \mathbb{S}^{\intercal n} \cap \rho) ; \mathbb{T}\big]$ destroy and append; π is total
$= \mathtt{sup}_{n\geq 0}\, (\pi ; \mathbb{S}^{\intercal n} \cap \rho) ; (\mathbb{T} \cap \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n})$ masking
$= \mathtt{sup}_{n\geq 0}\, (\mathbb{S}^{\intercal n} \between \mathbb{I}) ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n}$ first version
$= \mathtt{sup}_{n\geq 0}\, (\mathbb{I} \between \mathbb{S}^{n}) ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n}$ second version via Prop. 2.1.1.iii

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Alternative version with projection migration:

$\mathbb{A}^{\hookleftarrow} = (\,\mathbb{A}^{\intercal} \between \rho^{\intercal}) ; \pi = (\big[\,\mathtt{sup}_{n\geq 0}\, (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \between \mathbb{S}^{n})\,\big]^{\intercal} \between \rho^{\intercal}) ; \pi$ Prop. 6.2.1.i.a
$= \mathtt{sup}_{n\geq 0}\, (\,(\mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n} \between \mathbb{S}^{\intercal n}) \between \rho^{\intercal}) ; \pi$
$= \mathtt{sup}_{n\geq 0}\, (\pi' \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n} \between (\mathbb{S}^{\intercal n} \between \mathbb{I})\,) ; \pi$ projection migration Prop. 3.1.2.ii
$= \mathtt{sup}_{n\geq 0}\, \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n} \cap (\mathbb{S}^{\intercal n} \between \mathbb{I}) ; \mathbb{T}$ destroy and append; π' total
$= \mathtt{sup}_{n\geq 0}\, (\mathbb{S}^{\intercal n} \between \mathbb{I}) ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n}$ masking gives first version
$= \mathtt{sup}_{n\geq 0}\, (\mathbb{I} \between \mathbb{S}^{n}) ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n}$ second version via Prop. 2.1.1.iii

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$^{\hookrightarrow}\mathbb{A} = (\pi^{\intercal} \between \mathbb{A}^{\intercal}) ; \rho = (\pi^{\intercal} \between \big[\,\mathtt{sup}_{n\geq 0}\, (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \between \mathbb{S}^{n})\,\big]^{\intercal}) ; \rho$ Prop. 6.2.1.i.a
$= (\pi^{\intercal} \between \big[\,\mathtt{sup}_{n\geq 0}\, (\mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n} \between \mathbb{S}^{\intercal n})\,\big]) ; \rho$
$= \mathtt{sup}_{n\geq 0}\, (\pi^{\intercal} \between (\mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n} \between \mathbb{S}^{\intercal n})\,) ; \rho$
$= \mathtt{sup}_{n\geq 0}\, \big[\pi ; \pi^{\intercal} \cap \rho ; (\mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n} \between \mathbb{S}^{\intercal n})\,\big] ; \rho$
$= \mathtt{sup}_{n\geq 0}\, \big[\pi ; \pi^{\intercal} \cap \rho ; (\mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n} ; \pi^{\intercal} \cap \mathbb{S}^{\intercal n} ; \rho^{\intercal})\big] ; \rho$
$= \mathtt{sup}_{n\geq 0}\, \big[(\pi \cap \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n}) ; \pi^{\intercal} \cap \rho ; \mathbb{S}^{\intercal n} ; \rho^{\intercal}\big] ; \rho$
$= \mathtt{sup}_{n\geq 0} (\pi \cap \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n}) ; \mathbb{T} \cap \rho ; \mathbb{S}^{\intercal n}$ destroy and append
$= \mathtt{sup}_{n\geq 0}\, \pi ; (\mathbb{T} \cap \mathbb{S}^{\intercal n} ; z ; \mathbb{T}) \cap \rho ; \mathbb{S}^{\intercal n}$ mask shifting
$= \mathtt{sup}_{n\geq 0}\, (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \between \mathbb{S}^{\intercal n})$

Alternative version with projection migration:

$^{\hookrightarrow}\mathbb{A} = (\pi^{\intercal} \between \mathbb{A}^{\intercal}) ; \rho = (\pi^{\intercal} \between \big[\,\mathtt{sup}_{n\geq 0}\, (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \between \mathbb{S}^{n})\,\big]^{\intercal}) ; \rho$ Prop. 6.2.1.i.a
$= \mathtt{sup}_{n\geq 0}\, (\pi^{\intercal} \between (\mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n} \between \mathbb{S}^{\intercal n})\,) ; \rho$
$= \mathtt{sup}_{n\geq 0}\, (\,(\mathbb{I} \between \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n}) \between \rho ; \mathbb{S}^{\intercal n}) ; \rho$ projection migration Prop. 3.1.2.i right to left
$= \mathtt{sup}_{n\geq 0}\, (\mathbb{I} \between \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n}) ; \mathbb{T} \cap \rho ; \mathbb{S}^{\intercal n}$ destroy and append
$= \mathtt{sup}_{n\geq 0}\, (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \between \mathbb{I}) ; \mathbb{T} \cap \rho ; \mathbb{S}^{\intercal n}$ due to Prop. 2.1.1.iii
$= \mathtt{sup}_{n\geq 0}\, \pi ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \rho ; \mathbb{S}^{\intercal n}$ masking and ρ total
$= \mathtt{sup}_{n\geq 0}\, (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \between \mathbb{S}^{\intercal n})$

$\hookrightarrow\!\mathcal{B} = (\pi^\intercal \circledcirc B^\intercal)\fatsemi\rho = (\pi^\intercal \circledcirc \big[\,\mathtt{sup}_{n\geq 0}\,(\mathbb{S}^n \circledcirc \mathbb{S}^{\intercal n}\fatsemi z\fatsemi\mathbb{T})\,\big]^\intercal)\fatsemi\rho$ Prop. 6.2.1.i.a
$= (\pi^\intercal \circledcirc \big[\,\mathtt{sup}_{n\geq 0}\,(\mathbb{S}^{\intercal n} \circledast \mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^n)\,\big])\fatsemi\rho$
$= \mathtt{sup}_{n\geq 0}\,(\pi^\intercal \circledcirc (\mathbb{S}^{\intercal n} \circledast \mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^n))\fatsemi\rho$
$= \mathtt{sup}_{n\geq 0}\,\big[\pi\fatsemi\pi^\intercal \cap \rho\fatsemi(\mathbb{S}^{\intercal n} \circledast \mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^n)\big]\fatsemi\rho$
$= \mathtt{sup}_{n\geq 0}\,\big[\pi\fatsemi\pi^\intercal \cap \rho\fatsemi(\mathbb{S}^{\intercal n}\fatsemi\pi^\intercal \cap \mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^n\fatsemi\rho^\intercal)\big]\fatsemi\rho$
$= \mathtt{sup}_{n\geq 0}\,\big[(\pi \cap \rho\fatsemi\mathbb{S}^{\intercal n})\fatsemi\pi^\intercal \cap \rho\fatsemi\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^n\fatsemi\rho^\intercal\big]\fatsemi\rho$
$= \mathtt{sup}_{n\geq 0}(\pi \cap \rho\fatsemi\mathbb{S}^{\intercal n})\fatsemi\mathbb{T} \cap \mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^n$
$= \mathtt{sup}_{n\geq 0}\,(\mathbb{I} \circledcirc \mathbb{S}^{\intercal n})\fatsemi\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^n$ first version
$= \mathtt{sup}_{n\geq 0}\,(\mathbb{S}^n \circledcirc \mathbb{I})\fatsemi\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^n$ second version via Prop. 2.1.1.iii

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Alternative version with projection migration:

$\hookrightarrow\!\mathcal{B} = (\pi^\intercal \circledcirc B^\intercal)\fatsemi\rho = (\pi^\intercal \circledcirc \big[\,\mathtt{sup}_{n\geq 0}\,(\mathbb{S}^n \circledcirc \mathbb{S}^{\intercal n}\fatsemi z\fatsemi\mathbb{T})\,\big]^\intercal)\fatsemi\rho$ Prop. 6.2.1.i.a
$= \mathtt{sup}_{n\geq 0}\,(\pi^\intercal \circledcirc (\mathbb{S}^{\intercal n} \circledast \mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^n))\fatsemi\rho$
$= \mathtt{sup}_{n\geq 0}\,((\mathbb{I} \circledcirc \mathbb{S}^{\intercal n}) \circledast \rho\fatsemi\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^n)\fatsemi\rho$ projection migration Prop. 3.1.2.i
$= \mathtt{sup}_{n\geq 0}\,(\mathbb{I} \circledcirc \mathbb{S}^{\intercal n})\fatsemi\mathbb{T} \cap \mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^n$ destroy and append; ρ total
$= \mathtt{sup}_{n\geq 0}\,(\mathbb{I} \circledcirc \mathbb{S}^{\intercal n})\fatsemi\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^n$ masking gives first version
$= \mathtt{sup}_{n\geq 0}\,(\mathbb{S}^n \circledcirc \mathbb{I})\fatsemi\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^n$ second version via Prop. 2.1.1.iii

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Alternative version with projection migration:

$B^{\hookleftarrow} = (B^\intercal \circledcirc \rho^\intercal)\fatsemi\pi = (\big[\,\mathtt{sup}_{n\geq 0}\,(\mathbb{S}^n \circledcirc \mathbb{S}^{\intercal n}\fatsemi z\fatsemi\mathbb{T})\,\big]^\intercal \circledcirc \rho^\intercal)\fatsemi\pi$ Prop. 6.2.1.i.a
$= \mathtt{sup}_{n\geq 0}\,((\mathbb{S}^{\intercal n} \circledast \mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^n) \circledcirc \rho^\intercal)\fatsemi\pi$
$= \mathtt{sup}_{n\geq 0}\,(\pi'\fatsemi\mathbb{S}^{\intercal n} \circledast (\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^n \circledcirc \mathbb{I}))\fatsemi\pi$ projection migration Prop. 3.1.2.ii
$= \mathtt{sup}_{n\geq 0}\,\pi'\fatsemi\mathbb{S}^{\intercal n} \cap (\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^n \circledcirc \mathbb{I})\fatsemi\mathbb{T}$ destroy and append
$= \mathtt{sup}_{n\geq 0}\,\pi'\fatsemi\mathbb{S}^{\intercal n} \cap (\mathbb{I} \circledcirc \mathbb{S}^{\intercal n}\fatsemi z\fatsemi\mathbb{T})\fatsemi\mathbb{T}$ due to Prop. 2.1.1.iii
$= \mathtt{sup}_{n\geq 0}\,\pi'\fatsemi\mathbb{S}^{\intercal n} \cap \rho'\fatsemi\mathbb{S}^{\intercal n}\fatsemi z\fatsemi\mathbb{T}$ masking; π is total
$= \mathtt{sup}_{n\geq 0}\,(\mathbb{S}^{\intercal n} \circledcirc \mathbb{S}^{\intercal n}\fatsemi z\fatsemi\mathbb{T})$

---

ii.a) In order to prove $\mathbb{A} = B$, we show just $b(\mathbb{A}) \subseteq \mathbb{A}$, implying $B \subseteq \mathbb{A}$. The proof of the other direction, finally leading to $\mathbb{A} \subseteq B$, is then completely symmetric.

$b(\mathbb{A}) = (\mathbb{I} \circledcirc z\fatsemi\mathbb{T}) \cup (\mathbb{I} \otimes \mathbb{S}^\intercal)\fatsemi\mathbb{A}\fatsemi\mathbb{S}$
$= (\mathbb{I} \circledcirc z\fatsemi\mathbb{T}) \cup (\mathbb{I} \otimes \mathbb{S}^\intercal)\fatsemi(\mathbb{I} \otimes \mathbb{S})\fatsemi\mathbb{A}$ due to (i.b)
$= (\mathbb{I} \circledcirc z\fatsemi\mathbb{T}) \cup (\mathbb{I} \otimes \mathbb{S}^\intercal\fatsemi\mathbb{S})\fatsemi\mathbb{A} \subseteq (\mathbb{I} \circledcirc z\fatsemi\mathbb{T}) \cup (\mathbb{I} \otimes \mathbb{I})\fatsemi\mathbb{A} = (\mathbb{I} \circledcirc z\fatsemi\mathbb{T}) \cup \mathbb{A} \subseteq \mathbb{A}$

The last step needs a justification, for which we use Prop. 5.2.2.ii, and, thus, regressive finiteness:

$(\mathbb{I} \circledcirc z\fatsemi\mathbb{T}) = (D \circledcirc z\fatsemi\mathbb{T}) = (\mathtt{sup}_{k\geq 0}\,\mathbb{S}^{\intercal k}\fatsemi z\fatsemi\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^k \circledcirc z\fatsemi\mathbb{T}) \subseteq \mathbb{A}$

To prove this, we start an induction with $k = 0$:

$(\mathbb{S}^{\intercal 0}\fatsemi z\fatsemi\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^0 \circledcirc z\fatsemi\mathbb{T}) = (z\fatsemi\mathbb{T}\fatsemi z^\intercal \circledcirc z\fatsemi\mathbb{T})$
$= (z\fatsemi\mathbb{T} \circledcirc z\fatsemi\mathbb{T})\fatsemi\mathbb{T}\fatsemi z^\intercal$ masking a join
$= \mathbb{A}\fatsemi z\fatsemi\mathbb{T}\fatsemi z^\intercal$ Prop. 6.2.1.i.c
$\subseteq \mathbb{A}\fatsemi\mathbb{I} = \mathbb{A}$ since z is injective

Now the step $k \mapsto k+1$:

$(\mathbb{S}^{\intercal k+1}\fatsemi z\fatsemi\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^{k+1} \circledcirc z\fatsemi\mathbb{T})$
$= (\mathbb{S}^\intercal \otimes \mathbb{I})\fatsemi(\mathbb{S}^{\intercal k}\fatsemi z\fatsemi\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^{k+1} \circledcirc z\fatsemi\mathbb{T})$
$= (\mathbb{S}^\intercal \otimes \mathbb{I})\fatsemi(\mathbb{S}^{\intercal k}\fatsemi z\fatsemi\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^k \circledcirc z\fatsemi\mathbb{T})\fatsemi\mathbb{S}$ masking a join
$\subseteq (\mathbb{S}^\intercal \otimes \mathbb{I})\fatsemi\mathbb{A}\fatsemi\mathbb{S}$ induction hypothesis
$\subseteq \mathbb{A}$ Prop. 6.2.1.i.a

---

ii.b) Following Prop. 5.2.2.ii.a, we have $D = \mathbb{I}$ in (i.c).

---

ii.c) Regressive finiteness leads to $\mathbb{V} = \mathbb{S}^*$, using Prop. 5.3.1.ii.a. Therefore with $\mathbb{A} = B$ and using (i.g)

$\pi^\intercal ; \mathbb{A} = \mathbb{V} = \rho^\intercal ; B = \mathbb{S}^*$

---

ii.d) With $\mathbb{A} = B$, the results $(\mathbb{S} \otimes \mathbb{I}) ; B = B ; \mathbb{S}$ and $(\mathbb{I} \otimes \mathbb{S}) ; \mathbb{A} = \mathbb{A} ; \mathbb{S}$ of (i.b) suffice for the proof.

---

ii.e) $\mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\, (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash \mathbb{S}^{\intercal i} ; z ; \mathbb{T}) ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{n+i}$

$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\, \big[\pi ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \rho ; \mathbb{S}^{\intercal i} ; z ; \mathbb{T}\big] ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{n+i}$

$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\, \pi ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \rho ; \mathbb{S}^{\intercal i} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{n+i}$ masking

$= \mathtt{sup}_{n\geq 0}\, \pi ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \rho ; \big[\mathtt{sup}_{i\geq 0}\, \mathbb{S}^{\intercal i} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{i+n}\big]$

$= \mathtt{sup}_{n\geq 0}\, \pi ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \rho ; \big[\mathtt{sup}_{i\geq 0}\, \mathbb{S}^{\intercal i} ; z ; \mathbb{T} \cap \mathbb{S}^n\big]$ Prop. 5.2.1.i.g

$= \mathtt{sup}_{n\geq 0}\, \pi ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \rho ; \big[\mathbb{S}^n \cap \mathtt{sup}_{i\geq 0}\, \mathbb{S}^{\intercal i} ; z ; \mathbb{T}\big]$

$= \mathtt{sup}_{n\geq 0}\, \pi ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \rho ; \big[\mathbb{S}^n \cap \mathbb{T}\big]$ Prop. 5.2.1.ii.a, with regressive finiteness

$= \mathtt{sup}_{n\geq 0}\, (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash \mathbb{S}^n) = \mathbb{A}$

$\mathbb{A}^\intercal ; (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash \mathbb{S}^{\intercal i} ; z ; \mathbb{T})$

$= \big[\mathtt{sup}_{k\geq 0}\, \mathtt{sup}_{l\geq 0}\, (\mathbb{S}^{\intercal k} ; z ; \mathbb{T} \oslash \mathbb{S}^{\intercal l} ; z ; \mathbb{T}) ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{k+l}\big]^\intercal ; (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash \mathbb{S}^{\intercal i} ; z ; \mathbb{T})$ (ii.e — first)

$= \mathtt{sup}_{k\geq 0}\, \mathtt{sup}_{l\geq 0}\, \mathbb{S}^{\intercal k+l} ; z ; \mathbb{T} ; (\mathbb{T} ; z^\intercal ; \mathbb{S}^k \oslash \mathbb{T} ; z^\intercal ; \mathbb{S}^l) ; (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash \mathbb{S}^{\intercal i} ; z ; \mathbb{T})$

$= \mathtt{sup}_{k\geq 0}\, \mathtt{sup}_{l\geq 0}\, \mathbb{S}^{\intercal k+l} ; z ; \mathbb{T} ; \big[\mathbb{T} ; z^\intercal ; \mathbb{S}^k ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \mathbb{T} ; z^\intercal ; \mathbb{S}^l ; \mathbb{S}^{\intercal i} ; z ; \mathbb{T}\big]$ implying $k := n, l := i$

$= \mathbb{S}^{\intercal n+i} ; z ; \mathbb{T} ; \big[\mathbb{T} ; z^\intercal ; \mathbb{S}^n ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \mathbb{T} ; z^\intercal ; \mathbb{S}^i ; \mathbb{S}^{\intercal i} ; z ; \mathbb{T}\big]$

$= \begin{cases} \mathbb{S}^{\intercal n+i} ; z ; \mathbb{T} ; [\mathbb{T} \cap \mathbb{T}] & \text{when } \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \neq \mathbb{\perp} \quad \textbf{and} \quad \mathbb{S}^{\intercal i} ; z ; \mathbb{T} \neq \mathbb{\perp} \\ \mathbb{\perp} & \text{when } \mathbb{S}^{\intercal n} ; z ; \mathbb{T} = \mathbb{\perp} \quad \textbf{or} \quad \mathbb{S}^{\intercal i} ; z ; \mathbb{T} = \mathbb{\perp} \end{cases}$

$= \mathbb{S}^{\intercal n+i} ; z ; \mathbb{T}$ since, e.g., $\mathbb{S}^{\intercal i} ; z ; \mathbb{T} = \mathbb{\perp} \implies \mathbb{S}^{\intercal n+i} ; z ; \mathbb{T} = \mathbb{S}^{\intercal n} ; \mathbb{S}^{\intercal i} ; z ; \mathbb{T} = \mathbb{\perp}$

For a proof via recursion, we use (ii.b) or even (i.c) at $n = 0$, and proceed $n \mapsto n + 1$ with

$\mathbb{A}^\intercal ; (\mathbb{S}^{\intercal n+1} ; z ; \mathbb{T} \oslash \mathbb{I}) = \mathbb{A}^\intercal ; (\mathbb{S}^\intercal \otimes \mathbb{I}) ; (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash \mathbb{I})$

$= \mathbb{S}^\intercal ; \mathbb{A}^\intercal ; (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash \mathbb{I})$ via (ii.d)

$= \mathbb{S}^\intercal ; \mathbb{S}^{\intercal n} = \mathbb{S}^{\intercal n+1}$ hypothesis

---

Variant proof using first variant of (ii.e):

$\mathbb{A}^\intercal ; (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash \mathbb{I})$

$= \big[\mathtt{sup}_{k\geq 0}\, \mathtt{sup}_{l\geq 0}\, (\mathbb{S}^{\intercal k} ; z ; \mathbb{T} \oslash \mathbb{S}^{\intercal l} ; z ; \mathbb{T}) ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{k+l}\big]^\intercal ; (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash \mathbb{I})$

$= \mathtt{sup}_{k\geq 0}\, \mathtt{sup}_{l\geq 0}\, \mathbb{S}^{\intercal k+l} ; z ; \mathbb{T} ; (\mathbb{T} ; z^\intercal ; \mathbb{S}^k \oslash \mathbb{T} ; z^\intercal ; \mathbb{S}^l) ; (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash \mathbb{I})$

$= \mathtt{sup}_{k\geq 0}\, \mathtt{sup}_{l\geq 0}\, \mathbb{S}^{\intercal k+l} ; z ; \mathbb{T} ; \big[\mathbb{T} ; z^\intercal ; \mathbb{S}^k ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \mathbb{T} ; z^\intercal ; \mathbb{S}^l\big]$ implying $k := n$

$= \mathtt{sup}_{l\geq 0}\, \mathbb{S}^{\intercal n+l} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^l$

$= \mathbb{S}^{\intercal n} ; \mathtt{sup}_{l\geq 0}\, \mathbb{S}^{\intercal l} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^l = \mathbb{S}^{\intercal n} ; \mathbb{I} = \mathbb{S}^{\intercal n}$

$\mathfrak{T}_{21\to 12} ; (\mathbb{I} \otimes \mathbb{A}) ; \mathbb{A} = (\pi' ; \pi \oslash (\rho \otimes \mathbb{I})) ; (\mathbb{I} \otimes \mathbb{A}) ; \mathbb{A}$ [15, Def. 3.3.3]

$= (\pi' ; \pi \oslash (\rho \otimes \mathbb{I}) ; \mathbb{A}) ; \mathbb{A}$

$= (\pi' ; \pi \oslash (\rho \otimes \mathbb{I}) ; \big[\mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\, (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash \mathbb{S}^{\intercal i} ; z ; \mathbb{T}) ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{n+i}\big]) ; \mathbb{A}$ (ii.e — first)

$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\ (\pi' ; \pi \oslash\ (\rho ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash \mathbb{S}^{\intercal i} ; z ; \mathbb{T}) ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n+i}) ; \mathbb{A}$
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\ (\pi' ; \pi \oslash\ (\rho ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash \mathbb{S}^{\intercal i} ; z ; \mathbb{T}) ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n+i}) ;$
$\qquad \big[\, \mathtt{sup}_{l\geq 0}\, \mathtt{sup}_{s\geq 0}\ (\mathbb{S}^{\intercal l} ; z ; \mathbb{T} \oslash \mathbb{S}^{\intercal s} ; z ; \mathbb{T}) ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{l+s} \big]$ again (ii.e — first)
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\, \mathtt{sup}_{l\geq 0}\, \mathtt{sup}_{s\geq 0}$
$\qquad \big(\pi' ; \pi ; \mathbb{S}^{\intercal l} ; z ; \mathbb{T} \cap\ (\rho ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash \mathbb{S}^{\intercal i} ; z ; \mathbb{T}) ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n+i} ; \mathbb{S}^{\intercal s} ; z ; \mathbb{T}\big) ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{l+s}$
$\qquad\qquad$ wherein $n+i \neq s$ implies $= \mathbb{\perp}$
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\, \mathtt{sup}_{l\geq 0}$
$\qquad \big(\pi' ; \pi ; \mathbb{S}^{\intercal l} ; z ; \mathbb{T} \cap\ (\rho ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash \mathbb{S}^{\intercal i} ; z ; \mathbb{T})\ \big) ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{l+(n+i)}$
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\, \mathtt{sup}_{l\geq 0}$
$\qquad \big(\pi' ; \pi ; \mathbb{S}^{\intercal l} ; z ; \mathbb{T} \cap \big[\pi' ; \rho ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \rho' ; \mathbb{S}^{\intercal i} ; z ; \mathbb{T}\big]\big) ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{(l+n)+i}$
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\, \mathtt{sup}_{l\geq 0}$
$\qquad \big(\pi' ; \big[\pi ; \mathbb{S}^{\intercal l} ; z ; \mathbb{T} \cap \rho ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T}\big] \cap \rho' ; \mathbb{S}^{\intercal i} ; z ; \mathbb{T}\big) ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{(l+n)+i}$
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\, \mathtt{sup}_{l\geq 0}$
$\qquad \big(\pi' ; (\mathbb{S}^{\intercal l} ; z ; \mathbb{T} \oslash \mathbb{S}^{\intercal n} ; z ; \mathbb{T}) \cap \rho' ; \mathbb{S}^{\intercal i} ; z ; \mathbb{T}\big) ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{(l+n)+i}$
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\, \mathtt{sup}_{l\geq 0}$
$\qquad \big(\, (\mathbb{S}^{\intercal l} ; z ; \mathbb{T} \oslash \mathbb{S}^{\intercal n} ; z ; \mathbb{T})\ ; \mathbb{T} \oslash \mathbb{S}^{\intercal i} ; z ; \mathbb{T}\big) ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{(l+n)+i}$
$= \mathtt{sup}_{m\geq 0}\, \mathtt{sup}_{i\geq 0}\, \mathtt{sup}_{l\geq 0}\, \mathtt{sup}_{n\geq 0}$ introduce $m := l+n$ with $\mathbb{T} ; z^{\intercal} ; \mathbb{S}^{l+n} ; \mathbb{S}^{\intercal m} ; z ; \mathbb{T} = \mathbb{T}$ or $= \mathbb{\perp}$
$\qquad \big(\, (\mathbb{S}^{\intercal l} ; z ; \mathbb{T} \oslash \mathbb{S}^{\intercal n} ; z ; \mathbb{T}) ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{l+n} ; \mathbb{S}^{\intercal m} ; z ; \mathbb{T} \oslash \mathbb{S}^{\intercal i} ; z ; \mathbb{T}\big) ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{m+i}$
$= \mathtt{sup}_{m\geq 0}\, \mathtt{sup}_{i\geq 0}$
$\qquad \big(\big[\, \mathtt{sup}_{l\geq 0}\, \mathtt{sup}_{n\geq 0}\ (\mathbb{S}^{\intercal l} ; z ; \mathbb{T} \oslash \mathbb{S}^{\intercal n} ; z ; \mathbb{T}) ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{l+n}\big] ; \mathbb{S}^{\intercal m} ; z ; \mathbb{T} \oslash \mathbb{S}^{\intercal i} ; z ; \mathbb{T}\big) ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{m+i}$
$= \mathtt{sup}_{m\geq 0}\, \mathtt{sup}_{i\geq 0}\ (\mathbb{A} ; \mathbb{S}^{\intercal m} ; z ; \mathbb{T} \oslash \mathbb{S}^{\intercal i} ; z ; \mathbb{T}) ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{m+i}$ due to (ii.e — first)
$= \mathtt{sup}_{m\geq 0}\, \mathtt{sup}_{i\geq 0}\ (\mathbb{A} \otimes \mathbb{I}) ;\ (\mathbb{S}^{\intercal m} ; z ; \mathbb{T} \oslash \mathbb{S}^{\intercal i} ; z ; \mathbb{T}) ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{m+i}$
$= (\mathbb{A} \otimes \mathbb{I}) ; \big[\, \mathtt{sup}_{m\geq 0}\, \mathtt{sup}_{i\geq 0}\ (\mathbb{S}^{\intercal m} ; z ; \mathbb{T} \oslash \mathbb{S}^{\intercal i} ; z ; \mathbb{T}) ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{m+i}\big]$
$= (\mathbb{A} \otimes \mathbb{I}) ; \mathbb{A}$ due to (ii.e — first)

---

ii.f) $\mathbb{A}^{\intercal} ; (\mathtt{eve} ; \mathbb{T} \oslash \mathtt{eve} ; \mathbb{T}) = \mathtt{sup}_{n\geq 0}\ (\mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n} \oslash \mathbb{S}^{\intercal n}) ; (\mathtt{eve} ; \mathbb{T} \oslash \mathtt{eve} ; \mathbb{T})$
$= \mathtt{sup}_{n\geq 0}\, \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n} ; \mathtt{eve} ; \mathbb{T} \cap \mathbb{S}^{\intercal n} ; \mathtt{eve} ; \mathbb{T}$ since $\mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n}$ and $\mathbb{S}^{\intercal n}$ are both univalent
$= \mathtt{sup}_{n\geq 0}\, \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n} ; (\mathtt{sup}_{i\geq 0} (\mathbb{S}^{\intercal 2})^{i} ; \mathbb{T}) \cap \mathbb{S}^{\intercal n} ; (\mathbb{S}^{\intercal 2})^{*} ; z ; \mathbb{T}$ defined in Def. 5.2.1
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\, \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n} ; \mathbb{S}^{\intercal 2*i} ; \mathbb{T} \cap \mathbb{S}^{\intercal n} ; (\mathbb{S}^{\intercal 2})^{*} ; z ; \mathbb{T}$ implying $n := 2*i$ to be $\neq \mathbb{\perp}$
$= \mathtt{sup}_{i\geq 0}\, \mathbb{S}^{\intercal 2*i} ; (\mathbb{S}^{\intercal 2})^{*} ; z ; \mathbb{T} = \mathtt{eve} ; \mathbb{T}$
The others are to be proved in a similar way.

---

iii.a) See Prop. 5.2.1.iii.e

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iv.a) $\mathbb{A} ; \mathbb{T} = B ; \mathbb{T} = \mathtt{sup}_{n\geq 0}\ (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash \mathbb{S}^{n})\ ; \mathbb{T}$ due to (i.a) and (ii.b)
$= \mathtt{sup}_{n\geq 0}\, \pi ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \rho ; \mathbb{S}^{n} ; \mathbb{T}$ masking
$= \mathtt{sup}_{n\geq 0}\, \pi ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T}$ since $\mathbb{S}$ is total
$= \pi ; \big[\, \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\intercal n} ; z ; \mathbb{T}\big] = \pi ; V ; \mathbb{T} = \pi ; \mathbb{T} = \mathbb{T}$ Prop. 5.2.1.i,ii.a

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iv.b) Follows from (i.b) since $\mathbb{S}$ is totally defined. □

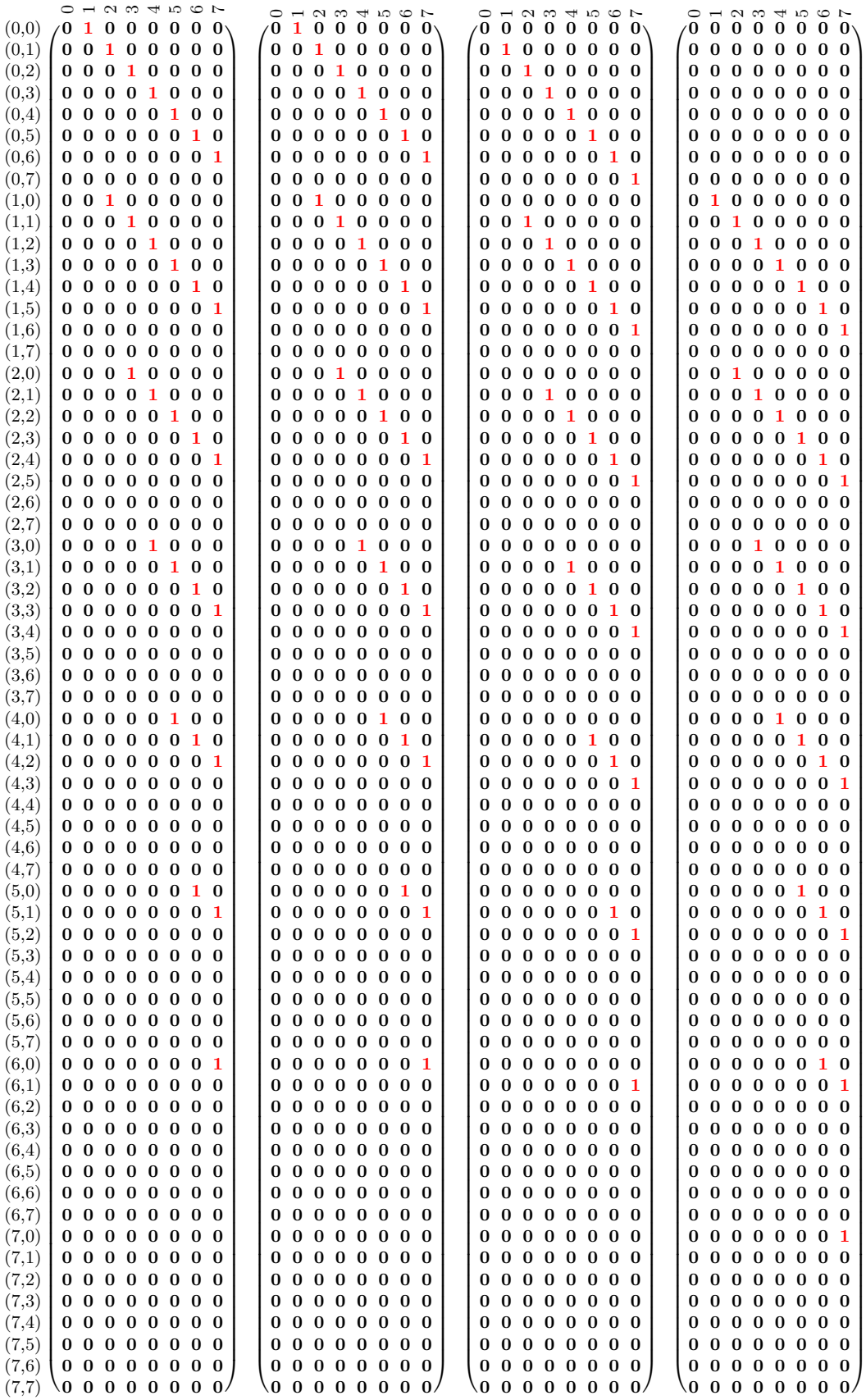


Fig. 6.2.5 Illustrating $(\mathbb{S} \otimes \mathbb{I})_{;}\mathbb{A}$, $\mathbb{A}_{;}\mathbb{S}$, $(\mathbb{S} \otimes \mathbb{S}^{\intercal})_{;}\mathbb{A}$ $(\mathbb{S}^{\intercal} \otimes \mathbb{S})_{;}\mathbb{A}$ of Prop. 6.2.1.i.b

We illustrate what happens when $\mathbb{S}$ is not regressively finite with Fig. 6.2.6:

$$(\mathbb{S} \otimes \mathbb{I}) ; \mathbb{A} = \begin{array}{c} \\ (0,0)\\(0,1)\\(0,2)\\(0,3)\\(0,4)\\(1,0)\\(1,1)\\(1,2)\\(1,3)\\(1,4)\\(2,0)\\(2,1)\\(2,2)\\(2,3)\\(2,4)\\(3,0)\\(3,1)\\(3,2)\\(3,3)\\(3,4)\\(4,0)\\(4,1)\\(4,2)\\(4,3)\\(4,4) \end{array} \begin{pmatrix} 0&1&0&0&0\\0&0&1&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix} \subseteq \mathbb{A} ; \mathbb{S} = \begin{pmatrix} 0&1&0&0&0\\0&0&1&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix} = (\mathbb{I} \otimes \mathbb{S}) ; \mathbb{A}$$

Fig. 6.2.6 Formulae in Prop. 6.2.1.i.b interpreted in the non-regressively finite case of Fig. 5.2.3

The first result of (i.d) says that $\mathbb{A}$ will deliver z only when both arguments are equal to z. When $\mathbb{S}$ is not yet regressively finite, it may look as already presented in Fig. 5.2.3 or in Fig. 5.2.6.

We have us, with Prop. 6.2.1.ii.a, convinced that $B = \mathbb{A}$ and only the way of their recursion runs over different components of their argument pair.

6.3 Subtraction

We will also need the partial function $\mathfrak{S} : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ representing subtraction. There is obviously more than one possibility to introduce it. The first two would be inverting addition to $\mathbb{A}^{\leftarrow}$, but we may also define it in the iterative form resembling more or less directly

$$x - 0 = x \quad \text{and} \quad (x+1) - (y+1) = x - y,$$

and then prove that these versions coincide. For this, we check being zero with $(\mathbb{I} \oslash z ; \mathbb{T})$ on the second component; just for mutual control, we mention also the subtraction $(x, y) \mapsto y - x$, changed only in the starting value. Finally, we will also arrive at the symmetric difference $(x, y) \mapsto |x - y|$ — that will turn out to be a mapping.

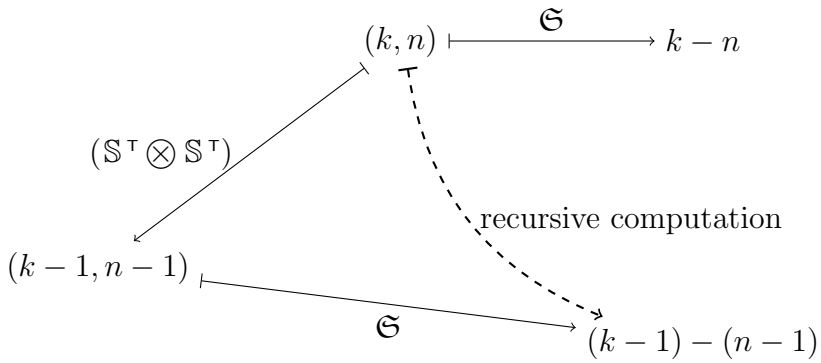


Fig. 6.3.1 Defining the subtraction relation on a possibly finite initial segment of the naturals

6.3.1 Definition and Proposition. We define as **subtractions** related to $\mathbb{S}$ the constructs

$\mathfrak{S} := \mathfrak{S}_{\mathbb{S}} := \mathtt{inf}\,\{\,X \mid (\mathbb{I} \oslash z\,;\mathbb{T}) \cup (\mathbb{S}^\intercal \otimes \mathbb{S}^\intercal)\,;X \subseteq X\},$
$\mathtt{S} := \mathtt{S}_{\mathbb{S}} := \mathtt{inf}\,\{\,X \mid (z\,;\mathbb{T} \oslash \mathbb{I}) \cup (\mathbb{S}^\intercal \otimes \mathbb{S}^\intercal)\,;X \subseteq X\},$

where we have in view of Lemma 4.2.1

$s(X) := \mathfrak{S}_0 \cup s_H(X)$ with $\mathfrak{S}_0 = (\mathbb{I} \oslash z\,;\mathbb{T})$ resp.
$\mathtt{s}(X) := \mathtt{S}_0 \cup s_H(X)$ with $\mathtt{S}_0 = (z\,;\mathbb{T} \oslash \mathbb{I})$ and $s_H(X) = (\mathbb{S}^\intercal \otimes \mathbb{S}^\intercal)\,;X.$

Respecting possible further properties of $\mathbb{S}$, the subtractions $\mathfrak{S}, \mathtt{S}$ satisfy

$$\mathfrak{S} = \begin{array}{c} \\ (0,0)\\(0,1)\\(0,2)\\(0,3)\\(0,4)\\(1,0)\\(1,1)\\(1,2)\\(1,3)\\(1,4)\\(2,0)\\(2,1)\\(2,2)\\(2,3)\\(2,4)\\(3,0)\\(3,1)\\(3,2)\\(3,3)\\(3,4)\\(4,0)\\(4,1)\\(4,2)\\(4,3)\\(4,4) \end{array} \begin{array}{c} \begin{array}{ccccc} 0&1&2&3&4 \end{array} \\ \begin{pmatrix} 1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&1&0&0&0\\1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&1&0&0\\0&1&0&0&0\\1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\\1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\\1&0&0&0&0 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{ccccc} 0&1&2&3&4 \end{array} \\ \begin{pmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&0&0&0\\1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0 \end{pmatrix} \end{array} = \mathtt{S}$$

$$\mathbb{X} := \mathfrak{S} \cup \mathtt{S} = \begin{array}{c} \\ (0,0)\\(0,1)\\(0,2)\\(0,3)\\(0,4)\\(1,0)\\(1,1)\\(1,2)\\(1,3)\\(1,4)\\(2,0)\\(2,1)\\(2,2)\\(2,3)\\(2,4)\\(3,0)\\(3,1)\\(3,2)\\(3,3)\\(3,4)\\(4,0)\\(4,1)\\(4,2)\\(4,3)\\(4,4) \end{array} \begin{array}{c} \begin{array}{ccccc} 0&1&2&3&4 \end{array} \\ \begin{pmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&1&0&0&0\\1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\\1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\\1&0&0&0&0\\0&1&0&0&0\\0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\\1&0&0&0&0 \end{pmatrix} \end{array}$$

Fig. 6.3.2 $\mathfrak{S}$ and $\mathtt{S}$ are necessarily different

i) in general:

a) $\mathfrak{S} = s(\mathfrak{S}) = (\mathbb{I} \oslash z ; \mathbb{T}) \cup (\mathbb{S}^\intercal \otimes \mathbb{S}^\intercal) ; \mathfrak{S} = \mathtt{sup}_{n\geq 0}\, s_H^n(\mathfrak{S}_0)$
$= \mathtt{sup}_{n\geq 0}\, (\mathbb{S}^{\intercal n} \oslash \mathbb{S}^{\intercal n} ; z ; \mathbb{T}) \subseteq \rho ; V$

$\mathtt{S} = \mathtt{s}(\mathtt{S}) = (z ; \mathbb{T} \oslash \mathbb{I}) \cup (\mathbb{S}^\intercal \otimes \mathbb{S}^\intercal) ; \mathtt{S} = \mathtt{sup}_{n\geq 0}\, s_H^n(\mathtt{S}_0)$
$= \mathtt{sup}_{n\geq 0}\, (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash \mathbb{S}^{\intercal n}) \subseteq \pi ; V$

b) $\mathfrak{S}$ and $\mathtt{S}$ are univalent and surjective; they satisfy $\mathfrak{P} ; \mathfrak{S} = \mathtt{S}$

c) $\mathfrak{S}^\intercal ; (P \oslash z ; \mathbb{T}) = P \qquad \mathfrak{S}^\intercal ; (Q \oslash e ; \mathbb{T}) = \mathbb{S} ; Q \qquad \mathfrak{S}^\intercal ; (z ; \mathbb{T} \oslash \mathbb{I}) = z ; \mathbb{T} ; z^\intercal$

$\mathtt{S}^\intercal ; (z ; \mathbb{T} \oslash P) = P \qquad \mathtt{S}^\intercal ; (e ; \mathbb{T} \oslash Q) = \mathbb{S} ; Q$

d) $\mathfrak{S} ; z ; \mathbb{T} = (\mathbb{I} \oslash \mathbb{I}) ; V \qquad (\mathbb{I} \oslash \mathbb{I}) ; \mathfrak{S} = \mathbb{S}^{\intercal *} \cap \mathbb{T} ; z^\intercal \qquad (\mathbb{S}^{\intercal +} \oslash \mathbb{I}) ; \mathfrak{S} = \perp\!\!\!\perp$

$\mathtt{S} ; z ; \mathbb{T} = (\mathbb{I} \oslash \mathbb{I}) ; V \qquad (\mathbb{I} \oslash \mathbb{I}) ; \mathtt{S} = \mathbb{S}^{\intercal *} \cap \mathbb{T} ; z^\intercal \qquad (\mathbb{I} \oslash \mathbb{S}^{\intercal +}) ; \mathtt{S} = \perp\!\!\!\perp$

$(\mathbb{S} \otimes \mathbb{S}) ; \mathfrak{S} = \mathfrak{S} \cap (\mathbb{S} ; \mathbb{T} \oslash \mathbb{S} ; \mathbb{T})$

$(\mathbb{S}^\intercal \otimes \mathbb{I}) ; \mathfrak{S} ; \mathbb{S} = \mathfrak{S} \cap \mathbb{T} ; \mathbb{S}$

e) $\hookrightarrow\!\mathfrak{S} = \mathbb{A}^{\hookleftarrow} \qquad \mathfrak{S}^{\hookleftarrow} = B$

$\mathtt{S}^{\hookleftarrow} = \hookrightarrow\! B \qquad \hookrightarrow\!\mathtt{S} = \mathbb{A}$

$\pi^\intercal ; \mathfrak{S} \subseteq (\mathbb{S}^\intercal)^* \qquad \rho^\intercal ; \mathtt{S} \subseteq (\mathbb{S}^\intercal)^* \qquad \rho^\intercal ; \mathfrak{S} = \boxslash = \pi^\intercal ; \mathtt{S}$

f) $\mathbb{X} := (\mathbb{I} \cup \mathfrak{P}) ; \mathfrak{S} = \mathfrak{S} \cup \mathtt{S}$ is a function showing the **absolute difference** — with a notation $\mathbb{X}$ that resembles via "/" the minus in one and via "\" the minus in the other direction.

g) $\mathfrak{S} ; \mathbb{T} = (\mathbb{I} \oslash \mathbb{V}) ; \mathbb{T} \qquad \mathtt{S} ; \mathbb{T} = (\mathbb{V} \oslash \mathbb{I}) ; \mathbb{T}$

h) $\mathfrak{P} ; \mathbb{X} = \mathbb{X}$

i) For $l \geq 0$ we have $\mathtt{rel}(\mathfrak{S} ; \mathbb{S}^{\intercal l} ; z ; \mathbb{T}) = \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\intercal n+l} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^n$.

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \quad \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array} \quad \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array} \quad \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array} \quad \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \quad \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Fig. 6.3.3 Illustrating (i.e) and (ii.j): lower left diagonals $\mathbb{S}^{\intercal n}$; cf. also $\mathfrak{S}_{table}$ in Fig. 6.3.10

j) $(z ; \mathbb{T} ; z^\intercal \otimes z ; \mathbb{T} ; z^\intercal) ; \mathfrak{S} = (z ; \mathbb{T} ; z^\intercal \oslash z ; \mathbb{T})$

$(z ; \mathbb{T} ; z^\intercal \otimes z ; \mathbb{T} ; z^\intercal) ; \rho = (z ; \mathbb{T} \oslash z ; \mathbb{T} ; z^\intercal)$

$(z ; \mathbb{T} ; z^\intercal \otimes z ; \mathbb{T} ; z^\intercal) ; (\mathfrak{S} \oslash \rho) = (z ; \mathbb{T} ; z^\intercal \otimes z ; \mathbb{T} ; z^\intercal)$

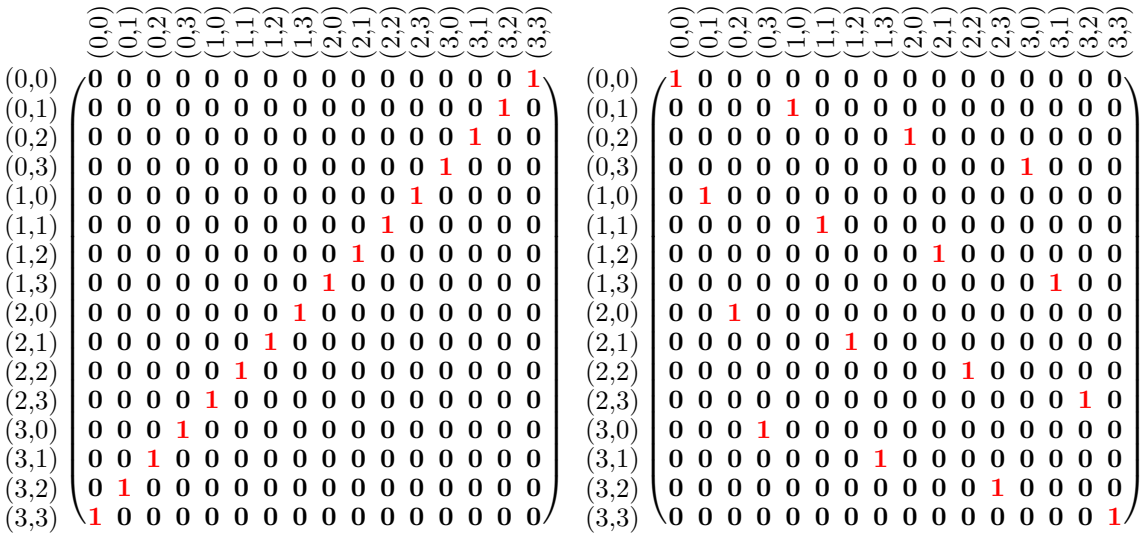


Fig. 6.3.4 $(\texttt{FlipTD} \otimes \texttt{FlipTD})$, existing only for non-total $\mathbb{S}$, as opposed to $\mathfrak{P}$

ii) additionally, if $\mathbb{S}$ is regressively finite:

a) $\mathfrak{S} ; z ; \mathbb{T} = (\mathbb{I} \circledcirc \mathbb{I}) ; \mathbb{T} \qquad \mathtt{S} ; z ; \mathbb{T} = (\mathbb{I} \circledcirc \mathbb{I}) ; \mathbb{T}$

b) $(\mathbb{I} \circledcirc \mathbb{I}) ; \mathfrak{S} = \mathbb{T} ; z^{\mathsf{T}} \qquad (\mathbb{I} \circledcirc \mathbb{I}) ; \mathtt{S} = \mathbb{T} ; z^{\mathsf{T}} \qquad (\mathbb{I} \circledcirc \mathbb{I}) ; \mathbb{X} = \mathbb{T} ; z^{\mathsf{T}}$

c) $(\mathbb{I} \otimes \mathbb{S}) ; \mathfrak{S} = \mathfrak{S} ; \mathbb{S}^{\mathsf{T}} \qquad (\mathbb{S} \otimes \mathbb{I}) ; \mathtt{S} = \mathtt{S} ; \mathbb{S}^{\mathsf{T}} \qquad (\mathbb{S}^{+\mathsf{T}} \circledcirc \mathbb{I}) ; \mathfrak{S} = \mathbb{\perp}$

d) $\mathfrak{S} ; \mathbb{T} = (\mathbb{I} \circledcirc \mathbb{S}^*) ; \mathbb{T} = (\mathbb{S}^{\mathsf{T}*} \circledcirc \mathbb{I}) ; \mathbb{T} = (\mathfrak{S} \circledcirc \rho) ; \mathbb{T}$

$\mathtt{S} ; \mathbb{T} = (\mathbb{S}^* \circledcirc \mathbb{I}) ; \mathbb{T} = (\mathbb{I} \circledcirc \mathbb{S}^{\mathsf{T}*}) ; \mathbb{T}$

e) $\hookrightarrow\!\mathfrak{S} = \mathfrak{S} \qquad \mathfrak{S}^{\hookleftarrow} = \mathbb{A} = {}^{\hookrightarrow}\!\mathtt{S} \qquad \mathtt{S}^{\hookleftarrow} = \mathtt{S}$

$\mathtt{S} ; \mathbb{S}^{\mathsf{T}i} ; z ; \mathbb{T} = \mathtt{vec}(\mathbb{S}^i) \qquad \mathfrak{S} ; \mathbb{S}^{\mathsf{T}i} ; z ; \mathbb{T} = \mathtt{vec}(\mathbb{S}^{\mathsf{T}i})$

f) $\mathbb{X} := (\mathbb{I} \cup \mathfrak{P}) ; \mathfrak{S} = \mathfrak{S} \cup \mathtt{S}$ is a mapping.

g) $\mathfrak{S} = \mathtt{sup}_{n \geq 0} \, \mathtt{sup}_{i \geq 0} \, (\mathbb{S}^{\mathsf{T}n} ; z ; \mathbb{T} \circledcirc \mathbb{S}^{\mathsf{T}i} ; z ; \mathbb{T}) ; \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^n ; \mathbb{S}^{\mathsf{T}i}$

h) $\mathfrak{S}^{\mathsf{T}} ; (\mathbb{S}^q ; \mathbb{S}^{\mathsf{T}r} ; z ; \mathbb{T} \circledcirc \mathbb{S}^{\mathsf{T}s} ; z ; \mathbb{T}) = \mathbb{S}^{s+q} ; \mathbb{S}^{\mathsf{T}r} ; z ; \mathbb{T}$ for arbitrary $q, r, s \geq 0$

i) $\mathfrak{S}^{\mathsf{T}} ; (\mathbb{S}^{\mathsf{T}n} ; z ; \mathbb{T} \circledcirc \mathbb{S}^{\mathsf{T}i} ; z ; \mathbb{T}) = \mathbb{S}^i ; \mathbb{S}^{\mathsf{T}n} ; z ; \mathbb{T}$

$$= \begin{cases} \mathbb{S}^{\mathsf{T}n-i} ; z ; \mathbb{T} & \text{if } n \geq i \text{ and } \mathbb{S}^{\mathsf{T}n} ; z ; \mathbb{T} \neq \mathbb{\perp}, \text{ implying } \mathbb{\perp} \neq \mathbb{S}^{\mathsf{T}i} ; z ; \mathbb{T} \\ \mathbb{\perp} & \text{otherwise} \end{cases}$$

j) $\pi^{\mathsf{T}} ; \mathfrak{S} = \mathbb{S}^{\mathsf{T}*} = \rho^{\mathsf{T}} ; \mathtt{S}$

k) $(\mathbb{S} \otimes \mathbb{I}) ; \mathfrak{S} \cap \mathbb{T} ; \mathbb{S} = \mathfrak{S} ; \mathbb{S} \cap \pi ; \mathbb{S} ; \mathbb{T}$

l) $(\mathbb{S} \otimes \mathbb{S}) ; \mathfrak{S} = \mathfrak{S} \cap \pi ; \mathbb{S} ; \mathbb{T}$

m) For any point $x ; \mathbb{T}$ and $n \geq 0$ always $(\mathbb{T} ; x^{\mathsf{T}} \circledcirc \mathbb{T} ; e^{\mathsf{T}}) ; (\mathfrak{S} \circledcirc \rho)^n = (\mathbb{T} ; x^{\mathsf{T}} ; \mathbb{S}^{\mathsf{T}n} \circledcirc \mathbb{T} ; e^{\mathsf{T}})$

iii) If $\mathbb{S}$ happens to be regressively finite but not total, i.e. $\mathbb{S} ; \mathbb{T} \neq \mathbb{T}$,

a) $\mathfrak{S} = \mathtt{sup}_{n_{\mathbb{S}} \geq n \geq 0} \, (\mathbb{S}^{\intercal n} \circledcirc \mathbb{S}^{\intercal n} ; z ; \mathbb{T})$

b) $\mathtt{FlipTD} = (\mathbb{T} ; g^{\intercal}_{\mathbb{S}} \circledcirc \mathbb{I}) ; \mathfrak{S}$

c) $(\mathtt{FlipTD} \otimes \mathtt{FlipTD}) ; \mathfrak{S} = \mathfrak{P} ; \mathfrak{S} = \mathtt{S}$

d) $(\mathtt{FlipTD} \otimes \mathtt{FlipTD}) ; \mathbb{X} = \mathbb{X}$

e) $\rho^{\intercal} ; \mathfrak{S} = \mathbb{S}^* ; \mathtt{FlipTD} = \pi^{\intercal} ; \mathtt{S}$

iv) In case $\mathbb{S}$ is regressively finite and also total, i.e. $\mathbb{S} ; \mathbb{T} = \mathbb{T}$,

a) $\rho^{\intercal} ; \mathfrak{S} = \mathbb{T} = \pi^{\intercal} ; \mathtt{S}$

Proof: The applicability of Lemma 4.2.1 is obvious.

---

i.a) resembles Lemma 4.2.1.i. The last containment is obtained by
$\mathtt{sup}_{n \geq 0} \, (\mathbb{S}^{\intercal n} \circledcirc \mathbb{S}^{\intercal n} ; z ; \mathbb{T}) = \mathtt{sup}_{n \geq 0} \, \pi ; \mathbb{S}^{\intercal n} \cap \rho ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \subseteq \mathtt{sup}_{n \geq 0} \, \rho ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T}$
$= \rho ; \mathtt{sup}_{n \geq 0} \, \mathbb{S}^{\intercal n} ; z ; \mathbb{T} = \rho ; V$ Prop. 5.2.1.i.a

Concerning $\mathtt{S}$, everything works analogously.

---

i.b) We focus on $\mathfrak{S} = \mathtt{sup}_{n \geq 0} \, (\mathbb{S}^{\intercal n} \circledcirc \mathbb{S}^{\intercal n} ; z ; \mathbb{T})$. In case the summation indices in the construct $\mathfrak{S}^{\intercal} ; \mathfrak{S}$ differ, we will always somehow obtain $\mathbb{S} ; z = \perp\!\!\!\perp$. It remains, therefore, to consider
$\mathfrak{S}^{\intercal} ; \mathfrak{S} = \mathtt{sup}_{n \geq 0} \, (\mathbb{S}^{n} \circledcirc \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n}) ; (\mathbb{S}^{\intercal n} \circledcirc \mathbb{S}^{\intercal n} ; z ; \mathbb{T})$
$= \mathtt{sup}_{n \geq 0} \, \mathbb{S}^{n} ; \mathbb{S}^{\intercal n} \cap \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n} ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T}$
$\subseteq \mathtt{sup}_{n \geq 0} \, \mathbb{I} \cap \mathbb{T} \subseteq \mathbb{I}$ since $\mathbb{S}$ is injective and z a point

---

For surjectivity, we focus on $\mathfrak{S}_0$ alone:
$\mathbb{T} ; \mathfrak{S} \supseteq \mathbb{T} ; \mathfrak{S}_0 = \mathbb{T} ; (\mathbb{I} \circledcirc z ; \mathbb{T}) = \mathbb{T} ; \pi^{\intercal} ; (\mathbb{I} \circledcirc z ; \mathbb{T}) = \mathbb{T} ; (\mathbb{I} \cap \mathbb{T} ; z ; \mathbb{T}) = \mathbb{T} ; (\mathbb{I} \cap \mathbb{T}) = \mathbb{T}$

Concerning $\mathtt{S}$, everything works similarly. $\mathfrak{P} ; \mathfrak{S} = \mathtt{S}$ is obvious.

---

i.c) $\mathfrak{S}^{\intercal} ; (P \circledcirc z ; \mathbb{T}) = \mathtt{sup}_{n \geq 0} \, (\mathbb{S}^{n} \circledcirc \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n}) ; (P \circledcirc z ; \mathbb{T})$
$= P \cap \mathbb{T} ; z^{\intercal} ; z ; \mathbb{T} = P \cap \mathbb{T} = P$ because $\mathbb{S} ; z = \perp\!\!\!\perp$ occurs for $n > 0$ and since z is a point

---

$\mathfrak{S}^{\intercal} ; (Q \circledcirc e ; \mathbb{T}) = \mathtt{sup}_{n \geq 0} \, (\mathbb{S}^{n} \circledcirc \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n}) ; (Q \circledcirc e ; \mathbb{T})$
$= (Q \cap \mathbb{T} ; z^{\intercal} ; e ; \mathbb{T}) \cup (\mathbb{S} ; Q \cap \mathbb{T} ; z^{\intercal} ; \mathbb{S} ; e ; \mathbb{T}) \cup \mathtt{sup}_{n \geq 2} \, (\mathbb{S}^{n} \circledcirc \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n}) ; (Q \circledcirc e ; \mathbb{T})$
$= \perp\!\!\!\perp \cup (\mathbb{S} ; Q \cap \mathbb{T} ; z^{\intercal} ; \mathbb{S} ; e ; \mathbb{T}) \cup \perp\!\!\!\perp$ since z, e are different points and $\mathbb{S}^{n} ; e ; \mathbb{T} = \perp\!\!\!\perp$ for $n \geq 2$
$= \mathbb{S} ; Q \cap \mathbb{T} ; z^{\intercal} ; z ; \mathbb{T}$ because $\mathbb{S} ; e = z$
$= \mathbb{S} ; Q \cap \mathbb{T} = \mathbb{S} ; Q$ since z is a point

Concerning $\mathtt{S}$, everything works similarly.

$\mathfrak{S}^{\intercal} ; (z ; \mathbb{T} \circledcirc \mathbb{I})$
$= \mathtt{sup}_{n \geq 0} \, (\mathbb{S}^{n} \circledcirc \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n}) ; (z ; \mathbb{T} \circledcirc \mathbb{I}) = \mathtt{sup}_{n \geq 0} \, \mathbb{S}^{n} ; z ; \mathbb{T} \cap \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n}$ implying $n = 0$
$= z ; \mathbb{T} \cap \mathbb{T} ; z^{\intercal} = z ; \mathbb{T} ; z^{\intercal}$

---

i.d) $\mathfrak{S}\fatsemi z\fatsemi\mathbb{T} = \mathtt{sup}_{n\geq 0}\ (\mathbb{S}^{\intercal n} \circledcirc \mathbb{S}^{\intercal n}\fatsemi z\fatsemi\mathbb{T})\fatsemi z\fatsemi\mathbb{T}$
$= \mathtt{sup}_{n\geq 0}\ (\mathbb{S}^{\intercal n}\fatsemi z\fatsemi\mathbb{T} \circledcirc \mathbb{S}^{\intercal n}\fatsemi z\fatsemi\mathbb{T})$ masking a join
$= \big[\mathtt{sup}_{n\geq 0}\ (\mathbb{S}^{\intercal n} \circledcirc \mathbb{S}^{\intercal n})\big]\fatsemi z\fatsemi\mathbb{T}$ since $z\fatsemi\mathbb{T}$ is injective
$= \big[\mathtt{sup}_{n\geq 0}\ (\mathbb{I} \circledcirc \mathbb{I})\fatsemi\mathbb{S}^{\intercal n}\big]\fatsemi z\fatsemi\mathbb{T}$ since $\mathbb{S}^{\intercal}$ is injective
$= (\mathbb{I} \circledcirc \mathbb{I})\fatsemi\big[\mathtt{sup}_{n\geq 0}\ \mathbb{S}^{\intercal n}\big]\fatsemi z\fatsemi\mathbb{T}$
$= (\mathbb{I} \circledcirc \mathbb{I})\fatsemi\mathbb{S}^{\intercal *}\fatsemi z\fatsemi\mathbb{T}$ based on Prop. 4.3.1.iii
$= (\mathbb{I} \circledcirc \mathbb{I})\fatsemi V$ due to Prop. 5.2.1.i.a

---

$(\mathbb{I} \circledcirc \mathbb{I})\fatsemi\mathfrak{S} = (\mathbb{I} \circledcirc \mathbb{I})\fatsemi\mathtt{sup}_{n\geq 0}\ (\mathbb{S}^{\intercal n} \circledcirc \mathbb{S}^{\intercal n}\fatsemi z\fatsemi\mathbb{T})$
$= \mathtt{sup}_{n\geq 0}\ \mathbb{S}^{\intercal n} \cap \mathbb{S}^{\intercal n}\fatsemi z\fatsemi\mathbb{T} = \mathtt{sup}_{n\geq 0}\ \mathbb{S}^{\intercal n}\fatsemi(\mathbb{I} \cap z\fatsemi\mathbb{T})$
$= \mathtt{sup}_{n\geq 0}\ \mathbb{S}^{\intercal n}\fatsemi(\mathbb{I} \cap \mathbb{T}\fatsemi z^{\intercal})$ transposing a partial identity
$= \mathtt{sup}_{n\geq 0}\ \mathbb{S}^{\intercal n} \cap \mathbb{T}\fatsemi z^{\intercal}$ masking
$= \big[\mathtt{sup}_{n\geq 0}\ \mathbb{S}^{\intercal n}\big] \cap \mathbb{T}\fatsemi z^{\intercal} = \mathbb{S}^{\intercal *} \cap \mathbb{T}\fatsemi z^{\intercal}$

See Fig. 5.2.6, e.g., for the unexpected first term. Concerning $\mathtt{S}$, everything works similarly.

$(\mathbb{S}^{\intercal +} \circledcirc \mathbb{I})\fatsemi\mathfrak{S} = (\mathbb{S}^{\intercal +} \circledcirc \mathbb{I})\fatsemi\mathtt{sup}_{n\geq 0}\ (\mathbb{S}^{\intercal n} \circledcirc \mathbb{S}^{\intercal n}\fatsemi z\fatsemi\mathbb{T})$
$= \mathtt{sup}_{n\geq 0}\ (\mathbb{S}^{\intercal +} \circledcirc \mathbb{I})\fatsemi(\mathbb{S}^{\intercal n} \circledcirc \mathbb{S}^{\intercal n}\fatsemi z\fatsemi\mathbb{T})$
$\subseteq \mathtt{sup}_{n\geq 0}\ \mathbb{S}^{\intercal +}\fatsemi\mathbb{S}^{\intercal n} \cap \mathbb{S}^{\intercal n}\fatsemi z\fatsemi\mathbb{T}$
$= \mathtt{sup}_{n\geq 0}\ \mathbb{S}^{\intercal n}\fatsemi\mathbb{S}^{\intercal +} \cap \mathbb{S}^{\intercal n}\fatsemi z\fatsemi\mathbb{T}$
$= \mathtt{sup}_{n\geq 0}\ \mathbb{S}^{\intercal n}\fatsemi\big[\mathbb{S}^{\intercal +} \cap z\fatsemi\mathbb{T}\big]$
$= \mathtt{sup}_{n\geq 0}\ \mathbb{S}^{\intercal n}\fatsemi\big[\mathbb{S}^{\intercal +} \cap \overline{\mathbb{S}^{\intercal}\fatsemi\mathbb{T}}\big] = \perp\!\!\!\perp$

$(\mathbb{S} \otimes \mathbb{S})\fatsemi\mathfrak{S} = (\mathbb{S} \otimes \mathbb{S})\fatsemi\big[\ (\mathbb{I} \circledcirc z\fatsemi\mathbb{T}) \cup (\mathbb{S}^{\intercal} \otimes \mathbb{S}^{\intercal})\fatsemi\mathfrak{S}\big]$ see (i.a)
$= (\mathbb{S} \otimes \mathbb{S})\fatsemi(\mathbb{I} \circledcirc z\fatsemi\mathbb{T}) \cup (\mathbb{S} \otimes \mathbb{S})\fatsemi(\mathbb{S}^{\intercal} \otimes \mathbb{S}^{\intercal})\fatsemi\mathfrak{S}$
$= \perp\!\!\!\perp \cup (\mathbb{S}\fatsemi\mathbb{S}^{\intercal} \otimes \mathbb{S}\fatsemi\mathbb{S}^{\intercal})\fatsemi\mathfrak{S}$ since $\mathbb{S}\fatsemi z\fatsemi\mathbb{T} = \perp\!\!\!\perp$
$= (\mathbb{I} \cap \mathbb{S}\fatsemi\mathbb{T} \otimes \mathbb{I} \cap \mathbb{S}\fatsemi\mathbb{T})\fatsemi\mathfrak{S} = \big[\pi\fatsemi(\mathbb{I} \cap \mathbb{S}\fatsemi\mathbb{T})\fatsemi\pi^{\intercal} \cap \rho\fatsemi(\mathbb{I} \cap \mathbb{S}\fatsemi\mathbb{T})\fatsemi\rho^{\intercal}\big]\fatsemi\mathfrak{S}$
$= \big[\pi\fatsemi\pi^{\intercal} \cap \pi\fatsemi\mathbb{S}\fatsemi\mathbb{T}\fatsemi\pi^{\intercal} \cap \rho\fatsemi\rho^{\intercal} \cap \rho\fatsemi\mathbb{S}\fatsemi\mathbb{T}\fatsemi\rho^{\intercal}\big]\fatsemi\mathfrak{S}$
$= \big[\mathbb{I} \cap \pi\fatsemi\mathbb{S}\fatsemi\mathbb{T} \cap \rho\fatsemi\mathbb{S}\fatsemi\mathbb{T}\big]\fatsemi\mathfrak{S}$ properties of the direct product
$= \mathfrak{S} \cap \pi\fatsemi\mathbb{S}\fatsemi\mathbb{T} \cap \rho\fatsemi\mathbb{S}\fatsemi\mathbb{T}$ masking
$= \mathfrak{S} \cap (\mathbb{S}\fatsemi\mathbb{T} \circledcirc \mathbb{S}\fatsemi\mathbb{T})$

$(\mathbb{S}^{\intercal} \otimes \mathbb{I})\fatsemi\mathfrak{S}\fatsemi\mathbb{S} = (\mathbb{S}^{\intercal} \otimes \mathbb{I})\fatsemi\mathtt{sup}_{n\geq 0}\ (\mathbb{S}^{\intercal n} \circledcirc \mathbb{S}^{\intercal n}\fatsemi z\fatsemi\mathbb{T})\fatsemi\mathbb{S}$
$= \mathtt{sup}_{n\geq 0}\ (\mathbb{S}^{\intercal}\fatsemi\mathbb{S}^{\intercal n} \circledcirc \mathbb{S}^{\intercal n}\fatsemi z\fatsemi\mathbb{T})\fatsemi\mathbb{S} = \mathtt{sup}_{n\geq 0}\ (\mathbb{S}^{\intercal}\fatsemi\mathbb{S}^{\intercal n}\fatsemi\ \mathbb{S} \circledcirc \mathbb{S}^{\intercal n}\fatsemi z\fatsemi\mathbb{T})$ masking a join
$= \mathtt{sup}_{n\geq 0}\ (\mathbb{S}^{\intercal n}\fatsemi\ \mathbb{S}^{\intercal}\fatsemi\mathbb{S} \circledcirc \mathbb{S}^{\intercal n}\fatsemi z\fatsemi\mathbb{T})$
$= \mathtt{sup}_{n\geq 0}\ (\mathbb{S}^{\intercal n}\fatsemi(\mathbb{I} \cap \mathbb{T}\fatsemi\mathbb{S}) \circledcirc \mathbb{S}^{\intercal n}\fatsemi z\fatsemi\mathbb{T})$ due to Prop. 5.2.1.i.b
$= \mathtt{sup}_{n\geq 0}\ (\mathbb{S}^{\intercal n} \cap \mathbb{T}\fatsemi\mathbb{S} \circledcirc \mathbb{S}^{\intercal n}\fatsemi z\fatsemi\mathbb{T}) = \mathtt{sup}_{n\geq 0}\ (\mathbb{S}^{\intercal n} \circledcirc \mathbb{S}^{\intercal n}\fatsemi z\fatsemi\mathbb{T}) \cap \mathbb{T}\fatsemi\mathbb{S} = \mathfrak{S} \cap \mathbb{T}\fatsemi\mathbb{S}$

---

i.e) $\hookrightarrow\!\mathfrak{S} = (\pi^{\intercal} \circledcirc \mathfrak{S}^{\intercal})\fatsemi\rho = (\pi^{\intercal} \circledcirc \big[\ \mathtt{sup}_{n\geq 0}\ (\mathbb{S}^{\intercal n} \circledcirc \mathbb{S}^{\intercal n}\fatsemi z\fatsemi\mathbb{T})\ \big]^{\intercal})\fatsemi\rho$
$= \mathtt{sup}_{n\geq 0}\ (\pi^{\intercal} \circledcirc (\mathbb{S}^{n} \circledcirc \mathbb{T}\fatsemi z^{\intercal}\fatsemi\mathbb{S}^{n}))\fatsemi\rho$
$= \mathtt{sup}_{n\geq 0}\ \big[\pi\fatsemi\pi^{\intercal} \cap \rho\fatsemi(\mathbb{S}^{n} \circledcirc \mathbb{T}\fatsemi z^{\intercal}\fatsemi\mathbb{S}^{n})\big]\fatsemi\rho$
$= \mathtt{sup}_{n\geq 0}\ \big[\pi\fatsemi\pi^{\intercal} \cap \rho\fatsemi(\mathbb{S}^{n}\fatsemi\pi^{\intercal} \cap \mathbb{T}\fatsemi z^{\intercal}\fatsemi\mathbb{S}^{n}\fatsemi\rho^{\intercal})\big]\fatsemi\rho$
$= \mathtt{sup}_{n\geq 0}\ \big[(\pi \cap \rho\fatsemi\mathbb{S}^{n})\fatsemi\pi^{\intercal} \cap \mathbb{T}\fatsemi z^{\intercal}\fatsemi\mathbb{S}^{n}\fatsemi\rho^{\intercal}\big]\fatsemi\rho$ since ρ is a map
$= \mathtt{sup}_{n\geq 0}\ (\pi \cap \rho\fatsemi\mathbb{S}^{n})\fatsemi\mathbb{T} \cap \mathbb{T}\fatsemi z^{\intercal}\fatsemi\mathbb{S}^{n}$ destroy and append
$= \mathtt{sup}_{n\geq 0}\ (\mathbb{I} \circledcirc \mathbb{S}^{n})\fatsemi\mathbb{T}\fatsemi z^{\intercal}\fatsemi\mathbb{S}^{n}$ masking
$= \mathbb{A}^{\hookleftarrow}$ Prop. 6.2.1.i.l

Variant proof with projection migration:

$$
\begin{aligned}
&{}^{\hookrightarrow}\mathfrak{S} = (\pi^\intercal \circledcirc \mathfrak{S}^\intercal)\fatsemi \rho = (\pi^\intercal \circledcirc \big[\, \mathtt{sup}_{n\geq 0}\ (\mathbb{S}^{\intercal n} \circledcirc \mathbb{S}^{\intercal n}\fatsemi z \fatsemi \mathbb{T})\ \big]^\intercal)\fatsemi \rho \\
&= \mathtt{sup}_{n\geq 0}\ (\pi^\intercal \circledcirc\ (\mathbb{S}^n \circledcirc \mathbb{T}\fatsemi z^\intercal\fatsemi \mathbb{S}^n)\ )\fatsemi \rho \\
&= \mathtt{sup}_{n\geq 0}\ (\ (\mathbb{I} \circledcirc \mathbb{S}^n)\ \circledcirc \rho\fatsemi \mathbb{T}\fatsemi z^\intercal\fatsemi \mathbb{S}^n)\fatsemi \rho \quad \text{projection migration Prop. 3.1.2.i; right to left} \\
&= \mathtt{sup}_{n\geq 0}\ (\mathbb{I} \circledcirc \mathbb{S}^n)\fatsemi \mathbb{T} \cap \mathbb{T}\fatsemi z^\intercal\fatsemi \mathbb{S}^n \quad \text{destroy and append; } \rho \text{ is total} \\
&= \mathtt{sup}_{n\geq 0}\ (\mathbb{I} \circledcirc \mathbb{S}^n)\fatsemi \mathbb{T}\fatsemi z^\intercal\fatsemi \mathbb{S}^n \quad \text{masking} \\
&= \mathbb{A}^{\hookleftarrow} \quad \text{Prop. 6.2.1.i.l}
\end{aligned}
$$

$$
\begin{aligned}
&\mathfrak{S}^{\hookleftarrow} = (\mathfrak{S}^\intercal \circledcirc \rho^\intercal)\fatsemi \pi = (\mathtt{sup}_{n\geq 0}\ (\mathbb{S}^n \circledcirc \mathbb{T}\fatsemi z^\intercal\fatsemi \mathbb{S}^n) \circledcirc \rho^\intercal)\fatsemi \pi \\
&= \mathtt{sup}_{n\geq 0}\ \big[\pi\fatsemi (\mathbb{S}^n \circledcirc \mathbb{T}\fatsemi z^\intercal\fatsemi \mathbb{S}^n) \cap \rho\fatsemi \rho^\intercal\big]\fatsemi \pi \\
&= \mathtt{sup}_{n\geq 0}\ \big[\pi\fatsemi (\mathbb{S}^n\fatsemi \pi^\intercal \cap \mathbb{T}\fatsemi z^\intercal\fatsemi \mathbb{S}^n\fatsemi \rho^\intercal) \cap\ \rho\fatsemi \rho^\intercal\big]\fatsemi \pi \\
&= \mathtt{sup}_{n\geq 0}\ \big[\pi\fatsemi \mathbb{S}^n\fatsemi \pi^\intercal \cap \mathbb{T}\fatsemi z^\intercal\fatsemi \mathbb{S}^n\fatsemi \rho^\intercal \cap\ \rho\fatsemi \rho^\intercal\big]\fatsemi \pi \quad \text{masking} \\
&= \mathtt{sup}_{n\geq 0}\ \big[\pi\fatsemi \mathbb{S}^n\fatsemi \pi^\intercal \cap (\mathbb{T}\fatsemi z^\intercal\fatsemi \mathbb{S}^n \cap\ \rho)\fatsemi \rho^\intercal\big]\fatsemi \pi \quad \text{because } \rho \text{ is univalent} \\
&= \mathtt{sup}_{n\geq 0}\ \pi\fatsemi \mathbb{S}^n \cap (\mathbb{T}\fatsemi z^\intercal\fatsemi \mathbb{S}^n \cap\ \rho)\fatsemi \mathbb{T} \quad \text{destroy and append} \\
&= \mathtt{sup}_{n\geq 0}\ \pi\fatsemi \mathbb{S}^n \cap \rho\fatsemi (\mathbb{T} \cap \mathbb{S}^{\intercal n}\fatsemi z\fatsemi \mathbb{T}) \quad \text{mask shifting} \\
&= \mathtt{sup}_{n\geq 0}\ (\mathbb{S}^n \circledcirc \mathbb{S}^{\intercal n}\fatsemi z\fatsemi \mathbb{T}) \\
&= B \quad \text{due to Prop. 6.2.1.i.a}
\end{aligned}
$$

Variant proof with projection migration:

$$
\begin{aligned}
&\mathfrak{S}^{\hookleftarrow} = (\mathfrak{S}^\intercal \circledcirc \rho^\intercal)\fatsemi \pi = (\mathtt{sup}_{n\geq 0}\ (\mathbb{S}^n \circledcirc \mathbb{T}\fatsemi z^\intercal\fatsemi \mathbb{S}^n) \circledcirc \rho^\intercal)\fatsemi \pi \\
&= \mathtt{sup}_{n\geq 0}\ (\pi\fatsemi \mathbb{S}^n \circledcirc\ (\mathbb{T}\fatsemi z^\intercal\fatsemi \mathbb{S}^n \circledcirc \mathbb{I})\ )\fatsemi \pi \quad \text{projection migration Prop. 3.1.2.ii} \\
&= \mathtt{sup}_{n\geq 0}\ \pi\fatsemi \mathbb{S}^n \cap\ (\mathbb{T}\fatsemi z^\intercal\fatsemi \mathbb{S}^n \circledcirc \mathbb{I})\fatsemi \mathbb{T} \quad \text{destroy and append} \\
&= \mathtt{sup}_{n\geq 0}\ \pi\fatsemi \mathbb{S}^n \cap\ (\mathbb{I} \circledcirc \mathbb{S}^{\intercal n}\fatsemi z\fatsemi \mathbb{T})\fatsemi \mathbb{T} \quad \text{due to Prop. 2.1.1.iii} \\
&= \mathtt{sup}_{n\geq 0}\ \pi\fatsemi \mathbb{S}^n \cap \rho\fatsemi \mathbb{S}^{\intercal n}\fatsemi z\fatsemi \mathbb{T} \quad \text{masking a join} \\
&= \mathtt{sup}_{n\geq 0}\ (\mathbb{S}^n \circledcirc \mathbb{S}^{\intercal n}\fatsemi z\fatsemi \mathbb{T}) \\
&= B \quad \text{due to Prop. 6.2.1.i.a}
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{S}^{\hookleftarrow} = \big[\, \mathtt{sup}_{n\geq 0}\ (\mathbb{S}^{\intercal n}\fatsemi z\fatsemi \mathbb{T} \circledcirc \mathbb{S}^{\intercal n})\ \big]^{\hookleftarrow} = (\big[\, \mathtt{sup}_{n\geq 0}\ (\mathbb{S}^{\intercal n}\fatsemi z\fatsemi \mathbb{T} \circledcirc \mathbb{S}^{\intercal n})\ \big]^\intercal \circledcirc \rho^\intercal)\fatsemi \pi \\
&= \mathtt{sup}_{n\geq 0}\ (\ (\mathbb{T}\fatsemi z^\intercal\fatsemi \mathbb{S}^n \circledcirc \mathbb{S}^n)\ \circledcirc \rho^\intercal)\fatsemi \pi \\
&= \mathtt{sup}_{n\geq 0}\ (\pi\fatsemi \mathbb{T}\fatsemi z^\intercal\fatsemi \mathbb{S}^n \circledcirc\ (\mathbb{S}^n \circledcirc \mathbb{I})\ )\fatsemi \pi \quad \text{projection migration Prop. 3.1.2.ii} \\
&= \mathtt{sup}_{n\geq 0}\ \mathbb{T}\fatsemi z^\intercal\fatsemi \mathbb{S}^n \cap\ (\mathbb{S}^n \circledcirc \mathbb{I})\fatsemi \mathbb{T} \quad \text{since } \pi \text{ is total; destroy and append} \\
&= \mathtt{sup}_{n\geq 0}\ (\mathbb{S}^n \circledcirc \mathbb{I})\fatsemi \mathbb{T}\fatsemi z^\intercal\fatsemi \mathbb{S}^n \quad \text{masking} \\
&= {}^{\hookrightarrow}B \quad \text{Prop. 6.2.1.i.l}
\end{aligned}
$$

$$
\begin{aligned}
&{}^{\hookrightarrow}\mathbb{S} = {}^{\hookrightarrow}\big[\, \mathtt{sup}_{n\geq 0}\ (\mathbb{S}^{\intercal n}\fatsemi z\fatsemi \mathbb{T} \circledcirc \mathbb{S}^{\intercal n})\ \big] = (\pi^\intercal \circledcirc \big[\, \mathtt{sup}_{n\geq 0}\ (\mathbb{S}^{\intercal n}\fatsemi z\fatsemi \mathbb{T} \circledcirc \mathbb{S}^{\intercal n})\ \big]^\intercal)\fatsemi \rho \\
&= \mathtt{sup}_{n\geq 0}\ (\pi^\intercal \circledcirc\ (\mathbb{T}\fatsemi z^\intercal\fatsemi \mathbb{S}^n \circledcirc \mathbb{S}^n)\ )\fatsemi \rho \\
&= \mathtt{sup}_{n\geq 0}\ (\pi^\intercal \circledcirc \mathbb{T}\fatsemi z^\intercal\fatsemi \mathbb{S}^n\fatsemi \pi^\intercal \cap \mathbb{S}^n\fatsemi \rho^\intercal)\fatsemi \rho \\
&= \mathtt{sup}_{n\geq 0}(\pi\fatsemi \pi^\intercal \cap \rho\fatsemi [\mathbb{T}\fatsemi z^\intercal\fatsemi \mathbb{S}^n\fatsemi \pi^\intercal \cap \mathbb{S}^n\fatsemi \rho^\intercal])\fatsemi \rho \\
&= \mathtt{sup}_{n\geq 0}(\pi\fatsemi \pi^\intercal \cap \mathbb{T}\fatsemi z^\intercal\fatsemi \mathbb{S}^n\fatsemi \pi^\intercal \cap \rho\fatsemi \mathbb{S}^n\fatsemi \rho^\intercal)\fatsemi \rho \quad \text{masking} \\
&= \mathtt{sup}_{n\geq 0}\ \big[(\pi \cap \mathbb{T}\fatsemi z^\intercal\fatsemi \mathbb{S}^n)\fatsemi \pi^\intercal \cap \rho\fatsemi \mathbb{S}^n\fatsemi \rho^\intercal\big]\fatsemi \rho \\
&= \mathtt{sup}_{n\geq 0}(\pi \cap \mathbb{T}\fatsemi z^\intercal\fatsemi \mathbb{S}^n)\fatsemi \mathbb{T} \cap \rho\fatsemi \mathbb{S}^n \quad \text{destroy and append} \\
&= \mathtt{sup}_{n\geq 0}\ \pi\fatsemi (\mathbb{T} \cap \mathbb{S}^{\intercal n}\fatsemi z\fatsemi \mathbb{T}) \cap \rho\fatsemi \mathbb{S}^n \quad \text{mask shifting} \\
&= \mathtt{sup}_{n\geq 0}\ \pi\fatsemi \mathbb{S}^{\intercal n}\fatsemi z\fatsemi \mathbb{T} \cap \rho\fatsemi \mathbb{S}^n \\
&= \mathtt{sup}_{n\geq 0}\ (\mathbb{S}^{\intercal n}\fatsemi z\fatsemi \mathbb{T} \circledcirc \mathbb{S}^n) \\
&= \mathbb{A}
\end{aligned}
$$

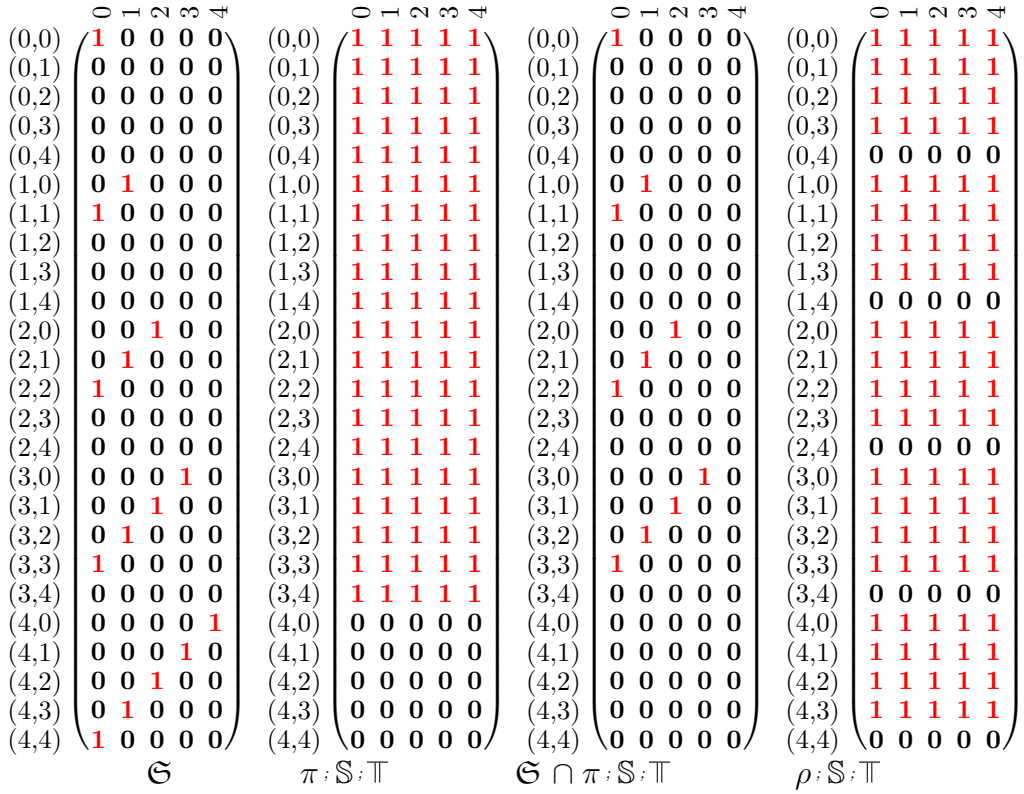


Fig. 6.3.5 Illustrating (i.d) and (ii.l): $(\mathbb{S} \otimes \mathbb{S})\fatsemi \mathfrak{S} = \mathfrak{S} \cap \pi \fatsemi \mathbb{S} \fatsemi \mathbb{T}$

Variant proof with projection migration:

$$
\begin{aligned}
&\hookrightarrow\!\mathtt{S} = {\hookrightarrow}\left[\,\mathtt{sup}_{n\geq 0}\ (\mathbb{S}^{\intercal n} \fatsemi z \fatsemi \mathbb{T} \oslash \mathbb{S}^{\intercal n})\,\right] = (\pi^\intercal \oslash \left[\,\mathtt{sup}_{n\geq 0}\ (\mathbb{S}^{\intercal n} \fatsemi z \fatsemi \mathbb{T} \oslash \mathbb{S}^{\intercal n})\,\right]^\intercal) \fatsemi \rho \\
&= \mathtt{sup}_{n\geq 0}\ (\pi^\intercal \oslash (\mathbb{T} \fatsemi z^\intercal \fatsemi \mathbb{S}^n \oslash \mathbb{S}^n)) \fatsemi \rho \\
&= \mathtt{sup}_{n\geq 0}\ ((\mathbb{I} \oslash \mathbb{T} \fatsemi z^\intercal \fatsemi \mathbb{S}^n) \oslash \rho \fatsemi \mathbb{S}^n) \fatsemi \rho \quad \text{projection migration Prop. 3.1.2.i reversed} \\
&= \mathtt{sup}_{n\geq 0}\ (\mathbb{I} \oslash \mathbb{T} \fatsemi z^\intercal \fatsemi \mathbb{S}^n) \fatsemi \mathbb{T} \cap \rho \fatsemi \mathbb{S}^n \quad \text{destroy and append} \\
&= \mathtt{sup}_{n\geq 0}\ (\mathbb{S}^{\intercal n} \fatsemi z \fatsemi \mathbb{T} \oslash \mathbb{I}) \fatsemi \mathbb{T} \cap \rho \fatsemi \mathbb{S}^n \quad \text{due to Prop. 2.1.1.iii} \\
&= \mathtt{sup}_{n\geq 0}\ \pi \fatsemi \mathbb{S}^{\intercal n} \fatsemi z \fatsemi \mathbb{T} \cap \rho \fatsemi \mathbb{S}^n \quad \text{masking} \\
&= \mathtt{sup}_{n\geq 0}\ (\mathbb{S}^{\intercal n} \fatsemi z \fatsemi \mathbb{T} \oslash \mathbb{S}^n) \\
&= \mathbb{A}
\end{aligned}
$$

$$
\begin{aligned}
&\pi^\intercal \fatsemi \mathfrak{S} = \pi^\intercal \fatsemi B^{\hookleftarrow} = \pi^\intercal \fatsemi (B^\intercal \oslash \rho^\intercal) \fatsemi \pi = \pi^\intercal \fatsemi \left[\pi \fatsemi B^\intercal \cap \rho \fatsemi \rho^\intercal\right] \fatsemi \pi \\
&= \left[B^\intercal \cap \mathbb{T}\right] \fatsemi \pi \quad \text{destroy and append; } \rho \text{ is total} \\
&= B^\intercal \fatsemi \pi \\
&\subseteq (\mathbb{S}^\intercal)^* \quad \text{Prop. 6.2.1.i.g}
\end{aligned}
$$

For $\rho^\intercal \fatsemi \mathtt{S}$ analogously.

$$
\begin{aligned}
&\rho^\intercal \fatsemi \mathfrak{S} = \rho^\intercal \fatsemi \mathtt{sup}_{n\geq 0}\ (\mathbb{S}^{\intercal n} \oslash \mathbb{S}^{\intercal n} \fatsemi z \fatsemi \mathbb{T}) = \mathtt{sup}_{n\geq 0}\, \rho^\intercal \fatsemi (\mathbb{S}^{\intercal n} \oslash \mathbb{S}^{\intercal n} \fatsemi z \fatsemi \mathbb{T}) \\
&= \mathtt{sup}_{n\geq 0}\, \mathbb{T} \fatsemi \mathbb{S}^{\intercal n} \cap \mathbb{S}^{\intercal n} \fatsemi z \fatsemi \mathbb{T} \quad \text{destroy and append} \\
&= \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\intercal n} \fatsemi z \fatsemi \mathbb{T} \fatsemi \mathbb{S}^{\intercal n} \quad \text{masking} \\
&= \mathbb{V} \quad \text{using Prop. 5.3.2.i.a}
\end{aligned}
$$

For $\pi^\intercal \fatsemi \mathtt{S}$ analogously.

---

i.f) $\mathbb{X}^\intercal \fatsemi \mathbb{X} = (\mathfrak{S}^\intercal \cup \mathfrak{S}^\intercal \fatsemi \mathfrak{P}) \fatsemi (\mathfrak{S} \cup \mathfrak{P} \fatsemi \mathfrak{S}) = \mathfrak{S}^\intercal \fatsemi \mathfrak{S} \cup \mathfrak{S}^\intercal \fatsemi \mathfrak{P} \fatsemi \mathfrak{S}$

Of this, the first term is certainly $\subseteq \mathbb{I}$ since $\mathfrak{S}$ is univalent due to (i.b). We also investigate the second:

$\mathfrak{S}^\intercal ; \mathfrak{P} ; \mathfrak{S} = \mathfrak{S}^\intercal ; \mathfrak{S}$ according to (i.b)
$= \big[\mathtt{sup}_{n\geq 0} \, (\mathbb{S}^n \between \mathbb{T} ; z^\intercal ; \mathbb{S}^n) \big] ; \big[\mathtt{sup}_{k\geq 0} \, (\mathbb{S}^{\intercal k} ; z ; \mathbb{T} \rightthreetimes \mathbb{S}^{\intercal k}) \big]$
$= \mathtt{sup}_{n\geq 0} \, \mathtt{sup}_{k\geq 0} \, \mathbb{S}^n ; \mathbb{S}^{\intercal k} ; z ; \mathbb{T} \cap \mathbb{T} ; z^\intercal ; \mathbb{S}^n ; \mathbb{S}^{\intercal k}$
$= \mathtt{sup}_{n\geq 0} \, \mathbb{S}^n ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \mathbb{T} ; z^\intercal ; \mathbb{S}^n ; \mathbb{S}^{\intercal n}$ since $n > k$ as well as $n < k$ imply $\ldots = \bot\!\!\!\bot$
$\subseteq \mathtt{sup}_{n\geq 0} \, z ; \mathbb{T} \cap \mathbb{T} ; z^\intercal = z ; \mathbb{T} \cap \mathbb{T} ; z^\intercal = z ; \mathbb{T} ; z^\intercal \subseteq \mathbb{I}$

---

i.g) $\mathfrak{S} ; \mathbb{T} = \mathtt{sup}_{n\geq 0} \, (\mathbb{S}^{\intercal n} \rightthreetimes \mathbb{S}^{\intercal n} ; z ; \mathbb{T}) ; \mathbb{T}$
$= \mathtt{sup}_{n\geq 0} \, (\mathbb{S}^{\intercal n} ; \mathbb{T} \rightthreetimes \mathbb{S}^{\intercal n} ; z ; \mathbb{T})$ masking
$= \mathtt{sup}_{n\geq 0} \, \pi ; \mathbb{S}^{\intercal n} ; \mathbb{T} \cap \rho ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T}$
$= \mathtt{sup}_{n\geq 0} (\pi \cap \rho ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T}) ; \mathbb{S}^{\intercal n} ; \mathbb{T}$ masking
$= \mathtt{sup}_{n\geq 0} (\pi \cap \rho ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T}) ; (\mathbb{T} \cap \mathbb{S}^{\intercal n} ; \mathbb{T})$
$= \mathtt{sup}_{n\geq 0} (\pi \cap \rho ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \mathbb{T} ; \mathbb{S}^n) ; \mathbb{T}$ mask shifting
$= \mathtt{sup}_{n\geq 0} (\pi \cap \rho ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; \mathbb{S}^n) ; \mathbb{T}$ masking
$= (\pi \cap \rho ; \mathtt{sup}_{n\geq 0} \, \mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; \mathbb{S}^n) ; \mathbb{T}$
$= (\pi \cap \rho ; \mathbb{V}) ; \mathbb{T}$ following Prop. 5.3.1.i.a
$= (\mathbb{I} \rightthreetimes \mathbb{V}) ; \mathbb{T}$

---

i.h) $\mathfrak{P} ; \mathbb{X} = \mathfrak{P} ; (\mathbb{I} \cup \mathfrak{P}) ; \mathfrak{S} = (\mathfrak{P} \cup \mathbb{I}) ; \mathfrak{S} = \mathbb{X}$

---

i.i) For $\mathbb{S}^{\intercal l} ; z ; \mathbb{T} = \bot\!\!\!\bot$, this is obviously satisfied. In case $\neq \bot\!\!\!\bot$ we have a point $\mathbb{S}^{\intercal l} ; z ; \mathbb{T}$:
$\mathtt{rel}(\mathfrak{S} ; \mathbb{S}^{\intercal l} ; z ; \mathbb{T}) = \mathtt{rel}(\big[\mathtt{sup}_{n\geq 0} \, (\mathbb{S}^{\intercal n} \rightthreetimes \mathbb{S}^{\intercal n} ; z ; \mathbb{T}) \big] ; \mathbb{S}^{\intercal l} ; z ; \mathbb{T})$
$= \mathtt{sup}_{n\geq 0} \, \mathtt{rel}((\mathbb{S}^{\intercal n} \rightthreetimes \mathbb{S}^{\intercal n} ; z ; \mathbb{T}) ; \mathbb{S}^{\intercal l} ; z ; \mathbb{T})$
$= \mathtt{sup}_{n\geq 0} \, \mathtt{rel}((\mathbb{S}^{\intercal n} ; \mathbb{S}^{\intercal l} ; z ; \mathbb{T} \rightthreetimes \mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; \mathbb{S}^{\intercal l} ; z ; \mathbb{T}))$ since $\mathbb{S}^{\intercal l} ; z ; \mathbb{T}$ is a point
$= \mathtt{sup}_{n\geq 0} \, \mathtt{rel}((\mathbb{S}^{\intercal n+l} ; z ; \mathbb{T} \rightthreetimes \mathbb{S}^{\intercal n} ; z ; \mathbb{T}))$ again since $\mathbb{S}^{\intercal l} ; z ; \mathbb{T}$ is a point
$= \mathtt{sup}_{n\geq 0} \, \pi^\intercal ; \big[(\mathbb{S}^{\intercal n+l} ; z ; \mathbb{T} \rightthreetimes \mathbb{S}^{\intercal n} ; z ; \mathbb{T}) \cap \rho \big]$ definition of $\mathtt{rel}$ in Def. 2.2.5
$= \mathtt{sup}_{n\geq 0} \, \pi^\intercal ; \big[\pi ; \mathbb{S}^{\intercal n+l} ; z ; \mathbb{T} \cap \rho ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \rho \big]$
$= \mathtt{sup}_{n\geq 0} \, \pi^\intercal ; \big[\pi ; \mathbb{S}^{\intercal n+l} ; z ; \mathbb{T} \cap \rho ; (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \mathbb{I}) \big]$
$= \mathtt{sup}_{n\geq 0} \, \mathbb{S}^{\intercal n+l} ; z ; \mathbb{T} \cap \mathbb{T} ; (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \mathbb{I})$ destroy and append
$= \mathtt{sup}_{n\geq 0} \, \mathbb{S}^{\intercal n+l} ; z ; \mathbb{T} \cap (\mathbb{T} \cap \mathbb{T} ; z^\intercal ; \mathbb{S}^n) ; \mathbb{I}$ mask shifting
$= \mathtt{sup}_{n\geq 0} \, \mathbb{S}^{\intercal n+l} ; z ; \mathbb{T} \cap \mathbb{T} ; z^\intercal ; \mathbb{S}^n$
$= \mathtt{sup}_{n\geq 0} \, \mathbb{S}^{\intercal n+l} ; z ; \mathbb{T} ; \mathbb{S}^n$ masking

---

i.j) $(z ; \mathbb{T} ; z^\intercal \otimes z ; \mathbb{T} ; z^\intercal) ; \mathfrak{S} = (z ; \mathbb{T} ; z^\intercal \otimes z ; \mathbb{T} ; z^\intercal) ; \mathtt{sup}_{n\geq 0} \, (\mathbb{S}^{\intercal n} \rightthreetimes \mathbb{S}^{\intercal n} ; z ; \mathbb{T})$
$= \mathtt{sup}_{n\geq 0} \, (z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{\intercal n} \rightthreetimes z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T})$ certainly $= \bot\!\!\!\bot$ in case $n > 0$
$= (z ; \mathbb{T} ; z^\intercal \rightthreetimes z ; \mathbb{T} ; z^\intercal ; z ; \mathbb{T}) = (z ; \mathbb{T} ; z^\intercal \rightthreetimes z ; \mathbb{T})$

$(z ; \mathbb{T} ; z^\intercal \otimes z ; \mathbb{T} ; z^\intercal) ; \rho = (z ; \mathbb{T} ; z^\intercal ; \mathbb{T} \rightthreetimes z ; \mathbb{T} ; z^\intercal) = (z ; \mathbb{T} \rightthreetimes z ; \mathbb{T} ; z^\intercal)$

$(z ; \mathbb{T} ; z^\intercal \otimes z ; \mathbb{T} ; z^\intercal) ; (\mathfrak{S} \between \rho) = ((z ; \mathbb{T} ; z^\intercal \otimes z ; \mathbb{T} ; z^\intercal) ; \mathfrak{S} \between (z ; \mathbb{T} ; z^\intercal \otimes z ; \mathbb{T} ; z^\intercal) ; \rho)$
$= ((z ; \mathbb{T} ; z^\intercal \rightthreetimes z ; \mathbb{T}) \between (z ; \mathbb{T} \rightthreetimes z ; \mathbb{T} ; z^\intercal))$ see earlier
$= (\pi ; z ; \mathbb{T} ; z^\intercal \cap \rho ; z ; \mathbb{T} \between \pi ; z ; \mathbb{T} \cap \rho ; z ; \mathbb{T} ; z^\intercal)$

$= \pi ; z ; \mathbb{T} ; z^\intercal ; \pi^\intercal \cap \rho ; z ; \mathbb{T} ; \pi^\intercal \cap \pi ; z ; \mathbb{T} ; \rho^\intercal \cap \rho ; z ; \mathbb{T} ; z^\intercal ; \rho^\intercal$
$= \pi ; z ; \mathbb{T} ; z^\intercal ; \pi^\intercal \cap \rho ; z ; \mathbb{T} \cap \pi ; z ; \mathbb{T} \cap \rho ; z ; \mathbb{T} ; z^\intercal ; \rho^\intercal$
$= \pi ; z ; \mathbb{T} ; z^\intercal ; \pi^\intercal \cap \rho ; z ; \mathbb{T} ; z^\intercal ; \rho^\intercal$
$= (z ; \mathbb{T} ; z^\intercal \otimes z ; \mathbb{T} ; z^\intercal)$

---

ii.a) Due to Prop. 5.2.1.ii.a, now $V = \mathbb{T}$ in (i.d).

---

ii.b) Remembering Prop. 5.2.1.ii.b, this follows immediately from (i.d).

---

ii.c) We start proving as a tricky auxiliary result that $(\mathbb{I} \otimes \mathbb{I} \cap \overline{\mathbb{S} ; \mathbb{T}}) ; \mathfrak{S} ; \mathbb{S}^\intercal = \bot\!\!\!\bot$. This will certainly hold in case $\mathbb{S}$ should be total. So we focus on the case with $\mathbb{S}$ not total; meaning that there exists according to Prop. 5.2.1.iii.a the point $g_{\mathbb{S}} ; \mathbb{T} := \overline{\mathbb{S} ; \mathbb{T}}$. Then

$(\mathbb{I} \otimes \mathbb{I} \cap \overline{\mathbb{S} ; \mathbb{T}}) ; \mathfrak{S} ; \mathbb{S}^\intercal = (\mathbb{I} \otimes g_{\mathbb{S}} ; g_{\mathbb{S}}^\intercal) ; \mathfrak{S} ; \mathbb{S}^\intercal = \mathtt{sup}_{n \geq 0} \, (\mathbb{I} \otimes g_{\mathbb{S}} ; g_{\mathbb{S}}^\intercal) ; (\mathbb{S}^{\intercal n} \oslash \mathbb{S}^{\intercal n} ; z ; \mathbb{T}) ; \mathbb{S}^\intercal$
$= \mathtt{sup}_{n \geq 0} \, (\mathbb{S}^{\intercal n} \oslash g_{\mathbb{S}} ; g_{\mathbb{S}}^\intercal ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T}) ; \mathbb{S}^\intercal$ implying $n = n_{\mathbb{S}}$ to be $\neq \bot\!\!\!\bot$
$= (\mathbb{S}^{\intercal n_{\mathbb{S}}} \oslash g_{\mathbb{S}} ; g_{\mathbb{S}}^\intercal ; \mathbb{S}^{\intercal n_{\mathbb{S}}} ; z ; \mathbb{T}) ; \mathbb{S}^\intercal = (\mathbb{S}^{\intercal n_{\mathbb{S}}} \oslash g_{\mathbb{S}} ; \mathbb{T}) ; \mathbb{S}^\intercal$
$= (\mathbb{S}^{\intercal n_{\mathbb{S}} + 1} \oslash g_{\mathbb{S}} ; \mathbb{T})$ masking a join
$= (\bot\!\!\!\bot \oslash g_{\mathbb{S}} ; \mathbb{T}) = \bot\!\!\!\bot$

Having established this in advance, we may proceed:

$(\mathbb{I} \otimes \mathbb{S}) ; \mathfrak{S} = \mathtt{sup}_{n \geq 0} \, (\mathbb{S}^{\intercal n} \oslash \mathbb{S} ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T})$
$= \bot\!\!\!\bot \cup \mathtt{sup}_{n \geq 1} \, (\mathbb{S}^{\intercal n-1} ; \mathbb{S}^\intercal \oslash \mathbb{S} ; \mathbb{S}^\intercal ; \mathbb{S}^{\intercal n-1} ; z ; \mathbb{T})$ since $\mathbb{S} ; z = \bot\!\!\!\bot$
$= \mathtt{sup}_{n \geq 1} \, (\mathbb{S}^{\intercal n-1} \oslash \mathbb{S} ; \mathbb{S}^\intercal ; \mathbb{S}^{\intercal n-1} ; z ; \mathbb{T}) ; \mathbb{S}^\intercal$ masking a join
$= (\mathbb{I} \otimes \mathbb{S} ; \mathbb{S}^\intercal) ; \mathtt{sup}_{n \geq 1} \, (\mathbb{S}^{\intercal n-1} \oslash \mathbb{S}^{\intercal n-1} ; z ; \mathbb{T}) ; \mathbb{S}^\intercal$
$= (\mathbb{I} \otimes \mathbb{S} ; \mathbb{S}^\intercal) ; \mathtt{sup}_{k \geq 0} \, (\mathbb{S}^{\intercal k} \oslash \mathbb{S}^{\intercal k} ; z ; \mathbb{T}) ; \mathbb{S}^\intercal$
$= (\mathbb{I} \otimes \mathbb{S} ; \mathbb{S}^\intercal) ; \mathfrak{S} ; \mathbb{S}^\intercal$
$= (\mathbb{I} \otimes \mathbb{I} \cap \mathbb{S} ; \mathbb{T}) ; \mathfrak{S} ; \mathbb{S}^\intercal$ due to Prop. 5.2.1.i.b
$= (\mathbb{I} \otimes \mathbb{I} \cap \mathbb{S} ; \mathbb{T}) ; \mathfrak{S} ; \mathbb{S}^\intercal \cup \bot\!\!\!\bot = (\mathbb{I} \otimes \mathbb{I} \cap \mathbb{S} ; \mathbb{T}) ; \mathfrak{S} ; \mathbb{S}^\intercal \cup (\mathbb{I} \otimes \mathbb{I} \cap \overline{\mathbb{S} ; \mathbb{T}}) ; \mathfrak{S} ; \mathbb{S}^\intercal$ see above
$= (\mathbb{I} \otimes \mathbb{I}) ; \mathfrak{S} ; \mathbb{S}^\intercal$
$= \mathbb{I} ; \mathfrak{S} ; \mathbb{S}^\intercal = \mathfrak{S} ; \mathbb{S}^\intercal$

Concerning $\mathtt{S}$, everything works similarly.

$(\mathbb{S}^{+\intercal} \oslash \mathbb{I}) ; \mathfrak{S} = (\mathbb{S}^{+\intercal} \oslash \mathbb{I}) ; \mathtt{sup}_{n \geq 0} \, (\mathbb{S}^{\intercal n} \oslash \mathbb{S}^{\intercal n} ; z ; \mathbb{T})$
$= \mathtt{sup}_{n \geq 0} \, (\mathbb{S}^{+\intercal} \oslash \mathbb{I}) ; (\mathbb{S}^{\intercal n} \oslash \mathbb{S}^{\intercal n} ; z ; \mathbb{T}) \subseteq \mathtt{sup}_{n \geq 0} \, \mathbb{S}^{+\intercal} ; \mathbb{S}^{\intercal n} \cap \mathbb{S}^{\intercal n} ; z ; \mathbb{T}$
$= \mathtt{sup}_{n \geq 0} \, \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \mathbb{S}^{\intercal n} ; \mathbb{S}^{+\intercal}$ from here with the Dedekind formula
$\subseteq \mathtt{sup}_{n \geq 0} (\mathbb{S}^{\intercal n} \cap \ldots) ; (z ; \mathbb{T} \cap \mathbb{S}^n ; \mathbb{S}^{\intercal n} ; \mathbb{S}^{+\intercal}) \subseteq \mathtt{sup}_{n \geq 0} (\ldots) ; (z ; \mathbb{T} \cap \mathbb{S}^{+\intercal}) = \bot\!\!\!\bot$

---

ii.d) From (i.g) we have $\mathfrak{S} ; \mathbb{T} = (\mathbb{I} \oslash \mathbb{V}) ; \mathbb{T}$. Herein holds $\mathbb{V} = \mathbb{S}^*$ according to Prop. 5.3.1.ii.a. The second variant follows via Prop. 2.1.1.iii. Furthermore

$(\mathfrak{S} \oslash \rho) ; \mathbb{T} = (\mathfrak{S} ; \pi^\intercal \cap \rho ; \rho^\intercal) ; \mathbb{T} = (\mathfrak{S} ; \pi^\intercal \cap \rho ; \rho^\intercal) ; \rho ; \mathbb{T} = (\mathfrak{S} ; \pi^\intercal ; \rho \cap \rho) ; \mathbb{T}$
$= (\mathfrak{S} ; \mathbb{T} \cap \rho) ; \mathbb{T} = \rho ; \mathbb{T} \cap \mathfrak{S} ; \mathbb{T} = \mathfrak{S} ; \mathbb{T}$ with ρ total

---

ii.e) We start from (i.e) in combination with Prop. 6.2.1.i.l and prove

$\hookrightarrow \mathfrak{S} = \mathtt{sup}_{n \geq 0} \, (\mathbb{I} \oslash \mathbb{S}^n) ; \mathbb{T} ; z^\intercal ; \mathbb{S}^n \subseteq \mathtt{sup}_{k \geq 0} \, (\mathbb{S}^{\intercal k} \oslash \mathbb{S}^{\intercal k} ; z ; \mathbb{T}) = \mathfrak{S}.$

For this, it suffices to show that for all $n \geq 0$

$(\mathbb{I} \circledcirc \mathbb{S}^n) ; \mathbb{T} ; z^\intercal ; \mathbb{S}^n \subseteq \mathtt{sup}_{k\geq 0} \, (\mathbb{S}^{\intercal k} \circledcirc \mathbb{S}^{\intercal k} ; z ; \mathbb{T})$.

Either $\mathbb{S}^{\intercal n} ; z ; \mathbb{T} = \mathbb{\perp}$, then it will obviously hold, or it is a point that may be shunted so as to obtain

$(\mathbb{I} \circledcirc \mathbb{S}^n) \subseteq \mathtt{sup}_{k\geq 0} \, (\mathbb{S}^{\intercal k} \circledcirc \mathbb{S}^{\intercal k} ; z ; \mathbb{T}) ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T}$
$= \mathtt{sup}_{k\geq 0} \, (\mathbb{S}^{\intercal k} ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \circledcirc \mathbb{S}^{\intercal k} ; z ; \mathbb{T} ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T})$ injectivity of a point
$= \mathtt{sup}_{k\geq 0} \, (\mathbb{S}^{\intercal n} ; \mathbb{S}^{\intercal k} ; z ; \mathbb{T} \circledcirc \mathbb{S}^{\intercal k} ; z ; \mathbb{T})$ surjectivity of a point
$= \mathtt{sup}_{k\geq 0} \, (\mathbb{S}^{\intercal n} \circledcirc \mathbb{I}) ; \mathbb{S}^{\intercal k} ; z ; \mathbb{T}$
$= (\mathbb{S}^{\intercal n} \circledcirc \mathbb{I}) ; \mathtt{sup}_{k\geq 0} \, \mathbb{S}^{\intercal k} ; z ; \mathbb{T}$
$= (\mathbb{S}^{\intercal n} \circledcirc \mathbb{I}) ; \mathbb{T}$ Prop. 5.2.1.ii.a
$= (\mathbb{I} \circledcirc \mathbb{S}^n) ; \mathbb{T}$ due to Prop. 2.1.1.iii

which holds indeed.

Following Prop. 3.2.3.iv, we then have also equality because

$\hookrightarrow \mathfrak{S} \subseteq \mathfrak{S} \quad \Longleftrightarrow \quad \hookrightarrow (\hookrightarrow \mathfrak{S}) = \mathfrak{S} \subseteq \hookrightarrow \mathfrak{S}$.

From (i.e) together with Prop. 6.2.1.ii.a, we get $\mathfrak{S}^{\hookleftarrow} = \mathbb{A}$.

$\mathbb{S} ; \mathbb{S}^{\intercal i} ; z ; \mathbb{T} = \hookrightarrow \mathbb{A} ; \mathbb{S}^{\intercal i} ; z ; \mathbb{T} = \hookrightarrow B ; \mathbb{S}^{\intercal i} ; z ; \mathbb{T}$ due to (i.e) together with Prop. 6.2.1.ii.a
$= \mathtt{sup}_{k\geq 0} \, (\mathbb{S}^k \circledcirc \mathbb{I}) ; \mathbb{T} ; z^\intercal ; \mathbb{S}^k ; \mathbb{S}^{\intercal i} ; z ; \mathbb{T}$ Prop. 6.2.1.i.l and using $B = \mathbb{A}$
$= (\mathbb{S}^i \circledcirc \mathbb{I}) ; \mathbb{T}$ implies $k := i$; whenever $k \neq i$ this will vanish
$= \mathtt{vec}(\mathbb{S}^i)$ Prop. 2.2.5

The second variant follows analogously.

---

ii.f) With (i.f), it has already been shown that $\mathbb{X}$ is univalent. It is now also total:

$\mathbb{X} ; \mathbb{T} = \big[\mathfrak{S} \cup \mathbb{S}\big] ; \mathbb{T} = \mathfrak{S} ; \mathbb{T} \cup \mathbb{S} ; \mathbb{T}$
$= (\mathbb{I} \circledcirc \mathbb{V}) ; \mathbb{T} \cup (\mathbb{V} \circledcirc \mathbb{I}) ; \mathbb{T}$ due to (i.g)
$= (\mathbb{I} \circledcirc \mathbb{V}) ; \mathbb{T} \cup (\mathbb{I} \circledcirc \mathbb{V}^\intercal) ; \mathbb{T}$ Prop. 2.1.1.ii
$= (\mathbb{I} \circledcirc \mathbb{V} \cup \mathbb{V}^\intercal) ; \mathbb{T}$
$= (\mathbb{I} \circledcirc \mathbb{S}^* \cup \mathbb{S}^{\intercal *}) ; \mathbb{T}$ Prop. 5.3.1.ii.a
$= (\mathbb{I} \circledcirc \mathbb{T}) ; \mathbb{T} = \mathbb{T}$

---

ii.g) $\mathtt{sup}_{n\geq 0} \, \mathtt{sup}_{i\geq 0} \, (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \circledcirc \mathbb{S}^{\intercal i} ; z ; \mathbb{T}) ; \mathbb{T} ; z^\intercal ; \mathbb{S}^n ; \mathbb{S}^{\intercal i}$
$= \mathtt{sup}_{i\geq 0} \, \mathtt{sup}_{n\geq 0} \, \big[\pi ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \rho ; \mathbb{S}^{\intercal i} ; z ; \mathbb{T}\big] ; \mathbb{T} ; z^\intercal ; \mathbb{S}^n ; \mathbb{S}^{\intercal i}$
$= \mathtt{sup}_{i\geq 0} \, \mathtt{sup}_{n\geq 0} \, \pi ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^n ; \mathbb{S}^{\intercal i} \cap \rho ; \mathbb{S}^{\intercal i} ; z ; \mathbb{T}$ masking
$= \mathtt{sup}_{i\geq 0} \, \pi ; \big[\mathtt{sup}_{n\geq 0} \, \mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^n\big] ; \mathbb{S}^{\intercal i} \cap \rho ; \mathbb{S}^{\intercal i} ; z ; \mathbb{T}$
$= \mathtt{sup}_{i\geq 0} \, \pi ; \mathbb{I} ; \mathbb{S}^{\intercal i} \cap \rho ; \mathbb{S}^{\intercal i} ; z ; \mathbb{T} = \mathtt{sup}_{i\geq 0} \, (\mathbb{S}^{\intercal i} \circledcirc \mathbb{S}^{\intercal i} ; z ; \mathbb{T}) = \mathfrak{S}$

---

ii.h) $\mathfrak{S}^\intercal ; (\mathbb{S}^q ; \mathbb{S}^{\intercal r} ; z ; \mathbb{T} \circledcirc \mathbb{S}^{\intercal s} ; z ; \mathbb{T})$
$= \mathtt{sup}_{m\geq 0} \, (\mathbb{S}^m \circledcirc \mathbb{T} ; z^\intercal ; \mathbb{S}^m) ; (\mathbb{S}^q ; \mathbb{S}^{\intercal r} ; z ; \mathbb{T} \circledcirc \mathbb{S}^{\intercal s} ; z ; \mathbb{T})$ expanded
$= \mathtt{sup}_{m\geq 0} \, \mathbb{S}^m ; \mathbb{S}^q ; \mathbb{S}^{\intercal r} ; z ; \mathbb{T} \cap \mathbb{T} ; z^\intercal ; \mathbb{S}^m ; \mathbb{S}^{\intercal s} ; z ; \mathbb{T}$ where the terms will be $\mathbb{\perp}$ if $m \neq s$
$= \mathbb{S}^{s+q} ; \mathbb{S}^{\intercal r} ; z ; \mathbb{T}$

A remark seems necessary here concerning the highly general quantification $q, r, s \geq 0$: If $q > r$, there will be $\mathbb{S}^q ; \mathbb{S}^{\intercal r} ; z ; \mathbb{T} = \mathbb{\perp} = \mathbb{S}^{s+q} ; \mathbb{S}^{\intercal r} ; z ; \mathbb{T}$. It is directly visible that $\mathbb{S}^{\intercal r} ; z ; \mathbb{T} = \mathbb{\perp}$ would

annihilate both sides. Should be $\mathbb{S}^{\intercal s};z;\mathbb{T}=\mathbb{L}$, this will precisely happen when $\mathbb{S}$ is not total and $s>n_{\mathbb{S}}$; then, however, also $\mathbb{S}^{s+q}=\mathbb{L}$.

---

ii.i) $\mathfrak{S}^{\intercal};(\mathbb{S}^{\intercal n};z;\mathbb{T}\oslash\mathbb{S}^{\intercal i};z;\mathbb{T})$
$=\big[\,\mathtt{sup}_{a\geq 0}\,\mathtt{sup}_{b\geq 0}\,(\mathbb{S}^{\intercal a};z;\mathbb{T}\oslash\mathbb{S}^{\intercal b};z;\mathbb{T});\mathbb{T};z^{\intercal};\mathbb{S}^{a};\mathbb{S}^{\intercal b}\big]^{\intercal};(\mathbb{S}^{\intercal n};z;\mathbb{T}\oslash\mathbb{S}^{\intercal i};z;\mathbb{T})$ due to (ii.g)
$=\mathtt{sup}_{a\geq 0}\,\mathtt{sup}_{b\geq 0}\,\mathbb{S}^{b};\mathbb{S}^{\intercal a};z;\mathbb{T};(\mathbb{T};z^{\intercal};\mathbb{S}^{a}\oslash\mathbb{T};z^{\intercal};\mathbb{S}^{b});(\mathbb{S}^{\intercal n};z;\mathbb{T}\oslash\mathbb{S}^{\intercal i};z;\mathbb{T})$
$=\mathtt{sup}_{a\geq 0}\,\mathtt{sup}_{b\geq 0}\,\mathbb{S}^{b};\mathbb{S}^{\intercal a};z;\mathbb{T};\big[\mathbb{T};z^{\intercal};\mathbb{S}^{a};\mathbb{S}^{\intercal n};z;\mathbb{T}\cap\mathbb{T};z^{\intercal};\mathbb{S}^{b};\mathbb{S}^{\intercal i};z;\mathbb{T}\big]$ implying $a:=n,b:=i$
$=\mathbb{S}^{i};\mathbb{S}^{\intercal n};z;\mathbb{T};\big[\mathbb{T};z^{\intercal};\mathbb{S}^{n};\mathbb{S}^{\intercal n};z;\mathbb{T}\cap\mathbb{T};z^{\intercal};\mathbb{S}^{i};\mathbb{S}^{\intercal i};z;\mathbb{T}\big]$
$=\mathbb{S}^{i};\mathbb{S}^{\intercal n};z;\mathbb{T}$

The version with variants is now more or less evident.

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ii.j) $\pi^{\intercal};\mathfrak{S}=B^{\intercal};\pi$ shown during the proof of (i.e)
$=\mathbb{A}^{\intercal};\pi$ Prop. 6.2.1.ii.a
$=\mathbb{S}^{\intercal *}$ Prop. 6.2.1.ii.c

---

ii.k) $(\mathbb{S}\otimes\mathbb{I});\mathfrak{S}\cap\mathbb{T};\mathbb{S}=(\mathbb{S}\otimes\mathbb{I});\mathtt{sup}_{n\geq 0}\,(\mathbb{S}^{\intercal n}\oslash\mathbb{S}^{\intercal n};z;\mathbb{T})\cap\mathbb{T};\mathbb{S}$
$=\mathtt{sup}_{n\geq 0}\,(\mathbb{S};\mathbb{S}^{\intercal n}\oslash\mathbb{S}^{\intercal n};z;\mathbb{T})\cap\mathbb{T};\mathbb{S}$
$=\mathtt{sup}_{n\geq 0}\,\pi;\mathbb{S};\mathbb{S}^{\intercal n}\cap\rho;\mathbb{S}^{\intercal n};z;\mathbb{T}\cap\mathbb{T};\mathbb{S}=\mathtt{sup}_{n\geq 0}\,\pi;[\mathbb{S};\mathbb{S}^{\intercal n}\cap\mathbb{T};\mathbb{S}]\cap\rho;\mathbb{S}^{\intercal n};z;\mathbb{T};\mathbb{S}$
$=\mathtt{sup}_{n>0}\,\pi;[\mathbb{S}^{\intercal n};\mathbb{S}\cap\mathbb{S};\mathbb{T}]\cap\rho;\mathbb{S}^{\intercal n};z;\mathbb{T};\mathbb{S}$ Prop. 5.2.1.i.b
$=\mathtt{sup}_{n\geq 0}\,\pi;[\mathbb{S}^{\intercal n};\mathbb{S}\cap\mathbb{S};\mathbb{T}]\cap\rho;\mathbb{S}^{\intercal n};z;\mathbb{T};\mathbb{S}$
$=\mathtt{sup}_{n\geq 0}\,\pi;\mathbb{S}^{\intercal n};\mathbb{S}\cap\rho;\mathbb{S}^{\intercal n};z;\mathbb{T};\mathbb{S}\cap\pi;\mathbb{S};\mathbb{T}$ shuffled; π is univalent
$=\big[\,\mathtt{sup}_{n\geq 0}\,\pi;\mathbb{S}^{\intercal n}\cap\rho;\mathbb{S}^{\intercal n};z;\mathbb{T}\big];\mathbb{S}\cap\pi;\mathbb{S};\mathbb{T}$
$=\big[\,\mathtt{sup}_{n\geq 0}\,(\mathbb{S}^{\intercal n}\oslash\mathbb{S}^{\intercal n};z;\mathbb{T})\,\big];\mathbb{S}\cap\pi;\mathbb{S};\mathbb{T}$
$=\mathfrak{S};\mathbb{S}\cap\pi;\mathbb{S};\mathbb{T}$

---

ii.l) From (i.d) we have already an equation as a result, that will now be sharpened to:
$(\mathbb{S}\otimes\mathbb{S});\mathfrak{S}=\mathfrak{S}\cap\pi;\mathbb{S};\mathbb{T}\cap\rho;\mathbb{S};\mathbb{T}=\mathfrak{S}\cap\pi;\mathbb{S};\mathbb{T}$

We are, namely, able to prove that the intersection of the first two terms is contained in the last. While in view of Fig. 6.3.5 this is immediate, the formal proof is rather demanding:

$\mathfrak{S}\cap\pi;\mathbb{S};\mathbb{T}=\mathtt{sup}_{n\geq 0}\,(\mathbb{S}^{\intercal n}\oslash\mathbb{S}^{\intercal n};z;\mathbb{T})\cap\pi;\mathbb{S};\mathbb{T}\subseteq\rho;\mathbb{S};\mathbb{T}$
$\iff\quad(\mathbb{S}^{\intercal n}\oslash\mathbb{S}^{\intercal n};z;\mathbb{T})\cap\pi;\mathbb{S};\mathbb{T}\subseteq\rho;\mathbb{S};\mathbb{T}$ for all $n\geq 0$
$\iff\quad\rho^{\intercal};\big[\rho;\mathbb{S}^{\intercal n};z;\mathbb{T}\cap\pi;(\mathbb{S}^{\intercal n}\cap\mathbb{S};\mathbb{T})\big]\subseteq\mathbb{S};\mathbb{T}$ expanded and ρ shunted
$\iff\quad\mathbb{S}^{\intercal n};z;\mathbb{T}\cap\mathbb{T};(\mathbb{S}^{\intercal n}\cap\mathbb{S};\mathbb{T})\subseteq\mathbb{S};\mathbb{T}$ destroy and append with $\rho^{\intercal};\pi=\mathbb{T}$

Now either $\mathbb{S}$ is total, in which case we are done, or it is not and we have the greatest point $\overline{\mathbb{S};\mathbb{T}}=g_{\mathbb{S}};\mathbb{T}$ according to Prop. 5.2.1.iii.a. Using it, we may proceed as follows
$\iff\quad g_{\mathbb{S}};\mathbb{T}\cap\mathbb{S}^{\intercal n};z;\mathbb{T}\cap\mathbb{T};(\mathbb{S}^{\intercal n}\cap\overline{g_{\mathbb{S}};\mathbb{T}})\subseteq\mathbb{L}$.
In the first case the points $g_{\mathbb{S}};\mathbb{T}$ and $\mathbb{S}^{\intercal n};z;\mathbb{T}$ are unequal, so that $g_{\mathbb{S}};\mathbb{T}\cap\mathbb{S}^{\intercal n};z;\mathbb{T}=\mathbb{L}$. In the other case, they may coincide $g_{\mathbb{S}};\mathbb{T}=\mathbb{S}^{\intercal n_{\mathbb{S}}};z;\mathbb{T}$; then however according to Prop. 5.2.1.iii.d
$\iff\quad\mathbb{S}^{\intercal n_{\mathbb{S}}}=g_{\mathbb{S}};z^{\intercal}\subseteq g_{\mathbb{S}};\mathbb{T}$, so that $\mathbb{S}^{\intercal n_{\mathbb{S}}}\cap\overline{g_{\mathbb{S}};\mathbb{T}}=\mathbb{L}$.

---

ii.m) We convince us that $(\mathbb{T};x^{\intercal}\oslash\mathbb{T};e^{\intercal})$ is a transposed point, showing that it is univalent:
$(\mathbb{T};x^{\intercal}\oslash\mathbb{T};e^{\intercal})^{\intercal};(\mathbb{T};x^{\intercal}\oslash\mathbb{T};e^{\intercal})=(x;\mathbb{T}\oslash e;\mathbb{T});(\mathbb{T};x^{\intercal}\oslash\mathbb{T};e^{\intercal})$

$= (x ; \mathbb{T} ; x^\intercal \otimes e ; \mathbb{T} ; e^\intercal)$ Prop. 2.1.1.ii
$\subseteq (\mathbb{I} \otimes \mathbb{I}) = \mathbb{I}$

Now the inductive proof starts trivially for $n = 0$ and proceeds as follows
$(\mathbb{T} ; x^\intercal \oslash \mathbb{T} ; e^\intercal) ; (\mathfrak{S} \oslash \rho)^{n+1} = (\mathbb{T} ; x^\intercal \oslash \mathbb{T} ; e^\intercal) ; (\mathfrak{S} \oslash \rho)^n ; (\mathfrak{S} \oslash \rho)$
$= (\mathbb{T} ; x^\intercal ; \mathbb{S}^{\intercal n} \oslash \mathbb{T} ; e^\intercal) ; (\mathfrak{S} \oslash \rho)$ hypothesis
$= ((\mathbb{T} ; x^\intercal ; \mathbb{S}^{\intercal n} \oslash \mathbb{T} ; e^\intercal) ; \mathfrak{S} \oslash (\mathbb{T} ; x^\intercal ; \mathbb{S}^{\intercal n} \oslash \mathbb{T} ; e^\intercal) ; \rho)$ point property shown above
$= ((\mathbb{T} ; x^\intercal ; \mathbb{S}^{\intercal n} \oslash \mathbb{T} ; e^\intercal) ; \mathtt{sup}_{k \geq 0} (\mathbb{S}^{\intercal k} \oslash \mathbb{S}^{\intercal k} ; z ; \mathbb{T}) \oslash \mathbb{T} ; x^\intercal ; \mathbb{S}^n ; \mathbb{T} \cap \mathbb{T} ; e^\intercal)$
$= (\mathtt{sup}_{k \geq 0} (\mathbb{T} ; x^\intercal ; \mathbb{S}^{\intercal n} \oslash \mathbb{T} ; e^\intercal) ; (\mathbb{S}^{\intercal k} \oslash \mathbb{S}^{\intercal k} ; z ; \mathbb{T}) \oslash \mathbb{T} ; x^\intercal ; \mathbb{S}^n ; \mathbb{T} \cap \mathbb{T} ; e^\intercal)$
$= (\mathtt{sup}_{k \geq 0} \mathbb{T} ; x^\intercal ; \mathbb{S}^{\intercal n} ; \mathbb{S}^{\intercal k} \cap \mathbb{T} ; e^\intercal ; \mathbb{S}^{\intercal k} ; z ; \mathbb{T} \oslash \mathbb{T} ; x^\intercal ; \mathbb{S}^n ; \mathbb{T} \cap \mathbb{T} ; e^\intercal)$ implying $k = 1$ to be $\neq \mathbb{\perp}$
$= (\mathbb{T} ; x^\intercal ; \mathbb{S}^{\intercal n} ; \mathbb{S}^\intercal \cap \mathbb{T} ; e^\intercal ; \mathbb{S}^\intercal ; z ; \mathbb{T} \oslash \mathbb{T} ; x^\intercal ; \mathbb{S}^n ; \mathbb{T} \cap \mathbb{T} ; e^\intercal)$

Now either $\mathbb{T} ; x^\intercal ; \mathbb{S}^{\intercal n} \neq \mathbb{\perp}$, in which case
$= (\mathbb{T} ; x^\intercal ; \mathbb{S}^{\intercal n+1} \cap \mathbb{T} ; z^\intercal ; z ; \mathbb{T} \oslash \mathbb{T} ; x^\intercal ; \mathbb{S}^n ; \mathbb{T} \cap \mathbb{T} ; e^\intercal)$
$= (\mathbb{T} ; x^\intercal ; \mathbb{S}^{n+1} \oslash \mathbb{T} ; e^\intercal)$
or everything is $= \mathbb{\perp}$.

---

iii.a) Due to Prop. 5.2.1.iii.c always $\mathbb{S}^{\intercal n} ; z ; \mathbb{T} = \mathbb{\perp}$ when $n > n_{\mathbb{S}}$

---

iii.b) $(\mathbb{T} ; g_{\mathbb{S}}^\intercal \oslash \mathbb{I}) ; \mathfrak{S} = \mathtt{sup}_{n_{\mathbb{S}} \geq n \geq 0} (\mathbb{T} ; g_{\mathbb{S}}^\intercal \oslash \mathbb{I}) ; (\mathbb{S}^{\intercal n} \oslash \mathbb{S}^{\intercal n} ; z ; \mathbb{T})$
$= \mathtt{sup}_{n_{\mathbb{S}} \geq n \geq 0} \mathbb{T} ; g_{\mathbb{S}}^\intercal ; \mathbb{S}^{\intercal n} \cap \mathbb{S}^{\intercal n} ; z ; \mathbb{T}$
$= \mathtt{sup}_{n_{\mathbb{S}} \geq n \geq 0} \mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; g_{\mathbb{S}}^\intercal ; \mathbb{S}^{\intercal n} = \mathtt{FlipTD}$ masking, Prop. 5.2.1.iii.f

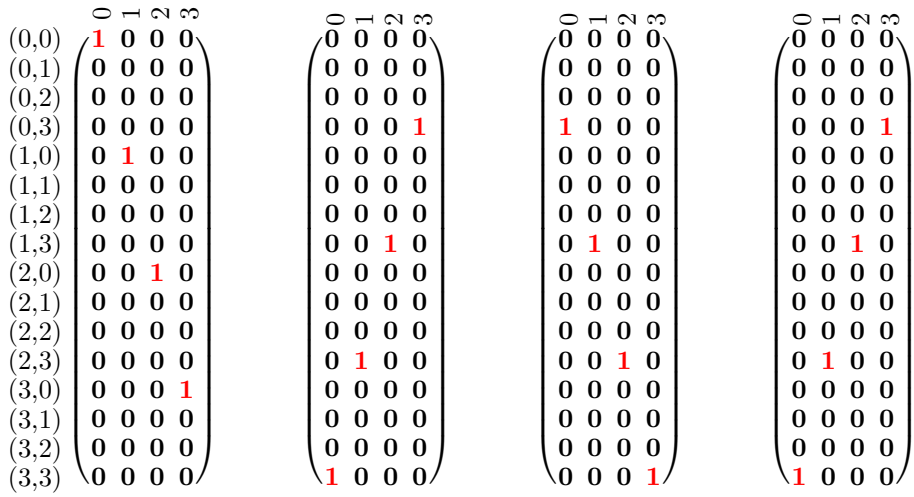


Fig. 6.3.6 $(\mathbb{I} \oslash z ; \mathbb{T})$ $(\mathtt{FlipTD} \otimes \mathtt{FlipTD}) ; (\mathbb{I} \oslash z ; \mathbb{T})$ $(\mathbb{I} \oslash g_{\mathbb{S}} ; \mathbb{T})$ $(\mathbb{I} \oslash g_{\mathbb{S}} ; \mathbb{T}) ; \mathtt{FlipTD}$

---

iii.c) The following proof is still a horrible one. There is mainly one difficult point when working with the symmetries involved.

$(\mathtt{FlipTD} \otimes \mathtt{FlipTD}) ; (\mathbb{I} \oslash z ; \mathbb{T}) = (\mathtt{FlipTD} \oslash \mathtt{FlipTD} ; z ; \mathbb{T})$
$= (\mathtt{FlipTD} \oslash \mathtt{FlipTD} ; z ; \mathbb{T} ; \mathtt{FlipTD})$ since FlipTD is surjective
$= (\mathbb{I} \oslash \mathtt{FlipTD} ; z ; \mathbb{T}) ; \mathtt{FlipTD}$ since FlipTD is injective
$= (\mathbb{I} \oslash g_{\mathbb{S}} ; \mathbb{T}) ; \mathtt{FlipTD}$ Prop. 5.2.1.iii.g

The illustration Fig. 6.3.6 above may provide some help in understanding.

$$
\begin{array}{ll}
& (\mathtt{FlipTD} \otimes \mathtt{FlipTD}) ; \mathfrak{S} \\
= & (\mathtt{FlipTD} \otimes \mathtt{FlipTD}) ; \mathtt{sup}_{n_{\mathbb{S}} \geq n \geq 0} \, (\mathbb{S}^{\intercal n} \otimes \mathbb{S}^{\intercal n}) ; (\mathbb{I} \oslash z ; \mathbb{T}) \\
= & \mathtt{sup}_{n_{\mathbb{S}} \geq n \geq 0} \, (\mathbb{S}^{n} \otimes \mathbb{S}^{n}) ; (\mathtt{FlipTD} \otimes \mathtt{FlipTD}) ; (\mathbb{I} \oslash z ; \mathbb{T}) \quad \text{Prop. 5.2.1.iii.i} \\
= & \mathtt{sup}_{n_{\mathbb{S}} \geq n \geq 0} \, (\mathbb{S}^{n} \otimes \mathbb{S}^{n}) ; (\mathbb{I} \oslash g_{\mathbb{S}} ; \mathbb{T}) ; \mathtt{FlipTD} \quad \text{see above} \\
= & \mathtt{sup}_{n_{\mathbb{S}} \geq n \geq 0} \, (\mathbb{S}^{n} \otimes \mathbb{S}^{n}) ; \mathfrak{P} ; \mathfrak{P} ; (\mathbb{I} \oslash g_{\mathbb{S}} ; \mathbb{T}) ; \mathtt{FlipTD} \\
= & \mathfrak{P} ; \mathtt{sup}_{n_{\mathbb{S}} \geq n \geq 0} \, (\mathbb{S}^{n} \otimes \mathbb{S}^{n}) ; (g_{\mathbb{S}} ; \mathbb{T} \oslash \mathbb{I}) ; \mathtt{FlipTD} \quad \text{due to Prop. 3.1.1} \\
= & \mathfrak{P} ; \mathtt{sup}_{n_{\mathbb{S}} \geq n \geq 0} \, (\mathbb{S}^{n} \otimes \mathbb{S}^{n}) ; (g_{\mathbb{S}} ; \mathbb{T} \oslash \mathbb{I}) ; (\mathtt{FlipTD} \cap \mathbb{T}) \\
= & \mathfrak{P} ; \mathtt{sup}_{n_{\mathbb{S}} \geq n \geq 0} \, (\mathbb{S}^{n} \otimes \mathbb{S}^{n}) ; (g_{\mathbb{S}} ; \mathbb{T} \oslash \mathbb{I}) ; (\mathtt{FlipTD} \cap \mathtt{sup}_{n_{\mathbb{S}} \geq k \geq 0} \, \mathbb{T} ; g_{\mathbb{S}}^{\intercal} ; \mathbb{S}^{\intercal k}) \\
= & \mathfrak{P} ; \mathtt{sup}_{n_{\mathbb{S}} \geq n \geq 0} \, (\mathbb{S}^{n} \otimes \mathbb{S}^{n}) ; (g_{\mathbb{S}} ; \mathbb{T} \oslash \mathbb{I}) ; (\mathtt{sup}_{n_{\mathbb{S}} \geq k \geq 0} \, \mathtt{FlipTD} \cap \mathbb{T} ; g_{\mathbb{S}}^{\intercal} ; \mathbb{S}^{\intercal k}) \\
= & \mathfrak{P} ; \mathtt{sup}_{n_{\mathbb{S}} \geq n \geq 0} \, (\mathbb{S}^{n} \otimes \mathbb{S}^{n}) ; (g_{\mathbb{S}} ; \mathbb{T} \oslash \mathbb{I}) ; (\mathtt{sup}_{n_{\mathbb{S}} \geq k \geq 0} \, \mathbb{S}^{\intercal k} ; z ; \mathbb{T} ; g_{\mathbb{S}}^{\intercal} ; \mathbb{S}^{\intercal k}) \quad \text{Prop. 5.2.1.iii.g} \\
\subseteq & \mathfrak{P} ; \mathtt{sup}_{n_{\mathbb{S}} \geq n \geq 0} \, (\mathbb{S}^{n} \otimes \mathbb{S}^{n}) ; (\mathtt{sup}_{n_{\mathbb{S}} \geq k \geq 0} \, (\mathbb{S}^{\intercal k} \oslash \mathbb{S}^{\intercal k} ; z ; \mathbb{T})) \\
& \qquad \text{because } \mathbb{S}^{\intercal k} ; z ; \mathbb{T} ; g_{\mathbb{S}}^{\intercal} ; \mathbb{S}^{\intercal k} \text{ is injective as well as } g_{\mathbb{S}}, \text{ so that} \\
& \qquad \text{and } g_{\mathbb{S}} ; \mathbb{T} ; \mathbb{S}^{\intercal k} ; z ; \mathbb{T} ; g_{\mathbb{S}}^{\intercal} ; \mathbb{S}^{\intercal k} \subseteq g_{\mathbb{S}} ; \mathbb{T} ; g_{\mathbb{S}}^{\intercal} ; \mathbb{S}^{\intercal k} \subseteq \mathbb{S}^{\intercal k} \\
= & \mathfrak{P} ; \mathtt{sup}_{n_{\mathbb{S}} \geq n \geq 0} (\mathtt{sup}_{n_{\mathbb{S}} \geq k \geq 0} \, (\mathbb{S}^{n} \otimes \mathbb{S}^{n}) ; (\mathbb{S}^{\intercal k} \oslash \mathbb{S}^{\intercal k} ; z ; \mathbb{T})) \\
= & \mathfrak{P} ; \mathtt{sup}_{n_{\mathbb{S}} \geq n \geq 0} (\mathtt{sup}_{n_{\mathbb{S}} \geq k \geq 0} \, (\mathbb{S}^{n} ; \mathbb{S}^{\intercal k} \oslash \mathbb{S}^{n} ; \mathbb{S}^{\intercal k} ; z ; \mathbb{T})) \quad \text{now } n > k \text{ implies } \ldots = \mathbb{\perp} \\
\subseteq & \mathfrak{P} ; \mathtt{sup}_{n_{\mathbb{S}} \geq k \geq 0} (\mathtt{sup}_{n_{\mathbb{S}} \geq n \geq 0} \, (\mathbb{S}^{\intercal k - n} \oslash \mathbb{S}^{\intercal k - n} ; z ; \mathbb{T})) \quad \text{summation changed; } \mathbb{S} \text{ injective} \\
= & \mathfrak{P} ; \mathtt{sup}_{n_{\mathbb{S}} \geq m \geq 0} \, (\mathbb{S}^{\intercal m} \oslash \mathbb{S}^{\intercal m} ; z ; \mathbb{T})) \quad \text{where } m := k - n \\
= & \mathfrak{P} ; \mathfrak{S}
\end{array}
$$

The other direction:

$$
\begin{array}{ll}
& \mathfrak{P} ; \mathfrak{S} = \mathfrak{P} ; (\mathtt{FlipTD} \otimes \mathtt{FlipTD}) ; (\mathtt{FlipTD} \otimes \mathtt{FlipTD}) ; \mathfrak{S} \quad \text{Prop. 5.2.1.iii.j} \\
\subseteq & \mathfrak{P} ; (\mathtt{FlipTD} \otimes \mathtt{FlipTD}) ; \mathfrak{P} ; \mathfrak{S} \quad \text{see above} \\
\subseteq & \mathfrak{P} ; \mathfrak{P} ; (\mathtt{FlipTD} \otimes \mathtt{FlipTD}) ; \mathfrak{S} = (\mathtt{FlipTD} \otimes \mathtt{FlipTD}) ; \mathfrak{S} \quad \text{Prop. 5.2.1.iii.j}
\end{array}
$$

---

iii.d) $(\mathtt{FlipTD} \otimes \mathtt{FlipTD}) ; \mathbb{X} = (\mathtt{FlipTD} \otimes \mathtt{FlipTD}) ; \mathfrak{S} \cup (\mathtt{FlipTD} \otimes \mathtt{FlipTD}) ; \mathfrak{P} ; \mathfrak{S}$
$= \mathfrak{P} ; \mathfrak{S} \cup \mathfrak{P} ; (\mathtt{FlipTD} \otimes \mathtt{FlipTD}) ; \mathfrak{S}$ (iii.c) and Prop. 5.2.1.iii.j
$= \mathfrak{P} ; \mathfrak{S} \cup \mathfrak{P} ; \mathfrak{P} ; \mathfrak{S} = \mathfrak{P} ; \mathfrak{S} \cup \mathfrak{S} = \mathbb{X}$ see (iii.c)

---

iii.e) We start proving the first formula in case $\mathbb{S}$ is not total with

$$
\begin{array}{ll}
& \rho^{\intercal} ; \mathfrak{S} = \rho^{\intercal} ; \mathtt{sup}_{n \geq 0} \, (\mathbb{S}^{\intercal n} \oslash \mathbb{S}^{\intercal n} ; z ; \mathbb{T}) \quad \text{expanded} \\
= & \mathtt{sup}_{n \geq 0} \, \mathbb{T} ; \mathbb{S}^{\intercal n} \cap \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \quad \text{destroy and append} \\
= & \mathtt{sup}_{n \geq 0} \, \mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; \mathbb{S}^{\intercal n} \quad \text{masking} \\
= & \mathtt{sup}_{n \geq 0} \, \mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; \mathbb{S}^{\intercal n} ; \mathtt{FlipTD} ; \mathtt{FlipTD} \quad \text{since } \mathtt{FlipTD} \text{ is a bijective map} \\
= & \mathtt{sup}_{n \geq 0} \, \mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; \mathtt{FlipTD} ; \mathbb{S}^{n} ; \mathtt{FlipTD} \quad \text{Prop. 5.2.1.iii.i} \\
= & \mathtt{sup}_{n \geq 0} \, \mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; \mathbb{S}^{n} ; \mathtt{FlipTD} \quad \text{since } \mathtt{FlipTD} \text{ is a bijective map} \\
= & \mathbb{V} ; \mathtt{FlipTD} = \mathbb{S}^{*} ; \mathtt{FlipTD} \quad \text{Prop. 5.3.1.ii.a}
\end{array}
$$

$$
\begin{array}{c|ccccc}
 & 0 & 1 & 2 & 3 & 4 \\
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
2 & 1 & 1 & 1 & 0 & 0 \\
3 & 1 & 1 & 1 & 1 & 0 \\
4 & 1 & 1 & 1 & 1 & 1
\end{array}
\qquad
\begin{array}{c|ccccc}
 & 0 & 1 & 2 & 3 & 4 \\
0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 \\
2 & 1 & 1 & 1 & 0 & 0 \\
3 & 1 & 1 & 0 & 0 & 0 \\
4 & 1 & 0 & 0 & 0 & 0
\end{array}
\qquad
\begin{array}{c|ccccc}
 & 0 & 1 & 2 & 3 & 4 \\
0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 \\
2 & 1 & 1 & 1 & 0 & 0 \\
3 & 1 & 1 & 1 & 1 & 0 \\
4 & 1 & 1 & 1 & 1 & 1
\end{array}
$$

$$
\pi^{\intercal} ; \mathfrak{S} = \left[\mathbb{S}^{*}\right]^{\intercal} = \rho^{\intercal} ; \mathtt{S} \qquad \rho^{\intercal} ; \mathfrak{S} = \mathbb{\nabla} = \pi^{\intercal} ; \mathtt{S} \qquad \pi^{\intercal} ; \mathbb{X} = \rho^{\intercal} ; \mathbb{X}
$$

Fig. 6.3.7 Visualizing (i.e) and (ii.j)

iv.a) This is (i.e) together with Prop. 5.3.2.iv.a. It may also be proved in the following way.
$\rho^{\mathsf{T}}{;}\,\mathfrak{S} = \rho^{\mathsf{T}}{;}\mathbb{A}^{\hookleftarrow} = \rho^{\mathsf{T}}{;}\,(\,\mathbb{A}^{\mathsf{T}} \circledcirc \rho^{\mathsf{T}}){;}\,\pi = \rho^{\mathsf{T}}{;}(\pi{;}\,\mathbb{A}^{\mathsf{T}} \cap \rho\,{;}\rho^{\mathsf{T}})\,{;}\,\pi$
$= (\mathbb{T}{;}\,\mathbb{A}^{\mathsf{T}} \cap \rho^{\mathsf{T}})\,{;}\,\pi$ destroy and append
$= (\mathbb{T} \cap \rho^{\mathsf{T}})\,{;}\,\pi$ since $\mathbb{A}$ is then total, Prop. 6.2.1.iv.a
$= \rho^{\mathsf{T}}{;}\,\pi = \mathbb{T}$ □

Results in (i.c) are easily interpreted: Subtracting z does not change the minuend, while subtracting e means transition to the predecessor wrt. to $\mathbb{S}$.

The following figures show how addition, subtraction and their supremum representations are mutually related:

$$\mathtt{sup}_{n\geq 0}\ (\mathbb{S}^{\mathsf{T}n}{;}\,z\,{;}\mathbb{T} \circledcirc \mathbb{S}^{n})$$

$$\Vert$$

$$\mathtt{S} = \mathtt{sup}_{n\geq 0}\ (\mathbb{S}^{\mathsf{T}n}{;}\,z\,{;}\mathbb{T} \circledcirc \mathbb{S}^{\mathsf{T}n}) \mathrel{=\!\!\hookrightarrow} \mathbb{A} \qquad \mathbb{A} \qquad \mathbb{A}^{\hookleftarrow} = \mathtt{sup}_{n\geq 0}\ (\mathbb{S}^{\mathsf{T}n} \circledcirc \mathbb{I})\,{;}\mathbb{T}\,{;}z^{\mathsf{T}}{;}\,\mathbb{S}^{n}$$

$$\Vert \iff \Vert \quad \text{if regressively finite} \quad \Vert \implies \Vert$$

$$\mathtt{sup}_{n\geq 0}\ (\mathbb{I} \circledcirc \mathbb{S}^{\mathsf{T}n})\,{;}\mathbb{T}\,{;}z^{\mathsf{T}}{;}\,\mathbb{S}^{n} \mathrel{=\!\!\hookrightarrow} B \qquad B \qquad B^{\hookleftarrow} = \mathtt{sup}_{n\geq 0}\ (\mathbb{S}^{\mathsf{T}n} \circledcirc \mathbb{S}^{\mathsf{T}n}{;}\,z\,{;}\mathbb{T}) = \mathfrak{S}$$

$$\Vert$$

$$\mathtt{sup}_{n\geq 0}\ (\mathbb{S}^{n} \circledcirc \mathbb{S}^{\mathsf{T}n}{;}\,z\,{;}\mathbb{T})$$

Fig. 6.3.8 Addition inversions on possibly finite naturals

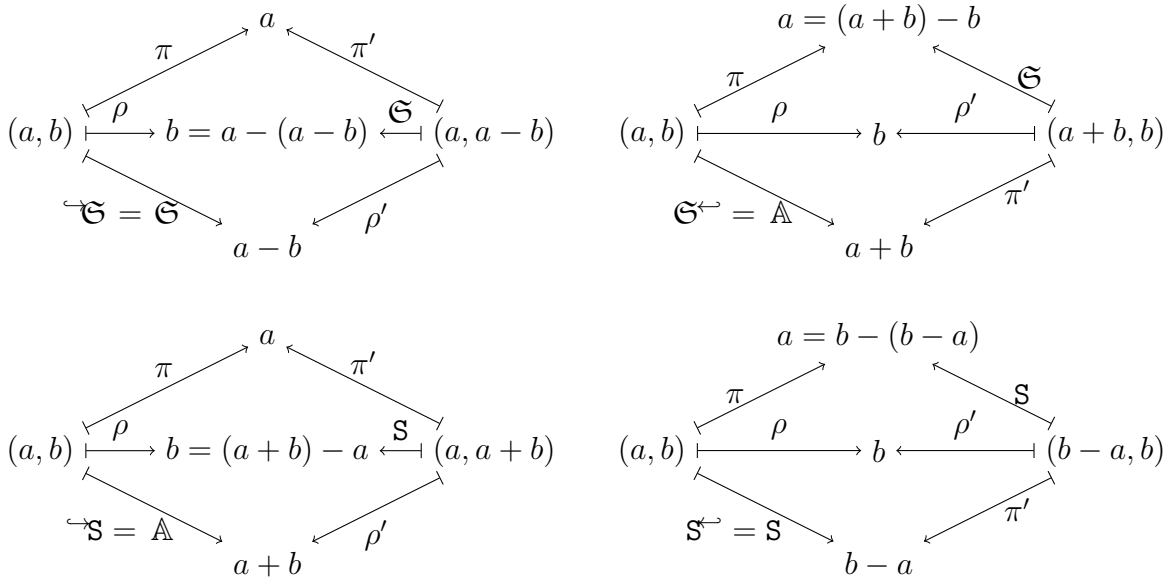


Fig. 6.3.9 Illustrating left and right inversion of subtractions and their idempotence

The following behaves slightly differently with regard to $\mathbb{S}$ being total or not:

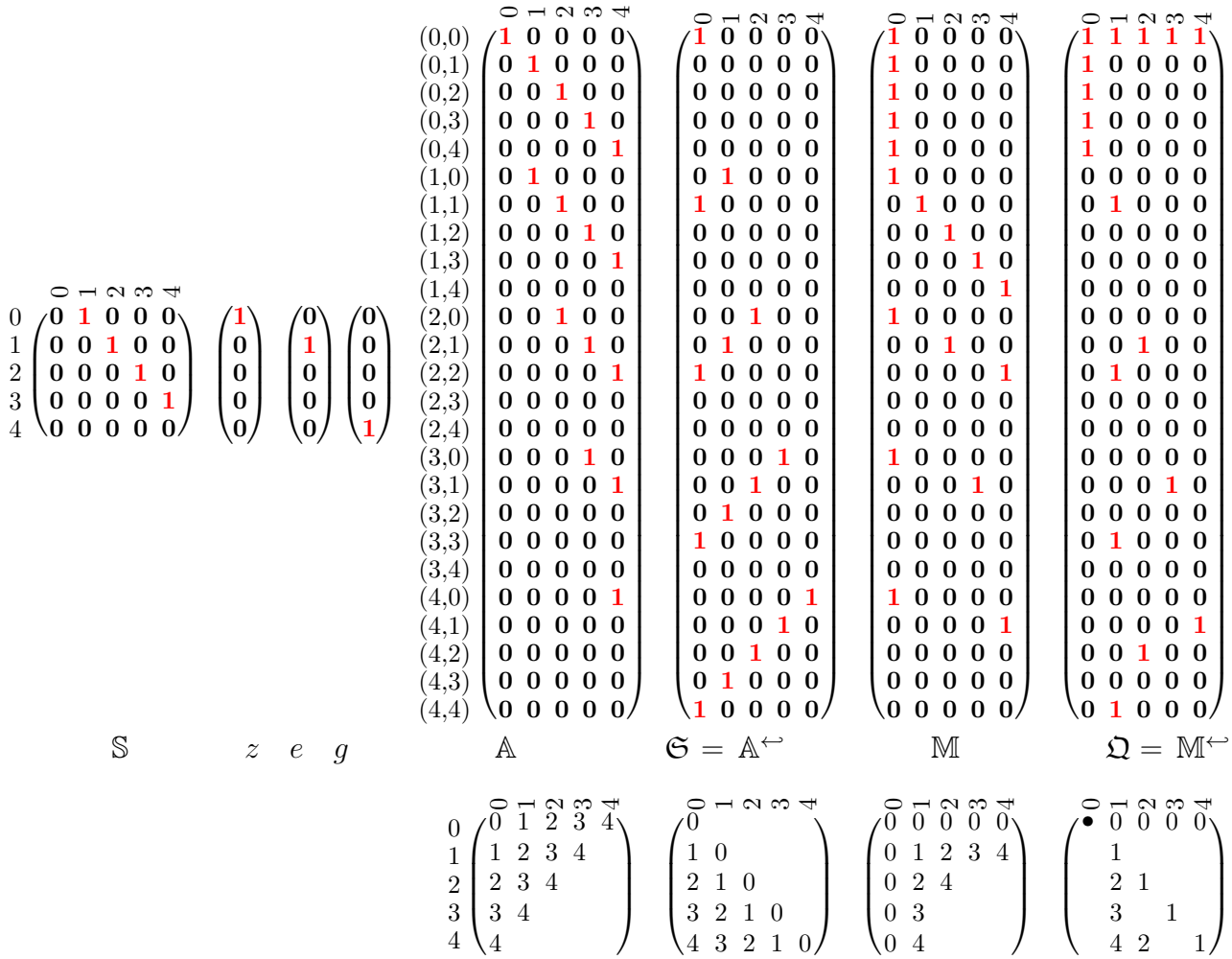


Fig. 6.3.10 Operations on an initial segment of the naturals $\mathbb{N}$

To show how this works, we drop the requirement of being total in the definition of $\mathbb{S}$. Then limited integers become possible as examples, shown with `[0..4]` in Fig. 6.3.10 — the denotation $\mathbb{N}$ has been preserved. The greatest of the numbers so restricted is obviously described by the point $g_{\mathbb{S}} := \overline{\mathbb{S}\,;\mathbb{T}}$. When $\mathbb{S}$ is demanded to be total, $g_{\mathbb{S}}$ will no longer be a point but an empty relation.

The construct $(\mathfrak{S} \oslash \rho)$ allows a major degree of self similarity:

6.3.2 Proposition. We assume $\mathbb{S}$ to be regressively finite.

i) $(\mathfrak{S} \oslash \rho)\,;(z\,;\mathbb{T} \oslash z\,;\mathbb{T}) = (z\,;\mathbb{T} \oslash z\,;\mathbb{T}) \qquad (\mathfrak{S} \oslash \rho)\,;\pi = \mathfrak{S}$

ii) $(\mathbb{S}^{+\intercal} \oslash \mathbb{I})\,;(\mathfrak{S} \oslash \rho) = \mathbb{\perp}$

iii) $(\mathbb{A}^{\intercal} \oslash \rho^{\intercal}) = (\mathfrak{S} \oslash \rho) \subseteq (\mathfrak{S} \oslash \rho)^{*} \subseteq (\mathbb{S}^{\intercal *} \otimes \mathbb{I}) \subseteq \mathbb{I} \cup \pi\,;\mathbb{S}^{\intercal +}\,;\pi^{\intercal}$

iv) $(\mathbb{S}^{\intercal} \otimes \mathbb{I})\,;(\mathfrak{S} \oslash \rho)\,;(\mathbb{S} \otimes \mathbb{I}) = (\mathfrak{S} \oslash \rho) \cap \mathbb{T}\,;\mathbb{S}\,;\pi^{\intercal} \subseteq (\mathfrak{S} \oslash \rho)$

$(\mathbb{S} \otimes \mathbb{I})\,;(\mathfrak{S} \oslash \rho)\,;(\mathbb{S}^{\intercal} \otimes \mathbb{I}) = (\mathfrak{S} \oslash \rho) \cap \pi\,;\mathbb{S}\,;\mathbb{T} \subseteq (\mathfrak{S} \oslash \rho)$

v) $\mathfrak{S}^\intercal ; (\mathfrak{S}^\intercal \oslash \rho^\intercal)^{l} ; (\mathbb{S}^{q} ; \mathbb{S}^{\intercal r} ; z ; \mathbb{T} \oslash \mathbb{S}^{\intercal s} ; z ; \mathbb{T}) = \mathbb{S}^{s*(l+1)+q} ; \mathbb{S}^{\intercal r} ; z ; \mathbb{T}$ for arbitrary l, q, r, s

$\mathfrak{S}^\intercal ; (\mathfrak{S}^\intercal \oslash \rho^\intercal)^{*} ; (\mathbb{S}^{q} ; \mathbb{S}^{\intercal r} ; z ; \mathbb{T} \oslash \mathbb{S}^{\intercal s} ; z ; \mathbb{T}) = (\mathbb{S}^{s})^{+} ; \mathbb{S}^{q} ; \mathbb{S}^{\intercal r} ; z ; \mathbb{T}$ for arbitrary q, r, s

vi) $(\mathfrak{S} \oslash \rho)^{n} ; (\mathbb{S}^{+} \oslash \mathbb{I}) = (\mathfrak{S} \oslash \rho)^{n-1} ; \mathfrak{S} ; \mathbb{S}^{+} \cap \rho$ for $n > 0$

$(\mathfrak{S} \oslash \rho)^{*} ; (\mathbb{S}^{+} \oslash \mathbb{I}) = \rho \cap \mathbb{T} ; \mathbb{S} \qquad (\mathfrak{S} \oslash \rho)^{*} ; (\mathbb{S}^{*} \oslash \mathbb{I}) = \rho \cap \big[\pi \cup \mathbb{T} ; \mathbb{S}\big]$

vii) $(\mathfrak{S} \oslash \rho)^{n} ; \mathfrak{S} = \mathtt{sup}_{i \geq 0} \, (\mathbb{S}^{\intercal (n+1)*i} \oslash \mathbb{S}^{\intercal i} ; z ; \mathbb{T})$

viii) $(\mathbb{T} ; z^\intercal \oslash \mathbb{T} ; z^\intercal) ; (\mathfrak{S} \oslash \rho) = (\mathbb{T} ; z^\intercal \oslash \mathbb{T} ; z^\intercal)$

$(\mathbb{I} \oslash \mathbb{I}) ; (\mathfrak{S} \oslash \rho) = (\mathbb{T} ; z^\intercal \oslash \mathbb{I})$

$(\mathbb{I} \oslash \mathbb{I}) ; (\mathfrak{S} \oslash \rho)^{n} = (z ; \mathbb{T} ; z^\intercal \oslash \mathbb{I})$ for $n > 1$

ix) $(z ; \mathbb{T} \oslash z ; \mathbb{T}) ; \mathbb{T} ; z^\intercal = (z ; \mathbb{T} ; z^\intercal \oslash \mathbb{I}) = (\mathbb{I} \oslash z ; \mathbb{T} ; z^\intercal)$

Proof: i) $(\mathfrak{S} \oslash \rho) ; (z ; \mathbb{T} \oslash z ; \mathbb{T}) = \mathfrak{S} ; z ; \mathbb{T} \cap \rho ; z ; \mathbb{T}$
$= (\mathbb{I} \oslash \mathbb{I}) ; \mathbb{T} \cap \rho ; z ; \mathbb{T}$ following Prop. 6.3.1.ii.a
$= (\pi \cap \rho) ; \mathbb{T} \cap \rho ; z ; \mathbb{T} = (\pi \cap \rho \cap \rho ; z ; \mathbb{T}) ; \mathbb{T}$ masking
$= (\pi \cap \rho ; [\mathbb{I} \cap z ; \mathbb{T}]) ; \mathbb{T} = (\pi \cap \rho ; [\mathbb{I} \cap \mathbb{T} ; z^\intercal]) ; \mathbb{T}$ transposing a partial identity
$= (\pi \cap \rho \cap \rho ; \mathbb{T} ; z^\intercal) ; \mathbb{T} = (\pi \cap \rho \cap \mathbb{T} ; z^\intercal) ; \mathbb{T}$ since ρ is a map
$= (\pi \cap \rho) ; [\mathbb{T} \cap z ; \mathbb{T}]$ mask shifting
$= (\mathbb{I} \oslash \mathbb{I}) ; z ; \mathbb{T} = (z ; \mathbb{T} \oslash z ; \mathbb{T})$ because $z ; \mathbb{T}$ is injective

---

The second statement is proved with the destroy and append rule:
$(\mathfrak{S} \oslash \rho) ; \pi = (\mathfrak{S} ; \pi^\intercal \cap \rho ; \rho^\intercal) ; \pi = \mathfrak{S} \cap \rho ; \rho^\intercal ; \pi = \mathfrak{S}$

---

ii) $(\mathbb{S}^{+\intercal} \oslash \mathbb{I}) ; (\mathfrak{S} \oslash \rho) \subseteq ((\mathbb{S}^{+\intercal} \oslash \mathbb{I}) ; \mathfrak{S} \oslash (\mathbb{S}^{+\intercal} \oslash \mathbb{I}) ; \rho)$
$= (\mathbb{\perp} \oslash \ldots) = \mathbb{\perp}$ due to Prop. 6.3.1.ii.c

---

iii) Due to Prop. 6.2.1.ii.a, we now have $\mathbb{A} = B$ in Prop. 6.3.1.i.e together with Prop. 6.3.1.ii.e. Furthermore

$\mathbb{A}^{\hookleftarrow} = (\mathbb{A}^\intercal \oslash \rho^\intercal) ; \pi = \mathfrak{S}$
$\Longrightarrow \quad (\mathbb{A}^\intercal \oslash \rho^\intercal) ; \pi \subseteq \mathfrak{S} \quad \Longleftrightarrow \quad (\mathbb{A}^\intercal \oslash \rho^\intercal) \subseteq \mathfrak{S} ; \pi^\intercal,$

but also

$\mathfrak{S}^{\hookleftarrow} = (\mathfrak{S}^\intercal \oslash \rho^\intercal) ; \pi = \mathbb{A}$
$\Longrightarrow \quad (\mathfrak{S}^\intercal \oslash \rho^\intercal) ; \pi \subseteq \mathbb{A} \quad \Longleftrightarrow \quad (\mathfrak{S}^\intercal \oslash \rho^\intercal) \subseteq \mathbb{A} ; \pi^\intercal \quad \Longleftrightarrow \quad (\mathfrak{S} \oslash \rho) \subseteq \pi ; \mathbb{A}^\intercal$

Collecting containments in both directions for the middle equality, thus indeed

$(\mathbb{A}^\intercal \oslash \rho^\intercal) = \pi ; \mathbb{A}^\intercal \cap \rho ; \rho^\intercal = \mathfrak{S} ; \pi^\intercal \cap \rho ; \rho^\intercal = (\mathfrak{S} \oslash \rho) \cap \rho ; \rho^\intercal$

Containment of the powers $(\mathfrak{S} \oslash \rho)^{n}$ is obvious in case $n = 0$. It holds also for $n = 1$:

$(\mathfrak{S} \oslash \rho) \subseteq (\mathbb{S}^{\intercal *} \otimes \mathbb{I})$

means in expanded form

$\mathfrak{S} ; \pi^\intercal \cap \rho ; \rho^\intercal \subseteq \pi ; \mathbb{S}^{\intercal *} ; \pi^\intercal \cap \rho ; \rho^\intercal,$

and follows from Prop. 6.3.1.i.e, $\pi^\intercal ; \mathfrak{S} \subseteq \mathbb{S}^{\intercal *}$, shunting π.

Now we prove it by induction for $n \mapsto n+1$:

$(\mathfrak{S} \circledcirc \rho)^{n+1} = (\mathfrak{S} \circledcirc \rho) ; (\mathfrak{S} \circledcirc \rho)^{n} \subseteq (\mathfrak{S} \circledcirc \rho) ; (\mathbb{S}^{\intercal *} \otimes \mathbb{I})$ hypothesis
$= (\mathfrak{S} ; \mathbb{S}^{\intercal *} \circledcirc \rho)$
$\subseteq (\pi ; \mathbb{S}^{\intercal *} ; \mathbb{S}^{\intercal *} \circledcirc \rho) = (\pi ; \mathbb{S}^{\intercal *} \circledcirc \rho)$ once again since $\pi^{\intercal} ; \mathfrak{S} \subseteq \mathbb{S}^{\intercal *} \iff \mathfrak{S} \subseteq \pi ; \mathbb{S}^{\intercal *}$
$= (\mathbb{S}^{\intercal *} \otimes \mathbb{I})$

The final containment is trivially satisfied

---

iv) $(\mathbb{S}^{\intercal} \otimes \mathbb{I}) ; (\mathfrak{S} \circledcirc \rho) ; (\mathbb{S} \otimes \mathbb{I}) = (\mathbb{S}^{\intercal} \otimes \mathbb{I}) ; (\mathfrak{S} ; \mathbb{S} \circledcirc \rho)$
$= ((\mathbb{S}^{\intercal} \otimes \mathbb{I}) ; \mathfrak{S} ; \mathbb{S} \circledcirc (\mathbb{S}^{\intercal} \otimes \mathbb{I}) ; \rho)$ since the factor on the left is univalent
$= (\mathfrak{S} \cap \mathbb{T} ; \mathbb{S} \circledcirc (\mathbb{S}^{\intercal} ; \mathbb{T} \circledcirc \mathbb{I}))$ due to Prop. 6.3.1.i.d
$= \mathfrak{S} ; \pi^{\intercal} \cap \mathbb{T} ; \mathbb{S} ; \pi^{\intercal} \cap \pi ; \mathbb{S}^{\intercal} ; \mathbb{T} ; \rho^{\intercal} \cap \rho ; \rho^{\intercal}$
$= \mathfrak{S} ; \pi^{\intercal} \cap \rho ; \rho^{\intercal} \cap \mathbb{T} ; \mathbb{S} ; \pi^{\intercal} \cap \pi ; \mathbb{S}^{\intercal} ; \mathbb{T}$ shuffled; ρ is total
$= (\mathfrak{S} \circledcirc \rho) \cap \mathbb{T} ; \mathbb{S} ; \pi^{\intercal} \cap \pi ; \mathbb{S}^{\intercal} ; \mathbb{T}$
$= (\mathfrak{S} \circledcirc \rho) \cap \mathbb{T} ; \mathbb{S} ; \pi^{\intercal} \subseteq (\mathfrak{S} \circledcirc \rho)$ see below

It has yet to be shown that $(\mathfrak{S} \circledcirc \rho) \cap \mathbb{T} ; \mathbb{S} ; \pi^{\intercal} \subseteq \pi ; \mathbb{S}^{\intercal} ; \mathbb{T}$. This, however, follows from (iv):

$(\mathfrak{S} \circledcirc \rho) \cap \mathbb{T} ; \mathbb{S} ; \pi^{\intercal} \subseteq \left[\mathbb{I} \cup \pi ; \mathbb{S}^{\intercal +} ; \pi^{\intercal} \right] \cap \mathbb{T} ; \mathbb{S} ; \pi^{\intercal}$
$= \left[\mathbb{I} \cap \mathbb{T} ; \mathbb{S} ; \pi^{\intercal} \right] \cup \left[\pi ; \mathbb{S}^{\intercal +} ; \pi^{\intercal} \cap \mathbb{T} ; \mathbb{S} ; \pi^{\intercal} \right]$
$= \left[\mathbb{I} \cap \pi ; \mathbb{S}^{\intercal} ; \mathbb{T} \right] \cup \left[\pi ; \mathbb{S}^{\intercal +} ; \pi^{\intercal} \cap \mathbb{T} ; \mathbb{S} ; \pi^{\intercal} \right]$ transposing a partial identity
$\subseteq \pi ; \mathbb{S}^{\intercal} ; \mathbb{T}$

$(\mathbb{S} \otimes \mathbb{I}) ; (\mathfrak{S} \circledcirc \rho) ; (\mathbb{S}^{\intercal} \otimes \mathbb{I}) = (\mathbb{S} \otimes \mathbb{I}) ; (\mathfrak{S} ; \mathbb{S}^{\intercal} \circledcirc \rho)$
$= (\mathbb{S} \otimes \mathbb{I}) ; ((\mathbb{I} \otimes \mathbb{S}) ; \mathfrak{S} \circledcirc \rho)$ due to Prop. 6.3.1.ii.c
$= ((\mathbb{S} \otimes \mathbb{I}) ; (\mathbb{I} \otimes \mathbb{S}) ; \mathfrak{S} \circledcirc (\mathbb{S} \otimes \mathbb{I}) ; \rho)$ since the factor $(\mathbb{S} \otimes \mathbb{I})$ is univalent
$= ((\mathbb{S} \otimes \mathbb{S}) ; \mathfrak{S} \circledcirc (\mathbb{S} ; \mathbb{T} \circledcirc \mathbb{I}))$ destroy and append
$= (\mathfrak{S} \cap \pi ; \mathbb{S} ; \mathbb{T} \circledcirc (\mathbb{S} ; \mathbb{T} \circledcirc \mathbb{I}))$ Prop. 6.3.1.ii.l
$= (\mathfrak{S} \cap \pi ; \mathbb{S} ; \mathbb{T}) ; \pi^{\intercal} \cap (\pi ; \mathbb{S} ; \mathbb{T} \cap \rho) ; \rho^{\intercal}$
$= \mathfrak{S} ; \pi^{\intercal} \cap \pi ; \mathbb{S} ; \mathbb{T} ; \pi^{\intercal} \cap \pi ; \mathbb{S} ; \mathbb{T} ; \rho^{\intercal} \cap \rho ; \rho^{\intercal} = \mathfrak{S} ; \pi^{\intercal} \cap \pi ; \mathbb{S} ; \mathbb{T} \cap \pi ; \mathbb{S} ; \mathbb{T} \cap \rho ; \rho^{\intercal}$
$= \mathfrak{S} ; \pi^{\intercal} \cap \rho ; \rho^{\intercal} \cap \pi ; \mathbb{S} ; \mathbb{T}$
$= (\mathfrak{S} \circledcirc \rho) \cap \pi ; \mathbb{S} ; \mathbb{T} \subseteq (\mathfrak{S} \circledcirc \rho)$

---

v) Induction over l:

$\mathfrak{S}^{\intercal} ; (\mathfrak{S}^{\intercal} \circledcirc \rho^{\intercal})^{0} ; (\mathbb{S}^{q} ; \mathbb{S}^{\intercal r} ; z ; \mathbb{T} \circledcirc \mathbb{S}^{\intercal s} ; z ; \mathbb{T})$
$= \mathfrak{S}^{\intercal} ; (\mathbb{S}^{q} ; \mathbb{S}^{\intercal r} ; z ; \mathbb{T} \circledcirc \mathbb{S}^{\intercal s} ; z ; \mathbb{T})$
$= \mathbb{S}^{s+q} ; \mathbb{S}^{\intercal r} ; z ; \mathbb{T}$ due to Prop. 6.3.1.ii.h
$= \mathbb{S}^{s*(0+1)+q} ; \mathbb{S}^{\intercal r} ; z ; \mathbb{T}$

$\mathfrak{S}^{\intercal} ; (\mathfrak{S}^{\intercal} \circledcirc \rho^{\intercal})^{l+1} ; (\mathbb{S}^{q} ; \mathbb{S}^{\intercal r} ; z ; \mathbb{T} \circledcirc \mathbb{S}^{\intercal s} ; z ; \mathbb{T})$
$= \mathfrak{S}^{\intercal} ; (\mathfrak{S}^{\intercal} \circledcirc \rho^{\intercal})^{l} ; (\mathfrak{S}^{\intercal} \circledcirc \rho^{\intercal}) ; (\mathbb{S}^{q} ; \mathbb{S}^{\intercal r} ; z ; \mathbb{T} \circledcirc \mathbb{S}^{\intercal s} ; z ; \mathbb{T})$
$= \mathfrak{S}^{\intercal} ; (\mathfrak{S}^{\intercal} \circledcirc \rho^{\intercal})^{l} ; (\mathfrak{S}^{\intercal} ; (\mathbb{S}^{q} ; \mathbb{S}^{\intercal r} ; z ; \mathbb{T} \circledcirc \mathbb{S}^{\intercal s} ; z ; \mathbb{T}) \circledcirc \rho^{\intercal} ; (\mathbb{S}^{q} ; \mathbb{S}^{\intercal r} ; z ; \mathbb{T} \circledcirc \mathbb{S}^{\intercal s} ; z ; \mathbb{T}))$
$= \mathfrak{S}^{\intercal} ; (\mathfrak{S}^{\intercal} \circledcirc \rho^{\intercal})^{l} ; (\mathbb{S}^{s+q} ; \mathbb{S}^{\intercal r} ; z ; \mathbb{T} \circledcirc \mathbb{S}^{\intercal s} ; z ; \mathbb{T})$ due to Prop. 6.3.1.ii.h
$= \mathbb{S}^{s*(l+1)+s+q} ; \mathbb{S}^{\intercal r} ; z ; \mathbb{T}$ hypothesis
$= \mathbb{S}^{s*(l+2)+q} ; \mathbb{S}^{\intercal r} ; z ; \mathbb{T}$

$\mathfrak{S}^{\intercal} ; (\mathfrak{S}^{\intercal} \circledcirc \rho^{\intercal})^{*} ; (\mathbb{S}^{q} ; \mathbb{S}^{\intercal r} ; z ; \mathbb{T} \circledcirc \mathbb{S}^{\intercal s} ; z ; \mathbb{T})$
$= \mathfrak{S}^{\intercal} ; \mathtt{sup}_{l \geq 0} \, (\mathfrak{S}^{\intercal} \circledcirc \rho^{\intercal})^{l} ; (\mathbb{S}^{q} ; \mathbb{S}^{\intercal r} ; z ; \mathbb{T} \circledcirc \mathbb{S}^{\intercal s} ; z ; \mathbb{T})$
$= \mathtt{sup}_{l \geq 0} \, \mathfrak{S}^{\intercal} ; (\mathfrak{S}^{\intercal} \circledcirc \rho^{\intercal})^{l} ; (\mathbb{S}^{q} ; \mathbb{S}^{\intercal r} ; z ; \mathbb{T} \circledcirc \mathbb{S}^{\intercal s} ; z ; \mathbb{T})$

$= \mathtt{sup}_{l\geq 0}\, \mathbb{S}^{\,s*(l+1)+q}; \mathbb{S}^{\,\intercal^r}; z; \mathbb{T} = \big[\mathtt{sup}_{j\geq 1}(\mathbb{S}^{\,s})^j\big]; \mathbb{S}^{\,q}; \mathbb{S}^{\,\intercal^r}; z; \mathbb{T}$ with $j := l+1$
$= (\mathbb{S}^{\,s})^+; \mathbb{S}^{\,q}; \mathbb{S}^{\,\intercal^r}; z; \mathbb{T}$

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vi) The start $n = 1$ of the induction is fairly trivial:

$(\mathfrak{S} \circledcirc \rho); (\mathbb{S}^+ \circledcirc \mathbb{I}) = \mathfrak{S}; \mathbb{S}^+ \cap \rho$ because $\mathfrak{S}, \rho$ are univalent

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$(\mathfrak{S} \circledcirc \rho)^{\,n+1}; (\mathbb{S}^+ \circledcirc \mathbb{I}) = (\mathfrak{S} \circledcirc \rho); (\mathfrak{S} \circledcirc \rho)^{\,n}; (\mathbb{S}^+ \circledcirc \mathbb{I})$
$= (\mathfrak{S} \circledcirc \rho); \big[(\mathfrak{S} \circledcirc \rho)^{\,n-1}; \mathfrak{S}; \mathbb{S}^+ \cap \rho\big]$ hypothesis
$= (\mathfrak{S} \circledcirc \rho)^{\,n}; \mathfrak{S}; \mathbb{S}^+ \cap (\mathfrak{S} \circledcirc \rho); \rho$ since $(\mathfrak{S} \circledcirc \rho)$ is univalent
$= (\mathfrak{S} \circledcirc \rho)^{\,n}; \mathfrak{S}; \mathbb{S}^+ \cap \mathfrak{S}; \mathbb{T} \cap \rho$ destroy and append
$= (\mathfrak{S} \circledcirc \rho)^{\,n}; \mathfrak{S}; \mathbb{S}^+ \cap \rho$ because $(\mathfrak{S} \circledcirc \rho) \subseteq \mathfrak{S}; \pi^\intercal \subseteq \mathfrak{S}; \mathbb{T}$

$(\mathfrak{S} \circledcirc \rho)^*; (\mathbb{S}^+ \circledcirc \mathbb{I}) = \big[\mathtt{sup}_{n\geq 0}\, (\mathfrak{S} \circledcirc \rho)^{\,n}\big]; (\mathbb{S}^+ \circledcirc \mathbb{I})$
$= \mathtt{sup}_{n\geq 0}\, (\mathfrak{S} \circledcirc \rho)^{\,n}; (\mathbb{S}^+ \circledcirc \mathbb{I})$
$= (\mathbb{S}^+ \circledcirc \mathbb{I}) \cup \mathtt{sup}_{n\geq 1}\big[(\mathfrak{S} \circledcirc \rho)^{\,n-1}; \mathfrak{S}; \mathbb{S}^+ \cap \rho\big]$ separated to apply the result above
$= \big[\pi; \mathbb{S}^+ \cap \rho\big] \cup \big[\mathtt{sup}_{i\geq 0}\, (\mathfrak{S} \circledcirc \rho)^{\,i}; \mathfrak{S}; \mathbb{S}^+ \cap \rho\big]$ renumbered with $i := n-1$
$= \big[\pi; \mathbb{S}^+ \cup (\mathfrak{S} \circledcirc \rho)^*; \mathfrak{S}; \mathbb{S}^+\big] \cap \rho$
$= \big[\pi \cup (\mathfrak{S} \circledcirc \rho)^*; \mathfrak{S}\big]; \mathbb{S}^+ \cap \rho$

The latter term is supposed to be equal to $\rho \cap \mathbb{T}; \mathbb{S}$, so that just one piece remains to be proved:

$\rho \cap \mathbb{T}; \mathbb{S} \subseteq \big[\pi \cup (\mathfrak{S} \circledcirc \rho)^*; \mathfrak{S}\big]; \mathbb{S}^+$

Pictorially, in view of the entries $(1,1), (2,1), (3,1), (4,1), (5,1)$ in Fig. 6.3.11, an attempt should be made to prove containment row-wise with every row in two portions. The first of which shall consist for $n \geq 0$ of

$\rho \cap \mathbb{T}; \mathbb{S} \cap \pi; \mathbb{S}^{\,\intercal^n}; z; \mathbb{T} \cap \rho; (\mathbb{S}^{\,\intercal})^+; \mathbb{S}^{\,\intercal^n}; z; \mathbb{T} \subseteq \pi; \mathbb{S}^+.$
$\iff \quad \pi^\intercal; \big\{\pi; \mathbb{S}^{\,\intercal^n}; z; \mathbb{T} \cap \rho; \big[\mathbb{I} \cap \mathbb{T}; \mathbb{S} \cap (\mathbb{S}^{\,\intercal})^+; \mathbb{S}^{\,\intercal^n}; z; \mathbb{T}\big]\big\} \subseteq \mathbb{S}^+$ shunting π and rearranged

Now the left side may, starting with destroy and append, be evaluated as follows:

$\ldots = \mathbb{S}^{\,\intercal^n}; z; \mathbb{T} \cap \mathbb{T}; \big[\mathbb{I} \cap \mathbb{T}; \mathbb{S} \cap (\mathbb{S}^{\,\intercal})^+; \mathbb{S}^{\,\intercal^n}; z; \mathbb{T}\big]$
$\subseteq \mathbb{S}^{\,\intercal^n}; z; \mathbb{T} \cap \mathbb{T}; \big[\mathbb{I} \cap (\mathbb{S}^{\,\intercal})^+; \mathbb{S}^{\,\intercal^n}; z; \mathbb{T}\big]$
$= \mathbb{S}^{\,\intercal^n}; z; \mathbb{T} \cap \mathbb{T}; \big[\mathbb{I} \cap \mathbb{T}; z^\intercal; \mathbb{S}^{\,n}; \mathbb{S}^+\big]$ transposing a partial identity
$= \mathbb{S}^{\,\intercal^n}; z; \mathbb{T} \cap \mathbb{T}; \mathbb{I} \cap \mathbb{T}; z^\intercal; \mathbb{S}^{\,n}; \mathbb{S}^+$ masking
$= \mathbb{S}^{\,\intercal^n}; z; \mathbb{T}; \mathbb{T}; z^\intercal; \mathbb{S}^{\,n}; \mathbb{S}^+$ masking again
$\subseteq \mathbb{I}; \mathbb{S}^+$ since $\mathbb{S}^{\,\intercal^n}; z; \mathbb{T}$ is injective

The other portion is supposed to be contained in the second term:

$\rho \cap \mathbb{T}; \mathbb{S} \cap \pi; \mathbb{S}^{\,\intercal^n}; z; \mathbb{T} \cap \overline{\rho; (\mathbb{S}^{\,\intercal})^+; \mathbb{S}^{\,\intercal^n}; z; \mathbb{T}} \subseteq (\mathfrak{S} \circledcirc \rho)^*; \mathfrak{S}; \mathbb{S}^+$

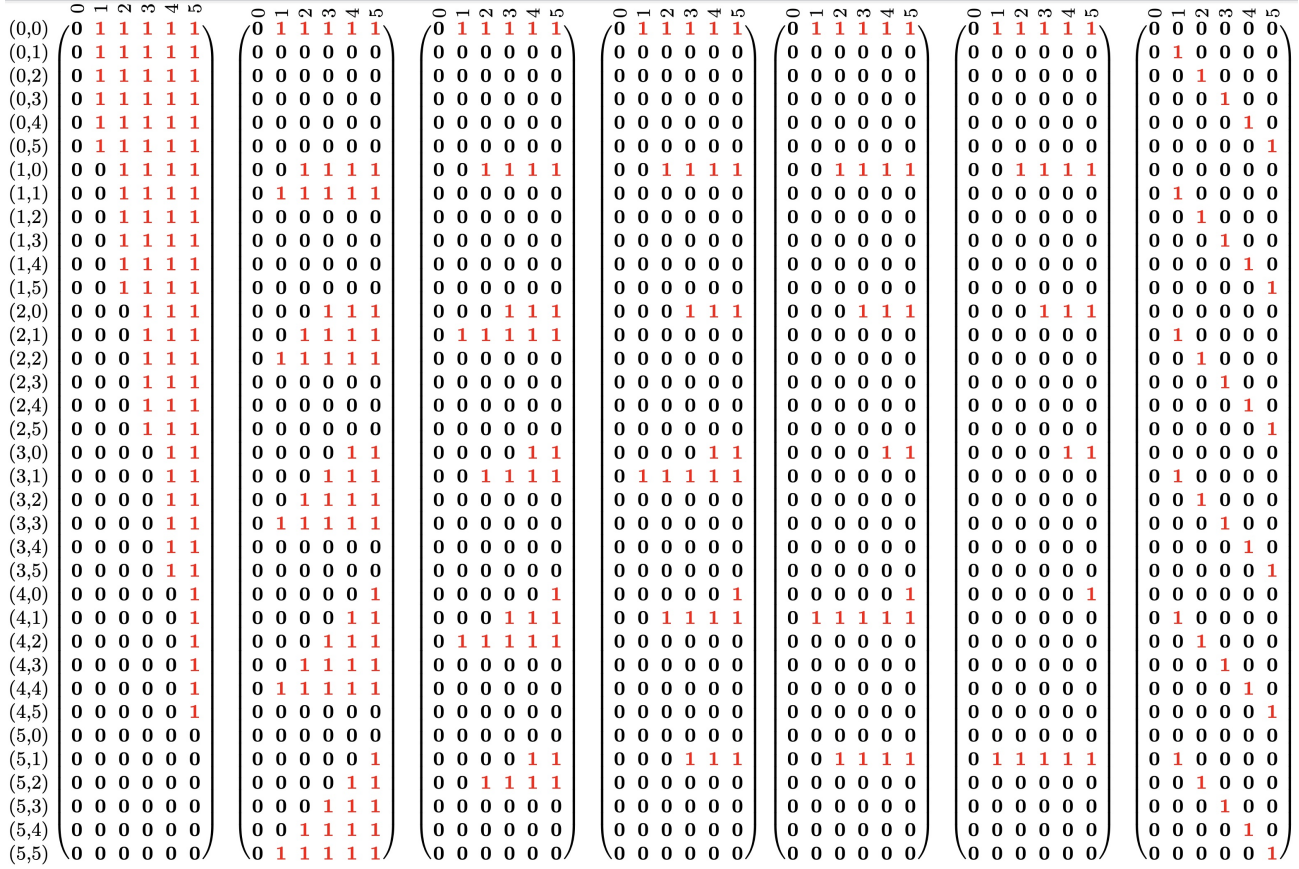


Fig. 6.3.11 $\pi; \mathbb{S}^+$, $(\mathfrak{S} \circledcirc \rho)^n; \mathfrak{S}; \mathbb{S}^+$ for $n = 0, 1, 2, 3, 4$, $\rho \cap \mathbb{T}; \mathbb{S}$

A first observation shows

$\overline{\rho; (\mathbb{S}^\intercal)^+; \mathbb{S}^{\intercal n}; z; \mathbb{T}} = \rho; \overline{(\mathbb{S}^\intercal)^+}; \mathbb{S}^{\intercal n}; z; \mathbb{T}$ since ρ is a map and $\mathbb{S}^{\intercal n}; z; \mathbb{T}$ is a point
$= \rho; \mathbb{S}^*; \mathbb{S}^{\intercal n}; z; \mathbb{T}$ because $(\mathbb{S}^\intercal)^+$ is a linear strictorder

so that we will — recalling entries such as $(3, 1)$, e.g. — try to prove the rearranged

$\pi; \mathbb{S}^{\intercal n}; z; \mathbb{T} \cap \rho; \big[\mathbb{I} \cap \mathbb{S}^*; \mathbb{S}^{\intercal n}; z; \mathbb{T}\big] \cap \mathbb{T}; \mathbb{S} \subseteq (\mathfrak{S} \circledcirc \rho)^*; \mathfrak{S}; \mathbb{S}^+$.

Aiming at a proof by contradiction, let us assume three points $p_i := \mathbb{S}^{\intercal i}; z; \mathbb{T}, p_k := \mathbb{S}^{\intercal k}; z; \mathbb{T}$ and $p_p := \mathbb{S}^{\intercal p}; z; \mathbb{T}$. Then we form an entry $(p_i \circledcirc p_k); p_p^\intercal$ and assume that is contained in the left but not in the right term. This would mean

$(p_i \circledcirc p_k); p_p^\intercal \subseteq \pi; \mathbb{S}^{\intercal n}; z; \mathbb{T}$
$(p_i \circledcirc p_k); p_p^\intercal \subseteq \rho; \big[\mathbb{I} \cap \mathbb{S}^*; \mathbb{S}^{\intercal n}; z; \mathbb{T}\big]$
$(p_i \circledcirc p_k); p_p^\intercal \subseteq \mathbb{T}; \mathbb{S}$
$(p_i \circledcirc p_k); p_p^\intercal \subseteq \overline{(\mathfrak{S} \circledcirc \rho)^*; \mathfrak{S}; \mathbb{S}^+}$.

Shunting in these four conditions the projection maps π, ρ and the points appropriately, we are going to derive a contradiction:

$\pi^\intercal; (p_i \circledcirc p_k); \mathbb{T} = p_i \subseteq \mathbb{S}^{\intercal n}; z; \mathbb{T}; p_p = \mathbb{S}^{\intercal n}; z; \mathbb{T}$ implies $i := n$
$\rho^\intercal; (\mathbb{S}^{\intercal n}; z; \mathbb{T} \circledcirc p_k); \mathbb{T} = \mathbb{S}^{\intercal k}; z; \mathbb{T} \subseteq \big[\mathbb{I} \cap \mathbb{S}^*; \mathbb{S}^{\intercal n}; z; \mathbb{T}\big]; p_p$
$= p_p \cap \mathbb{S}^*; \mathbb{S}^{\intercal n}; z; \mathbb{T}; p_p = p_p \cap \mathbb{S}^*; \mathbb{S}^{\intercal n}; z; \mathbb{T}$ implies $k := p \leq n$
$(\mathbb{S}^{\intercal n}; z; \mathbb{T} \circledcirc p_p); p_p^\intercal \subseteq \mathbb{T}; \mathbb{S}$ implies $p \neq 0$
$\mathbb{T} \subseteq (\mathbb{T}; z^\intercal; \mathbb{S}^n \circledcirc p_p^\intercal); \overline{(\mathfrak{S} \circledcirc \rho)^*; \mathfrak{S}; \mathbb{S}^+}; p_p$ points shunted on both sides

$$= \overline{(\mathbb{T};z^\intercal;\mathbb{S}^n \oslash p_p^\intercal);(\mathfrak{S}\oslash\rho)^*;\mathfrak{S};\mathbb{S}^+;p_p}$$
$$\iff \quad (\mathbb{T};z^\intercal;\mathbb{S}^n \oslash p_p^\intercal);(\mathfrak{S}\oslash\rho)^*;\mathfrak{S};\mathbb{S}^+;p_p \subseteq \mathbb{\perp}$$
$$\iff \quad \mathfrak{S}^\intercal;(\mathfrak{S}^\intercal\oslash\rho^\intercal)^*;(\mathbb{S}^{\intercal n};z;\mathbb{T}\oslash p_p);\mathbb{T}$$
$$\subseteq \overline{\mathbb{S}^+;p_p} = \overline{\mathbb{S}^+};p_p = (\mathbb{S}^\intercal)^*;p_p \quad \text{Schröder rule, linear order}$$

Evaluating now the left side with Prop. 6.3.1.ii.h, we find out that this means

$(\mathbb{S}^p)^+;\mathbb{S}^{\intercal n};z;\mathbb{T} \subseteq (\mathbb{S}^\intercal)^*;p_p$.

It is not easy to recognize the contradiction herein. The right side describes all the rows equal to or above p. Now consider the decreasing sequence $n, n-p, n-2*p, n-3*p, \ldots$ as long as it stays ≥ 0. Let l be the maximum factor, i.e. $n-l*p \geq 0$ but $n-(l+1)*p < 0$. Then obviously $n-l*p-p<0$ or else $n-l*p<p$. The corresponding row could, thus, not be contained in the right side.

The second formula is only a slight variation of the first concerning $(\mathbb{I}\oslash\mathbb{I})$ added to the right factor. We prove by induction that for all $n\geq 0$

$(\mathfrak{S}\oslash\rho)^n;(\mathbb{I}\oslash\mathbb{I}) \subseteq \rho\cap\big[\pi\cup\mathbb{T};\mathbb{S}\big]$.

The case $n=0$ brings $\mathbb{I};(\mathbb{I}\oslash\mathbb{I}) = \pi\cap\rho\subseteq\rho\cap\big[\pi\cup\mathbb{T};\mathbb{S}\big]$. Then the induction step:

$(\mathfrak{S}\oslash\rho)^{n+1};(\mathbb{I}\oslash\mathbb{I}) = (\mathfrak{S}\oslash\rho);(\mathfrak{S}\oslash\rho)^n;(\mathbb{I}\oslash\mathbb{I}) \subseteq (\mathfrak{S}\oslash\rho);\big(\rho\cap[\pi\cup\mathbb{T};\mathbb{S}]\big)$ hypoth.
$= (\mathfrak{S}\oslash\rho);\rho\cap(\mathfrak{S}\oslash\rho);[\pi\cup\mathbb{T};\mathbb{S}]$
$= (\mathfrak{S};\mathbb{T}\cap\rho)\cap[(\mathfrak{S}\oslash\rho);\pi\cup(\mathfrak{S}\oslash\rho);\mathbb{T};\mathbb{S}]$
$\subseteq \rho\cap[(\mathfrak{S}\cap\mathbb{T})\cup\mathbb{T};\mathbb{S}]$

It remains to show

$\rho\cap\mathfrak{S} = \rho\cap\mathtt{sup}_{n\geq0}(\mathbb{S}^{\intercal n}\oslash\mathbb{S}^{\intercal n};z;\mathbb{T}) = \mathtt{sup}_{n\geq0}\rho\cap(\mathbb{S}^{\intercal n}\oslash\mathbb{S}^{\intercal n};z;\mathbb{T})$
$= \mathtt{sup}_{n\geq0}\pi;\mathbb{S}^{\intercal n}\cap\rho;[\mathbb{I}\cap\mathbb{S}^{\intercal n};z;\mathbb{T}]$
$= \mathtt{sup}_{n\geq0}\pi;\mathbb{S}^{\intercal n}\cap\rho;[\mathbb{I}\cap\mathbb{T};z^\intercal;\mathbb{S}^n]$ transposing a partial identity
$= \big\{\pi;\mathbb{S}^{\intercal 0}\cap\rho;[\mathbb{I}\cap\mathbb{T};z^\intercal;\mathbb{S}^0]\big\}\cup\mathtt{sup}_{i\geq1}\pi;\mathbb{S}^{\intercal i}\cap\rho;[\mathbb{I}\cap\mathbb{T};z^\intercal;\mathbb{S}^i]$
$\subseteq \pi\cup\mathtt{sup}_{i\geq1}\rho\cap\mathbb{T};z^\intercal;\mathbb{S}^i$
$\subseteq \pi\cup\mathtt{sup}_{i\geq1}\mathbb{T};z^\intercal;\mathbb{S}^i \subseteq \pi\cup\mathbb{T};\mathbb{S}$

---

vii) Proof by induction. The start at $n=0$ is trivial according to Prop. 6.3.1.i.a:

$(\mathfrak{S}\oslash\rho)^{n+1};\mathfrak{S} = (\mathfrak{S}\oslash\rho);(\mathfrak{S}\oslash\rho)^n;\mathfrak{S}$
$= (\mathfrak{S}\oslash\rho);\mathtt{sup}_{i\geq0}(\mathbb{S}^{\intercal(n+1)*i}\oslash\mathbb{S}^{\intercal i};z;\mathbb{T})$ hypothesis
$= \mathtt{sup}_{i\geq0}\mathfrak{S};\mathbb{S}^{\intercal(n+1)*i}\cap\rho;\mathbb{S}^{\intercal i};z;\mathbb{T}$ since $\mathfrak{S},\rho$ are univalent
$= \mathtt{sup}_{i\geq0}\big[\mathtt{sup}_{k\geq0}(\mathbb{S}^{\intercal k}\oslash\mathbb{S}^{\intercal k};z;\mathbb{T})\big];\mathbb{S}^{\intercal(n+1)*i}\cap\rho;\mathbb{S}^{\intercal i};z;\mathbb{T}$ expanding $\mathfrak{S}$
$= \mathtt{sup}_{i\geq0}\mathtt{sup}_{k\geq0}(\pi;\mathbb{S}^{\intercal k}\cap\rho;\mathbb{S}^{\intercal k};z;\mathbb{T});\mathbb{S}^{\intercal(n+1)*i}\cap\rho;\mathbb{S}^{\intercal i};z;\mathbb{T}$
$= \mathtt{sup}_{i\geq0}\mathtt{sup}_{k\geq0}\pi;\mathbb{S}^{\intercal k};\mathbb{S}^{\intercal(n+1)*i}\cap\rho;\mathbb{S}^{\intercal k};z;\mathbb{T}\cap\rho;\mathbb{S}^{\intercal i};z;\mathbb{T}$ masking
$= \mathtt{sup}_{i\geq0}\pi;\mathbb{S}^{\intercal i};\mathbb{S}^{\intercal(n+1)*i}\cap\rho;\mathbb{S}^{\intercal i};z;\mathbb{T}$ implying $k:=i$, otherwise $\ldots=\mathbb{\perp}$
$= \mathtt{sup}_{i\geq0}\pi;\mathbb{S}^{\intercal(n+2)*i}\cap\rho;\mathbb{S}^{\intercal i};z;\mathbb{T}$
$= \mathtt{sup}_{i\geq0}(\mathbb{S}^{\intercal(n+2)*i}\oslash\mathbb{S}^{\intercal i};z;\mathbb{T})$

---

viii) $(\mathbb{T};z^\intercal\oslash\mathbb{T};z^\intercal);(\mathfrak{S}\oslash\rho) = ((\mathbb{T};z^\intercal\oslash\mathbb{T};z^\intercal);\mathfrak{S}\oslash(\mathbb{T};z^\intercal\oslash\mathbb{T};z^\intercal);\rho)$
$= (\mathbb{T};z^\intercal\oslash\mathbb{T};z^\intercal)$

$(\mathbb{I}\oslash\mathbb{I});(\mathfrak{S}\oslash\rho) = ((\mathbb{I}\oslash\mathbb{I});\mathfrak{S}\oslash(\mathbb{I}\oslash\mathbb{I});\rho)$

$= (\mathbb{T} ; z^\mathsf{T} \oslash \mathbb{I})$ using Prop. 6.3.1.ii.b and the destroy and append rule

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Proof by induction beginning with

$(\mathbb{I} \oslash \mathbb{I}) ; (\mathfrak{S} \oslash \rho)^2 = (\mathbb{T} ; z^\mathsf{T} \oslash \mathbb{I}) ; (\mathfrak{S} \oslash \rho)^1$ see before

$= ((\mathbb{T} ; z^\mathsf{T} \oslash \mathbb{I}) ; \mathfrak{S} \oslash (\mathbb{T} ; z^\mathsf{T} \oslash \mathbb{I}) ; \rho)$

$= (z ; \mathbb{T} ; z^\mathsf{T} \oslash \mathbb{I})$ Prop. 6.3.1.i.c

---

ix) $(z ; \mathbb{T} \ominus z ; \mathbb{T}) ; \mathbb{T} ; z^\mathsf{T} = [\pi ; z ; \mathbb{T} \cap \rho ; z ; \mathbb{T}] ; \mathbb{T} ; z^\mathsf{T} = \pi ; z ; \mathbb{T} ; z^\mathsf{T} \cap \rho ; z ; \mathbb{T}$ masking

while on the other side

$(z ; \mathbb{T} ; z^\mathsf{T} \ominus \mathbb{I}) = \pi ; z ; \mathbb{T} ; z^\mathsf{T} \cap \rho$

Equality now follows mainly observing that twice via the Dedekind rule

$\pi ; z ; \mathbb{T} ; z^\mathsf{T} \cap \rho ; z ; \mathbb{T} \subseteq (\pi \cap \rho ; z ; \mathbb{T} ; z ; \mathbb{T} ; z^\mathsf{T}) ; (z ; \mathbb{T} ; z^\mathsf{T} \cap \pi^\mathsf{T} ; \rho ; z ; \mathbb{T}) \subseteq (\ldots \cap \rho ; \mathbb{I}) ; (\mathbb{I} \cap \ldots)$

$\pi ; z ; \mathbb{T} ; z^\mathsf{T} \cap \rho \subseteq (\pi \cap \rho ; z ; \mathbb{T} ; z^\mathsf{T}) ; (z ; \mathbb{T} ; z^\mathsf{T} \cap \pi^\mathsf{T} ; \rho) \subseteq \rho ; z ; \mathbb{T} ; z^\mathsf{T} ; \mathbb{T} = \rho ; z ; \mathbb{T}$

The other variant is proved analogously. □

6.3.3 Proposition. We assume $\mathbb{S}$ to be regressively finite.

i) $(\mathbb{S} \otimes \mathbb{I}) ; (\mathfrak{S} \oslash \rho) ; (\mathbb{S}^\mathsf{T} \otimes \mathbb{I}) = (\mathfrak{S} \oslash \rho) \cap \pi ; \mathbb{S} ; \mathbb{T} \subseteq (\mathfrak{S} \oslash \rho)$

ii) $(\mathbb{S}^\mathsf{T} \otimes \mathbb{I}) ; (\mathfrak{S} \oslash \rho) ; (\mathbb{S} \otimes \mathbb{I}) = (\mathfrak{S} \oslash \rho) \cap \mathbb{T} ; \mathbb{S} ; \pi^\mathsf{T} \subseteq (\mathfrak{S} \oslash \rho)$

Proof: i) $(\mathbb{S} \otimes \mathbb{I}) ; (\mathfrak{S} \oslash \rho) ; (\mathbb{S}^\mathsf{T} \otimes \mathbb{I}) = (\mathbb{S} \otimes \mathbb{I}) ; (\mathfrak{S} ; \mathbb{S}^\mathsf{T} \oslash \rho)$

$= (\mathbb{S} \otimes \mathbb{I}) ; ((\mathbb{I} \otimes \mathbb{S}) ; \mathfrak{S} \oslash \rho)$ due to Prop. 6.3.1.ii.c

$= ((\mathbb{S} \otimes \mathbb{I}) ; (\mathbb{I} \otimes \mathbb{S}) ; \mathfrak{S} \oslash (\mathbb{S} \otimes \mathbb{I}) ; \rho)$ since the factor $(\mathbb{S} \otimes \mathbb{I})$ is univalent

$= ((\mathbb{S} \otimes \mathbb{S}) ; \mathfrak{S} \oslash (\mathbb{S} ; \mathbb{T} \ominus \mathbb{I}))$ destroy and append

$= (\mathfrak{S} \cap \pi ; \mathbb{S} ; \mathbb{T} \oslash (\mathbb{S} ; \mathbb{T} \ominus \mathbb{I}))$ Prop. 6.3.1.ii.l

$= (\mathfrak{S} \cap \pi ; \mathbb{S} ; \mathbb{T}) ; \pi^\mathsf{T} \cap (\pi ; \mathbb{S} ; \mathbb{T} \cap \rho) ; \rho^\mathsf{T}$

$= \mathfrak{S} ; \pi^\mathsf{T} \cap \pi ; \mathbb{S} ; \mathbb{T} ; \pi^\mathsf{T} \cap \pi ; \mathbb{S} ; \mathbb{T} ; \rho^\mathsf{T} \cap \rho ; \rho^\mathsf{T} = \mathfrak{S} ; \pi^\mathsf{T} \cap \pi ; \mathbb{S} ; \mathbb{T} \cap \pi ; \mathbb{S} ; \mathbb{T} \cap \rho ; \rho^\mathsf{T}$

$= \mathfrak{S} ; \pi^\mathsf{T} \cap \rho ; \rho^\mathsf{T} \cap \pi ; \mathbb{S} ; \mathbb{T}$

$= (\mathfrak{S} \oslash \rho) \cap \pi ; \mathbb{S} ; \mathbb{T} \subseteq (\mathfrak{S} \oslash \rho)$

---

ii) $(\mathbb{S}^\mathsf{T} \otimes \mathbb{I}) ; (\mathfrak{S} \oslash \rho) ; (\mathbb{S} \otimes \mathbb{I}) = (\mathbb{S}^\mathsf{T} \otimes \mathbb{I}) ; (\mathfrak{S} ; \mathbb{S} \oslash \rho)$

$= ((\mathbb{S}^\mathsf{T} \otimes \mathbb{I}) ; \mathfrak{S} ; \mathbb{S} \oslash (\mathbb{S}^\mathsf{T} \otimes \mathbb{I}) ; \rho)$ since the factor on the left is univalent

$= (\mathfrak{S} \cap \mathbb{T} ; \mathbb{S} \oslash (\mathbb{S}^\mathsf{T} ; \mathbb{T} \ominus \mathbb{I}))$ due to Prop. 6.3.1.i.d

$= \mathfrak{S} ; \pi^\mathsf{T} \cap \mathbb{T} ; \mathbb{S} ; \pi^\mathsf{T} \cap \pi ; \mathbb{S}^\mathsf{T} ; \mathbb{T} ; \rho^\mathsf{T} \cap \rho ; \rho^\mathsf{T}$

$= \mathfrak{S} ; \pi^\mathsf{T} \cap \rho ; \rho^\mathsf{T} \cap \mathbb{T} ; \mathbb{S} ; \pi^\mathsf{T} \cap \pi ; \mathbb{S}^\mathsf{T} ; \mathbb{T}$

$= (\mathfrak{S} \oslash \rho) \cap \mathbb{T} ; \mathbb{S} ; \pi^\mathsf{T} \cap \pi ; \mathbb{S}^\mathsf{T} ; \mathbb{T}$

$= (\mathfrak{S} \oslash \rho) \cap \mathbb{T} ; \mathbb{S} ; \pi^\mathsf{T} \subseteq (\mathfrak{S} \oslash \rho)$ see below

It has yet to be shown that $(\mathfrak{S} \oslash \rho) \cap \mathbb{T} ; \mathbb{S} ; \pi^\mathsf{T} \subseteq \pi ; \mathbb{S}^\mathsf{T} ; \mathbb{T}$. This, however, follows from Prop. 6.3.2:

$(\mathfrak{S} \oslash \rho) \cap \mathbb{T} ; \mathbb{S} ; \pi^\mathsf{T} \subseteq \left[\mathbb{I} \cup \pi ; \mathbb{S}^{\mathsf{T}+} ; \pi^\mathsf{T}\right] \cap \mathbb{T} ; \mathbb{S} ; \pi^\mathsf{T}$

$= \left[\mathbb{I} \cap \mathbb{T} ; \mathbb{S} ; \pi^\mathsf{T}\right] \cup \left[\pi ; \mathbb{S}^{\mathsf{T}+} ; \pi^\mathsf{T} \cap \mathbb{T} ; \mathbb{S} ; \pi^\mathsf{T}\right]$

$= \left[\mathbb{I} \cap \pi ; \mathbb{S}^\mathsf{T} ; \mathbb{T}\right] \cup \left[\pi ; \mathbb{S}^{\mathsf{T}+} ; \pi^\mathsf{T} \cap \mathbb{T} ; \mathbb{S} ; \pi^\mathsf{T}\right]$ transposing a partial identity

$\subseteq \pi ; \mathbb{S}^\mathsf{T} ; \mathbb{T}$ □

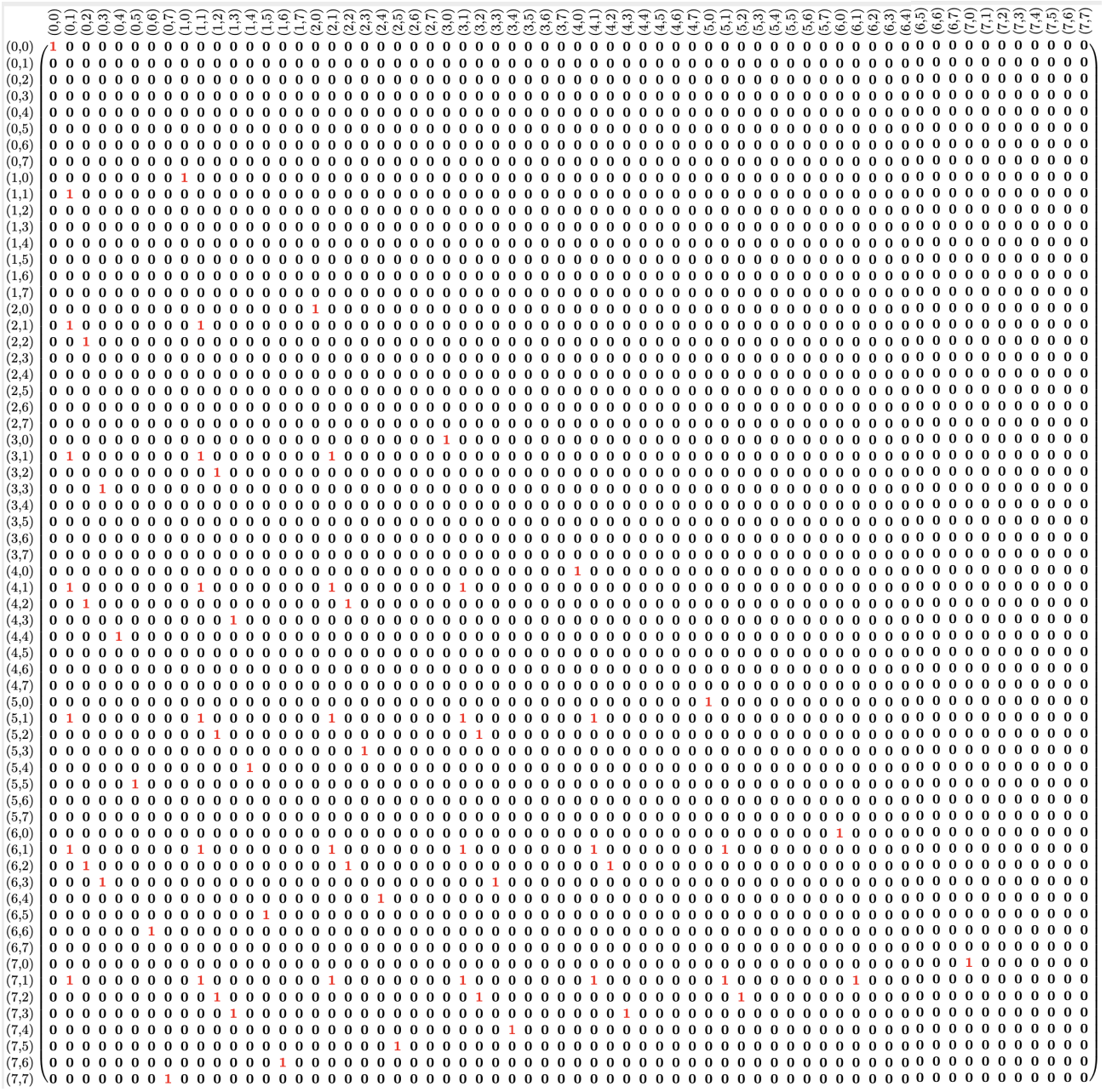


Fig. 6.3.12 $(\mathfrak{S} \oslash \rho)^+ = \mathtt{sup}_{n>0}\, (\mathfrak{S} \oslash \rho)^n \subseteq (\mathbb{S}^{\intercal *} \otimes \mathbb{I})$

An observation is immediate: There are diagonals parallel to the identity and below it. The pattern of the room between them, here corresponding to the sequence $1, \ldots, 7$, seems to depend on the limit chosen. Therefore, number theoreticians may not be aware of them.

These observations seem to hold prior to the definition of primes or even of multiplication:

$$\begin{aligned}\mathtt{SubtrRhoRed} &:= (\mathfrak{S} \oslash \rho)^+ \cap \mathbb{T} ; z^\intercal ; \pi^\intercal \cap \overline{\mathbb{T} ; e^\intercal} ; \rho^\intercal \cap \overline{(\mathbb{I} \oslash \mathbb{I}) ; \mathbb{T}} \\ &= (\mathfrak{S} \oslash \rho)^+ \cap (\mathbb{T} ; z^\intercal \oslash \overline{\mathbb{T} ; e^\intercal}) \cap \overline{(\mathbb{I} \oslash \mathbb{I}) ; \mathbb{T}}\end{aligned}$$

In notation introduced only later, this might provide the definition

$$\wp_1 ; \mathbb{T} := \overline{z ; \mathbb{T} \cup \pi^\intercal ; \mathtt{SubtrRhoRed} ; \mathbb{T}}.$$

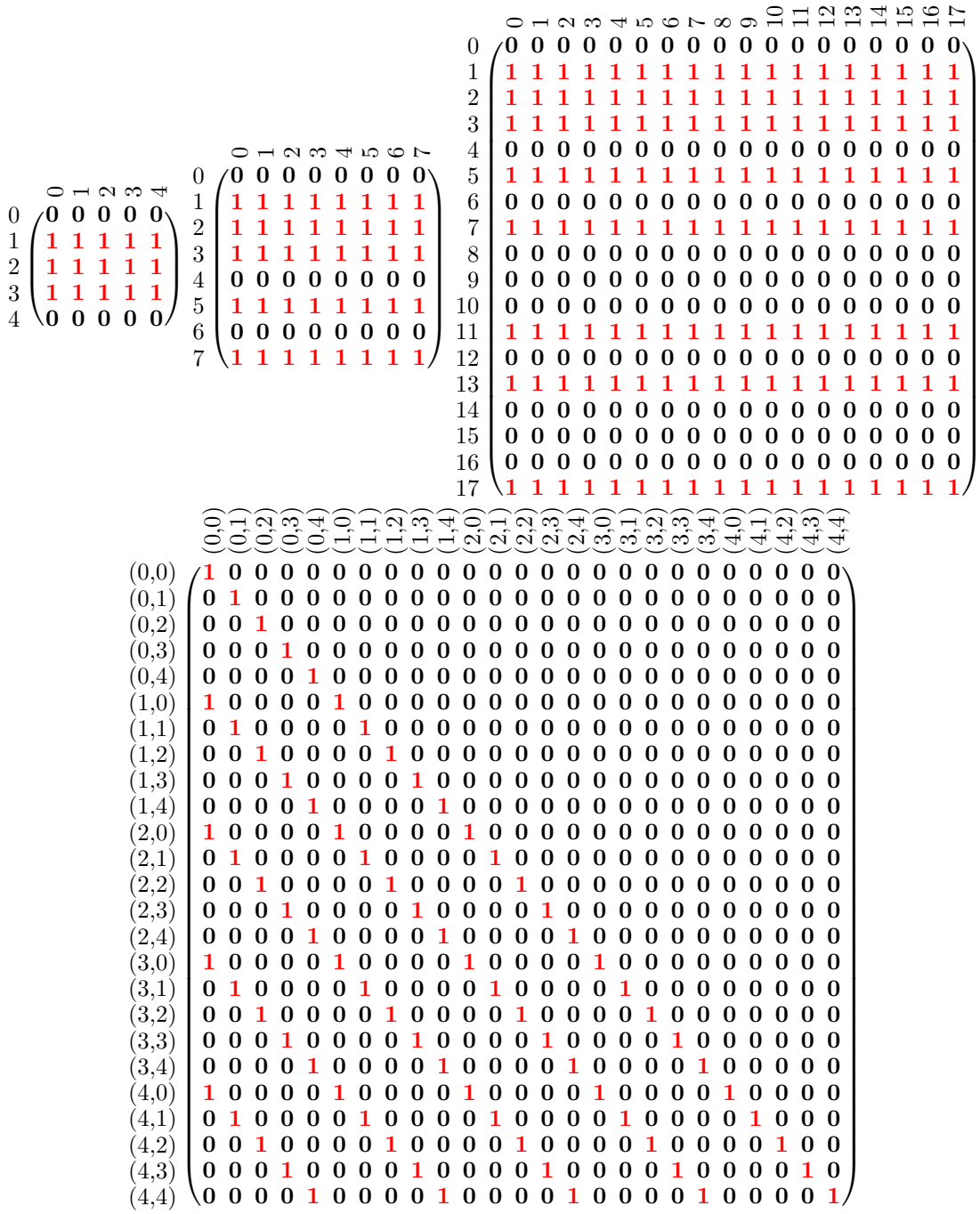


Fig. 6.3.13 Primes derived as $\wp_1 ; \mathbb{T} = \overline{\pi^\mathsf{T} ; \texttt{SubtrRhoRed} ; \mathbb{T}} \cap \overline{z ; \mathbb{T}}$ — when limited by 4, 7, 17 and also the shape $(\mathbb{S}^{\mathsf{T}*} \otimes \mathbb{I})$ in order to be recognized in Fig. 6.3.12

When we conceive — prior to their formal definition — the primes $1, 2, 3, 5, 7, 11 \ldots$ as a vector $\wp_1 ; \mathbb{T}$ along the naturals, then apparently $\wp_1 ; \mathbb{T} = \overline{\pi^\mathsf{T} ; \texttt{SubtrRhoRed} ; \mathbb{T}} \cap \overline{z ; \mathbb{T}}$. That this is likely to be correct, shows the following comment concerning numbers 15 and 17 when we subtract iteratedly:

$$(15,5)\quad (10,5)\quad (5,5)\quad (0,5)\qquad \text{and}\qquad (15,3)\quad (12,3)\quad (9,3)\quad (6,3)\quad (3,3)\quad (0,3)$$
$$(17,17)\quad (0,17)\qquad\qquad\qquad \text{and}\qquad (17,1)\quad (16,1)\quad \ldots\quad (1,1)\quad (0,1)$$

In the formula above it is taken care with $\mathbb{T} ; z^\mathsf{T} ; \pi^\mathsf{T}$ that the resulting pair has first component

0, with $\overline{\mathbb{T}\,;e^{\mathsf{T}}}\,;\rho^{\mathsf{T}}$ that the second component is not the 1 and with $\overline{(\mathbb{I}\oslash\mathbb{I})\,;\mathbb{T}}$ that the argument pair has not equal components. Thus, only for primes there are no intermediate pairs.

There is a closely related variant form:

$$\wp_1\,;\mathbb{T} = \overline{\pi^{\mathsf{T}}\,;\,(\mathfrak{S}\oslash\rho)\,;\,(\mathfrak{S}\oslash\rho)^{+}\,;\,(z\,;\mathbb{T}\oslash\overline{e\,;\mathbb{T}})}$$

One will observe that, starting from some pair (n,d), iterated subtraction of ever the same d is executed at least two times — thus avoiding (n,n). Only pairs formed of 0 and $\neq 1$ are accepted as results — meaning that d divides n, however, not in the special situation $d=1$, so that $n = n*1$. Of course, this is not a proof, but it delivers relations such as in the upper row of Fig. 6.3.13.

We do not even need subtraction $\mathfrak{S}$; via the inversion rules also this holds

$$\wp_1\,;\mathbb{T} = \overline{\pi^{\mathsf{T}}\,;\,(\,\mathbb{A}^{\mathsf{T}}\oslash\rho^{\mathsf{T}})\,;\,(\,\mathbb{A}^{\mathsf{T}}\oslash\rho^{\mathsf{T}})^{+}\,;\,(z\,;\mathbb{T}\oslash e\,;\mathbb{T})}\,.$$

	0	1	2	3	4	5
(0,0)	**1**	0	0	0	0	0
(0,1)	0	**1**	0	0	0	0
(0,2)	0	0	**1**	0	0	0
(0,3)	0	0	0	**1**	0	0
(0,4)	0	0	0	0	**1**	0
(0,5)	0	0	0	0	0	**1**
(1,0)	0	**1**	0	0	0	0
(1,1)	**1**	0	0	0	0	0
(1,2)	0	0	**1**	0	0	0
(1,3)	0	0	0	**1**	0	0
(1,4)	0	0	0	0	**1**	0
(1,5)	0	0	0	0	0	**1**
(2,0)	0	0	**1**	0	0	0
(2,1)	0	**1**	0	0	0	0
(2,2)	**1**	0	0	0	0	0
(2,3)	0	0	0	**1**	0	0
(2,4)	0	0	0	0	**1**	0
(2,5)	0	0	0	0	0	**1**
(3,0)	0	0	0	**1**	0	0
(3,1)	0	0	**1**	0	0	0
(3,2)	0	**1**	0	0	0	0
(3,3)	**1**	0	0	0	0	0
(3,4)	0	0	0	0	**1**	0
(3,5)	0	0	0	0	0	**1**
(4,0)	0	0	0	0	**1**	0
(4,1)	0	0	0	**1**	0	0
(4,2)	0	0	**1**	0	0	0
(4,3)	0	**1**	0	0	0	0
(4,4)	**1**	0	0	0	0	0
(4,5)	0	0	0	0	0	**1**
(5,0)	0	0	0	0	0	**1**
(5,1)	0	0	0	0	**1**	0
(5,2)	0	0	0	**1**	0	0
(5,3)	0	0	**1**	0	0	0
(5,4)	0	**1**	0	0	0	0
(5,5)	**1**	0	0	0	0	0

Fig. 6.3.14 The mapping $\mathfrak{S}\cup(\mathbb{S}^{+}\oslash\mathbb{I})$

6.3.4 Proposition. i) $\mathfrak{S}\cup(\mathbb{S}^{+}\oslash\mathbb{I})$ is a mapping; it executes

$(a,b)\mapsto a-b$ in case $a\geq b$ and $(a,b)\mapsto b$ if $a<b$.

ii) $\overline{(\mathfrak{S}\oslash\rho)^{+}\,;\mathbb{T}} = \overline{(\mathfrak{S}\oslash\rho)\,;\mathbb{T}} = \overline{\mathfrak{S}\,;\mathbb{T}} = (\mathbb{S}^{+}\oslash\mathbb{I})\,;\mathbb{T}$

Proof: i) $\mathfrak{S}$ as well as $(\mathbb{S}^+ \oslash \mathbb{I})$ are certainly univalent since $(\mathbb{S}^+ \oslash \mathbb{I}) \subseteq \rho$. It remains to show that they have complementary regions of definition. To this end, we assume points $a := \mathbb{S}^{\intercal i}; z; \mathbb{T}$ and $b := \mathbb{S}^{\intercal k}; z; \mathbb{T}$ leading to a point $(a \oslash b); \mathbb{T}$. Then with shunting

$$\begin{array}{lll} (a \oslash b); \mathbb{T} \subseteq \mathfrak{S}; \mathbb{T} & \iff & \mathbb{T} \subseteq (a^\intercal \otimes b^\intercal); \mathfrak{S}; \mathbb{T} \quad \iff \quad i \geq k \\ (a \oslash b); \mathbb{T} \subseteq (\mathbb{S}^+ \oslash \mathbb{I}); \mathbb{T} & \iff & \mathbb{T} \subseteq (a^\intercal \otimes b^\intercal); (\mathbb{S}^+ \oslash \mathbb{I}); \mathbb{T} \quad \iff \quad i < k \end{array}$$

---

ii) "$\supseteq$" $\iff$ $(\mathbb{S}^+ \oslash \mathbb{I})^\intercal; (\mathfrak{S} \otimes \rho)^+; \mathbb{T} \subseteq \mathbb{1}$ Schröder rule

$\impliedby$ $(\mathbb{S}^\intercal \otimes \mathbb{I}); (\mathfrak{S} \otimes \rho) \subseteq \mathbb{1}$

$\impliedby$ $(\mathbb{S}^\intercal \otimes \mathbb{I}); \mathfrak{S} \subseteq \mathbb{1}$ due to Prop. 6.3.1.ii.c

"$\subseteq$" follows via the point-wise reasoning in (i); and these are equivalent:

$$\overline{(\mathfrak{S} \otimes \rho)^+; \mathbb{T}} = \overline{(\mathfrak{S} \otimes \rho); (\mathfrak{S} \otimes \rho)^*; \mathbb{T}} = \overline{(\mathfrak{S} \otimes \rho); \mathbb{T}} = \overline{(\mathfrak{S} \otimes \rho); \pi; \mathbb{T}} = \overline{\mathfrak{S}; \mathbb{T}} \qquad \square$$

$(\mathfrak{S} \otimes \rho) =$

	0	1	2	3	4	5
0	0					
1	6	1				
2	12	7	2			
3	18	13	8	3		
4	24	19	14	9	4	
5	30	25	20	15	10	5

, based on the following

	(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(1,0)	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(2,0)	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(3,0)	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(4,0)	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(5,0)	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)
(0,0)	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(0,1)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(0,2)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(0,3)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(0,4)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(0,5)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(1,0)	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(1,1)	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(1,2)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(1,3)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(1,4)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(1,5)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(2,0)	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(2,1)	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(2,2)	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(2,3)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(2,4)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(2,5)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(3,0)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(3,1)	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(3,2)	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(3,3)	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(3,4)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(3,5)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(4,0)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
(4,1)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
(4,2)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(4,3)	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(4,4)	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(4,5)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(5,0)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
(5,1)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
(5,2)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
(5,3)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
(5,4)	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
(5,5)	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

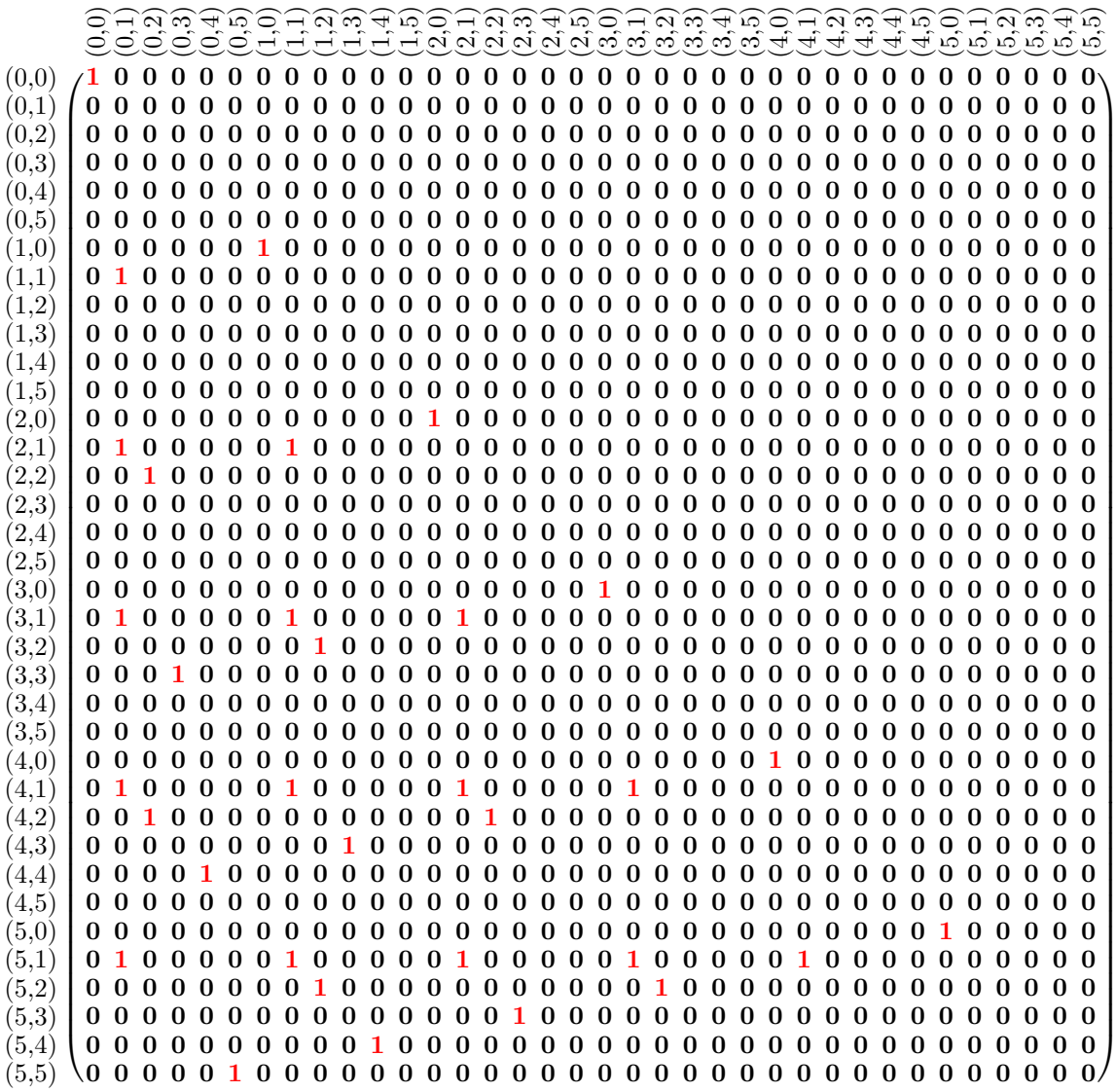
$(\mathfrak{S} \circledcirc \rho)^2 =$

	0	1	2	3	4	5
0	0					
1	6					
2	12	1				
3	18	7				
4	24	13	2			
5	30	19	8			

, based on the following

	(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(1,0)	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(2,0)	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(3,0)	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(4,0)	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(5,0)	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)
(0,0)	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(0,1)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(0,2)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(0,3)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(0,4)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(0,5)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(1,0)	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(1,1)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(1,2)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(1,3)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(1,4)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(1,5)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(2,0)	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(2,1)	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(2,2)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(2,3)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(2,4)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(2,5)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(3,0)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(3,1)	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(3,2)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(3,3)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(3,4)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(3,5)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(4,0)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
(4,1)	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(4,2)	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(4,3)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(4,4)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(4,5)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(5,0)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
(5,1)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(5,2)	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(5,3)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(5,4)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(5,5)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$(\mathfrak{S} \circledcirc \rho)^+ =$



$(\mathfrak{S} \oslash \rho)^* =$

	(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(1,0)	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(2,0)	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(3,0)	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(4,0)	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(5,0)	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)
(0,0)	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(0,1)	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(0,2)	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(0,3)	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(0,4)	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(0,5)	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(1,0)	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(1,1)	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(1,2)	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(1,3)	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(1,4)	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(1,5)	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(2,0)	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(2,1)	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(2,2)	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(2,3)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(2,4)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(2,5)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(3,0)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(3,1)	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(3,2)	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(3,3)	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(3,4)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
(3,5)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
(4,0)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
(4,1)	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
(4,2)	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
(4,3)	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
(4,4)	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
(4,5)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
(5,0)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
(5,1)	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0
(5,2)	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
(5,3)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
(5,4)	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
(5,5)	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

6.4 Iterated addition

Having in view the introduction of multiplication, we will first iterate addition as follows:

6.4.1 Definition. We define the n-**fold sum** relations *iterated over the first component* by

$$k_0 := \mathbb{T} \mathbin{;} z^\intercal \qquad k_{n+1} := (k_n \oslash \mathbb{I}) \mathbin{;} \mathbb{A}\,.$$

Correspondingly, we define the n-**fold sum** relations *iterated over the second component* by

$$l_0 := \mathbb{T} \mathbin{;} z^\intercal \qquad l_{n+1} := (\mathbb{I} \oslash l_n) \mathbin{;} B. \qquad \square$$

So k_0, l_0, considered as mappings, will always deliver 0,

$$k_1 = (k_0 \oslash \mathbb{I}) \mathbin{;} \mathbb{A} = (\mathbb{T} \mathbin{;} z^\intercal \oslash \mathbb{I}) \mathbin{;} \mathbb{A} = \mathbb{I} \quad \text{Prop. 6.2.1.i.c,}$$

and one will grasp the meaning of

$$k_2 = (k_1 \oslash \mathbb{I}) \mathbin{;} \mathbb{A} = (\mathbb{I} \oslash \mathbb{I}) \mathbin{;} \mathbb{A}\,,$$
$$k_3 = (k_2 \oslash \mathbb{I}) \mathbin{;} \mathbb{A} = (\,(\mathbb{I} \oslash \mathbb{I}) \mathbin{;} \mathbb{A} \oslash \mathbb{I}) \mathbin{;} \mathbb{A}\,,$$

as visualized in Fig. 6.4.1.

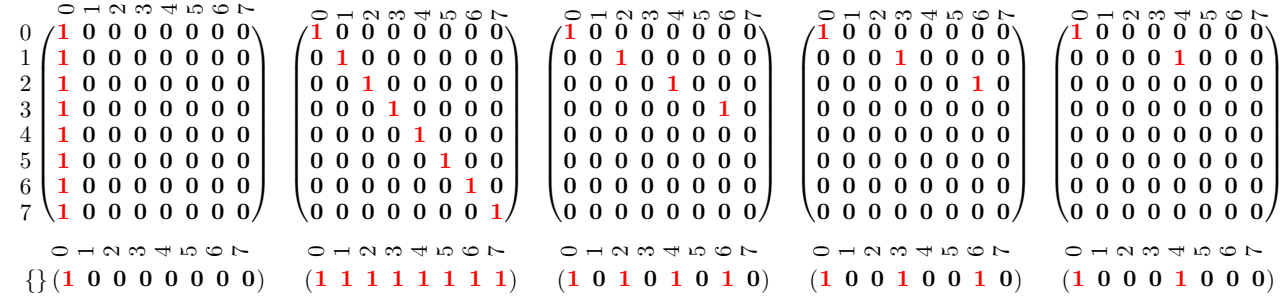


Fig. 6.4.1 n-fold sums k_0, k_1, k_2, k_3, k_4, together with row vectors $\mathbb{T} ; k_n$

The following lemma relates the n-fold sum with successor forming.

6.4.2 Lemma. Concerning this n-fold sum, the following will hold

i) in general:

a) The k_n are univalent and satisfy $k_n = l_n$ for all $n \geq 0$.

$k_n \cap \overline{k_n ; \overline{\mathbb{I}}} = k_n$ for all $n \geq 0$

b) $k_n^\mathsf{T} ; z ; \mathbb{T} = z ; \mathbb{T}, \qquad k_n^\mathsf{T} ; e ; \mathbb{T} = \mathbb{S}^{\mathsf{T} n} ; z ; \mathbb{T} \qquad$ for all $n \geq 0 \qquad$ and more general:

$$k_0^\mathsf{T} ; \mathbb{S}^{\mathsf{T} i} ; z ; \mathbb{T} = \begin{cases} \perp\!\!\!\perp & \text{if } \mathbb{S}^{\mathsf{T} i} ; z ; \mathbb{T} = \perp\!\!\!\perp \\ z ; \mathbb{T} & \text{otherwise} \end{cases}$$

For all $n \geq 1, i \geq 0$: $\quad k_n^\mathsf{T} ; \mathbb{S}^{\mathsf{T} i} ; z ; \mathbb{T} = \mathbb{S}^{\mathsf{T} n * i} ; z ; \mathbb{T} \qquad$ (satisfied also for $\mathbb{S}^{\mathsf{T} i} ; z ; \mathbb{T} = \perp\!\!\!\perp$)

$z ; \mathbb{T} ; z^\mathsf{T} \subseteq k_n$ for all $n \geq 0$

c) Assuming $\mathbb{S}^{\mathsf{T} m} ; z ; \mathbb{T} \neq \perp\!\!\!\perp$:

$$k_n ; \mathbb{S}^{\mathsf{T} m} ; z ; \mathbb{T} = \begin{cases} \mathbb{T} & n = 0 = m \\ \perp\!\!\!\perp & n \nmid m \qquad \text{or} \qquad n = 0 \text{ and } m > 0 \\ \mathbb{S}^{\mathsf{T} i} ; z ; \mathbb{T} & n \mid m, \qquad \text{abbreviating } i := \frac{m}{n} \end{cases}$$

In particular: $\quad k_n ; z ; \mathbb{T} = \begin{cases} \mathbb{T} & \text{when } n = 0 \\ z ; \mathbb{T} & \text{when } n > 0 \end{cases} \qquad k_n ; e ; \mathbb{T} = \begin{cases} e ; \mathbb{T} & \text{when } n = 1 \\ \perp\!\!\!\perp & \text{otherwise} \end{cases}$

$$k_m ; \mathbb{S}^{\mathsf{T} m} ; z ; \mathbb{T} = \begin{cases} \mathbb{T} & \text{if } m = 0 \\ \mathbb{S}^\mathsf{T} ; z ; \mathbb{T} = e ; \mathbb{T} & \text{otherwise} \end{cases}$$

For arbitrary m, n always $\qquad n \nmid m \implies \mathbb{S}^{\mathsf{T} m} ; z ; \mathbb{T} \subseteq \overline{k_n^\mathsf{T} ; \overline{\mathbb{I}}}$.

$$k_n ; \overline{e ; \mathbb{T}} = \begin{cases} \mathbb{T} & \text{if } n = 0 \\ z ; \mathbb{T} \cup e ; \mathbb{T} & \text{if } 0 < n \leq n_\mathbb{S} \\ z ; \mathbb{T} & n > n_\mathbb{S} \end{cases}$$

d) $k_n \cap e ; \mathbb{T} = e ; \mathbb{T} ; e^\mathsf{T} ; k_n = e ; \mathbb{T} ; z^\mathsf{T} ; \mathbb{S}^n = \mathbb{S}^\mathsf{T} ; \mathbb{S}^n \cap e ; \mathbb{T}$

e) $k_0 ; k_0^\intercal = \mathbb{T} \qquad (k_0 \circledcirc \mathbb{I}) ; (k_0^\intercal \circledcirc \mathbb{I}) = \mathbb{I} \qquad (k_0^\intercal \circledcirc \mathbb{I}) ; (k_0 \circledcirc \mathbb{I}) = (z ; \mathbb{T} ; z^\intercal \otimes \mathbb{I})$

f) $k_0^\intercal \cap \overline{\mathbb{I}} = k_0^\intercal \cap \overline{z ; \mathbb{T} ; z^\intercal} = z ; \mathbb{T} ; \overline{\mathbb{T} ; z^\intercal} \qquad k_1^\intercal \cap \overline{\mathbb{I}} = \perp\!\!\!\perp$

$k_n^\intercal \cap \overline{\mathbb{I}} = k_n^\intercal \cap \overline{z ; \mathbb{T} ; z^\intercal} = k_n^\intercal \cap \overline{z ; \mathbb{T}}$ for $n \geq 2$

g) $m \leq n \implies k_n ; \mathbb{T} \subseteq k_m ; \mathbb{T}$

ii) In case $\mathbb{S}$ is regressively finite, there hold in addition

a) $k_n = \mathtt{sup}_{i \geq 0} \, \mathbb{S}^{\intercal i} ; z ; \mathbb{T} \cap \mathbb{S}^{i*(n-1)}$ for $n \geq 1$

$= \mathtt{sup}_{i \geq 0} \, \mathbb{S}^{\intercal i} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{i*n}$ for $n \geq 0$

$k_0 ; k_0^\intercal = \mathbb{T} \qquad k_1 ; k_1^\intercal = \mathbb{I} \qquad k_n ; k_n^\intercal \subseteq \mathbb{I} \qquad k_n$ injective for $n > 0$

$$k_n^\intercal \cap \overline{k_i^\intercal ; \overline{\mathbb{I}}} = \begin{cases} k_n^\intercal & \text{if } n = i \\ z ; \mathbb{T} ; z^\intercal \cup \left[k_n^\intercal \cap \overline{k_i^\intercal ; \mathbb{T}} \right] & \text{if } n \neq i \end{cases} \qquad \text{for all } n \geq 1 \text{ and } i \geq 1$$

$k_n \cap \mathbb{T} ; z^\intercal ; \mathbb{S}^n = e ; \mathbb{T} ; z^\intercal ; \mathbb{S}^n \subseteq e ; \mathbb{T}$ for all $n \geq 1$

$$k_n^\intercal \cap \mathbb{S}^{\intercal n} ; z ; \mathbb{T} = \begin{cases} z ; \mathbb{T} & n = 0 \\ \mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; \mathbb{T} ; e^\intercal & \text{otherwise} \end{cases}$$

$$k_n^\intercal ; \overline{e ; \mathbb{T}} = \begin{cases} z ; \mathbb{T} & \text{if } n = 0 \\ \overline{e ; \mathbb{T}} & \text{if } n = 1 \\ k_n^\intercal ; \mathbb{T} \cap \overline{\mathbb{S}^{\intercal n} ; z ; \mathbb{T}} & \text{if } n > 1 \end{cases}$$

$\mathbb{S}^\intercal ; k_2^\intercal ; \mathbb{T} = \mathtt{odd} ; \mathbb{T} = \overline{\mathtt{eve} ; \mathbb{T}} = \overline{k_2^\intercal ; \mathbb{T}}$

b) $\mathbb{T} ; k_n = \mathbb{T} ; z^\intercal ; (\mathbb{S}^n)^*$ for all $n \geq 0$.

$k_i^\intercal ; \mathbb{T} \cap k_l^\intercal ; \mathbb{T} = k_{\mathtt{lcm}(i,l)}^\intercal ; \mathbb{T}$ for arbitrary $i, l \geq 0$

$k_i^\intercal ; \mathbb{T} \cup k_l^\intercal ; \mathbb{T} \subseteq k_{\mathtt{gcd}(i,l)}^\intercal ; \mathbb{T}$ for arbitrary $i, l \geq 0$

c) $\mathbb{S} ; k_0 = k_0 \cap \mathbb{S} ; \mathbb{T}$ and $\mathbb{S} ; k_n = k_n ; \mathbb{S}^n$ for all $n \geq 1$

d) $\mathbb{S}^\intercal ; k_0 \cup z ; \mathbb{T} ; z^\intercal = k_0$ and $\mathbb{S}^\intercal ; k_n \cap k_n ; \mathbb{T} = k_n ; \mathbb{S}^{\intercal n}$ for all $n \geq 1$

e) $\mathtt{inf}_{n \geq 2} \, \overline{k_n^\intercal ; \overline{\mathbb{I}}} ; z ; \mathbb{T} = z ; \mathbb{T} \cup e ; \mathbb{T} \qquad \mathtt{inf}_{n \geq 1} \, \overline{k_n^\intercal ; \overline{\mathbb{I}}} = z ; \mathbb{T} ; z^\intercal \cup e ; \mathbb{T} ; e^\intercal$

f) $(\mathfrak{S} \circledcirc \rho)^{i} ; (k_n^\intercal \circledcirc \mathbb{I}) = (k_{i+n}^\intercal \circledcirc \mathbb{I})$ for $i, n \geq 0$

This includes for $n = 1$ the special case $(\mathfrak{S} \circledcirc \rho)^{i} ; (\mathbb{I} \circledcirc \mathbb{I}) = (k_{i+1}^\intercal \circledcirc \mathbb{I})$.

$(\pi \circledcirc \mathtt{S})^{i} ; (\mathbb{I} \circledcirc k_n^\intercal) = (\mathbb{I} \circledcirc k_{i+n}^\intercal)$ for $i, n \geq 0$

g) $(\mathfrak{S} \circledcirc \rho)^{i} ; \mathtt{vec}(k_n^\intercal) = \mathtt{vec}(k_{n+i}^\intercal)$ for $n \geq 0$

h) $k_n^\intercal \cap \overline{\mathbb{I}} = k_n^\intercal \cap \overline{z ; \mathbb{T}} = k_n^\intercal \cap \overline{z ; \mathbb{T} ; z^\intercal} \subseteq \mathbb{S}^+$ for $n \geq 2$

i) $k_i ; k_n = \begin{cases} k_0 & \text{for } i = 0, n \geq 0 \\ k_0 \cap k_i ; \mathbb{T} & \text{for } i > 0, n = 0 \\ k_{i*n} & \text{for } i > 0, n > 0 \end{cases}$

j) $k_n ; \mathbb{S} \cap k_l = \begin{cases} e ; \mathbb{T} ; e^\intercal ; \mathbb{S}^n & \text{if } l = n+1 \\ \bot\!\!\!\bot & \text{otherwise} \end{cases}$

k) $k_0^\intercal \cap \overline{\overline{k_0^\intercal ; \overline{e ; \mathbb{T}}}} = \bot\!\!\!\bot$

$k_n^\intercal \cap \overline{\overline{k_n^\intercal ; \overline{e ; \mathbb{T}}}} = k_n^\intercal \cap \mathbb{S}^{\intercal n} ; z ; \mathbb{T} = \mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; e^\intercal \quad \text{for } n \geq 1$

l) $\mathbb{S}^{\intercal n} ; z ; \mathbb{T} = \bot\!\!\!\bot \quad \Longleftrightarrow \quad z ; \mathbb{T} ; z^\intercal = k_n$

m) For n, i with $i \neq n$ always $\quad k_i \cap k_n = z ; \mathbb{T} ; z^\intercal$

n) $\mathbb{T} ; e^\intercal \subseteq \overline{k_n^\intercal ; \overline{\mathbb{I}}} \cup \mathtt{sup}_{i \geq 0, i \neq 1} \mathbb{S}^{\intercal i*n} ; z ; \mathbb{T} \quad \text{for all } n \geq 0$

iii) If $\mathbb{S}$ happens to be regressively finite but not total, i.e. $\mathbb{S} ; \mathbb{T} \neq \mathbb{T}$,

a) $k_0 ; \mathbb{T} = \mathbb{T} \qquad k_1 ; \mathbb{T} = \mathbb{T} \qquad k_n ; \mathbb{T} = \mathtt{sup}_{0 \leq i \leq \mathtt{quot}\ n_\mathbb{S}\ n} \mathbb{S}^{\intercal i} ; z ; \mathbb{T}$

b) $\mathbb{S} ; k_0 = k_0 \cap \overline{g_\mathbb{S} ; \mathbb{T}}$

iv) In case $\mathbb{S}$ is regressively finite and also total, i.e. $\mathbb{S} ; \mathbb{T} = \mathbb{T}$,

a) k_n is total

b) $\mathbb{S}^\intercal ; k_n = k_n ; \mathbb{S}^{\intercal n}$

c) $\mathbb{S} ; k_0 = k_0$

Proof: i.a) Univalency is obtained by induction started $k_0^\intercal ; k_0 = z ; \mathbb{T} ; z^\intercal \subseteq \mathbb{I}$ with z injective:
$k_{n+1}^\intercal ; k_{n+1} = \mathbb{A}^\intercal ; (k_n^\intercal \rhd \mathbb{I}) ; (k_n \lhd \mathbb{I}) ; \mathbb{A} \subseteq \mathbb{A}^\intercal ; (k_n^\intercal ; k_n \otimes \mathbb{I}) ; \mathbb{A}$ using Prop. 2.1.1.i
$\subseteq \mathbb{A}^\intercal ; (\mathbb{I} \otimes \mathbb{I}) ; \mathbb{A}$ induction hypothesis
$= \mathbb{A}^\intercal ; \mathbb{A}$ property of the direct product
$\subseteq \mathbb{I}$ since $\mathbb{A}$ is already known to be univalent

---

We also get $k_n = l_n$ by induction starting with the obvious $k_0 = l_0$:
$k_{n+1} = (k_n \lhd \mathbb{I}) ; \mathbb{A} = (k_n \lhd \mathbb{I}) ; \mathfrak{P} ; \mathfrak{P} ; \mathbb{A}$ see the introduction of $\mathfrak{P}$ in Prop. 3.1.1
$= (k_n \lhd \mathbb{I}) ; \mathfrak{P} ; B$ Prop. 6.2.1.i.d
$= (\mathbb{I} \lhd k_n) ; B$
$= (\mathbb{I} \lhd l_n) ; B$ induction hypothesis $k_n = l_n$
$= l_{n+1}$
Observe, that there has not been made any use of regressive finiteness.

The second statement follows simply from univalency established above.

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i.b) $k_0^\intercal ; z ; \mathbb{T} = z ; \mathbb{T} ; z ; \mathbb{T} = z ; \mathbb{T}$ since z is a point

$k_{n+1}^{\intercal};z;\mathbb{T} = \mathbb{A}^{\intercal};(k_n^{\intercal} \oslash \mathbb{I});z;\mathbb{T} = \mathbb{A}^{\intercal};(k_n^{\intercal};z;\mathbb{T} \oslash z;\mathbb{T})$
$= \mathbb{A}^{\intercal};(z;\mathbb{T} \oslash z;\mathbb{T})$ hypothesis
$= z;\mathbb{T}$ Prop. 6.2.1.i.c

$k_0^{\intercal};e;\mathbb{T} = z;\mathbb{T};e;\mathbb{T} = z;\mathbb{T} = \mathbb{S}^{\intercal 0};z;\mathbb{T}$ since e is a point
$k_{n+1}^{\intercal};e;\mathbb{T} = \mathbb{A}^{\intercal};(k_n^{\intercal} \oslash \mathbb{I});e;\mathbb{T} = \mathbb{A}^{\intercal};(k_n^{\intercal};e;\mathbb{T} \oslash e;\mathbb{T})$
$= \mathbb{A}^{\intercal};(\mathbb{S}^{\intercal n};z;\mathbb{T} \oslash \mathbb{S}^{\intercal};z;\mathbb{T})$ hypothesis and expansion of e
$= \mathbb{S}^{\intercal n+1};z;\mathbb{T}$ Prop. 6.2.1.i.j

For the general case (beyond the trivial $n = 0$), we take $n = 1$ as the start of an induction:

$k_1^{\intercal};\mathbb{S}^{\intercal i};z;\mathbb{T} = \mathbb{I};\mathbb{S}^{\intercal i};z;\mathbb{T} = \mathbb{S}^{\intercal 1*i};z;\mathbb{T}$
$k_{n+1}^{\intercal};\mathbb{S}^{\intercal i};z;\mathbb{T} = \mathbb{A}^{\intercal};(k_n^{\intercal} \oslash \mathbb{I});\mathbb{S}^{\intercal i};z;\mathbb{T}$
$= \mathbb{A}^{\intercal};(k_n^{\intercal};\mathbb{S}^{\intercal i};z;\mathbb{T} \oslash \mathbb{S}^{\intercal i};z;\mathbb{T})$ since $\mathbb{S}^{\intercal i};z;\mathbb{T}$ is either a point or else $\mathbb{S}^{\intercal i};z;\mathbb{T} = \mathbb{\perp\!\!\!\perp}$
$= \mathbb{A}^{\intercal};(\mathbb{S}^{\intercal n*i};z;\mathbb{T} \oslash \mathbb{S}^{\intercal i};z;\mathbb{T})$ hypothesis
$= \mathbb{S}^{\intercal n*i+i};z;\mathbb{T}$ Prop. 6.2.1.i.j
$= \mathbb{S}^{\intercal (n+1)*i};z;\mathbb{T}$

The next proof is by induction: $z;\mathbb{T};z^{\intercal} \subseteq \mathbb{T};z^{\intercal} = k_0$ is obvious. Furthermore

$k_{n+1} = (k_n \oslash \mathbb{I});\mathbb{A} \supseteq (z;\mathbb{T};z^{\intercal} \oslash z;\mathbb{T};z^{\intercal});\mathbb{A}_0$ hypothesis and z injective
$= (z;\mathbb{T};z^{\intercal} \oslash z;\mathbb{T};z^{\intercal});(z;\mathbb{T} \oslash \mathbb{I})$ Prop. 6.2.1
$= z;\mathbb{T};z^{\intercal};z;\mathbb{T} \cap z;\mathbb{T};z^{\intercal} = z;\mathbb{T} \cap z;\mathbb{T};z^{\intercal} = z;\mathbb{T};z^{\intercal}$

---

i.c) We assume $\mathbb{S}^{\intercal m};z;\mathbb{T} \neq \mathbb{\perp\!\!\!\perp}$, meaning that it is a point. By induction on n, we obtain

$$k_0;\mathbb{S}^{\intercal m};z;\mathbb{T} = \mathbb{T};z^{\intercal};\mathbb{S}^{\intercal m};z;\mathbb{T} = \begin{cases} \mathbb{T} & m = 0, \text{ since } z \text{ is a point} \\ \mathbb{\perp\!\!\!\perp} & m > 0, \text{ since } \mathbb{S};z = \mathbb{\perp\!\!\!\perp} \end{cases}$$

$k_1;\mathbb{S}^{\intercal m};z;\mathbb{T} = \mathbb{I};\mathbb{S}^{\intercal m};z;\mathbb{T} = \mathbb{S}^{\intercal i};z;\mathbb{T}$ with $i := \frac{m}{1}$

Furthermore

$k_{n+1};\mathbb{S}^{\intercal m};z;\mathbb{T} = (k_n \oslash \mathbb{I});\mathbb{A};\mathbb{S}^{\intercal m};z;\mathbb{T}$
$= (k_n \oslash \mathbb{I});\big[\,\mathtt{sup}_{a\geq 0}\ (\mathbb{S}^{\intercal a};z;\mathbb{T} \oslash \mathbb{S}^{a})\,\big];\mathbb{S}^{\intercal m};z;\mathbb{T}$ Prop. 6.2.1.i.a
$= \mathtt{sup}_{a\geq 0}\ (k_n \oslash \mathbb{I});(\mathbb{S}^{\intercal a};z;\mathbb{T} \oslash \mathbb{S}^{a});\mathbb{S}^{\intercal m};z;\mathbb{T}$
$= \mathtt{sup}_{a\geq 0}\ (k_n;\mathbb{S}^{\intercal a};z;\mathbb{T} \cap \mathbb{S}^{a});\mathbb{S}^{\intercal m};z;\mathbb{T}$ because k_n is univalent
$= \mathtt{sup}_{m\geq a\geq 0}\ \mathbb{S}^{a};\mathbb{S}^{\intercal m};z;\mathbb{T} \cap k_n;\mathbb{S}^{\intercal a};z;\mathbb{T}$ masking; $a \leq m$ since otherwise $\mathbb{S};z = \mathbb{\perp\!\!\!\perp}$

Always two terms are thus intersected. Since $\mathbb{S}^{\intercal m};z;\mathbb{T} \neq \mathbb{\perp\!\!\!\perp}$ as assumed and $a \leq m$, we have with $\mathbb{S}^{a};\mathbb{S}^{\intercal m};z;\mathbb{T} = \mathbb{S}^{\intercal m-a};z;\mathbb{T} \neq \mathbb{\perp\!\!\!\perp}$ also a point. For the second term, we are going to use the hypothesis: This term is either

– equal to $\mathbb{\perp\!\!\!\perp}$ in case $n \nmid a$ or $n = 0$ and $a > 0$ or
– the point $\mathbb{S}^{\intercal \frac{a}{n}};z;\mathbb{T}$ if $n \mid a$.

Therefore, the intersection will mostly be

– equal to $\mathbb{\perp\!\!\!\perp}$, but
– a point when exponents $i := \frac{a}{n}$ and $m - a$ happen to be equal.

This means finally $m = i*(n+1)$ — so as to meet the requirement of the induction. Of course, there exists just one such a, resp. i, namely $a = \frac{n*m}{n+1}$, resp. $i = \frac{m}{n+1}$.

The next three statements are specializations, for $m = 0$, $m = 1$, or $m = n$.

The final statement is then easy to prove starting from $k_n;\mathbb{S}^{\intercal m};z;\mathbb{T} = \mathbb{\perp\!\!\!\perp}$ if $n \nmid m$:

$\mathbb{S}^{\intercal m};z;\mathbb{T} \subseteq \overline{k_n^\intercal;\overline{\mathbb{I}}} \iff k_n^\intercal;\overline{\mathbb{I}} \subseteq \overline{\mathbb{S}^{\intercal m};z;\mathbb{T}} \iff k_n;\mathbb{S}^{\intercal m};z;\mathbb{T} = \mathbb{\perp} \subseteq \mathbb{I}$

$k_n;\overline{e;\mathbb{T}} = k_n;\left[z;\mathbb{T} \cup \mathtt{sup}_{i\geq 2}\,\mathbb{S}^{\intercal i};z;\mathbb{T}\right] = k_n;z;\mathbb{T} \cup \mathtt{sup}_{i\geq 2}\,k_n;\mathbb{S}^{\intercal i};z;\mathbb{T}$

$$= \begin{cases} \mathbb{T} & \text{if } n = 0 \\ z;\mathbb{T} \cup e;\mathbb{T} & \text{if } 0 < n \text{ and } e;\mathbb{T} \subseteq k_n;\mathbb{T} \quad \text{see immediately before} \\ z;\mathbb{T} & \text{otherwise} \end{cases}$$

The last alternative means that $n > n_{\mathbb{S}}$, the greatest number in the initial segment of naturals; see Prop. 5.2.1.iii.c.

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i.d) For the first equation "$\supseteq$" is obvious with e injective, while in the other direction

$e;\mathbb{T};\mathbb{T} \cap k_n \subseteq (e;\mathbb{T} \cap k_n;\mathbb{T});(\mathbb{T} \cap \mathbb{T};e^\intercal;k_n) \subseteq e;\mathbb{T};e^\intercal;k_n$

and $\ldots = e;\mathbb{T};z^\intercal;\mathbb{S}^n$ follows from (i.b). The third equality follows with $\mathbb{S};e = z$:

$e;\mathbb{T};\mathbb{T} \cap \mathbb{S}^\intercal;\mathbb{S}^n \subseteq (e;\mathbb{T} \cap \mathbb{S}^\intercal;\mathbb{S}^n;\mathbb{T});(\mathbb{T} \cap \mathbb{T};e^\intercal;\mathbb{S}^\intercal;\mathbb{S}^n) \subseteq e;\mathbb{T};z^\intercal;\mathbb{S}^n$

$e;\mathbb{T};z^\intercal;\mathbb{S}^n = \mathbb{S}^\intercal;z;\mathbb{T};z^\intercal;\mathbb{S}^n \subseteq \mathbb{S}^\intercal;\mathbb{S}^n$

---

i.e) The first is trivial; the second:

$(k_0 \circledcirc \mathbb{I});(k_0^\intercal \circledcirc \mathbb{I}) = k_0;k_0^\intercal \cap \mathbb{I} = \mathbb{T};z^\intercal;z;\mathbb{T} \cap \mathbb{I} = \mathbb{I}$ since z is a point

$(k_0^\intercal \circledcirc \mathbb{I});(k_0 \circledcirc \mathbb{I}) = (z;\mathbb{T} \circledcirc \mathbb{I});(\mathbb{T};z^\intercal \circledcirc \mathbb{I}) = (z;\mathbb{T};z^\intercal \otimes \mathbb{I})$ using Prop. 2.1.1.ii

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i.f) Some are trivial: $k_n^\intercal \cap \overline{z;\mathbb{T}} \subseteq k_n^\intercal \cap \overline{z;\mathbb{T};z^\intercal}$ and $k_n^\intercal \cap \overline{\mathbb{I}} \subseteq k_n^\intercal \cap \overline{z;\mathbb{T};z^\intercal}$ since $z;\mathbb{T}$ is injective.

For $n > 0$ also $k_n^\intercal \cap \overline{z;\mathbb{T};z^\intercal} \subseteq \overline{z;\mathbb{T}} \iff k_n^\intercal \cap z;\mathbb{T} \subseteq z;\mathbb{T};z^\intercal$ so that with the Dedekind rule

$z;\mathbb{T};\mathbb{T} \cap k_n^\intercal \subseteq (z;\mathbb{T} \cap k_n^\intercal;\mathbb{T});(\mathbb{T} \cap \mathbb{T};z^\intercal;k_n^\intercal) \subseteq z;\mathbb{T};z^\intercal$,

using $k_n;z;\mathbb{T} = z;\mathbb{T}$ from (i.c).

The final part $k_n^\intercal \cap \overline{z;\mathbb{T};z^\intercal} \subseteq \overline{\mathbb{I}} \iff k_n^\intercal \cap \mathbb{I} \subseteq z;\mathbb{T};z^\intercal$ is proved by something that is not an induction; rather it uses the structure of the definition in case $n > 0$ together with (i.c):

$k_{n+1}^\intercal \cap \mathbb{I} = \mathbb{A}^\intercal;(k_n^\intercal \circledcirc \mathbb{I}) \cap \mathbb{I} = \mathtt{sup}_{i\geq 0}\,(\mathbb{T};z^\intercal;\mathbb{S}^i \circledcirc \mathbb{S}^i);(k_n^\intercal \circledcirc \mathbb{I}) \cap \mathbb{I}$

$= \mathtt{sup}_{i\geq 0}\left[\mathbb{T};z^\intercal;\mathbb{S}^i;k_n^\intercal \cap \mathbb{S}^i\right] \cap \mathbb{I}$

$= \mathbb{T};z^\intercal;k_n^\intercal \cap \mathbb{I}$ since $\mathbb{S}^i \subseteq \overline{\mathbb{I}}$ for $i \geq 1$

$= \mathbb{T};z^\intercal \cap \mathbb{I} = z;\mathbb{T};z^\intercal$ since $k_n;z;\mathbb{T} = z;\mathbb{T}$ following (i.c), when $n > 0$

---

i.g) The proof starts with the obvious fact $k_1;\mathbb{T} = \mathbb{T} = k_0;\mathbb{T}$ and shows

$k_{n+1};\mathbb{T} = (k_n \circledcirc \mathbb{I});\mathbb{A};\mathbb{T} = (k_n;\pi^\intercal \cap \rho^\intercal);\mathbb{A};\mathbb{T} \subseteq k_n;\pi^\intercal;\mathbb{A};\mathbb{T} \subseteq k_n;\mathbb{T}$.

---

ii.a) The first version is correct for $n = 1$, since with Prop. 5.2.1.ii.a indeed

$k_1 = \mathbb{I} = \mathbb{T} \cap \mathbb{I} = \mathtt{sup}_{i\geq 0}\,\mathbb{S}^{\intercal i};z;\mathbb{T} \cap \mathbb{S}^0 = \mathtt{sup}_{i\geq 0}\,\mathbb{S}^{\intercal i};z;\mathbb{T} \cap \mathbb{S}^{i*(1-1)}$.

Now we prove $n \mapsto n+1$, using (i.c) for $n \geq 1$ and distinguishing as to $\mathbb{S}^{\intercal k};z;\mathbb{T} \neq \mathbb{\perp}$ or not:

$k_{n+1} = (k_n \circledcirc \mathbb{I});\mathbb{A} = (k_n \circledcirc \mathbb{I});\mathtt{sup}_{k\geq 0}\,(\mathbb{S}^{\intercal k};z;\mathbb{T} \circledcirc \mathbb{S}^k) = \mathtt{sup}_{k\geq 0}\,k_n;\mathbb{S}^{\intercal k};z;\mathbb{T} \cap \mathbb{S}^k$

$$= \mathtt{sup}_{k\geq 0}\,\mathbb{S}^k \cap \begin{cases} \mathbb{\perp} & n \nmid k \quad \text{or} \quad \mathbb{S}^{\intercal^k};z;\mathbb{T} = \mathbb{\perp} \\ \mathbb{S}^{\intercal i};z;\mathbb{T} & n \mid k \text{ and } \mathbb{S}^{\intercal^k};z;\mathbb{T} \neq \mathbb{\perp}, \quad \text{denoting } i := \frac{k}{n} \end{cases}$$

$= \mathsf{sup}_{i\geq 0}\, \mathbb{S}^{i*n} \cap \mathbb{S}^{\intercal i} ; z ; \mathbb{T}$

The second version follows with Prop. 5.2.1.i.g and ii.a; it applies also to the case $n = 0$, since
$\mathsf{sup}_{i\geq 0}\, \mathbb{S}^{\intercal i} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^0 = \big[\mathsf{sup}_{i\geq 0}\, \mathbb{S}^{\intercal i} ; z ; \mathbb{T}\big] ; \mathbb{T} ; z^\intercal = \mathbb{T} ; \mathbb{T} ; z^\intercal = k_0$

The cases $n = 0$ and $n = 1$ are trivial. For $n > 0$ we use the preceding result:
$k_n ; k_n^\intercal = \big[\mathsf{sup}_{i\geq 0}\ \mathbb{S}^{\intercal i} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{i*n}\big] ; \big[\mathsf{sup}_{l\geq 0}\ \mathbb{S}^{\intercal l*n} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{l}\big]$
$= \mathsf{sup}_{i\geq 0}\, \mathsf{sup}_{l\geq 0}\ \mathbb{S}^{\intercal i} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{i*n} ; \mathbb{S}^{\intercal l*n} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{l}$ which terms are $= \mathbb{\bot}$ for $i * n \neq l * n$
$= \mathsf{sup}_{i\geq 0}\ \mathbb{S}^{\intercal i} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{i*n} ; \mathbb{S}^{\intercal i*n} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{i}$
$\subseteq \mathsf{sup}_{i\geq 0}\ \mathbb{S}^{\intercal i} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{i} \subseteq \mathsf{sup}_{i\geq 0}\ \mathbb{S}^{\intercal i} ; \mathbb{S}^{i}$ since $z ; \mathbb{T}$ is injective
$\subseteq \mathbb{I}$ because $\mathbb{S}$ is univalent

The case $n = i$ is rather trivial and requires just to convince ourselves concerning
$k_n^\intercal \subseteq \overline{k_n^\intercal ; \overline{\mathbb{I}}} \quad \Longleftrightarrow \quad k_n^\intercal ; \overline{\mathbb{I}} \subseteq \overline{k_n^\intercal} \quad \Longleftrightarrow \quad k_n ; k_n^\intercal \subseteq \mathbb{I}$ since k_n is injective for $n \geq 1$

The case $n \neq i$:
$k_n^\intercal \cap \overline{k_i^\intercal ; \overline{\mathbb{I}}} = k_n^\intercal \cap \overline{\overline{k_i}^\intercal \cap k_i^\intercal ; \mathbb{T}}$ since $k_i^\intercal$ is univalent for $i \geq 1$
$= k_n^\intercal \cap \big[k_i^\intercal \cup \overline{k_i^\intercal ; \mathbb{T}}\big] = \big[k_n^\intercal \cap k_i^\intercal\big] \cup \big[k_n^\intercal \cap \overline{k_i^\intercal ; \mathbb{T}}\big] = z ; \mathbb{T} ; z^\intercal \cup \big[k_n^\intercal \cap \overline{k_i^\intercal ; \mathbb{T}}\big]$ since $n \neq i$

This means that we may use — up to the minor correction in the upper left corner — the row-constant $\overline{k_i^\intercal ; \mathbb{T}}$ instead of the more complicated $\overline{k_i^\intercal ; \overline{\mathbb{I}}}$.

The next proof — already started with a masking observation — to obtain for $n \geq 1$
$e ; \mathbb{T} ; z^\intercal ; \mathbb{S}^n = e ; \mathbb{T} \cap \mathbb{T} ; z^\intercal ; \mathbb{S}^n = \mathbb{T} ; z^\intercal ; \mathbb{S}^n \cap k_n$
requires these two considerations:
$\mathbb{T} ; z^\intercal ; \mathbb{S}^n \cap k_n = \mathbb{T} ; e^\intercal ; k_n \cap k_n$ see (i.b)
$\subseteq (\mathbb{T} ; e^\intercal \cap k_n ; k_n^\intercal) ; (k_n \cap e ; \mathbb{T} ; k_n)$ Dedekind rule
$\subseteq \mathbb{I} ; e ; \mathbb{T} ; k_n \subseteq e ; \mathbb{T}$ since k_n is injective for $n \geq 1$

$e ; \mathbb{T} ; z^\intercal ; \mathbb{S}^n \subseteq k_n \quad \Longleftrightarrow \quad \mathbb{T} ; z^\intercal ; \mathbb{S}^n \subseteq \mathbb{T} ; e^\intercal ; k_n \quad \Longleftarrow \quad k_n^\intercal ; e ; \mathbb{T} = \mathbb{S}^{\intercal n} ; z ; \mathbb{T}$ due to (i.b)

Furthermore
$k_0^\intercal ; \overline{e ; \mathbb{T}} = z ; \mathbb{T} ; \overline{e ; \mathbb{T}} = z ; \mathbb{T} ; \overline{\mathbb{I}} ; e ; \mathbb{T} = z ; \mathbb{T} ; e ; \mathbb{T} = z ; \mathbb{T}$
$k_1^\intercal ; \overline{e ; \mathbb{T}} = \mathbb{I} ; \overline{e ; \mathbb{T}} = \overline{e ; \mathbb{T}}$
and for $n > 1$:
$k_n^\intercal ; \overline{e ; \mathbb{T}} = \overline{k_n^\intercal ; e ; \mathbb{T}} \cap k_n^\intercal ; \mathbb{T}$ since k_n is injective for $n > 0$
$= \overline{\mathbb{S}^{\intercal n} ; z ; \mathbb{T}} \cap k_n^\intercal ; \mathbb{T}$ following (i.b)

We use as an intermediate result
$k_2^\intercal ; \mathbb{T} = \big[\mathsf{sup}_{i\geq 0}\, \mathbb{T} ; z^\intercal ; \mathbb{S}^i \cap \mathbb{S}^{\intercal i}\big] ; \mathbb{T}$ following (ii.a)
$= \mathsf{sup}_{i\geq 0}\, \mathbb{S}^{\intercal i} ; (\mathbb{T} \cap \mathbb{S}^{\intercal i} ; z ; \mathbb{T})$ mask shifting
$= \mathsf{sup}_{i\geq 0}\, \mathbb{S}^{\intercal 2*i} ; z ; \mathbb{T}$
$\mathbb{T} = \mathsf{sup}_{i\geq 0}\, \mathbb{S}^{\intercal i} ; z ; \mathbb{T} = \mathsf{sup}_{m\geq 0}(\mathbb{S}^{\intercal 2*m} ; z ; \mathbb{T} \cup \mathbb{S}^{\intercal 2*m+1} ; z ; \mathbb{T})$
$= \big[\mathsf{sup}_{m\geq 0}\, \mathbb{S}^{\intercal 2*m} ; z ; \mathbb{T}\big] \cup \big[\mathsf{sup}_{m\geq 0}\, \mathbb{S}^{\intercal 2*m+1} ; z ; \mathbb{T}\big]$
$= \big[\mathsf{sup}_{m\geq 0}(\mathbb{S}^{\intercal 2})^m ; z ; \mathbb{T}\big] \cup \mathbb{S}^\intercal ; \big[\mathsf{sup}_{m\geq 0}(\mathbb{S}^{\intercal 2})^m ; z ; \mathbb{T}\big] = k_2^\intercal ; \mathbb{T} \cup \mathbb{S}^\intercal ; k_2^\intercal ; \mathbb{T}$
which is obviously a disjoint decomposition fitting into Def. 5.2.1.

---

ii.b) Using (ii.a), and the fact that $\mathbb{S}^{\tau i} ; z ; \mathbb{T}$ is a point or $= \mathbb{\perp}$, we proceed as follows:

$\mathbb{T} ; k_n = \mathbb{T} ; \mathtt{sup}_{i\geq 0}\, \mathbb{S}^{\tau i} ; z ; \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{i*n} = \mathtt{sup}_{i\geq 0}\, \mathbb{T} ; \mathbb{S}^{\tau i} ; z ; \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{n*i}$
$= \mathtt{sup}_{i\geq 0}\, \mathbb{T} ; z^{\mathsf{T}} ; (\mathbb{S}^n)^i = \mathbb{T} ; z^{\mathsf{T}} ; (\mathbb{S}^n)^*$

The other two follow from most elementary number theory:

$k_i^{\mathsf{T}} ; \mathbb{T} \cap k_l^{\mathsf{T}} ; \mathbb{T} = (\mathbb{S}^{\tau i})^* ; z ; \mathbb{T} \cap (\mathbb{S}^{\tau l})^* ; z ; \mathbb{T}$
$= \mathtt{sup}_{k\geq 0}\, \mathbb{S}^{\tau k*i} ; z ; \mathbb{T} \cap \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\tau n*l} ; z ; \mathbb{T}$ where $k*i \neq n*l$ implies $\ldots = \mathbb{\perp}$
$= \mathtt{sup}_{m\geq 0}\, \mathbb{S}^{\tau m*\mathtt{lcm}(i,l)} ; z ; \mathbb{T} = k^{\mathsf{T}}_{\mathtt{lcm}(i,l)} ; \mathbb{T}$

Now the third claim, starting with the first:

$k_i^{\mathsf{T}} ; \mathbb{T} = (\mathbb{S}^{\tau i})^* ; z ; \mathbb{T} = \mathtt{sup}_{k\geq 0}\, \mathbb{S}^{\tau k*i} ; z ; \mathbb{T}$
$k_l^{\mathsf{T}} ; \mathbb{T} = (\mathbb{S}^{\tau l})^* ; z ; \mathbb{T} = \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\tau n*l} ; z ; \mathbb{T}$

and therefore

$k_i^{\mathsf{T}} ; \mathbb{T} \cup k_l^{\mathsf{T}} ; \mathbb{T} = \mathtt{sup}_{k\geq 0}\, \mathbb{S}^{\tau k*i} ; z ; \mathbb{T} \cup \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\tau n*l} ; z ; \mathbb{T}$
$\subseteq \mathtt{sup}_{m\geq 0}\, \mathbb{S}^{\tau m*\mathtt{gcd}(i,l)} ; z ; \mathbb{T} = k^{\mathsf{T}}_{\mathtt{gcd}(i,l)} ; \mathbb{T}$

We prove $k_n^{\mathsf{T}} ; \mathbb{T} = (\mathbb{S}^{\tau n})^* ; z ; \mathbb{T}$ along the inductive definition of the k_n, starting with $n = 0$:

$k_0^{\mathsf{T}} ; \mathbb{T} = z ; \mathbb{T} ; \mathbb{T} = \mathbb{I}^* ; z ; \mathbb{T} = (\mathbb{S}^{\tau 0})^* ; z ; \mathbb{T}$

The case $n = 1$ needs regressive finiteness:

$k_1^{\mathsf{T}} ; \mathbb{T} = \mathbb{I} ; \mathbb{T} = \mathbb{T} = (\mathbb{S}^{\tau})^* ; z ; \mathbb{T}$ due to Prop. 5.2.1.ii.a

The following is not a recursive proof; it just reflects the method of construction:

$k^{\mathsf{T}}_{n+1} ; \mathbb{T} = \mathbb{A}^{\mathsf{T}} ; (k_n^{\mathsf{T}} \oslash \mathbb{I}) ; \mathbb{T} = \mathtt{sup}_{i\geq 0}\, (\mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\,i} \oslash \mathbb{S}^{\tau i}) ; (k_n^{\mathsf{T}} \oslash \mathbb{I}) ; \mathbb{T}$ definition of $\mathbb{A}$
$= \mathtt{sup}_{i\geq 0}\, \big[\mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\,i} ; k_n^{\mathsf{T}} \cap \mathbb{S}^{\tau i}\big] ; \mathbb{T} = \mathtt{sup}_{i\geq 0}\, \mathbb{S}^{\tau i} ; \big[\mathbb{T} \cap k_n ; \mathbb{S}^{\tau i} ; z ; \mathbb{T}\big]$ mask shifting
$= \mathtt{sup}_{i\geq 0}\, \mathbb{S}^{\tau i} ; k_n ; \mathbb{S}^{\tau i} ; z ; \mathbb{T}$ remembering now that $n \geq 1$ when using (i.c):
$= \mathtt{sup}_{i\geq 0}\, \mathbb{S}^{\tau i} ; \begin{cases} \mathbb{\perp} & \text{when } n \nmid i \\ \mathbb{S}^{\tau j} ; z ; \mathbb{T} & \text{when } n \,|\, i \text{ with } j := \frac{i}{n} \end{cases}$
$= \mathtt{sup}_{j\geq 0}\, \mathbb{S}^{\tau n*j} ; \mathbb{S}^{\tau j} ; z ; \mathbb{T} = \mathtt{sup}_{j\geq 0}\, \mathbb{S}^{\tau (n+1)*j} ; z ; \mathbb{T}$ focussing on non-vanishing contributions
$= \mathtt{sup}_{j\geq 0} (\mathbb{S}^{\tau n+1})^j ; z ; \mathbb{T} = \big[\mathtt{sup}_{j\geq 0} (\mathbb{S}^{\tau n+1})^j\big] ; z ; \mathbb{T} = (\mathbb{S}^{\tau n+1})^* ; z ; \mathbb{T}$

---

ii.c) We prove the case $n = 0$ separately:

$\mathbb{S} ; k_0 = \mathbb{S} ; \mathbb{T} ; z^{\mathsf{T}}$
$= \mathbb{T} ; z^{\mathsf{T}} \cap \mathbb{S} ; \mathbb{T}$ masking
$= k_0 ; \mathbb{S}^{\,0} \cap \mathbb{S} ; \mathbb{T} = k_0 \cap \mathbb{S} ; \mathbb{T}$

Now the induction for $n \geq 1$ with hypothesis $\mathbb{S} ; k_n = k_n ; \mathbb{S}^{\,n}$ starting at

$\mathbb{S} ; k_1 = \mathbb{S} ; \mathbb{I} = \mathbb{I} ; \mathbb{S} = k_1 ; \mathbb{S}^{\,1}$
$\mathbb{S} ; k_{n+1} = \mathbb{S} ; (k_n \oslash \mathbb{I}) ; \mathbb{A} = (\mathbb{S} ; k_n \oslash \mathbb{S}) ; \mathbb{A}$ univalence of $\mathbb{S}$
$= (k_n ; \mathbb{S}^{\,n} \oslash \mathbb{S}) ; \mathbb{A}$ induction hypothesis
$= (k_n ; \mathbb{S}^{\,n} \oslash \mathbb{I}) ; (\mathbb{I} \otimes \mathbb{S}) ; \mathbb{A}$ injectivity of $\mathbb{S}$
$= (k_n ; \mathbb{S}^{\,n} \oslash \mathbb{I}) ; \mathbb{A} ; \mathbb{S}$ Prop. 6.2.1.i.b

$= (k_n \oslash \mathbb{I}) ; (\mathbb{S}^n \otimes \mathbb{I}) ; \mathbb{A} ; \mathbb{S}$ injectivity of $\mathbb{S}$
$= (k_n \oslash \mathbb{I}) ; \mathbb{A} ; \mathbb{S}^n ; \mathbb{S}$ Prop. 6.2.1.ii.d, which uses regressive finiteness
$= k_{n+1} ; \mathbb{S}^{n+1}$

---

ii.d) We start with the case $n = 0$:

$\mathbb{S}^\intercal ; k_0 \cup z ; \mathbb{T} ; z^\intercal = \mathbb{S}^\intercal ; \mathbb{T} ; z^\intercal \cup z ; \mathbb{T} ; z^\intercal$
$= \mathbb{S}^\intercal ; \mathbb{T} ; z^\intercal \cup \overline{\mathbb{S}^\intercal ; \mathbb{T}} ; z^\intercal = \left[\mathbb{S}^\intercal ; \mathbb{T} \cup \overline{\mathbb{S}^\intercal ; \mathbb{T}}\right] ; z^\intercal = \mathbb{T} ; z^\intercal = k_0$

Now let be $n \geq 1$, where the case $n = 1$ is trivial. Direction "$\subseteq$" of $\mathbb{S}^\intercal ; k_n \cap k_n ; \mathbb{T} = k_n ; \mathbb{S}^{\intercal n}$ is proved fairly easily via the Dedekind rule, thereby using (ii.c):

$k_n ; \mathbb{T} \cap \mathbb{S}^\intercal ; k_n \subseteq (k_n \cap \ldots) ; (\mathbb{T} \cap k_n^\intercal ; \mathbb{S}^\intercal ; k_n) \subseteq k_n ; k_n^\intercal ; \mathbb{S}^\intercal ; k_n = k_n ; \mathbb{S}^{\intercal n} ; k_n^\intercal ; k_n \subseteq k_n ; \mathbb{S}^{\intercal n}$

For the relevant portion $k_n ; \mathbb{S}^{\intercal n} \subseteq \mathbb{S}^\intercal ; k_n$ of the other direction "$\supseteq$", we reason for $n \geq 2$ as follows:

$k_n ; \mathbb{S}^{\intercal n} = \left[\mathtt{sup}_{i \geq 0} \, \mathbb{S}^{\intercal i} ; z ; \mathbb{T} \cap \mathbb{S}^{i*n-i}\right] ; \mathbb{S}^{\intercal n}$ for $n \geq 1$; using (ii.a)
$= \mathtt{sup}_{i \geq 0} \, \mathbb{S}^{\intercal i} ; z ; \mathbb{T} \cap \mathbb{S}^{i*n-i} ; \mathbb{S}^{\intercal n}$ masking
$= (z ; \mathbb{T} \cap \mathbb{S}^{\intercal n}) \cup (\mathbb{S}^\intercal ; z ; \mathbb{T} \cap \mathbb{S}^{n-1} ; \mathbb{S}^{\intercal n}) \cup \mathtt{sup}_{i \geq 2} \left[\mathbb{S}^{\intercal i} ; z ; \mathbb{T} \cap \mathbb{S}^{(i-1)*n-i} \cap \mathbb{T} ; \mathbb{S}^{\intercal n}\right]$
Prop. 5.2.1.i.b with $i * n - i \geq n$
$= \mathbb{\bot} \cup (\mathbb{S}^\intercal ; z ; \mathbb{T} \cap \mathbb{S}^\intercal \cap \mathbb{S}^{n-1} ; \mathbb{T}) \cup \mathtt{sup}_{i \geq 2} \left[\mathbb{S}^{\intercal i} ; z ; \mathbb{T} \cap \mathbb{S}^{(i-1)*n-i} \cap \mathbb{T} ; \mathbb{S}^{\intercal n}\right]$
Prop. 5.2.1.i.b with $n \geq n - 1$

This has to be contained in

$\mathbb{S}^\intercal ; k_n = \mathbb{S}^\intercal ; \left[\mathtt{sup}_{i \geq 0} \, \mathbb{S}^{\intercal i} ; z ; \mathbb{T} \cap \mathbb{S}^{i*n-i}\right]$ due to (ii.a)
$= \mathtt{sup}_{i \geq 0} \, \mathbb{S}^{\intercal i+1} ; z ; \mathbb{T} \cap \mathbb{S}^\intercal ; \mathbb{S}^{i*n-i} = (\mathbb{S}^\intercal ; z ; \mathbb{T} \cap \mathbb{S}^\intercal ; \mathbb{S}^0) \cup \mathtt{sup}_{i \geq 1} \, \mathbb{S}^{\intercal i+1} ; z ; \mathbb{T} \cap \mathbb{S}^\intercal ; \mathbb{S}^{i*n-i}$
$= (\mathbb{S}^\intercal ; z ; \mathbb{T} \cap \mathbb{S}^\intercal) \cup \mathtt{sup}_{j \geq 2} \, \mathbb{S}^{\intercal j} ; z ; \mathbb{T} \cap \mathbb{S}^\intercal ; \mathbb{S}^{(j-1)*n-(j-1)}$ defining $j := i + 1$
$= (\mathbb{S}^\intercal ; z ; \mathbb{T} \cap \mathbb{S}^\intercal) \cup \mathtt{sup}_{j \geq 2} \, \mathbb{S}^{\intercal j} ; z ; \mathbb{T} \cap \mathbb{S}^{(j-1)*n-j} \cap \mathbb{S}^\intercal ; \mathbb{T}$
Prop. 5.2.1.i.b with $(j-1) * n - (j-1) \geq 1$
$= (\mathbb{S}^\intercal ; z ; \mathbb{T} \cap \mathbb{S}^\intercal) \cup \mathtt{sup}_{j \geq 2} \, \mathbb{S}^{\intercal j} ; z ; \mathbb{T} \cap \mathbb{S}^{(j-1)*n-j}$ since $\mathbb{S}^{\intercal j} ; z ; \mathbb{T} \subseteq \mathbb{S}^\intercal ; \mathbb{T}$ for $j \geq 2$

which is certainly satisfied.

---

ii.e) The differently running indices are important. In both cases, the proof of "$\supseteq$" follows this scheme using (i.c):

$z ; \mathbb{T} \subseteq \mathtt{inf}_{n \geq 2} \, \overline{k_n^\intercal ; \overline{z ; \mathbb{T}}} = \mathtt{inf}_{n \geq 2} \, \overline{k_n^\intercal ; \overline{\mathbb{I}} ; z ; \mathbb{T}}$ shunting $z ; \mathbb{T}$ makes it part of the other task:

$z ; \mathbb{T} ; z^\intercal \subseteq \overline{k_n^\intercal ; \overline{\mathbb{I}}} \iff k_n^\intercal ; \overline{\mathbb{I}} \subseteq \overline{z ; \mathbb{T} ; z^\intercal} \iff k_n ; z ; \mathbb{T} ; z^\intercal = z ; \mathbb{T} ; z^\intercal \subseteq \mathbb{I}$ for $n \geq 1$

$e ; \mathbb{T} \subseteq \overline{k_n^\intercal ; \overline{z ; \mathbb{T}}} \iff k_n^\intercal ; \overline{z ; \mathbb{T}} \subseteq \overline{e ; \mathbb{T}} \impliedby k_n ; e ; \mathbb{T} = \mathbb{\bot} \subseteq z ; \mathbb{T}$ for $n \geq 2$; see (i.c)

$e ; \mathbb{T} ; e^\intercal \subseteq \overline{k_n^\intercal ; \overline{\mathbb{I}}} \iff k_n^\intercal ; \overline{\mathbb{I}} \subseteq \overline{e ; \mathbb{T} ; e^\intercal} \iff k_n ; e ; \mathbb{T} ; e^\intercal = \begin{cases} e ; \mathbb{T} ; e^\intercal & \text{when } n = 1 \\ \mathbb{\bot} & \text{otherwise} \end{cases} \subseteq \mathbb{I}$

To show containment in the other direction "$\subseteq$", we use a special representation of $\mathbb{T}$:

$\mathbb{T} = \mathtt{sup}_{i \geq 0} \, \mathbb{S}^{\intercal i} ; z ; \mathbb{T} = z ; \mathbb{T} \cup e ; \mathbb{T} \cup \mathtt{sup}_{i \geq 2} \, \mathbb{S}^{\intercal i} ; z ; \mathbb{T}$,

which to ensure required regressive finiteness in Prop. 5.2.1.ii.a. In the first case, we have, thus, to show — after the point $z ; \mathbb{T}$ has slipped below both negations —

$\mathtt{inf}_{n \geq 2} \, \overline{k_n^\intercal ; \overline{z ; \mathbb{T}}} \subseteq z ; \mathbb{T} \cup e ; \mathbb{T}$
$\iff \quad z ; \mathbb{T} \cup e ; \mathbb{T} \cup \mathtt{sup}_{i \geq 2} \, \mathbb{S}^{\intercal i} ; z ; \mathbb{T} = \mathbb{T} \subseteq \mathtt{sup}_{n \geq 2} \, k_n^\intercal ; \overline{z ; \mathbb{T}} \cup z ; \mathbb{T} \cup e ; \mathbb{T}.$

With regard to $z ; \mathbb{T}$ and $e ; \mathbb{T}$, this is obvious. Each $\mathbb{S}^{\intercal i} ; z ; \mathbb{T}$ is either equal to $\mathbb{L}$, in which case we are done, or it is a point that may be shunted. Choosing $n := i$, it suffices to establish

$\mathbb{S}^{\intercal i} ; z ; \mathbb{T} \subseteq k_i^\intercal ; \overline{z ; \mathbb{T}}$

$\iff \quad \mathbb{T} \subseteq \mathbb{T} ; z^\intercal ; \mathbb{S}^{i} ; k_i^\intercal ; \overline{z ; \mathbb{T}}$, shunting a point

which holds indeed, because of (i.c)

$\mathbb{T} ; z^\intercal ; \mathbb{S}^{i} ; k_i^\intercal ; \overline{z ; \mathbb{T}} = \mathbb{T} ; z^\intercal ; \mathbb{S} ; \overline{z ; \mathbb{T}} = \overline{\mathbb{T} ; z^\intercal ; \mathbb{S} ; z ; \mathbb{T}} = \overline{\mathbb{L}} = \mathbb{T}.$

The regressive finiteness mentioned above is helpful in the second case:

$\mathtt{inf}_{n \geq 1} \overline{k_n^\intercal ; \overline{\mathbb{I}}} \subseteq z ; \mathbb{T} ; z^\intercal \cup e ; \mathbb{T} ; e^\intercal$

$\iff \quad \mathtt{sup}_{i \geq 0} \mathbb{S}^{\intercal i} ; z ; \mathbb{T} = \mathbb{T} \subseteq \mathtt{sup}_{n \geq 1} k_n^\intercal ; \overline{\mathbb{I}} \cup z ; \mathbb{T} ; z^\intercal \cup e ; \mathbb{T} ; e^\intercal$

Now $i = 0, i = 1$ and $i > 1$ will get a separate treatment, starting with $i = 0$:

$\mathbb{S}^{\intercal 0} ; z ; \mathbb{T} = z ; \mathbb{T} = z ; \mathbb{T} ; \mathbb{T} ; z^\intercal \cup z ; \mathbb{T} ; \overline{\mathbb{T} ; z^\intercal}$,

where the second term requires to shunt the point z and use (i.c) when $n \geq 1$:

$z ; \mathbb{T} ; \overline{\mathbb{T} ; z^\intercal} \subseteq k_n^\intercal ; \overline{\mathbb{I}} \quad \iff \quad \overline{\mathbb{T} ; z^\intercal} \subseteq \mathbb{T} ; z^\intercal ; k_n^\intercal ; \overline{\mathbb{I}} = \mathbb{T} ; z^\intercal ; \overline{\mathbb{I}} = \overline{\mathbb{T} ; z^\intercal}$

Similarly for $i = 1$:

$\mathbb{S}^{\intercal 1} ; z ; \mathbb{T} = e ; \mathbb{T} = e ; \mathbb{T} ; \mathbb{T} ; e^\intercal \cup e ; \mathbb{T} ; \overline{\mathbb{T} ; e^\intercal}$,

where the Schröder rule is applied for the second term and $n := 1$ is chosen:

$e ; \mathbb{T} ; \overline{\mathbb{T} ; e^\intercal} \subseteq k_1 ; \overline{\mathbb{I}} = \overline{\mathbb{I}} \quad \iff \quad \mathbb{T} ; e^\intercal ; \mathbb{I} \subseteq \mathbb{T} ; e^\intercal$

In general, for $i > 1$, shunting again and use of (i.c):

$\mathbb{S}^{\intercal i} ; z ; \mathbb{T} \subseteq k_1^\intercal ; \overline{\mathbb{I}} \cup k_i^\intercal ; \overline{\mathbb{I}} = \overline{\mathbb{I}} \cup k_i^\intercal ; \overline{\mathbb{I}} \quad \iff \quad \mathbb{T} \subseteq \mathbb{T} ; z^\intercal ; \mathbb{S}^{i} ; \overline{\mathbb{I}} \cup \mathbb{T} ; z^\intercal ; \mathbb{S}^{i} ; k_i^\intercal ; \overline{\mathbb{I}}$

$= \overline{\mathbb{T} ; z^\intercal ; \mathbb{S}^{i}} \cup \mathbb{T} ; z^\intercal ; \mathbb{S}^{1} ; \overline{\mathbb{I}} = \overline{\mathbb{T} ; z^\intercal ; \mathbb{S}^{i}} \cup \overline{\mathbb{T} ; z^\intercal ; \mathbb{S}^{1}} = \overline{\mathbb{T} ; z^\intercal ; \mathbb{S}^{i} \cap \mathbb{T} ; z^\intercal ; \mathbb{S}^{1}} = \overline{\mathbb{L}}$

---

ii.f) This shall be proved via induction over i. The case $i = 0$ is trivial; then stepwise with

$(\mathfrak{S} \circledcirc \rho)^{i+1} ; (k_n^\intercal \circledcirc \mathbb{I}) = (\mathfrak{S} \circledcirc \rho) ; (\mathfrak{S} \circledcirc \rho)^{i} ; (k_n^\intercal \circledcirc \mathbb{I}) = (\mathfrak{S} \circledcirc \rho) ; (k_{n+i}^\intercal \circledcirc \mathbb{I})$ hypothesis

$= (\mathbb{A}^\intercal \circledcirc \rho^\intercal) ; (k_{n+i}^\intercal \circledcirc \mathbb{I})$ inversion rule Prop. 3.2.3.v with $\mathfrak{S}^{\hookleftarrow} = \mathbb{A}$ from Prop. 6.3.1.ii.e

$= (\mathbb{A}^\intercal ; (k_{n+i}^\intercal \circledcirc \mathbb{I}) \circledcirc \rho^\intercal ; (k_{n+i}^\intercal \circledcirc \mathbb{I}))$ since k_{n+i} is univalent

$= (\mathbb{A}^\intercal ; (k_{n+i}^\intercal \circledcirc \mathbb{I}) \circledcirc \mathbb{T} ; k_{n+i}^\intercal \cap \mathbb{I})$ destroy and append

$= (\mathbb{A}^\intercal ; (k_{n+i}^\intercal \cap \mathbb{T} ; k_{n+i}^\intercal \circledcirc \mathbb{I}) \circledcirc \mathbb{I}) = (\mathbb{A}^\intercal ; (k_{n+i}^\intercal \circledcirc \mathbb{I}) \circledcirc \mathbb{I})$ Prop. 2.1.1.iv

$= (k_{n+(i+1)}^\intercal \circledcirc \mathbb{I})$

The other proof is executed in full analogy.

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ii.g) $(\mathfrak{S} \circledcirc \rho)^{i} ; (k_n^\intercal \circledcirc \mathbb{I}) ; \mathbb{T} = (k_{n+i}^\intercal \circledcirc \mathbb{I}) ; \mathbb{T}$ follows from (ii.f)

$= \mathtt{vec}(k_{n+i}^\intercal)$ by the definition in Prop. 2.2.5

---

ii.h) The two equalities have already been proved as (i.f). For the containment, we use one version. Let be $n \geq 2$ and use (ii.a) in

$\overline{z ; \mathbb{T} ; z^\intercal} \cap k_n = \overline{z ; \mathbb{T} ; z^\intercal} \cap \left[\mathtt{sup}_{i \geq 0} \mathbb{S}^{\intercal i} ; z ; \mathbb{T} \cap \mathbb{S}^{i*(n-1)} \right]$

$\subseteq \overline{z ; \mathbb{T} ; z^\intercal} \cap \left[(z ; \mathbb{T} \cap \mathbb{I}) \cup \mathtt{sup}_{i \geq 1} \mathbb{S}^{i*(n-1)} \right]$ where $z ; \mathbb{T} \cap \mathbb{I} = z ; \mathbb{T} ; z^\intercal$

$= \mathtt{sup}_{i \geq 1} \overline{z ; \mathbb{T} ; z^\intercal} \cap \mathbb{S}^{i*(n-1)} \subseteq \mathbb{S}^{+}$ since always $i * (n-1) \geq 1$

---

ii.i) We start with an induction over the second factor with fixed first factor k_0:

$k_0 ; k_0 = \mathbb{T} ; z^\intercal ; \mathbb{T} ; z^\intercal = \mathbb{T} ; z^\intercal = k_0$ since z is a point
$k_0 ; k_{n+1} = k_0 ; (k_n \between \mathbb{I}) ; \mathbb{A} = (k_0 ; k_n \between k_0) ; \mathbb{A}$ since k_0 is a map
$= (k_0 \between k_0) ; \mathbb{A}$ hypothesis
$= (\mathbb{T} ; z^\intercal \between \mathbb{T} ; z^\intercal) ; \mathbb{A} = \mathbb{T} ; z^\intercal$ Prop. 6.2.1.i.c
$= k_0$

Now the situation with second factor fixed to k_0 and $i > 0$:
$k_i ; k_0 = k_i ; \mathbb{T} ; z^\intercal = \mathbb{T} ; z^\intercal \cap k_i ; \mathbb{T}$ masking
$= k_0 \cap k_i ; \mathbb{T}$

Finally an induction over the first factor — with second factor k_m fixed and $n > 0$. The start:
$i = 0$ see above: $k_0 ; k_n = k_0 = k_{0*n}$
and the step $i \mapsto i+1$:
$k_{i+1} ; k_n = (k_i \between \mathbb{I}) ; \mathbb{A} ; k_n = (k_i \between \mathbb{I}) ; B ; k_n$ Prop. 6.2.1.ii.a
$= (k_i \between \mathbb{I}) ; \big[\mathtt{sup}_{c \geq 0} \, (\mathbb{S}^{\,c} \between \mathbb{S}^{\,\intercal c} ; z ; \mathbb{T}) \, \big] ; k_n$ by definition in Prop. 6.2.1
$= \mathtt{sup}_{c \geq 0} \big[k_i ; \mathbb{S}^{\,c} \cap \mathbb{S}^{\,\intercal c} ; z ; \mathbb{T} \big] ; k_n$
$= \mathtt{sup}_{c \geq 0} \, k_i ; \mathbb{S}^{\,c} ; k_n \cap \mathbb{S}^{\,\intercal c} ; z ; \mathbb{T}$ masking
$= \mathtt{sup}_{c \geq 0} \, k_i ; k_n ; (\mathbb{S}^{\,n})^c \cap \mathbb{S}^{\,\intercal c} ; z ; \mathbb{T}$ following (ii.c)
$= \mathtt{sup}_{c \geq 0} \, k_{i*n} ; \mathbb{S}^{\,c*n} \cap \mathbb{S}^{\,\intercal c} ; z ; \mathbb{T}$ hypothesis
$= \mathtt{sup}_{c \geq 0} \big[\mathtt{sup}_{a \geq 0} \, \mathbb{S}^{\,\intercal a} ; z ; \mathbb{T} \cap \mathbb{S}^{\,a*(i*n-1)} \big] ; \mathbb{S}^{\,c*n} \cap \mathbb{S}^{\,\intercal c} ; z ; \mathbb{T}$ following (ii.a)
$= \mathtt{sup}_{c \geq 0} \, \mathtt{sup}_{a \geq 0} \, \mathbb{S}^{\,a*(i*n-1)} ; \mathbb{S}^{\,c*n} \cap \mathbb{S}^{\,\intercal a} ; z ; \mathbb{T} \cap \mathbb{S}^{\,\intercal c} ; z ; \mathbb{T}$ masking
$= \mathtt{sup}_{c \geq 0} \, \mathbb{S}^{\,c*(i*n-1)} ; \mathbb{S}^{\,c*n} \cap \mathbb{S}^{\,\intercal c} ; z ; \mathbb{T}$ since $\ldots = \bot\!\!\!\bot$ unless $a := c$
$= \mathtt{sup}_{c \geq 0} \, \mathbb{S}^{\,c*(i*n-1)+c*n} \cap \mathbb{S}^{\,\intercal c} ; z ; \mathbb{T}$
$= \mathtt{sup}_{c \geq 0} \, \mathbb{S}^{\,c*((i+1)*n-1)} \cap \mathbb{S}^{\,\intercal c} ; z ; \mathbb{T}$
$= k_{(i+1)*n}$ due to (ii.a)

---

ii.j) $k_n ; \mathbb{S} \cap k_l = \mathtt{sup}_{i \geq 0} \, \mathbb{S}^{\,\intercal i} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{\,i*n} ; \mathbb{S} \cap \mathtt{sup}_{j \geq 0} \, \mathbb{S}^{\,\intercal j} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{\,j*l}$ following (ii.a) twice
$= \mathtt{sup}_{i \geq 0} \, \mathbb{S}^{\,\intercal i} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{\,i*n+1} \cap \mathbb{S}^{\,\intercal i} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{\,i*l}$ certainly $\ldots = \bot\!\!\!\bot$ in case $j \neq i$
from the rear side implying $i*n+1 = i*l$, otherwise $\ldots = \bot\!\!\!\bot$
$= \mathbb{S}^{\,\intercal} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{\,n+1}$ since $1 = i*(l-n)$ required, i.e. $l = n+1$ and $i = 1$
$= e ; \mathbb{T} ; e^\intercal ; \mathbb{S}^{\,n}$

---

ii.k) The case $n = 0$:
$k_0^\intercal \cap \overline{k_0^\intercal ; \overline{e ; \mathbb{T}}} = z ; \mathbb{T} \cap \overline{z ; \mathbb{T}} ; \overline{e ; \mathbb{T}} = z ; \mathbb{T} \cap \overline{z ; \mathbb{T}}} = \bot\!\!\!\bot$

For $n \geq 1$:
$k_n^\intercal \cap \overline{k_n^\intercal ; \overline{e ; \mathbb{T}}} = k_n^\intercal \cap \overline{k_n^\intercal ; \mathbb{T} \cap \overline{k_n^\intercal ; e ; \mathbb{T}}}$ since $k_n^\intercal$ is univalent
$= k_n^\intercal \cap k_n^\intercal ; \mathbb{T} \cap \overline{\overline{\mathbb{S}^{\,\intercal n} ; z ; \mathbb{T}}}$ due to (i.b)
$= k_n^\intercal \cap \big[\overline{k_n^\intercal ; \mathbb{T}} \cup \mathbb{S}^{\,\intercal n} ; z ; \mathbb{T} \big]$
$= \big[k_n^\intercal \cap \overline{k_n^\intercal ; \mathbb{T}} \big] \cup \big[k_n^\intercal \cap \mathbb{S}^{\,\intercal n} ; z ; \mathbb{T} \big] = \bot\!\!\!\bot \cup \big[k_n^\intercal \cap \mathbb{S}^{\,\intercal n} ; z ; \mathbb{T} \big] = k_n^\intercal \cap \mathbb{S}^{\,\intercal n} ; z ; \mathbb{T}$
$= \big[\mathtt{sup}_{i \geq 0} \, \mathbb{S}^{\,\intercal i*n} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{\,i} \big] \cap \mathbb{S}^{\,\intercal n} ; z ; \mathbb{T}$ due to (ii.a); implying $i*n = n$ and thus $i := 1$
$= \mathbb{S}^{\,\intercal n} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S} \cap \mathbb{S}^{\,\intercal n} ; z ; \mathbb{T}$
$= \mathbb{S}^{\,\intercal n} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}$
$= \mathbb{S}^{\,\intercal n} ; z ; \mathbb{T} ; e^\intercal$

---

ii.l) We start proving $\Longrightarrow$ with (ii.a):

$k_n = \mathtt{sup}_{i\geq 0}\, \mathbb{S}^{\intercal i}; z; \mathbb{T}; z^{\intercal}; \mathbb{S}^{i*n} = z; \mathbb{T}; z^{\intercal} \cup \mathtt{sup}_{i\geq 1}\, \mathbb{S}^{\intercal i}; z; \mathbb{T}; z^{\intercal}; \mathbb{S}^{i*n} = z; \mathbb{T}; z^{\intercal} \cup \mathbb{\perp}$

since $\mathbb{S}^{\intercal n}; z; \mathbb{T} = \mathbb{\perp}$ implies $\mathbb{S}^{\intercal i*n}; z; \mathbb{T} = \mathbb{\perp}$ for $i \geq 1$

Now for the other direction:

$k_n = z; \mathbb{T}; z^{\intercal} \cup \mathtt{sup}_{i\geq 1}\, \mathbb{S}^{\intercal i}; z; \mathbb{T}; z^{\intercal}; \mathbb{S}^{i*n} = z; \mathbb{T}; z^{\intercal}$

implies in particular that for $i = 1$

$\mathbb{S}^{\intercal}; z; \mathbb{T}; z^{\intercal}; \mathbb{S}^{n} = e; \mathbb{T}; z^{\intercal}; \mathbb{S}^{n} \subseteq z; \mathbb{T}; z^{\intercal} \subseteq z; \mathbb{T}.$

Via shunting this is equivalent with $e; \mathbb{T}; z^{\intercal}; \mathbb{S}^{n}; \mathbb{T}; z^{\intercal} \subseteq \mathbb{I}$. The inner constituent $\mathbb{T}; z^{\intercal}; \mathbb{S}^{n}; \mathbb{T}$ is by construction either $= \mathbb{T}$ or $= \mathbb{\perp}$. Since certainly $e; \mathbb{T}; z^{\intercal} \not\subseteq \mathbb{I}$ we have $\mathbb{S}^{\intercal n}; z; \mathbb{T} = \mathbb{\perp}$.

---

ii.m) $k_i \cap k_n = \mathtt{sup}_{a\geq 0}\, \mathbb{S}^{\intercal a}; z; \mathbb{T}; z^{\intercal}; \mathbb{S}^{a*i} \cap \mathtt{sup}_{b\geq 0}\, \mathbb{S}^{\intercal b}; z; \mathbb{T}; z^{\intercal}; \mathbb{S}^{b*n}$ following (ii.a)

$= \mathtt{sup}_{a\geq 0}\, \mathbb{S}^{\intercal a}; z; \mathbb{T}; z^{\intercal}; \mathbb{S}^{a*i} \cap \mathbb{S}^{\intercal a}; z; \mathbb{T}; z^{\intercal}; \mathbb{S}^{a*n}$ since $\mathbb{S}^{\intercal a}; z; \mathbb{T} \cap \mathbb{S}^{\intercal b}; z; \mathbb{T} = \mathbb{\perp}$ for $a \neq b$

Now in a similar way $\mathbb{T}; z^{\intercal}; \mathbb{S}^{a*i} \cap \mathbb{T}; z^{\intercal}; \mathbb{S}^{a*n} = \mathbb{\perp}$ if $a * i \neq a * n$, thus leaving us with either $a = 0$ or $i = n$. The latter, however, is explicitly excluded, so that $a = 0$ remains, leading to $\ldots = z; \mathbb{T}; z^{\intercal}$.

---

ii.n) We prove this row-wise for rows $r \geq 0$. Since $\mathbb{S}^{\intercal r}; z; \mathbb{T} \cap \mathbb{T}; e^{\intercal} = \mathbb{S}^{\intercal r}; z; \mathbb{T}; e^{\intercal}$ is an entry, it will be contained as a whole in one (or more) constituents of the union on the right; so we first test whether

$\mathbb{S}^{\intercal r}; z; \mathbb{T}; e^{\intercal} \subseteq \mathbb{S}^{\intercal i*n}; z; \mathbb{T}$

for some $i \geq 0, i \neq 1$. Such a containment will hold when $r = i * n$, and the case of n being a proper divisor of r is, thus, finished. Now, we assume to the contrary that $n \nmid r$. In this case due to (i.c) above

$k_n; \mathbb{S}^{\intercal r}; z; \mathbb{T} = \mathbb{\perp} \quad\Longrightarrow\quad k_n; \mathbb{S}^{\intercal r}; z; \mathbb{T} \subseteq \mathbb{I} \quad\Longleftrightarrow\quad k_n^{\intercal}; \overline{\mathbb{I}} \subseteq \overline{\mathbb{S}^{\intercal r}; z; \mathbb{T}} \quad\Longleftrightarrow\quad \mathbb{S}^{\intercal r}; z; \mathbb{T} \subseteq \overline{k_n^{\intercal}; \overline{\mathbb{I}}}$

---

iii.a) The first two statements are trivial and don't need regressive finiteness. The others, with $n \geq 2$, are proved follows — it reflects the construction but is not a recursive proof:

$k_{n+1}; \mathbb{T} = (k_n \circledcirc \mathbb{I}); \mathbb{A}; \mathbb{T} = (k_n \circledcirc \mathbb{I}); \big[\, \mathtt{sup}_{k\geq 0}\, (\mathbb{S}^{\intercal k}; z; \mathbb{T} \circledcirc \mathbb{S}^{k}) \,\big]; \mathbb{T}$

$= \mathtt{sup}_{k\geq 0} \big[k_n; \mathbb{S}^{\intercal k}; z; \mathbb{T} \cap \mathbb{S}^{k}\big]; \mathbb{T} = \mathtt{sup}_{k\geq 0}\, k_n; \mathbb{S}^{\intercal k}; z; \mathbb{T} \cap \mathbb{S}^{k}; \mathbb{T}$ masking

$= \mathtt{sup}_{k\geq 0}\, \mathbb{S}^{k}; \mathbb{T} \cap \begin{cases} \mathbb{\perp} & n \nmid k \quad \text{or} \quad \mathbb{S}^{\intercal k}; z; \mathbb{T} = \mathbb{\perp} \\ \mathbb{S}^{\intercal i}; z; \mathbb{T} & n \,|\, k \text{ and } \mathbb{S}^{\intercal k}; z; \mathbb{T} \neq \mathbb{\perp}, \quad \text{denoting } i := \frac{k}{n} \end{cases}$ using (i.c)

$= \mathtt{sup}_{i\geq 0}\, \mathbb{S}^{i*n}; \mathbb{T} \cap \mathbb{S}^{\intercal i}; z; \mathbb{T}$ focussing on non-vanishing contributions

$= \mathtt{sup}_{i\geq 0}\, \mathbb{S}^{i*n}; \mathbb{T} \cap \mathbb{S}^{n_{\mathbb{S}} - i}; g_{\mathbb{S}}; \mathbb{T}$ implying $n_{\mathbb{S}} - i \geq i * n$ to be $\neq \mathbb{\perp}$, or $n_{\mathbb{S}} \geq i * (n + 1)$

Here Prop. 5.2.1.iii.d has been made use of.

---

iii.b) reflects (ii.c) together with $\mathbb{S}$ non-total.

---

iv.a) is proved by induction as follows:

$k_{n+1}; \mathbb{T} = (k_n \circledcirc \mathbb{I}); \mathbb{A}; \mathbb{T} = (k_n \circledcirc \mathbb{I}); \mathbb{T} = (k_n; \pi^{\intercal} \cap \rho^{\intercal}); \pi; \mathbb{T} = (k_n \cap \mathbb{T}); \mathbb{T} = k_n; \mathbb{T} = \mathbb{T}$

---

iv.b) follows from (ii.d) since k_n is then total.

---

iv.c) reflects (ii.c) together with $\mathbb{S}$ total. □

A remark seems necessary for (i.c): When restricting the naturals to the initial segment $0, 1, \ldots l$, it may well be that $i < l$, but $m > l$, making $k_n ; \mathbb{S}^{\tau m} ; z ; \mathbb{T} = \mathbb{1}$, but $\mathbb{S}^{\tau i} ; z ; \mathbb{T} \neq \mathbb{1}$; see Fig. 6.4.2. This is taken care of by qualifying the term $\mathbb{S}^{\tau m} ; z ; \mathbb{T}$ as being $= \mathbb{1}$ or not. To illustrate what may happen, we visualize the following for naturals restricted to 7:

$$k_4 ; \mathbb{S}^{\tau 12} ; z ; \mathbb{T} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{array} \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{array} = \mathbb{S}^{\tau 3} ; z ; \mathbb{T}$$

$$k_3 ; \mathbb{S}^{\tau 6} ; z ; \mathbb{T} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{array} \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{array} = \mathbb{S}^{\tau 2} ; z ; \mathbb{T}$$

Fig. 6.4.2 Illustrating Lemma 6.4.2.i.c for naturals restricted to 7

Yet another remark concerns (ii.n), i.e. the result that for all $n \geq 0$

$$\mathbb{T} ; e^{\tau} \subseteq \overline{k_n^{\tau} ; \mathbb{I}} \cup \mathtt{sup}_{i \geq 0, i \neq 1} \, \mathbb{S}^{\tau i * n} ; z ; \mathbb{T},$$

as reformulated when proving it row-wise: For all $r \geq 0$ and $n \geq 0$

$$\mathbb{S}^{\tau r} ; z ; \mathbb{T} ; e^{\tau} \subseteq \overline{k_n^{\tau} ; \mathbb{I}} \cup \mathtt{sup}_{i \geq 0, i \neq 1} \, \mathbb{S}^{\tau i * n} ; z ; \mathbb{T}.$$

When considering any fixed r with respect to all $n \geq 0$, three cases may be distinguished with respect to the terms on the right side:

Case 1: For $r = 0$, this is never contained in $\overline{k_n^{\tau} ; \mathbb{I}}$, but $\mathbb{S}^{\tau 0} ; z ; \mathbb{T} ; e^{\tau} \subseteq \mathbb{S}^{\tau 0 * n} ; z ; \mathbb{T}$ for all $n \geq 0$.

Case 2: For a prime $r \geq 1$, this will for all $n \geq 0$ be contained in $\overline{k_n^{\tau} ; \mathbb{I}}$, since it cannot be contained in any of the supremum-terms requiring $i * n = r$.

Case 3: For a non-prime $r \geq 1$, this cannot be contained in $\overline{k_n^{\tau} ; \mathbb{I}}$ for all $n \geq 0$. Assuming a factorization $r = a * b$ with $a > 1$ and $b > 1$, we provide, namely, the counterexample $n := a$ for which due to (i.c)

$$\mathbb{S}^{\tau a * b} ; z ; \mathbb{T} ; e^{\tau} \subseteq \overline{k_a^{\tau} ; \mathbb{I}} \quad \Longleftrightarrow \quad k_a ; \mathbb{S}^{\tau a * b} ; z ; \mathbb{T} = \mathbb{S}^{\tau b} ; z ; \mathbb{T} \neq e ; \mathbb{T}.$$

From this result, therefore, we get a dichotomy separating primes from non-primes that may be important to investigate.

7 Multiplicative structure

Mainly built on iterated addition, we introduce multiplication.

7.1 Multiplication

Starting from iterated additions of Def. 6.4.1, multiplication $\mathbb{M} : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ may be introduced according to

$(k+1) * n = k * n + n,$
$0 * n = 0.$

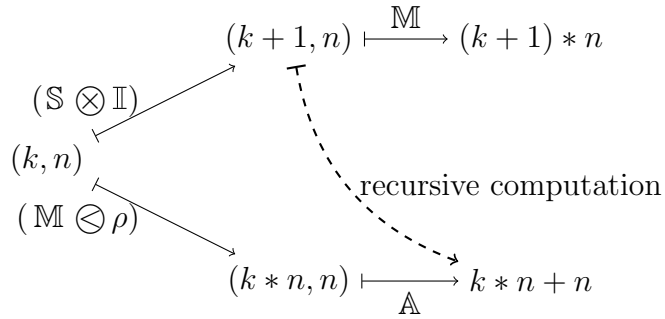


Fig. 7.1.1 Defining the multiplication relation on a possibly finite initial segment of the naturals

We again instantiate Lemma 4.2.1 with two functionals, now denoted $m(X)$ and $n(X)$, respectively, instead of f.

7.1.1 Definition and Proposition. We define as **multiplication** belonging to $\mathbb{S}$

$\mathbb{M} := \mathbb{M}_{\mathbb{S}} := \mathtt{inf}\,\{\, X \mid m(X) \subseteq X\},$
$N := N_{\mathbb{S}} := \mathtt{inf}\,\{\, X \mid n(X) \subseteq X\},$

where we have in view of Lemma 4.2.1

$m(X) := \mathbb{M}_0 \cup m_H(X)$ with $\mathbb{M}_0 = (z ; \mathbb{T} \oslash \mathbb{T} ; z^{\intercal})$ and $m_H(X) = (\mathbb{S}^{\intercal} \otimes \mathbb{I}) ; (X \oslash \rho) ; \mathbb{A}\,,$
$n(X) := N_0 \cup n_H(X)$ with $N_0 = (\mathbb{T} ; z^{\intercal} \oslash z ; \mathbb{T})$ and $n_H(X) = (\mathbb{I} \otimes \mathbb{S}^{\intercal}) ; (\pi \oslash X) ; B.$

Respecting possible further properties of $\mathbb{S}$, the versions $\mathbb{M}, N$ of multiplication satisfy

i) in general:

a) $\mathbb{M} = m(\mathbb{M}) = \mathtt{sup}_{n \geq 0}\, m_H^n(\mathbb{M}_0) = \mathtt{sup}_{n \geq 0}\, (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash k_n) = \mathbb{M}_0 \cup (\mathbb{S}^{\intercal} \otimes \mathbb{I}) ; (\mathbb{M} \oslash \rho) ; \mathbb{A}$

$N = n(N) = \mathtt{sup}_{n \geq 0}\, n_H^n(N_0) = \mathtt{sup}_{n \geq 0}\, (l_n \oslash \mathbb{S}^{\intercal n} ; z ; \mathbb{T}) = N_0 \cup (\mathbb{I} \otimes \mathbb{S}^{\intercal}) ; (\pi \oslash N) ; B.$

b) $m_H^n(\mathbb{M}_0);\mathbb{T} = (\mathbb{S}^{\intercal n};z;\mathbb{T} \oslash k_n;\mathbb{T})$ and

$m_H^n(\mathbb{M}_0) \subseteq \pi;\mathbb{S}^{\intercal n};z;\mathbb{T} = (\mathbb{S}^{\intercal n};z;\mathbb{T} \oslash \mathbb{T})$ for $n \geq 0$; i.e. they are all disjoint.

$n_H^n(N_0);\mathbb{T} = (l_n;\mathbb{T} \oslash \mathbb{S}^{\intercal n};z;\mathbb{T})$ and

$n_H^n(N_0) \subseteq \rho;\mathbb{S}^{\intercal n};z;\mathbb{T} = (\mathbb{T} \oslash \mathbb{S}^{\intercal n};z;\mathbb{T})$ for $n \geq 0$; i.e. they are all disjoint.

c) $\mathbb{M}$ and N need not be commutative or distributive.

d) $\mathbb{M}$ and N are both univalent.

e) $\mathbb{M}^{\intercal};(\mathbb{S}^{\intercal i};z;\mathbb{T} \oslash \mathbb{I}) = \begin{cases} k_i^{\intercal} & \text{if } \mathbb{S}^{\intercal i};z;\mathbb{T} \neq \mathbb{\perp} \\ \mathbb{\perp} & \text{otherwise} \end{cases}$ $\mathbb{M}^{\intercal};(e;\mathbb{T} \oslash R) = R$

$(\mathbb{M}_0 \oslash \rho);\mathbb{A} = (z;\mathbb{T} \oslash \mathbb{I})$ $\mathbb{M};e;\mathbb{T} = (\mathbb{I} \oslash \mathbb{I});e;\mathbb{T} = (\pi \cap \rho);\ e;\mathbb{T}$

$N^{\intercal};(\mathbb{I} \oslash \mathbb{S}^{\intercal i};z;\mathbb{T}) = \begin{cases} k_i^{\intercal} & \text{if } \mathbb{S}^{\intercal i};z;\mathbb{T} \neq \mathbb{\perp} \\ \mathbb{\perp} & \text{otherwise} \end{cases}$ $N^{\intercal};(R \oslash e;\mathbb{T}) = R$

$(\pi \oslash N_0);B = (\mathbb{I} \oslash z;\mathbb{T})$ $N;e;\mathbb{T} = (\mathbb{I} \oslash \mathbb{I});e;\mathbb{T} = (\pi \cap \rho);\ e;\mathbb{T}$

f) Neither $\mathbb{M}$ nor N has necessarily neutral elements.

g) $(\mathbb{S} \otimes \mathbb{I});\mathbb{M} = (\mathbb{M} \oslash \rho);\mathbb{A} \cap \pi;\mathbb{S};\mathbb{T}$ $(\mathbb{T};z^{\intercal} \oslash \mathbb{S});\mathbb{M} = \mathbb{S};\mathbb{T};z^{\intercal}$

$(\mathbb{I} \otimes \mathbb{S});N = (\pi \oslash N);\mathbb{A} \cap \rho;\mathbb{S};\mathbb{T}$ $(\mathbb{S} \oslash \mathbb{T};z^{\intercal});N = \mathbb{S};\mathbb{T};z^{\intercal}$

$\mathbb{M} \cap \pi;e;\mathbb{T} = (e;\mathbb{T} \oslash \mathbb{I})$ $N \cap \rho;e;\mathbb{T} = (\mathbb{I} \oslash e;\mathbb{T})$

$(\mathbb{M} \oslash \pi);\mathfrak{S} =$

	0	1	2	3	4
(0,0)	1	0	0	0	0
(0,1)	1	0	0	0	0
(0,2)	1	0	0	0	0
(0,3)	1	0	0	0	0
(0,4)	1	0	0	0	0
(1,0)	0	0	0	0	0
(1,1)	1	0	0	0	0
(1,2)	0	1	0	0	0
(1,3)	0	0	1	0	0
(1,4)	0	0	0	1	0
(2,0)	0	0	0	0	0
(2,1)	1	0	0	0	0
(2,2)	0	0	1	0	0
(2,3)	0	0	0	0	0
(2,4)	0	0	0	0	0
(3,0)	0	0	0	0	0
(3,1)	1	0	0	0	0
(3,2)	0	0	0	0	0
(3,3)	0	0	0	0	0
(3,4)	0	0	0	0	0
(4,0)	0	0	0	0	0
(4,1)	1	0	0	0	0
(4,2)	0	0	0	0	0
(4,3)	0	0	0	0	0
(4,4)	0	0	0	0	0

$\approx$

	0	1	2	3	4
(0,0)	0	0	0	0	0
(0,1)	1	0	0	0	0
(0,2)	1	0	0	0	0
(0,3)	1	0	0	0	0
(0,4)	1	0	0	0	0
(1,0)	0	0	0	0	0
(1,1)	1	0	0	0	0
(1,2)	0	1	0	0	0
(1,3)	0	0	1	0	0
(1,4)	0	0	0	1	0
(2,0)	0	0	0	0	0
(2,1)	1	0	0	0	0
(2,2)	0	0	1	0	0
(2,3)	0	0	0	0	1
(2,4)	0	0	0	0	0
(3,0)	0	0	0	0	0
(3,1)	1	0	0	0	0
(3,2)	0	0	0	1	0
(3,3)	0	0	0	0	0
(3,4)	0	0	0	0	0
(4,0)	0	0	0	0	0
(4,1)	1	0	0	0	0
(4,2)	0	0	0	0	1
(4,3)	0	0	0	0	0
(4,4)	0	0	0	0	0

$= (\mathbb{I} \otimes \mathbb{S}^{\intercal});\mathbb{M}$

Fig. 7.1.2 Situation of (ii.j) means up to border effects, that $a * b - a = a * (b - 1)$

h) $\mathfrak{P}\fatsemi \mathbb{M} = N \quad \text{and} \quad \mathfrak{P}\fatsemi N = \mathbb{M}$

i) For $l \geq 0$ we have $\mathtt{rel}(\,\mathbb{M}\fatsemi \mathbb{S}^{\intercal l}\fatsemi z\fatsemi \mathbb{T}) = \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\intercal n}\fatsemi z\fatsemi \mathbb{T}\fatsemi z^{\intercal}\fatsemi \mathbb{S}^{l}\fatsemi k_n^{\intercal}$

ii) additionally, in case $\mathbb{S}$ is regressively finite:

a) $\mathbb{M}$ and N need not necessarily be associative.

b) $\mathbb{M}$ and N are both commutative, i.e. $\mathfrak{P}\fatsemi \mathbb{M} = \mathbb{M} \quad \text{and} \quad \mathfrak{P}\fatsemi N = N$,

thus, together with (i.h), implying[1] $\mathbb{M} = N$.

c) $\mathbb{M}\fatsemi z\fatsemi \mathbb{T} = (z\fatsemi \mathbb{T} \circledcirc \mathbb{T}) \cup (\mathbb{T} \circledcirc z\fatsemi \mathbb{T}) = \overline{\overline{(z\fatsemi \mathbb{T} \circledcirc \mathbb{T})}\, \overline{(z\fatsemi \mathbb{T} \circledcirc \mathbb{T})}} = (\pi \cup \rho)\fatsemi z\fatsemi \mathbb{T}$

For an arbitrary point $\mathbb{S}^{\intercal r}\fatsemi z\fatsemi \mathbb{T}$ always $\quad \mathbb{M}\fatsemi \mathbb{S}^{\intercal r}\fatsemi z\fatsemi \mathbb{T} \supseteq (e\fatsemi \mathbb{T} \circledcirc \mathbb{T}) \cup (\mathbb{T} \circledcirc e\fatsemi \mathbb{T})$

d) $\mathbb{M}^{\intercal}\fatsemi (z\fatsemi \mathbb{T} \circledcirc R) = z\fatsemi \mathbb{T}\fatsemi R$

$\mathbb{M}^{\intercal}\fatsemi (z\fatsemi \mathbb{T} \circledcirc R) = z\fatsemi \mathbb{T}$ in case R is surjective

$\mathbb{M}^{\intercal}\fatsemi (e\fatsemi \mathbb{T} \circledcirc R) = R \qquad \mathbb{M}^{\intercal}\fatsemi (R \circledcirc e\fatsemi \mathbb{T}) = R \qquad \mathbb{M}^{\intercal}\fatsemi (e\fatsemi e^{\intercal} \circledcirc e\fatsemi e^{\intercal}) = e\fatsemi e^{\intercal}$

e) $\mathbb{M}$ has a left as well as right neutral element, namely $e := \mathbb{S}^{\intercal}\fatsemi z$:

$\mathbb{M}^{\intercal}\fatsemi (e\fatsemi \mathbb{T} \circledcirc \mathbb{I}) = \mathbb{M}^{\intercal}\fatsemi (\mathbb{I} \circledcirc e\fatsemi \mathbb{T}) = \mathbb{I}$

f) $\pi\fatsemi \pi^{\intercal} \cap \mathbb{M}\fatsemi \mathbb{M}^{\intercal} \subseteq \mathbb{I} \cup \pi\fatsemi z\fatsemi \mathbb{T}\fatsemi z^{\intercal}\fatsemi \pi^{\intercal}$ $\mathbb{M}$ is, thus "nearly" left-cancellative[2]

g) $\mathbb{M} = \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\, (\mathbb{S}^{\intercal n}\fatsemi z\fatsemi \mathbb{T} \circledcirc \mathbb{S}^{\intercal i}\fatsemi z\fatsemi \mathbb{T})\fatsemi \mathbb{T}\fatsemi z^{\intercal}\fatsemi \mathbb{S}^{n*i}$

$\mathbb{M}\fatsemi \mathbb{S}^{\intercal r}\fatsemi z\fatsemi \mathbb{T} = \mathtt{sup}_{a,b\geq 0,\, a*b=r}\, (\mathbb{S}^{\intercal a} \circledcirc \mathbb{S}^{\intercal b})\fatsemi z\fatsemi \mathbb{T}$

h) $\mathbb{M}^{\intercal}\fatsemi (\mathbb{S}^{\intercal i}\fatsemi z\fatsemi \mathbb{T} \circledcirc \mathbb{S}^{\intercal k}\fatsemi z\fatsemi \mathbb{T}) = \begin{cases} \perp\!\!\!\perp & \text{if} \quad \mathbb{S}^{\intercal i}\fatsemi z\fatsemi \mathbb{T} = \perp\!\!\!\perp \quad \text{or} \quad \mathbb{S}^{\intercal k}\fatsemi z\fatsemi \mathbb{T} = \perp\!\!\!\perp \\ \mathbb{S}^{\intercal i*k}\fatsemi z\fatsemi \mathbb{T} & \text{otherwise} \end{cases}$

i) $(\mathbb{T}\fatsemi z^{\intercal}\fatsemi \mathbb{S}^{\,n} \circledcirc \mathbb{I})\fatsemi \mathbb{M} \cap e\fatsemi \mathbb{T} = e\fatsemi \mathbb{T}\fatsemi z^{\intercal}\fatsemi \mathbb{S}^{\,n}$

j) $(\mathbb{M} \circledcirc \pi)\fatsemi \mathfrak{S} = (z\fatsemi \mathbb{T} \circledcirc \mathbb{T}\fatsemi z^{\intercal}) \cup \Big[(\mathbb{I} \otimes \mathbb{S}^{\intercal})\fatsemi \mathbb{M} \cap \mathbb{M}\fatsemi \mathbb{T}\Big]$

$(\mathbb{I} \otimes \mathbb{S}^{\intercal})\fatsemi (\mathbb{M} \circledcirc \pi)\fatsemi \mathbb{A} = \mathbb{M} \cap \rho\fatsemi \mathbb{S}^{\intercal}\fatsemi \mathbb{T}$

k) $\overline{\overline{\mathbb{M}}^{\intercal}\fatsemi \mathbb{M}} = \mathbb{I} \qquad \mathtt{syq}(\,\mathbb{M}\,,\, \mathbb{M}\,) = \mathbb{I}$

iii) If $\mathbb{S}$ happens to be regressively finite but not total, i.e. $\mathbb{S}\fatsemi \mathbb{T} \neq \mathbb{T}$,

...

iv) If $\mathbb{S}$ is regressively finite and also total, i.e. $\mathbb{S}\fatsemi \mathbb{T} = \mathbb{T}$, then since $\mathbb{M} = N$ with (ii.b)

a) $\mathbb{M}$ is total.

[1] See pictures Figs. 7.1.4, 7.1.5.
[2] See Fig. 7.1.3

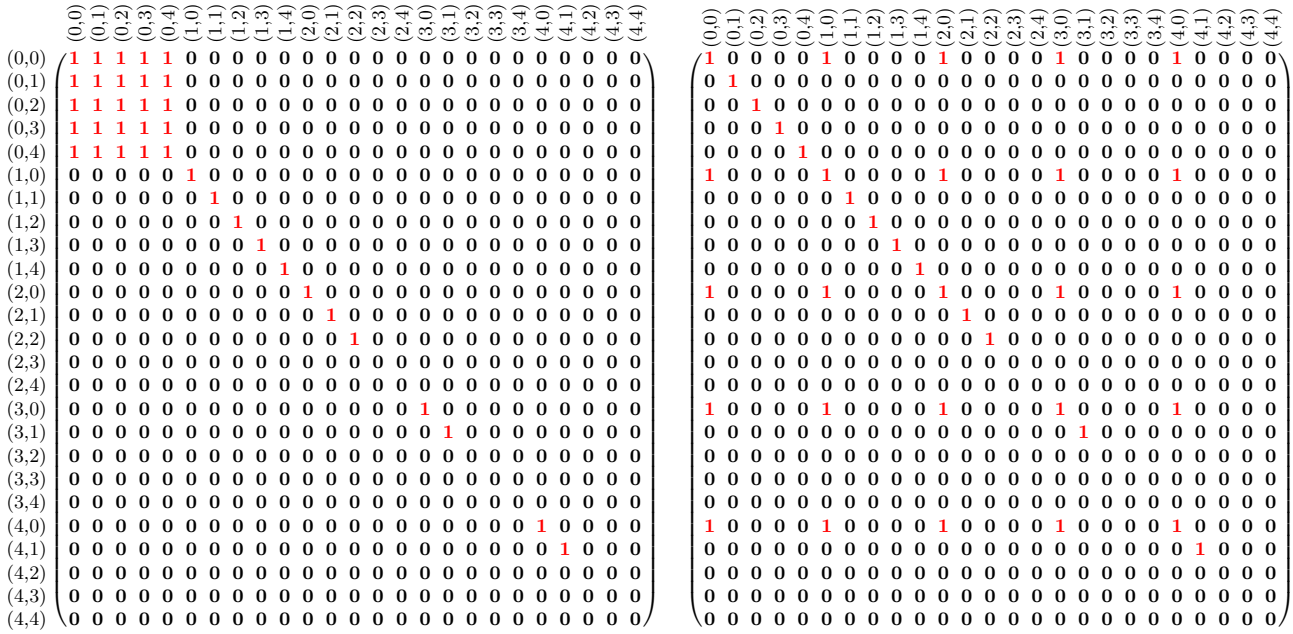
b) $\mathbb{M}$ is associative.

c) $(\mathbb{S} \otimes \mathbb{I})\,;\mathbb{M} \;=\; (\mathbb{M} \oslash \rho)\,;\mathbb{A}$

$(\mathbb{I} \otimes \mathbb{S})\,;\mathbb{M} \;=\; (\pi \oslash \mathbb{M})\,;\mathbb{A}$

d) $\pi;\pi^{\mathsf{T}} \cap \mathbb{M};\mathbb{M}^{\mathsf{T}} = \mathbb{I} \cup \pi;z;\mathbb{T};z^{\mathsf{T}};\pi^{\mathsf{T}}$

e) $\mathbb{M}^{\mathsf{T}};\ (\mathbb{S}^{\mathsf{T}i};\, z;\mathbb{T} \oslash \mathbb{I}) \;= k_i^{\mathsf{T}}$



$\pi;\pi^{\mathsf{T}} \cap \mathbb{M};\mathbb{M}^{\mathsf{T}}$ $\qquad$ $\mathbb{M};\mathbb{M}^{\mathsf{T}} \cap \rho;\rho^{\mathsf{T}}$

Fig. 7.1.3 $\mathbb{M}$ is "nearly" left-cancellative, resp. "nearly" right-cancellative

Proof: We observe in advance that Lemma 4.2.1 may be applied since the functional m_H is

isotonic, i.e. $A \subseteq B \implies (\mathbb{S}^{\mathsf{T}} \otimes \mathbb{I});(A \oslash \rho);\mathbb{A} \subseteq (\mathbb{S}^{\mathsf{T}} \otimes \mathbb{I});(B \oslash \rho);\mathbb{A}$,
i.e. $m_H(A) \subseteq m_H(B)$

additive, i.e. $m_H(X \cup Y) = (\mathbb{S}^{\mathsf{T}} \otimes \mathbb{I});(X \cup Y \oslash \rho);\mathbb{A}$
$= (\mathbb{S}^{\mathsf{T}} \otimes \mathbb{I});\big[\ (X \oslash \rho) \cup (Y \oslash \rho)\ \big];\mathbb{A} = m_H(X) \cup m_H(Y)$,

$\cap$-hereditary, i.e. $m_H(X \cap Y) = (\mathbb{S}^{\mathsf{T}} \otimes \mathbb{I});(X \cap Y \oslash \rho);\mathbb{A}$
$= (\mathbb{S}^{\mathsf{T}} \otimes \mathbb{I});\big[\ (X \oslash \rho) \cap (Y \oslash \rho)\ \big] = m_H(X) \cap m_H(Y)$.

This applies correspondingly to $n_H(X)$.

---

i.a) The claim collects the results of Lemma 4.2.1.i first. Where iteration becomes explicit, we prove by induction that $m_H^n(\mathbb{M}_0) = (\mathbb{S}^{\mathsf{T}n};\, z;\mathbb{T} \oslash k_n)$:

$m_H^0(\mathbb{M}_0) = \mathbb{M}_0 = (z;\mathbb{T} \oslash \mathbb{T};z^{\mathsf{T}}) = (\mathbb{S}^{\mathsf{T}0};z;\mathbb{T} \oslash k_0)$

$m_H^{n+1}(\mathbb{M}_0) = m_H(m_H^n(\mathbb{M}_0)) = m_H(\,(\mathbb{S}^{\mathsf{T}n};\, z;\mathbb{T} \oslash k_n)\,)$ induction hypothesis
$= (\mathbb{S}^{\mathsf{T}} \otimes \mathbb{I});(\,(\mathbb{S}^{\mathsf{T}n};\, z;\mathbb{T} \oslash k_n) \oslash \rho);\mathbb{A}$
$= (\mathbb{S}^{\mathsf{T}} \otimes \mathbb{I});(\,\mathbb{S}^{\mathsf{T}n};\, z;\mathbb{T};\pi^{\mathsf{T}} \oslash (k_n \oslash \mathbb{I})\,);\mathbb{A}$ projection migration Prop. 3.1.2.i

$= (\mathbb{S}^{\tau n+1}; z; \mathbb{T} \oslash (k_n \oslash \mathbb{I}); \mathbb{A})$ since π is total and masking a join
$= (\mathbb{S}^{\tau n+1}; z; \mathbb{T} \oslash k_{n+1})$

This would work completely analogous in the other case. We present, however, an alternative induction — using the notation l_n again, although $k_n = l_n$ has already been established in Lemma 6.4.2.i.a:

$n_H^0(N_0) = N_0 = (\mathbb{T}; z^\tau \oslash z; \mathbb{T}) = (l_0 \oslash \mathbb{S}^{\tau 0}; z; \mathbb{T})$.

$n_H^{n+1}(N_0) = n_H(n_H^n(N_0)) = n_H((l_n \oslash \mathbb{S}^{\tau n}; z; \mathbb{T}))$ induction hypothesis
$= (\mathbb{I} \otimes \mathbb{S}^\tau); (\pi \oslash (l_n \oslash \mathbb{S}^{\tau n}; z; \mathbb{T})); B$
$= (\mathbb{I} \otimes \mathbb{S}^\tau); (\pi \oslash \pi; l_n \cap \rho; \mathbb{S}^{\tau n}; z; \mathbb{T}); B$
$= (\mathbb{I} \otimes \mathbb{S}^\tau); [\pi; \pi^\tau \cap (\pi; l_n \cap \rho; \mathbb{S}^{\tau n}; z; \mathbb{T}); \rho^\tau]; B$
$= (\mathbb{I} \otimes \mathbb{S}^\tau); [\pi; (\mathbb{I} \oslash l_n) \cap \rho; \mathbb{S}^{\tau n}; z; \mathbb{T}]; B$
$= (\mathbb{I} \otimes \mathbb{S}^\tau); [\pi; (\mathbb{I} \oslash l_n); B \cap \rho; \mathbb{S}^{\tau n}; z; \mathbb{T}]$ masking
$= (\mathbb{I} \otimes \mathbb{S}^\tau); [\pi; l_{n+1} \cap \rho; \mathbb{S}^{\tau n}; z; \mathbb{T}]$ by Def. 6.4.1
$= (\mathbb{I} \otimes \mathbb{S}^\tau); (l_{n+1} \oslash \mathbb{S}^{\tau n}; z; \mathbb{T})$
$= (l_{n+1} \oslash \mathbb{S}^{\tau n+1}; z; \mathbb{T})$

---

i.b) $m_H^n(\mathbb{M}_0); \mathbb{T} = (\mathbb{S}^{\tau n}; z; \mathbb{T} \oslash k_n); \mathbb{T}$ see proof of (i.a)
$= (\mathbb{S}^{\tau n}; z; \mathbb{T} \oslash k_n; \mathbb{T})$ masking a join

The second claim is proved by induction with hypothesis $m_H^k(\mathbb{M}_0) \subseteq \pi; \mathbb{S}^{\tau n}; z; \mathbb{T}$ starting at
$m_H^0(\mathbb{M}_0) = \mathbb{M}_0 = (z; \mathbb{T} \oslash \mathbb{T}; z^\tau) \subseteq (z; \mathbb{T} \oslash \mathbb{T}) = \pi; z; \mathbb{T} = \pi; \mathbb{S}^{\tau 0}; z; \mathbb{T}$:
$m_H^{k+1}(\mathbb{M}_0) = m_H(m_H^k(\mathbb{M}_0)) \subseteq m_H(\pi; \mathbb{S}^{\tau k}; z; \mathbb{T})$ hypothesis with monotony
$= (\mathbb{S}^\tau \otimes \mathbb{I}); (\pi; \mathbb{S}^{\tau k}; z; \mathbb{T} \oslash \rho); \mathbb{A}$
$= (\mathbb{S}^\tau \otimes \mathbb{I}); (\mathbb{S}^{\tau k}; z; \mathbb{T} \otimes \mathbb{I}); \mathbb{A}$
$= (\mathbb{S}^{\tau k+1}; z; \mathbb{T} \otimes \mathbb{I}); \mathbb{A}$
$\subseteq [\pi; \mathbb{S}^{\tau k+1}; z; \mathbb{T}; \pi^\tau \cap \rho; \rho^\tau]; \mathbb{A} \subseteq \pi; \mathbb{S}^{\tau k+1}; z; \mathbb{T}$

The claim concerning iteration over the second component follows completely analogously.

---

i.c) Counter-example if not regressively finite: The product $(1 * 3) = 3$, but $3 * 1$ is undefined in Fig. 5.2.3.

Concerning distributivity: If naturals are restricted to 4, $0 * (2 + 3)$ delivers $\bot\!\!\!\bot$, but $(0 * 2 + 0 * 3)$ results in 0.

---

i.d) We abbreviate here temporarily $\mathbb{M}_n := m_H^n(\mathbb{M}_0)$. Following (i.b), all the $\mathbb{M}_n$ have disjoint domains. That they are univalent is shown by induction, starting with

$\mathbb{M}_0^\tau; \mathbb{M}_0 = (z; \mathbb{T} \oslash \mathbb{T}; z^\tau)^\tau; (z; \mathbb{T} \oslash \mathbb{T}; z^\tau) = (\mathbb{T}; z^\tau \oslash z; \mathbb{T}); (z; \mathbb{T} \oslash \mathbb{T}; z^\tau)$
$= \mathbb{T}; z^\tau; z; \mathbb{T} \cap z; \mathbb{T}; \mathbb{T}; z^\tau$
$= \mathbb{T} \cap z; \mathbb{T}; z^\tau = z; \mathbb{T}; z^\tau \subseteq \mathbb{I}$ since z is injective
$\mathbb{M}_{n+1}^\tau; \mathbb{M}_{n+1} = [(\mathbb{S}^\tau \otimes \mathbb{I}); (\mathbb{M}_n \oslash \rho); \mathbb{A}]^\tau; (\mathbb{S}^\tau \otimes \mathbb{I}); (\mathbb{M}_n \oslash \rho); \mathbb{A}$
$= \mathbb{A}^\tau; (\mathbb{M}_n^\tau \oslash \rho^\tau); (\mathbb{S} \otimes \mathbb{I}); (\mathbb{S}^\tau \otimes \mathbb{I}); (\mathbb{M}_n \oslash \rho); \mathbb{A}$
$\subseteq \mathbb{A}^\tau; (\mathbb{M}_n^\tau \oslash \rho^\tau); (\mathbb{M}_n \oslash \rho); \mathbb{A}$ since $\mathbb{S}$ is injective
$\subseteq \mathbb{A}^\tau; (\mathbb{M}_n^\tau; \mathbb{M}_n \otimes \rho^\tau; \rho); \mathbb{A}$ Prop. 2.1.1.i

$\subseteq \mathbb{A}^\intercal ; (\mathbb{I} \otimes \mathbb{I}) ; \mathbb{A}$ induction hypothesis and ρ univalent
$= \mathbb{A}^\intercal ; \mathbb{A} \subseteq \mathbb{I}$ since $\mathbb{A}$ is univalent

The second claim follows completely analogously.

---

i.e) $\mathbb{M}^\intercal ; (\mathbb{S}^{\intercal i} ; z ; \mathbb{T} \oslash \mathbb{I}) = \mathbf{sup}_{n\geq 0} (\mathbb{T} ; z^\intercal ; \mathbb{S}^n \oslash k_n^\intercal) ; (\mathbb{S}^{\intercal i} ; z ; \mathbb{T} \oslash \mathbb{I})$
$= \mathbf{sup}_{n\geq 0} \mathbb{T} ; z^\intercal ; \mathbb{S}^n ; \mathbb{S}^{\intercal i} ; z ; \mathbb{T} \cap k_n^\intercal$ will be $= \bot$ in case $n \neq i$
$= \mathbb{T} ; z^\intercal ; \mathbb{S}^i ; \mathbb{S}^{\intercal i} ; z ; \mathbb{T} \cap k_i^\intercal = \begin{cases} k_i^\intercal & \text{if } \mathbb{S}^{\intercal i} ; z ; \mathbb{T} \neq \bot \\ \bot & \text{otherwise} \end{cases}$

$\mathbb{M}^\intercal ; (e ; \mathbb{T} \oslash R) = \mathbf{sup}_{n\geq 0} (\mathbb{T} ; z^\intercal ; \mathbb{S}^n \oslash k_n^\intercal) ; (e ; \mathbb{T} \oslash R)$
$= \mathbf{sup}_{n\geq 0} \mathbb{T} ; z^\intercal ; \mathbb{S}^n ; e ; \mathbb{T} \cap k_n^\intercal ; R$ will be $= \bot$ unless $n = 1$
$= \mathbb{T} ; z^\intercal ; \mathbb{S} ; e ; \mathbb{T} \cap k_1^\intercal ; R = \mathbb{T} ; z^\intercal ; z ; \mathbb{T} \cap \mathbb{I} ; R = \mathbb{T} \cap R = R$

$(\mathbb{M}_0 \oslash \rho) ; \mathbb{A} = ((z ; \mathbb{T} \oslash \mathbb{T} ; z^\intercal) \oslash \rho) ; \mathbb{A} =$
$= (z ; \mathbb{T} ; \pi^\intercal \oslash (\mathbb{T} ; z^\intercal \oslash \mathbb{I})) ; \mathbb{A}$ projection migration Prop. 3.1.2.i
$= (z ; \mathbb{T} \oslash (\mathbb{T} ; z^\intercal \oslash \mathbb{I})) ; \mathbb{A} = (z ; \mathbb{T} \oslash (\mathbb{T} ; z^\intercal \oslash \mathbb{I}) ; \mathbb{A})$ since π is total
$= (z ; \mathbb{T} \oslash \mathbb{I})$ Prop. 6.2.1.i.c

$\mathbb{M} ; e ; \mathbb{T} = \mathbf{sup}_{n\geq 0} (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash k_n) ; e ; \mathbb{T}$
$= \mathbf{sup}_{n\geq 0} (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; e ; \mathbb{T} \oslash k_n ; e ; \mathbb{T})$ with $e ; \mathbb{T}$ a point
$= (\mathbb{S}^\intercal ; z ; \mathbb{T} \oslash e ; \mathbb{T})$ since $k_n ; e ; \mathbb{T} = \bot$ for $n \neq 1$ due to Lemma 6.4.2.i.c
$= (e ; \mathbb{T} \oslash e ; \mathbb{T}) = (\mathbb{I} \oslash \mathbb{I}) ; e ; \mathbb{T}$

The other claims follow completely analogously.

---

i.f) Witness for the first case is Fig. 5.2.7.

---

i.g) $(\mathbb{S} \otimes \mathbb{I}) ; \mathbb{M} = (\mathbb{S} \otimes \mathbb{I}) ; (\mathbb{S}^\intercal \otimes \mathbb{I}) ; (\mathbb{M} \oslash \rho) ; \mathbb{A}$ due to (i.a)
$= (\mathbb{S} ; \mathbb{S}^\intercal \otimes \mathbb{I}) ; (\mathbb{M} \oslash \rho) ; \mathbb{A}$ because $\mathbb{S}$ is univalent
$= (\mathbb{I} \cap \mathbb{S} ; \mathbb{T} \otimes \mathbb{I}) ; (\mathbb{M} \oslash \rho) ; \mathbb{A}$
$= (\mathbb{I} \otimes \mathbb{I}) ; (\mathbb{M} \oslash \rho) ; \mathbb{A} \cap \pi ; \mathbb{S} ; \mathbb{T}$ masking
$= (\mathbb{M} \oslash \rho) ; \mathbb{A} \cap \pi ; \mathbb{S} ; \mathbb{T}$

$(\mathbb{T} ; z^\intercal \oslash \mathbb{S}) ; \mathbb{M} = (\mathbb{T} ; z^\intercal \oslash \mathbb{S}) ; \mathbf{sup}_{n\geq 0} (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash k_n)$
$= \mathbf{sup}_{n\geq 0} \mathbb{T} ; z^\intercal ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \mathbb{S} ; k_n$ implying $n := 0$ since otherwise $\mathbb{S} ; z = \bot$
$= \mathbb{T} \cap \mathbb{S} ; \mathbb{T} ; z^\intercal = \mathbb{S} ; \mathbb{T} ; z^\intercal$

$\mathbb{M} \cap \pi ; e ; \mathbb{T} = \mathbf{sup}_{n\geq 0} (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash k_n) \cap \pi ; \mathbb{S}^\intercal ; z ; \mathbb{T}$
$= \mathbf{sup}_{n\geq 0} (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \mathbb{S}^\intercal ; z ; \mathbb{T} \oslash k_n)$ will be $= \bot$ unless $n = 1$
$= (\mathbb{S}^\intercal ; z ; \mathbb{T} \oslash k_1) = (e ; \mathbb{T} \oslash \mathbb{I})$

For N, the proofs run completely analogous.

---

i.h) We aim at $\mathfrak{P} ; \mathbb{M} = N$ and prove it by induction with hypothesis $\mathfrak{P} ; m_H^n(\mathbb{M}_0) = n_H^n(N_0)$.
$\mathfrak{P} ; m_H^0(\mathbb{M}_0) = \mathfrak{P} ; \mathbb{M}_0 = \mathfrak{P} ; (z ; \mathbb{T} \oslash \mathbb{T} ; z^\intercal) = (\mathbb{T} ; z^\intercal \oslash z ; \mathbb{T}) = N_0 = n_H^0(N_0)$

$\mathfrak{P} ; m_H^{n+1}(\mathbb{M}_0) = \mathfrak{P} ; m_H(m_H^n(\mathbb{M}_0))$
$= \mathfrak{P} ; (\mathbb{S}^\intercal \otimes \mathbb{I}) ; (m_H^n(\mathbb{M}_0) \oslash \rho) ; \mathbb{A}$ by definition

$= (\mathbb{I} \otimes \mathbb{S}^{\intercal}) ; \mathfrak{P} ; (m_H^n(\mathbb{M}_0) \oslash \rho) ; \mathbb{A}$
$= (\mathbb{I} \otimes \mathbb{S}^{\intercal}) ; (\mathfrak{P} ; m_H^n(\mathbb{M}_0) \oslash \mathfrak{P} ; \rho) ; \mathbb{A}$
$= (\mathbb{I} \otimes \mathbb{S}^{\intercal}) ; (n_H^n(N_0) \oslash \pi) ; \mathbb{A}$ induction hypothesis
$= (\mathbb{I} \otimes \mathbb{S}^{\intercal}) ; (\pi \oslash n_H^n(N_0)) ; \mathfrak{P} ; \mathbb{A}$
$= (\mathbb{I} \otimes \mathbb{S}^{\intercal}) ; (\pi \oslash n_H^n(N_0)) ; B$
$= n_H(n_H^n(N_0)) = n_H^{n+1}(N_0)$

Now also $\mathfrak{P} ; N = \mathfrak{P} ; (\mathfrak{P} ; \mathbb{M}) = \mathbb{M}$.

---

i.i) If $\mathbb{S}^{\intercal l} ; z ; \mathbb{T} = \mathbb{\perp}$, this is obviously satisfied. In case $\neq \mathbb{\perp}$ we have a point $\mathbb{S}^{\intercal l} ; z ; \mathbb{T}$:
$\mathtt{rel}(\mathbb{M} ; \mathbb{S}^{\intercal l} ; z ; \mathbb{T}) = \mathtt{rel}(\left[\mathtt{sup}_{n \geq 0} (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash k_n) \right] ; \mathbb{S}^{\intercal l} ; z ; \mathbb{T})$
$= \mathtt{sup}_{n \geq 0} \mathtt{rel}((\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash k_n) ; \mathbb{S}^{\intercal l} ; z ; \mathbb{T})$ see page 15: rel is additive
$= \mathtt{sup}_{n \geq 0} \mathtt{rel}((\mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; \mathbb{S}^{\intercal l} ; z ; \mathbb{T} \oslash k_n ; \mathbb{S}^{\intercal l} ; z ; \mathbb{T}))$ since $\mathbb{S}^{\intercal l} ; z ; \mathbb{T}$ is a point
$= \mathtt{sup}_{n \geq 0} \mathtt{rel}((\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash k_n ; \mathbb{S}^{\intercal l} ; z ; \mathbb{T}))$
$= \mathtt{sup}_{n \geq 0} \pi^{\intercal} ; \left[(\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash k_n ; \mathbb{S}^{\intercal l} ; z ; \mathbb{T}) \cap \rho \right]$ definition of rel
$= \mathtt{sup}_{n \geq 0} \pi^{\intercal} ; \left[\pi ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \rho ; k_n ; \mathbb{S}^{\intercal l} ; z ; \mathbb{T} \cap \rho \right]$
$= \mathtt{sup}_{n \geq 0} \pi^{\intercal} ; \left[\pi ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \rho ; (k_n ; \mathbb{S}^{\intercal l} ; z ; \mathbb{T} \cap \mathbb{I}) \right]$
$= \mathtt{sup}_{n \geq 0} \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \mathbb{T} ; (k_n ; \mathbb{S}^{\intercal l} ; z ; \mathbb{T} \cap \mathbb{I})$ destroy and append
$= \mathtt{sup}_{n \geq 0} \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap (\mathbb{T} \cap \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{l} ; k_n^{\intercal}) ; \mathbb{I}$ mask shifting
$= \mathtt{sup}_{n \geq 0} \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{l} ; k_n^{\intercal}$
$= \mathtt{sup}_{n \geq 0} \mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{l} ; k_n^{\intercal}$ masking

---

ii.a) Counter-example: When limited by 7 the product $(7 \cdot 2) \cdot 0$ is undefined, but $7 \cdot (2 \cdot 0) = 7 \cdot 0 = 0$. For the second case, a counter-example is provided with Fig. 7.1.5: When limited by 4 the product $(4 \cdot 2) \cdot 0$ is undefined, but $4 \cdot (2 \cdot 0) = 4 \cdot 0 = 0$.

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ii.b) We apply $\mathfrak{P}$ to the sup-version and remember that $k_n = l_n$, maintaining this notation:
$\mathfrak{P} ; \mathtt{sup}_{n \geq 0} (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash k_n) = \mathtt{sup}_{n \geq 0} \mathfrak{P} ; (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash k_n) = \mathtt{sup}_{n \geq 0} (l_n \oslash \mathbb{S}^{\intercal n} ; z ; \mathbb{T})$
The result obviously resembles recursion over the second component. It must then be shown that for $\mathbb{S}$ regressively finite always
$\mathtt{sup}_{n \geq 0} (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash k_n) = \mathtt{sup}_{n \geq 0} (l_n \oslash \mathbb{S}^{\intercal n} ; z ; \mathbb{T})$.

To prove that the second is contained in the first,
$\mathtt{sup}_{n \geq 0} (l_n \oslash \mathbb{S}^{\intercal n} ; z ; \mathbb{T}) \subseteq \mathbb{M}$,
it is required to show that for all $n \geq 0$
$(l_n \oslash \mathbb{S}^{\intercal n} ; z ; \mathbb{T}) \subseteq \mathbb{M}$.
Induction starts with the case $n = 0$, which needs regressive finiteness in Prop. 5.2.1.ii:
$(l_0 \oslash \mathbb{S}^{\intercal 0} ; z ; \mathbb{T}) = (\mathbb{T} ; z^{\intercal} \oslash z ; \mathbb{T})$
$= (\mathtt{sup}_{k \geq 0} \mathbb{S}^{\intercal k} ; z ; \mathbb{T} ; z^{\intercal} \oslash z ; \mathbb{T})$
$= \mathtt{sup}_{k \geq 0} (\mathbb{S}^{\intercal k} ; z ; \mathbb{T} ; z^{\intercal} \oslash z ; \mathbb{T})$
We show directly that for all $k \geq 0$
$(\mathbb{S}^{\intercal k} ; z ; \mathbb{T} ; z^{\intercal} \oslash z ; \mathbb{T}) = \pi ; \mathbb{S}^{\intercal k} ; z ; \mathbb{T} ; z^{\intercal} \cap \rho ; z ; \mathbb{T}$

$= (\pi ; \mathbb{S}^{\tau k} ; z ; \mathbb{T} \cap \rho ; z ; \mathbb{T}) ; z^\tau$ masking
$= \pi ; \mathbb{S}^{\tau k} ; z ; \mathbb{T} \cap \rho ; z ; \mathbb{T} ; z^\tau$ masking
$\subseteq \pi ; \mathbb{S}^{\tau k} ; z ; \mathbb{T} \cap \rho ; k_k$ due to Lemma 6.4.2.i.b
$= (\mathbb{S}^{\tau k} ; z ; \mathbb{T} \oslash k_k)$
$\subseteq \mathtt{sup}_{i \geq 0} \, (\mathbb{S}^{\tau i} ; z ; \mathbb{T} \oslash k_i) \; = \; \mathbb{M}$

We resume the initial induction with the step $n \mapsto n+1$:
$(l_{n+1} \oslash \mathbb{S}^{\tau n+1} ; z ; \mathbb{T}) \; = \; (\mathbb{I} \otimes \mathbb{S}^\tau) ; (l_{n+1} \oslash \mathbb{S}^{\tau n} ; z ; \mathbb{T})$
$= (\mathbb{I} \otimes \mathbb{S}^\tau) ; (\, (\mathbb{I} \oslash l_n) ; B \oslash \mathbb{S}^{\tau n} ; z ; \mathbb{T})$ expanding l_{n+1}
$= (\mathbb{I} \otimes \mathbb{S}^\tau) ; \big[\pi ; (\mathbb{I} \oslash l_n) ; B \cap \rho ; \mathbb{S}^{\tau n} ; z ; \mathbb{T} \big]$
$= (\mathbb{I} \otimes \mathbb{S}^\tau) ; \big[\pi ; (\mathbb{I} \oslash l_n) \cap \rho ; \mathbb{S}^{\tau n} ; z ; \mathbb{T} \big] ; B$ masking
$= (\mathbb{I} \otimes \mathbb{S}^\tau) ; \big[\pi ; \pi^\tau \cap \pi ; l_n ; \rho^\tau \cap \rho ; \mathbb{S}^{\tau n} ; z ; \mathbb{T} ; \rho^\tau \big] ; B$ expanded, ρ is total
$= (\mathbb{I} \otimes \mathbb{S}^\tau) ; \big[\pi ; \pi^\tau \cap (l_n \oslash \mathbb{S}^{\tau n} ; z ; \mathbb{T}) ; \rho^\tau \big] ; B$
$\subseteq (\mathbb{I} \otimes \mathbb{S}^\tau) ; \big[\pi ; \pi^\tau \cap \mathbb{M} ; \rho^\tau \big] ; B$ induction hypothesis
$= (\mathbb{I} \otimes \mathbb{S}^\tau) ; (\pi \oslash \mathbb{M}) ; B$
$= n_H(\mathbb{M})$ Prop. 7.1.1
$= \mathbb{M}$

---

ii.c) $\mathbb{M} ; z ; \mathbb{T} = \mathtt{sup}_{n \geq 0} \, (\mathbb{S}^{\tau n} ; z ; \mathbb{T} \oslash k_n) ; z ; \mathbb{T}$
$= (z ; \mathbb{T} \oslash k_0) ; z ; \mathbb{T} \cup \mathtt{sup}_{n \geq 1} \, (\mathbb{S}^{\tau n} ; z ; \mathbb{T} \oslash k_n) ; z ; \mathbb{T}$
$= (z ; \mathbb{T} ; z ; \mathbb{T} \oslash k_0 ; z ; \mathbb{T}) \cup \mathtt{sup}_{n \geq 1} \, (\mathbb{S}^{\tau n} ; z ; \mathbb{T} ; z ; \mathbb{T} \oslash k_n ; z ; \mathbb{T})$ since z is injective
$= (z ; \mathbb{T} \oslash \mathbb{T}) \cup \mathtt{sup}_{n \geq 1} \, (\mathbb{S}^{\tau n} ; z ; \mathbb{T} \oslash z ; \mathbb{T})$ Lemma 6.4.2.i.c, since here $n \geq 1$
$= (z ; \mathbb{T} \oslash \overline{z ; \mathbb{T}}) \cup (z ; \mathbb{T} \oslash z ; \mathbb{T}) \cup \mathtt{sup}_{n \geq 1} \, (\mathbb{S}^{\tau n} ; z ; \mathbb{T} \oslash z ; \mathbb{T})$
$= (\pi ; z ; \mathbb{T} \cap \rho ; \overline{z ; \mathbb{T}}) \cup (\mathbb{T} \oslash z ; \mathbb{T})$ needing regressive finiteness
$= (\pi ; z ; \mathbb{T} \cap \rho ; \overline{z ; \mathbb{T}}) \cup \rho ; z ; \mathbb{T}$
$= (\pi ; z ; \mathbb{T} \cup \rho ; z ; \mathbb{T}) \cap (\rho ; \overline{z ; \mathbb{T}} \cup \rho ; z ; \mathbb{T}) = (\pi ; z ; \mathbb{T} \cup \rho ; z ; \mathbb{T}) \cap \mathbb{T}$
$= \pi ; z ; \mathbb{T} \cup \rho ; z ; \mathbb{T} = (\pi \cup \rho) ; z ; \mathbb{T}$

We focus on just the first term — the other may be handled similarly.
$(e ; \mathbb{T} \oslash \mathbb{T}) \; = \pi ; e ; \mathbb{T} \cap \rho ; \mathbb{T} = \pi ; e ; \mathbb{T} \subseteq \; \mathbb{M} ; \mathbb{S}^{\tau r} ; z ; \mathbb{T}$
$\iff \quad e ; \mathbb{T} \subseteq \pi^\tau ; \mathbb{M} ; \mathbb{S}^{\tau r} ; z ; \mathbb{T} = \pi^\tau ; [\, \mathtt{sup}_{n \geq 0} \, (\mathbb{S}^{\tau n} ; z ; \mathbb{T} \oslash k_n) \,] ; \mathbb{S}^{\tau r} ; z ; \mathbb{T}$
$= \mathtt{sup}_{n \geq 0} [\pi^\tau ; (\mathbb{S}^{\tau n} ; z ; \mathbb{T} \oslash k_n) \,] ; \mathbb{S}^{\tau r} ; z ; \mathbb{T}$
$= \mathtt{sup}_{n \geq 0} [\mathbb{S}^{\tau n} ; z ; \mathbb{T} \cap \mathbb{T} ; k_n] ; \mathbb{S}^{\tau r} ; z ; \mathbb{T}$ destroy and append
$= \mathtt{sup}_{n \geq 0} \, \mathbb{S}^{\tau n} ; z ; \mathbb{T} ; \mathbb{S}^{\tau r} ; z ; \mathbb{T} \cap \mathbb{T} ; k_n ; \mathbb{S}^{\tau r} ; z ; \mathbb{T}$
$= \mathtt{sup}_{n \geq 0} \, \mathbb{S}^{\tau n} ; z ; \mathbb{T} \cap \mathbb{T} ; k_n ; \mathbb{S}^{\tau r} ; z ; \mathbb{T}$ since $\mathbb{S}^{\tau r} ; z ; \mathbb{T}$ is a point
$\iff \quad \mathbb{T} \subseteq \mathbb{T} ; e^\tau ; \mathtt{sup}_{n \geq 0} [\mathbb{S}^{\tau n} ; z ; \mathbb{T} \cap \mathbb{T} ; k_n ; \mathbb{S}^{\tau r} ; z ; \mathbb{T}]$
$= \mathtt{sup}_{n \geq 0} \, \overline{\mathbb{T}} ; e^\tau ; [\mathbb{S}^{\tau n} ; z ; \mathbb{T} \cap \mathbb{T} ; k_n ; \mathbb{S}^{\tau r} ; z ; \mathbb{T}]$
$= \mathtt{sup}_{n \geq 0} [\mathbb{T} ; e^\tau ; \mathbb{S}^{\tau n} ; z ; \mathbb{T} \cap \mathbb{T} ; e^\tau ; \mathbb{T} ; k_n ; \mathbb{S}^{\tau r} ; z ; \mathbb{T}]$
$= \mathtt{sup}_{n \geq 0} [\mathbb{T} ; e^\tau ; \mathbb{S}^{\tau n} ; z ; \mathbb{T} \cap \mathbb{T} ; k_n ; \mathbb{S}^{\tau r} ; z ; \mathbb{T}]$ since $e ; \mathbb{T}$ is a point
$= \mathbb{T} ; k_1 ; \mathbb{S}^{\tau r} ; z ; \mathbb{T}$ because $\mathbb{T} ; e^\tau ; \mathbb{S}^{\tau n} ; z ; \mathbb{T} = \mathbb{\perp}$ in case $n \neq 1$
$= \mathbb{T} ; \mathbb{S}^{\tau r} ; z ; \mathbb{T} = \mathbb{T}$ again since $\mathbb{S}^{\tau r} ; z ; \mathbb{T}$ is a point

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ii.d) $\mathbb{M}^\tau ; (z ; \mathbb{T} \oslash R) \; = \big[\, \mathtt{sup}_{n \geq 0} \, (\mathbb{S}^{\tau n} ; z ; \mathbb{T} \oslash k_n) \, \big]^\tau ; (z ; \mathbb{T} \oslash R)$
$= \mathtt{sup}_{n \geq 0} \, (\mathbb{T} ; z^\tau ; \mathbb{S}^n \oslash k_n^\tau) ; (z ; \mathbb{T} \oslash R)$
$= \mathtt{sup}_{n \geq 0} \, \mathbb{T} ; z^\tau ; \mathbb{S}^n ; z ; \mathbb{T} \cap k_n^\tau ; R$ certainly $= \mathbb{\perp}$ in case $n \neq 0$
$= \mathbb{T} ; z^\tau ; z ; \mathbb{T} \cap k_0^\tau ; R = \mathbb{T} \cap z ; \mathbb{T} ; R$

The second variant is obvious.

$\mathbb{M}^\intercal ; (e ; \mathbb{T} \circledcirc R) = \mathtt{sup}_{n\geq 0} (\mathbb{T} ; z^\intercal ; \mathbb{S}^n \circledcirc k_n^\intercal) ; (e ; \mathbb{T} \circledcirc R)$
$= (\mathbb{T} ; z^\intercal \circledcirc z ; \mathbb{T}) ; (e ; \mathbb{T} \circledcirc R) \cup \mathtt{sup}_{n\geq 1} \left[\mathbb{T} ; z^\intercal ; \mathbb{S}^n ; e ; \mathbb{T} \cap k_n^\intercal ; R\right]$ because $k_0^\intercal$ is not univalent
$= \left[(\mathbb{T} ; z^\intercal ; \pi^\intercal \cap z ; \mathbb{T} ; \rho^\intercal) ; (\pi ; e ; \mathbb{T} \cap \rho ; R)\right] \cup \left[\mathbb{T} ; z^\intercal ; \mathbb{S}^1 ; e ; \mathbb{T} \cap \mathbb{I} ; R\right]$ since $= \mathbb{\perp}$ for $n \neq 1$
$= \left[\mathbb{T} ; z^\intercal ; \pi^\intercal ; (\pi ; e ; \mathbb{T} \cap \rho ; R) \cap z ; \mathbb{T}\right] \cup \left[\mathbb{T} ; z^\intercal ; z ; \mathbb{T} \cap R\right]$ ρ is total and masking
$= \left[\mathbb{T} ; z^\intercal ; (e ; \mathbb{T} \cap \mathbb{T} ; R) \cap z ; \mathbb{T}\right] \cup \left[\mathbb{T} \cap R\right]$ destroy and append
$= \left[\mathbb{\perp} \cap z ; \mathbb{T}\right] \cup R = R$ since $\mathbb{T} ; z^\intercal ; e ; \mathbb{T} = \mathbb{\perp}$

$\mathbb{M}^\intercal ; (e ; e^\intercal \circledcirc e ; e^\intercal) = \mathbb{M}^\intercal ; (\mathbb{I} \cap \mathbb{T} ; e^\intercal \circledcirc \mathbb{I} \cap \mathbb{T} ; e^\intercal)$
$= \mathbb{M}^\intercal ; (\mathbb{I} \circledcirc \mathbb{I}) \cap \mathbb{T} ; e^\intercal$ masking
$= \left[\, \mathtt{sup}_{n\geq 0} (\mathbb{T} ; z^\intercal ; \mathbb{S}^n \circledcirc k_n^\intercal) ; (\mathbb{I} \circledcirc \mathbb{I}) \,\right] \cap \mathbb{T} ; e^\intercal$
$= \left[\, \mathtt{sup}_{n\geq 0} \mathbb{T} ; z^\intercal ; \mathbb{S}^n \cap k_n^\intercal\right] \cap \mathbb{T} ; e^\intercal$ certainly $\mathbb{T} ; z^\intercal ; \mathbb{S}^n \cap \mathbb{T} ; e^\intercal = \mathbb{\perp}$ in case $n \neq 1$
$= \mathbb{I} \cap \mathbb{T} ; e^\intercal = e ; e^\intercal$ since $k_1 = \mathbb{I}$

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ii.e) This is a consequence of (d).

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ii.f) $\pi ; \pi^\intercal \cap \mathbb{M} ; \mathbb{M}^\intercal = \pi ; \pi^\intercal \cap \left[\, \mathtt{sup}_{n\geq 0} (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \circledcirc k_n)\right] ; \left[\, \mathtt{sup}_{k\geq 0} (\mathbb{S}^{\intercal k} ; z ; \mathbb{T} \circledcirc k_k)\right]^\intercal$
$= \pi ; \pi^\intercal \cap \mathtt{sup}_{n\geq 0} \mathtt{sup}_{k\geq 0} (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \circledcirc k_n) ; (\mathbb{T} ; z^\intercal ; \mathbb{S}^k \circledcirc k_k^\intercal)$
$= \pi ; \pi^\intercal \cap \mathtt{sup}_{n\geq 0} \mathtt{sup}_{k\geq 0} (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^k \otimes k_n ; k_k^\intercal)$ Prop. 2.1.1.ii
$= \mathtt{sup}_{n\geq 0} \mathtt{sup}_{k\geq 0} (\mathbb{I} \cap \mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^k \otimes k_n ; k_k^\intercal)$ due to $\mathbb{I}$ this implies $n = k$
$\subseteq \mathtt{sup}_{n\geq 0} (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^n \otimes k_n ; k_n^\intercal)$
$\subseteq \left[\pi ; z ; \mathbb{T} ; z^\intercal ; \pi^\intercal \cap \rho ; \mathbb{T} ; \rho^\intercal\right] \cup \mathtt{sup}_{n\geq 1} (\mathbb{I} \otimes \mathbb{I})$ for $n \geq 1$ is k_n injective, Lemma 6.4.2.ii.a
$= \pi ; z ; \mathbb{T} ; z^\intercal ; \pi^\intercal \cup \mathbb{I}$

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ii.g) $\mathtt{sup}_{n\geq 0} \mathtt{sup}_{i\geq 0} (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \circledcirc \mathbb{S}^{\intercal i} ; z ; \mathbb{T}) ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{n*i}$
$= \mathtt{sup}_{n\geq 0} \mathtt{sup}_{i\geq 0} \left[\pi ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \rho ; \mathbb{S}^{\intercal i} ; z ; \mathbb{T}\right] ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{n*i}$
$= \mathtt{sup}_{n\geq 0} \mathtt{sup}_{i\geq 0} \pi ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \rho ; \mathbb{S}^{\intercal i} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{n*i}$ masking
$= \mathtt{sup}_{n\geq 0} \pi ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \rho ; \left[\, \mathtt{sup}_{i\geq 0} \mathbb{S}^{\intercal i} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{i*n}\right]$
$= \mathtt{sup}_{n\geq 0} (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \circledcirc k_n) = \mathbb{M}$ Lemma 6.4.2.ii.a; needing regressive finiteness

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$\mathbb{M} ; \mathbb{S}^{\intercal r} ; z ; \mathbb{T} = \mathtt{sup}_{n\geq 0} \mathtt{sup}_{i\geq 0} (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \circledcirc \mathbb{S}^{\intercal i} ; z ; \mathbb{T}) ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{n*i} ; \mathbb{S}^{\intercal r} ; z ; \mathbb{T}$
$= \mathtt{sup}_{i\geq 0} (\mathbb{S}^{\intercal \frac{r}{i}} ; z ; \mathbb{T} \circledcirc \mathbb{S}^{\intercal i} ; z ; \mathbb{T})$ because $\mathbb{T} ; z^\intercal ; \mathbb{S}^{n*i} ; \mathbb{S}^{\intercal r} ; z ; \mathbb{T} = \mathbb{\perp}$ in case $n * i \neq r$
$= \mathtt{sup}_{a,b\geq 0,\, a*b=r} (\mathbb{S}^{\intercal a} \circledcirc \mathbb{S}^{\intercal b}) ; z ; \mathbb{T}$

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ii.h) If $\mathbb{S}^{\intercal i} ; z ; \mathbb{T} = \mathbb{\perp}$ or $\mathbb{S}^{\intercal k} ; z ; \mathbb{T} = \mathbb{\perp}$ or both, we certainly have $\mathbb{M}^\intercal ; (\mathbb{S}^{\intercal i} ; z ; \mathbb{T} \circledcirc \mathbb{S}^{\intercal k} ; z ; \mathbb{T}) = \mathbb{\perp}$; so let us assume for the following that both are $\neq \mathbb{\perp}$:

$\mathbb{M}^\intercal ; (\mathbb{S}^{\intercal i} ; z ; \mathbb{T} \circledcirc \mathbb{S}^{\intercal k} ; z ; \mathbb{T}) = \mathtt{sup}_{n\geq 0} (\mathbb{T} ; z^\intercal ; \mathbb{S}^n \circledcirc k_n^\intercal) ; (\mathbb{S}^{\intercal i} ; z ; \mathbb{T} \circledcirc \mathbb{S}^{\intercal k} ; z ; \mathbb{T})$
$= \mathtt{sup}_{n\geq 0} \mathbb{T} ; z^\intercal ; \mathbb{S}^n ; \mathbb{S}^{\intercal i} ; z ; \mathbb{T} \cap k_n^\intercal ; \mathbb{S}^{\intercal k} ; z ; \mathbb{T}$ since the points on the right are injective
$= [\mathbb{T} ; z^\intercal ; \mathbb{S}^{\intercal i} ; z ; \mathbb{T} \cap k_0^\intercal ; \mathbb{S}^{\intercal k} ; z ; \mathbb{T}] \cup \mathtt{sup}_{n\geq 1} \mathbb{T} ; z^\intercal ; \mathbb{S}^n ; \mathbb{S}^{\intercal i} ; z ; \mathbb{T} \cap k_n^\intercal ; \mathbb{S}^{\intercal k} ; z ; \mathbb{T}$ separating $n = 0$
$= \begin{cases} [\mathbb{T} \cap z ; \mathbb{T} ; \mathbb{S}^{\intercal k} ; z ; \mathbb{T}] \cup \mathbb{\perp} & \text{if } i = 0 \text{ — and therefore } n > i \\ [\mathbb{\perp} \cap \ldots] \cup \left(\mathbb{T} \cap k_i^\intercal ; \mathbb{S}^{\intercal k} ; z ; \mathbb{T}\right) & \text{if } i > 0 \text{ since for } n \neq i \text{ every term is } = \mathbb{\perp} \end{cases}$
$= \begin{cases} z ; \mathbb{T} ; \mathbb{S}^{\intercal k} ; z ; \mathbb{T} & \text{if } i = 0 \\ k_i^\intercal ; \mathbb{S}^{\intercal k} ; z ; \mathbb{T} & \text{if } i > 0 \end{cases}$

$= \begin{cases} z;\mathbb{T} & \text{if } i=0 \text{ — since } \mathbb{S}^{\intercal k};z;\mathbb{T} \neq \mathbb{\perp} \text{ was assumed initially} \\ \mathbb{S}^{\intercal i*k};k_i^{\intercal};z;\mathbb{T} & \text{if } i>0 \quad \text{see Lemma 6.4.2.ii.c} \end{cases}$
$= \begin{cases} z;\mathbb{T} & \text{if } i=0 \\ \mathbb{S}^{\intercal i*k};z;\mathbb{T} & \text{if } i>0 \quad \text{see Lemma 6.4.2.i.b} \end{cases}$
$= \mathbb{S}^{\intercal i*k};z;\mathbb{T}$

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ii.i) The earlier item (i.e) allows to prove this for $\mathbb{S}^{\intercal n};z;\mathbb{T} \neq \mathbb{\perp}$:
$(\mathbb{T};z^{\intercal};\mathbb{S}^n \circledcirc \mathbb{I});\mathbb{M} \cap e;\mathbb{T} = k_n \cap e;\mathbb{T} = e;\mathbb{T};z^{\intercal};\mathbb{S}^n$ using Lemma 6.4.2.i.d

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ii.j) $(\mathbb{M} \circledcirc \pi);\mathfrak{S} = (\mathbb{M} \circledcirc \pi);\big[\,\mathtt{sup}_{n\geq 0}\ (\mathbb{S}^{\intercal n} \circledcirc \mathbb{S}^{\intercal n};z;\mathbb{T})\,\big]$ expanding $\mathfrak{S}$
$= \mathtt{sup}_{n\geq 0}\ \mathbb{M};\mathbb{S}^{\intercal n} \cap \pi;\mathbb{S}^{\intercal n};z;\mathbb{T}$
$= \mathtt{sup}_{n\geq 0}\ \big[\,\mathtt{sup}_{l\geq 0}\ (\mathbb{S}^{\intercal l};z;\mathbb{T} \circledcirc k_l)\,\big];\mathbb{S}^{\intercal n} \cap \pi;\mathbb{S}^{\intercal n};z;\mathbb{T}$ expanding $\mathbb{M}$
$= \mathtt{sup}_{n\geq 0}\,\mathtt{sup}_{l\geq 0}\,\big[\pi;\mathbb{S}^{\intercal l};z;\mathbb{T} \cap \rho;k_l\big];\mathbb{S}^{\intercal n} \cap \pi;\mathbb{S}^{\intercal n};z;\mathbb{T}$
$= \mathtt{sup}_{n\geq 0}\,\mathtt{sup}_{l\geq 0}\,\pi;\mathbb{S}^{\intercal l};z;\mathbb{T} \cap \rho;k_l;\mathbb{S}^{\intercal n} \cap \pi;\mathbb{S}^{\intercal n};z;\mathbb{T}$ masking
$= \mathtt{sup}_{n\geq 0}\,\mathtt{sup}_{l\geq 0}\,\pi;\big[\mathbb{S}^{\intercal l};z;\mathbb{T} \cap \mathbb{S}^{\intercal n};z;\mathbb{T}\big] \cap \rho;k_l;\mathbb{S}^{\intercal n}$
$= \mathtt{sup}_{n\geq 0}\,\pi;\mathbb{S}^{\intercal n};z;\mathbb{T} \cap \rho;k_n;\mathbb{S}^{\intercal n}$ since $n := l$ is implied by Prop. 5.2.1.i.d to be $\neq \mathbb{\perp}$
$= \mathtt{sup}_{n\geq 0}\ (\mathbb{S}^{\intercal n};z;\mathbb{T} \circledcirc k_n;\mathbb{S}^{\intercal n})$
$= (z;\mathbb{T} \circledcirc k_0) \cup \mathtt{sup}_{n\geq 1}\ (\mathbb{S}^{\intercal n};z;\mathbb{T} \circledcirc \mathbb{S}^{\intercal};k_n \cap k_n;\mathbb{T})$ Lemma 6.4.2.ii.d needs reg-fin!!!
$= (z;\mathbb{T} \circledcirc \mathbb{T};z^{\intercal}) \cup \mathtt{sup}_{n\geq 1}\,\big[\,(\mathbb{S}^{\intercal n};z;\mathbb{T} \circledcirc \mathbb{S}^{\intercal};k_n) \cap (\mathbb{S}^{\intercal n};z;\mathbb{T} \circledcirc k_n;\mathbb{T})\,\big]$
$= (z;\mathbb{T} \circledcirc \mathbb{T};z^{\intercal}) \cup \Big[\big[\mathtt{sup}_{n\geq 1}\ (\mathbb{S}^{\intercal n};z;\mathbb{T} \circledcirc \mathbb{S}^{\intercal};k_n)\,\big] \cap \big[\mathtt{sup}_{i\geq 1}\ (\mathbb{S}^{\intercal i};z;\mathbb{T} \circledcirc k_i;\mathbb{T})\,\big]\Big]$
because $\pi;(\mathbb{S}^{\intercal n};z;\mathbb{T} \cap \mathbb{S}^{\intercal i};z;\mathbb{T} \cap \ldots) = \mathbb{\perp}$ unless $n = i$
$= (z;\mathbb{T} \circledcirc \mathbb{T};z^{\intercal}) \cup \Big[\big[\mathtt{sup}_{n\geq 1}\ (\mathbb{I} \otimes \mathbb{S}^{\intercal});(\mathbb{S}^{\intercal n};z;\mathbb{T} \circledcirc k_n)\,\big] \cap \big[\mathtt{sup}_{i\geq 1}\ (\mathbb{S}^{\intercal i};z;\mathbb{T} \circledcirc k_i)\,\big];\mathbb{T}\Big]$
$= (z;\mathbb{T} \circledcirc \mathbb{T};z^{\intercal}) \cup \Big[\,(\mathbb{I} \otimes \mathbb{S}^{\intercal});\mathbb{M} \cap \mathbb{M};\mathbb{T}\Big]$

In case $\mathbb{S}$ is regressively finite, already $B = \mathbb{A}$. To prove $(\mathbb{I} \otimes \mathbb{S}^{\intercal});(\mathbb{M} \circledcirc \pi);B = \mathbb{M} \cap \rho;\mathbb{S}^{\intercal};\mathbb{T}$ we start with just the rear part of the left term:
$(\mathbb{M} \circledcirc \pi);B = \mathtt{sup}_{l\geq 0}\ (\mathbb{M} \circledcirc \pi);\ (\mathbb{S}^l \circledcirc \mathbb{S}^{\intercal l};z;\mathbb{T})$
$= \mathtt{sup}_{l\geq 0}\ \mathbb{M};\ \mathbb{S}^l \cap \pi;\mathbb{S}^{\intercal l};z;\mathbb{T}$
$= \mathtt{sup}_{l\geq 0}[\mathtt{sup}_{i\geq 0}\ (\mathbb{S}^{\intercal i};z;\mathbb{T} \circledcirc k_i)\,];\ \mathbb{S}^l \cap \pi;\mathbb{S}^{\intercal l};z;\mathbb{T}$ expanding $\mathbb{M}$
$= \mathtt{sup}_{l\geq 0}\,\mathtt{sup}_{i\geq 0}\,\pi;\ \mathbb{S}^{\intercal i};z;\mathbb{T} \cap \rho;k_i;\ \mathbb{S}^l \cap \pi;\mathbb{S}^{\intercal l};z;\mathbb{T}$ masking
$= \mathtt{sup}_{l\geq 0}\,\mathtt{sup}_{i\geq 0}\,\pi;\big[\ \mathbb{S}^{\intercal i};z;\mathbb{T} \cap \mathbb{S}^{\intercal l};z;\mathbb{T}\big] \cap \rho;k_i;\ \mathbb{S}^l$ certainly $= \mathbb{\perp}$ in case $l \neq i$
$= \mathtt{sup}_{i\geq 0}\,\pi;\mathbb{S}^{\intercal i};z;\mathbb{T} \cap \rho;k_i;\ \mathbb{S}^i$
$= \big[\pi;z;\mathbb{T} \cap \rho;k_0;\ \mathbb{S}^0\big] \cup \mathtt{sup}_{i\geq 1}\,\pi;\mathbb{S}^{\intercal i};z;\mathbb{T} \cap \rho;k_i;\ \mathbb{S}^i$
$= (z;\mathbb{T} \circledcirc \mathbb{T};z^{\intercal}) \cup \mathtt{sup}_{i\geq 1}\ (\mathbb{S}^{\intercal i};z;\mathbb{T} \circledcirc \mathbb{S};k_i)$ Lemma 6.4.2.ii.c
$= (z;\mathbb{T} \circledcirc [\mathbb{S};\mathbb{T} \cup \overline{\mathbb{S};\mathbb{T}}];z^{\intercal}) \cup \mathtt{sup}_{i\geq 1}\ (\mathbb{S}^{\intercal i};z;\mathbb{T} \circledcirc \mathbb{S};k_i)$
$= \mathtt{sup}_{i\geq 0}\ (\mathbb{S}^{\intercal i};z;\mathbb{T} \circledcirc \mathbb{S};k_i) \cup (z;\mathbb{T} \circledcirc \overline{\mathbb{S};\mathbb{T}};z^{\intercal})$ since $\mathbb{S};\mathbb{T};z^{\intercal} = \mathbb{S};k_0$
$= (\mathbb{I} \otimes \mathbb{S});\mathtt{sup}_{i\geq 0}\ (\mathbb{S}^{\intercal i};z;\mathbb{T} \circledcirc k_i) \cup (z;\mathbb{T} \circledcirc \overline{\mathbb{S};\mathbb{T}};z^{\intercal})$
$= (\mathbb{I} \otimes \mathbb{S});\mathbb{M} \cup (z;\mathbb{T} \circledcirc \overline{\mathbb{S};\mathbb{T}};z^{\intercal})$

and proceed with
$(\mathbb{I} \otimes \mathbb{S}^{\intercal});(\mathbb{M} \circledcirc \pi);B = (\mathbb{I} \otimes \mathbb{S}^{\intercal});\ (\mathbb{I} \otimes \mathbb{S});\mathbb{M} \cup (\mathbb{I} \otimes \mathbb{S}^{\intercal});\ (z;\mathbb{T} \circledcirc \overline{\mathbb{S};\mathbb{T}};z^{\intercal})$
$= (\mathbb{I} \otimes \mathbb{S}^{\intercal};\ \mathbb{S});\mathbb{M} \cup (z;\mathbb{T} \circledcirc \mathbb{S}^{\intercal};\ \overline{\mathbb{S};\mathbb{T}};z^{\intercal}) = (\mathbb{I} \otimes \mathbb{S}^{\intercal};\ \mathbb{S});\mathbb{M} \cup \mathbb{\perp}$
$= (\mathbb{I} \otimes \mathbb{I} \cap \mathbb{S}^{\intercal};\mathbb{T});\mathbb{M} = (\mathbb{I} \otimes \mathbb{I});\mathbb{M} \cap \rho;\mathbb{S}^{\intercal};\mathbb{T} = \mathbb{M} \cap \rho;\mathbb{S}^{\intercal};\mathbb{T}$

---

ii.k) Of the negated statement $\overline{\mathbb{M}^\intercal}\fatsemi\mathbb{M} = \overline{\mathbb{I}}$ direction $\overline{\mathbb{M}^\intercal}\fatsemi\mathbb{M} \subseteq \overline{\mathbb{I}}$ is trivial, because via the Schröder rule indeed $\mathbb{I}\fatsemi\mathbb{M}^\intercal \subseteq \mathbb{M}^\intercal$. The other direction seems to require a point-wise proof. So, let us assume points $\mathbb{S}^{\intercal r}\fatsemi z\fatsemi\mathbb{T}, \mathbb{S}^{\intercal c}\fatsemi z\fatsemi\mathbb{T}$ that satisfy $\mathbb{S}^{\intercal r}\fatsemi z\fatsemi\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^c \subseteq \overline{\mathbb{I}}$ — and are, thus, different — with the task to show that

$$\mathbb{S}^{\intercal r}\fatsemi z\fatsemi\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^c \subseteq \overline{\mathbb{M}^\intercal}\fatsemi\mathbb{M} \quad\Longleftrightarrow\quad \mathbb{T} \subseteq \mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^r\fatsemi\overline{\mathbb{M}^\intercal}\fatsemi\mathbb{M}\fatsemi\mathbb{S}^{\intercal c}\fatsemi z\fatsemi\mathbb{T}$$

So we try to show that the following term is $\supseteq \mathbb{T}$:

$$\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^r\fatsemi\overline{\mathbb{M}^\intercal}\fatsemi\mathbb{M}\fatsemi\mathbb{S}^{\intercal c}\fatsemi z\fatsemi\mathbb{T} \supseteq \mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^r\fatsemi\overline{\mathbb{M}^\intercal}\fatsemi\pi\fatsemi e\fatsemi\mathbb{T} \quad \text{due to (ii.c)}$$

Now we distinguish the three cases $r = 0, r = 1, r > 1$ as follows:

$$\begin{aligned}
&\mathbb{T}\fatsemi z^\intercal\fatsemi\overline{\mathbb{M}^\intercal}\fatsemi\pi\fatsemi e\fatsemi\mathbb{T} = \overline{\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{M}^\intercal}\fatsemi\pi\fatsemi e\fatsemi\mathbb{T} = \overline{(\mathbb{T}\fatsemi z^\intercal \oslash \mathbb{T}\fatsemi z^\intercal)}\fatsemi\pi\fatsemi e\fatsemi\mathbb{T} \quad \text{due to (ii.c)}\\
&= \overline{(\mathbb{T}\fatsemi z^\intercal \cap \mathbb{T}\fatsemi z^\intercal\fatsemi\rho^\intercal\fatsemi\pi)}\fatsemi e\fatsemi\mathbb{T} \quad \text{destroy and append for a join}\\
&= \overline{(\mathbb{T}\fatsemi z^\intercal \cap \mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{T})}\fatsemi e\fatsemi\mathbb{T}\\
&= \overline{\mathbb{T}\fatsemi z^\intercal}\fatsemi e\fatsemi\mathbb{T} \cap \overline{\mathbb{T}\fatsemi z^\intercal}\fatsemi\mathbb{T}\fatsemi e\fatsemi\mathbb{T} = \overline{\mathbb{T}\fatsemi z^\intercal\fatsemi e\fatsemi\mathbb{T}} \cap \overline{\mathbb{T}\fatsemi z^\intercal}\fatsemi\mathbb{T} = \overline{\mathbb{\perp}} \cap \mathbb{T} = \mathbb{T}
\end{aligned}$$

$$\begin{aligned}
&\mathbb{T}\fatsemi e^\intercal\fatsemi\overline{\mathbb{M}^\intercal}\fatsemi\pi\fatsemi e\fatsemi\mathbb{T} = \overline{\mathbb{T}\fatsemi e^\intercal\fatsemi\mathbb{M}^\intercal}\fatsemi\pi\fatsemi e\fatsemi\mathbb{T} = \overline{(\mathbb{T}\fatsemi e^\intercal \oslash \mathbb{T}\fatsemi e^\intercal)}\fatsemi\pi\fatsemi e\fatsemi\mathbb{T} \quad \text{due to (i.e)}\\
&\supseteq \overline{\mathbb{T}\fatsemi e^\intercal}\fatsemi\rho^\intercal\fatsemi\pi\fatsemi e\fatsemi\mathbb{T} = \overline{\mathbb{T}\fatsemi e^\intercal}\fatsemi\mathbb{T}\fatsemi e\fatsemi\mathbb{T} = \overline{\mathbb{T}\fatsemi e^\intercal}\fatsemi\mathbb{T} = \mathbb{T}
\end{aligned}$$

$$\begin{aligned}
&\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^r\fatsemi\overline{\mathbb{M}^\intercal}\fatsemi\pi\fatsemi e\fatsemi\mathbb{T} = \overline{\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^r\fatsemi\mathbb{M}^\intercal}\fatsemi\pi\fatsemi e\fatsemi\mathbb{T} \quad \text{now with } r > 1\\
&= \mathtt{sup}_{n\geq 0}\,\mathtt{sup}_{i\geq 0}\,\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^r\fatsemi\mathbb{S}^{\intercal n*i}\fatsemi z\fatsemi\mathbb{T}\fatsemi(\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^n \oslash \mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^i)\fatsemi\pi\fatsemi e\fatsemi\mathbb{T} \quad \text{following (ii.g)}\\
&\qquad\qquad \text{and implying } n*i = r \text{ to be } \neq \mathbb{\perp} \text{ and } i = 0 \text{ impossible}\\
&= \mathtt{sup}_{i\geq 1}\,(\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^{\frac{r}{i}} \oslash \mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^i)\fatsemi\pi\fatsemi e\fatsemi\mathbb{T}\\
&= [\mathtt{inf}_{i\geq 1}(\overline{\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^{\frac{r}{i}}}\pi^\intercal \cup \overline{\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^i}\rho^\intercal)]\fatsemi\pi\fatsemi e\fatsemi\mathbb{T} \quad \text{now } r*1 = r,\ 1*r = r \text{ and } a*b = r \text{ for } a,b > 1\\
&= \big[(\overline{\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^r}\pi^\intercal \cup \overline{\mathbb{T}\fatsemi e^\intercal}\rho^\intercal) \cap (\overline{\mathbb{T}\fatsemi e^\intercal}\pi^\intercal \cup \overline{\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^r}\rho^\intercal) \cap \mathtt{inf}_{a,b>1}(\overline{\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^a}\pi^\intercal \cup \overline{\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^b}\rho^\intercal)\big]\fatsemi\pi\fatsemi e\fatsemi\mathbb{T}\\
&\supseteq \big[\overline{\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^r}\pi^\intercal \cap \overline{\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^r}\rho^\intercal \cap \mathtt{inf}_{a,b>1}\,\overline{\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^a}\pi^\intercal\big]\fatsemi\pi\fatsemi e\fatsemi\mathbb{T}\\
&= \big[\big\{\overline{\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^r} \cap \mathtt{inf}_{a,b>1}\,\overline{\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^a}\big\}\fatsemi\pi^\intercal \cap \overline{\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^r}\rho^\intercal\big]\fatsemi\pi\fatsemi e\fatsemi\mathbb{T} \quad \text{rearranged}\\
&= \big[\big\{\overline{\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^r} \cap \mathtt{inf}_{a,b>1}\,\overline{\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^a}\big\} \cap \overline{\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^r}\rho^\intercal\fatsemi\pi\big]\fatsemi e\fatsemi\mathbb{T} \quad \text{destroy and append}\\
&= \overline{\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^r}\fatsemi e\fatsemi\mathbb{T} \cap \mathtt{inf}_{a,b>1}\,\overline{\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^a}\fatsemi e\fatsemi\mathbb{T} \cap \overline{\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^r}\fatsemi\mathbb{T}\fatsemi e\fatsemi\mathbb{T}\\
&= \overline{\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^r}\fatsemi e\fatsemi\mathbb{T} \cap \mathtt{inf}_{a,b>1}\,\overline{\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^a}\fatsemi e\fatsemi\mathbb{T} \cap \overline{\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^r}\fatsemi\mathbb{T}\\
&= \overline{\mathbb{\perp}} \cap \mathtt{inf}_{a,b>1}\,\overline{\mathbb{\perp}} \cap \mathbb{T} = \mathbb{T}
\end{aligned}$$

$$\mathtt{syq}(\mathbb{M},\mathbb{M}) = \overline{\overline{\mathbb{M}^\intercal}\fatsemi\mathbb{M}} \cap \overline{\mathbb{M}^\intercal\fatsemi\overline{\mathbb{M}}} = \mathbb{I} \cap \mathbb{I} = \mathbb{I}$$

---

iii) and (iv.a) are trivial while iv.b) is well-known — but probably difficult to prove relationally.

$$\begin{aligned}
&\text{iv.c)}\ (\mathbb{S} \otimes \mathbb{I})\fatsemi\mathbb{M} = (\mathbb{S} \otimes \mathbb{I})\fatsemi m(\mathbb{M}) = (\mathbb{S} \otimes \mathbb{I})\fatsemi\big[(z\fatsemi\mathbb{T} \oslash \mathbb{T}\fatsemi z^\intercal) \cup (\mathbb{S}^\intercal \otimes \mathbb{I})\fatsemi(\mathbb{M} \oslash \rho)\fatsemi\mathbb{A}\big]\\
&= \mathbb{\perp} \cup (\mathbb{S} \otimes \mathbb{I})\fatsemi(\mathbb{S}^\intercal \otimes \mathbb{I})\fatsemi(\mathbb{M} \oslash \rho)\fatsemi\mathbb{A} \quad \text{since } \mathbb{S}\fatsemi z = \mathbb{\perp}\\
&= (\mathbb{S}\fatsemi\mathbb{S}^\intercal \otimes \mathbb{I})\fatsemi(\mathbb{M} \oslash \rho)\fatsemi\mathbb{A}\\
&= (\mathbb{I} \otimes \mathbb{I})\fatsemi(\mathbb{M} \oslash \rho)\fatsemi\mathbb{A} = \mathbb{I}\fatsemi(\mathbb{M} \oslash \rho)\fatsemi\mathbb{A} \quad \text{Prop. 5.2.1.iv.a}
\end{aligned}$$

---

iv.d) follows from (ii.f) with $\mathbb{S}$ — and thus also $\mathbb{M}$ — total.

---

iv.e) is (i.e) without the consideration concerning $\mathbb{S}^{\intercal i}\fatsemi z\fatsemi\mathbb{T} = \mathbb{\perp}$. □

Interpreting (i.e): N with result 0 implies the first of the two factors being 0. Also: Multiplication with the factor 0 results in 0. The results (i,ii,iii) are illustrated with Fig. 7.1.4.

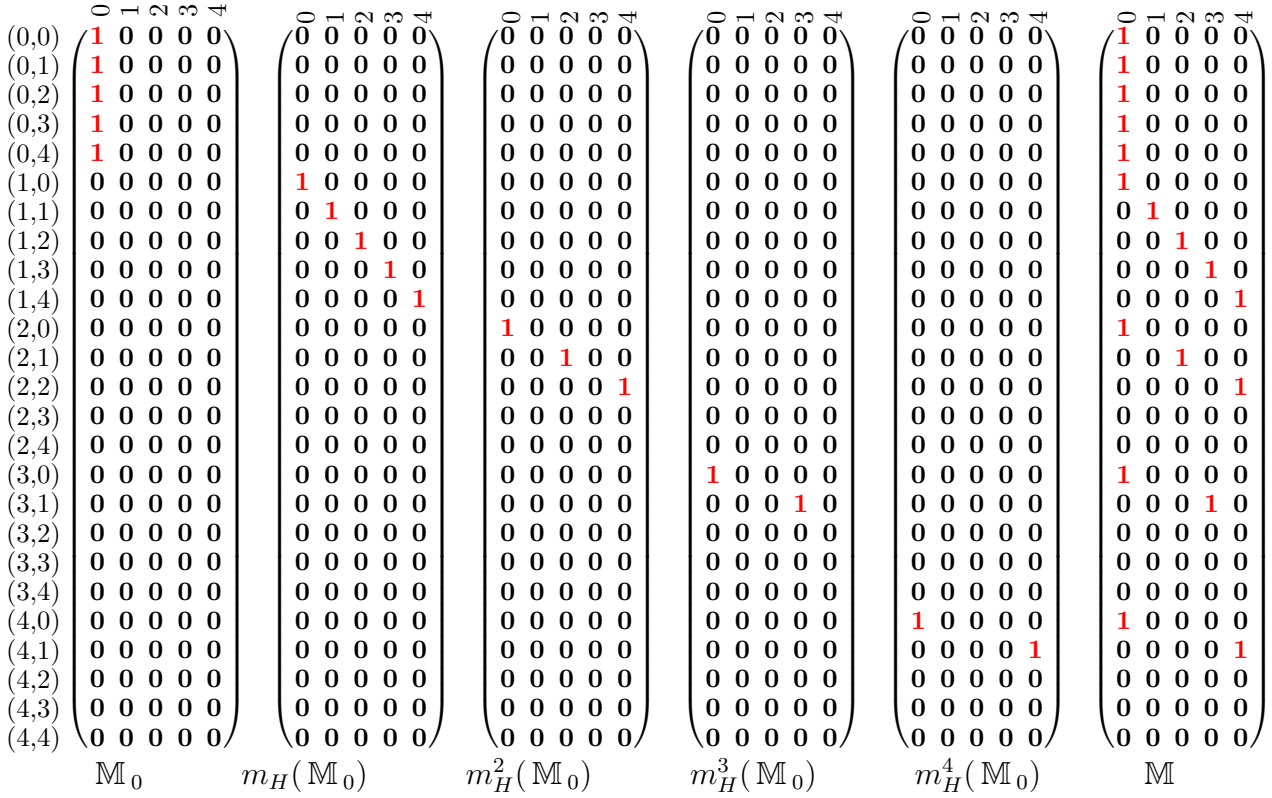


Fig. 7.1.4 Exhausting the multiplication relation iterating on the first component

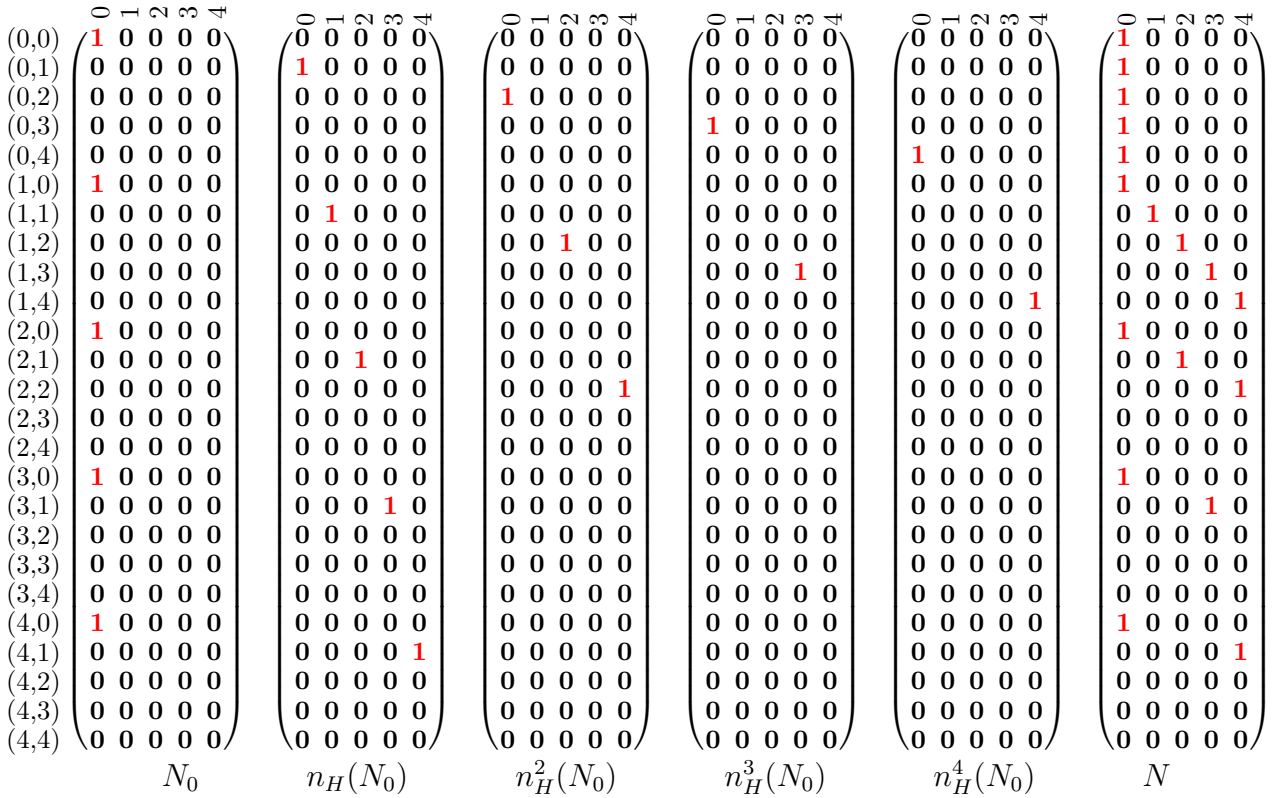
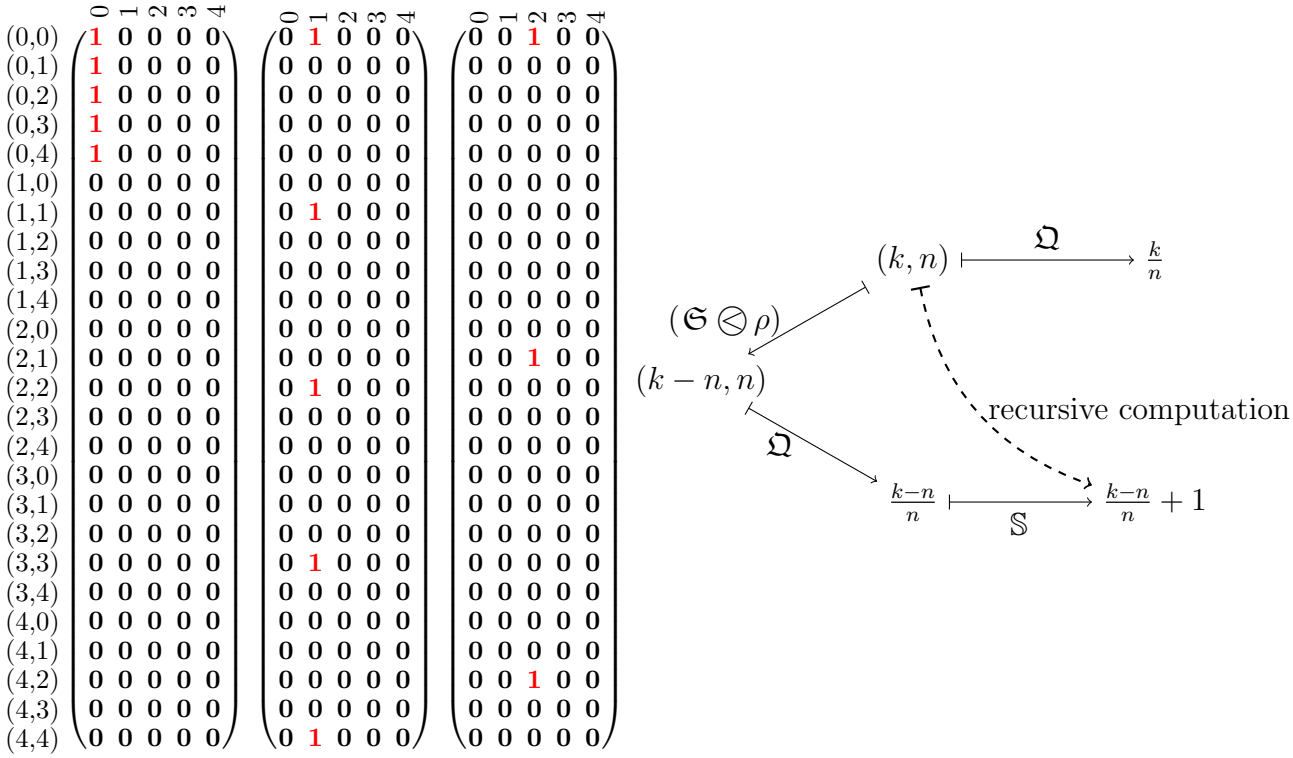


Fig. 7.1.5 Exhausting the multiplication relation iterating on the second component

The relation **root** $:= \mathbb{M}^{\intercal_2}(\mathbb{I} \oslash \mathbb{I})$ delivers the relation of row to column that is its root.

7.2 Division

We will — of course — need the relation $\mathfrak{Q} := \mathbb{M}^{\hookleftarrow} : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ representing division (see Fig. 6.3.10). It is also possible to obtain quotient forming directly by iteration. To this end, we build our $\mathfrak{Q}_0$ in view of Fig. 6.3.10 and design it as in Fig. 7.2.1. It consists of just a "straight line": The vertical part says that, if the dividend is 0, the result has to be 0 again.


0 1 2 3 4
(0,0) 1 0 0 0 0
(0,1) 1 0 0 0 0
(0,2) 1 0 0 0 0
(0,3) 1 0 0 0 0
(0,4) 1 0 0 0 0
(1,0) 0 0 0 0 0
(1,1) 0 0 0 0 0
(1,2) 0 0 0 0 0
(1,3) 0 0 0 0 0
(1,4) 0 0 0 0 0
(2,0) 0 0 0 0 0
(2,1) 0 0 0 0 0
(2,2) 0 0 0 0 0
(2,3) 0 0 0 0 0
(2,4) 0 0 0 0 0
(3,0) 0 0 0 0 0
(3,1) 0 0 0 0 0
(3,2) 0 0 0 0 0
(3,3) 0 0 0 0 0
(3,4) 0 0 0 0 0
(4,0) 0 0 0 0 0
(4,1) 0 0 0 0 0
(4,2) 0 0 0 0 0
(4,3) 0 0 0 0 0
(4,4) 0 0 0 0 0
0 1 2 3 4
0 1 0 0 0
0 0 0 0 0
0 0 0 0 0
0 0 0 0 0
0 0 0 0 0
0 0 0 0 0
0 1 0 0 0
0 0 0 0 0
0 0 0 0 0
0 0 0 0 0
0 0 0 0 0
0 0 0 0 0
0 0 0 0 0
0 1 0 0 0
0 0 0 0 0
0 0 0 0 0
0 0 0 0 0
0 1 0 0 0
0 0 0 0 0
0 0 0 0 0
0 0 0 0 0
0 0 0 0 0
0 1

$\mathtt{Q} = \mathtt{q}(\mathtt{Q}) = \mathtt{Q}_0 \cup (\pi \circledS \mathtt{S}) ; \mathtt{Q} ; \mathbb{S} = \mathtt{sup}_{n\geq 0}\, \mathtt{q}_{\mathtt{H}}{}^n(\mathtt{Q}_0) = \mathtt{sup}_{n\geq 0}\, (\pi \circledS \mathtt{S})^n ; \mathtt{Q}_0 ; \mathbb{S}^n$

b) $\mathfrak{Q} \cap \overline{(z ; \mathbb{T} \circledS z ; \mathbb{T})}$ as well as $\mathtt{Q} \cap \overline{(z ; \mathbb{T} \circledS z ; \mathbb{T})}$ is univalent

$\mathfrak{P} ; \mathfrak{Q} = \mathtt{Q}$

c) A injective $\implies$ $\mathfrak{Q}^{\intercal} ; (A \circledS e ; \mathbb{T}) \subseteq A$ $\qquad$ $\mathtt{Q}^{\intercal} ; (e ; \mathbb{T} \circledS A) \subseteq A$

d) $\mathfrak{Q} ; \mathbb{S}^{\intercal} = (\mathfrak{S} \circledS \rho) ; \mathfrak{Q} \cap \mathbb{T} ; \mathbb{S}^{\intercal}$ $\qquad$ $\mathtt{Q} ; \mathbb{S}^{\intercal} = (\pi \circledS \mathtt{S}) ; \mathtt{Q} \cap \mathbb{T} ; \mathbb{S}^{\intercal}$

e) $\mathfrak{Q} ; z ; \mathbb{T} = \pi ; z ; \mathbb{T}$ $\qquad$ $\mathtt{Q} ; z ; \mathbb{T} = \rho ; z ; \mathbb{T}$

f) In preparation for (ii.a):

$\hookrightarrow\!\mathbb{M} = \mathtt{sup}_{n\geq 0}\, (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \circledS k_n^{\intercal})$ $\qquad$ $\hookrightarrow\!N = \mathtt{sup}_{n\geq 0}\, \mathtt{vec}(k_n) ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^n = \mathfrak{P} ; \mathbb{M}^{\hookleftarrow}$

$N^{\hookleftarrow} = \mathtt{sup}_{n\geq 0}\, (k_n^{\intercal} \circledS \mathbb{S}^{\intercal n} ; z ; \mathbb{T})$ $\qquad$ $\mathbb{M}^{\hookleftarrow} = \mathtt{sup}_{n\geq 0}\, \mathtt{vec}(k_n^{\intercal}) ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^n = \mathfrak{P} ; \hookrightarrow\!N$

ii) additionally, if $\mathbb{S}$ is regressively finite: — implying $\mathbb{M} = N$

a) $\mathfrak{Q} = \mathtt{sup}_{n\geq 0}\, \mathtt{vec}(k_n^{\intercal}) ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^n$

$= \mathtt{sup}_{n\geq 0}\, (k_n^{\intercal} \circledS \mathbb{S}^{\intercal n} ; z ; \mathbb{T})$

$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\, (\mathbb{S}^{\intercal n * i} ; z ; \mathbb{T} \circledS \mathbb{S}^{\intercal i} ; z ; \mathbb{T}) ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^n = \mathbb{M}^{\hookleftarrow} = N^{\hookleftarrow}$

$\mathtt{Q} = \mathtt{sup}_{n\geq 0}\, \mathtt{vec}(k_n) ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^n$

$= \mathtt{sup}_{n\geq 0}\, (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \circledS k_n^{\intercal})$

$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\, (\mathbb{S}^{\intercal i} ; z ; \mathbb{T} \circledS \mathbb{S}^{\intercal n * i} ; z ; \mathbb{T}) ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^n = \hookrightarrow\!\mathbb{M} = \hookrightarrow\!N$

b) $\mathbb{S}^{\intercal k} ; z ; \mathbb{T}$ and $\mathbb{S}^{\intercal l} ; z ; \mathbb{T}$ are both points $\implies$

$$\mathfrak{Q}^{\intercal} ; (\mathbb{S}^{\intercal k} ; z ; \mathbb{T} \circledS \mathbb{S}^{\intercal l} ; z ; \mathbb{T}) = \begin{cases} \mathbb{T} & \text{when } k = l = 0 \\ \mathbb{\perp} & \text{when } l = 0 \text{ or } l \nmid k \\ \mathbb{S}^{\intercal q} ; z ; \mathbb{T} & \text{when } q := \frac{k}{l} \end{cases}$$

$(z ; \mathbb{T} \circledS z ; \mathbb{T}) = (\mathbb{I} \circledS \mathbb{I}) ; z ; \mathbb{T} \subseteq \mathfrak{Q}$ $\qquad$ $(z ; \mathbb{T} \circledS z ; \mathbb{T}) = (\mathbb{I} \circledS \mathbb{I}) ; z ; \mathbb{T} \subseteq \mathtt{Q}$

$\mathfrak{Q}$ and $\mathtt{Q}$ are surjective, i.e. $\mathbb{T} ; \mathfrak{Q} = \mathbb{T}$

c) $\hookrightarrow\!\mathfrak{Q} = \mathfrak{Q}$ $\qquad$ $\mathtt{Q} = \mathtt{Q}^{\hookleftarrow}$

$\mathfrak{Q}_0^{\hookleftarrow} = \mathfrak{Q}_0$ $\qquad$ $\hookrightarrow\!\mathtt{Q}_0 = \mathtt{Q}_0$ $\qquad$ $\hookrightarrow\!\mathfrak{Q}_0 = (z ; \mathbb{T} \circledS z ; \mathbb{T}) = \mathtt{Q}_0^{\hookleftarrow}$

d) $\mathfrak{Q} ; e ; \mathbb{T} = (\mathbb{I} \circledS \mathbb{I}) ; \mathbb{T}$

e) $\mathbb{S}^{\intercal l} ; z ; \mathbb{T}$ a point $\implies$ $\mathtt{rel}(\mathfrak{Q} ; \mathbb{S}^{\intercal l} ; z ; \mathbb{T}) = k_l^{\intercal}$

iii) If $\mathbb{S}$ happens to be regressively finite but not total, i.e. $\mathbb{S}\fatsemi\mathbb{T} \neq \mathbb{T}$,

a) with $g_{\mathbb{S}}$ the greatest point, (i.d) specializes to

$$\mathfrak{Q}\fatsemi\mathbb{S}^{\mathsf{T}} \cup (z\fatsemi\mathbb{T} \oslash z\fatsemi\mathbb{T} \cap \mathbb{T}\fatsemi g_{\mathbb{S}}^{\mathsf{T}}) = (\mathfrak{S} \oslash \rho)\fatsemi\mathfrak{Q}$$

$$\mathtt{Q}\fatsemi\mathbb{S}^{\mathsf{T}} \cup (z\fatsemi\mathbb{T} \oslash z\fatsemi\mathbb{T} \cap \mathbb{T}\fatsemi g_{\mathbb{S}}^{\mathsf{T}}) = (\pi \oslash \mathtt{S})\fatsemi\mathtt{Q}$$

iv) In case $\mathbb{S}$ is regressively finite and also total, i.e. $\mathbb{S}\fatsemi\mathbb{T} = \mathbb{T}$,

a) $\mathfrak{Q}\fatsemi\mathbb{S}^{\mathsf{T}} = (\mathfrak{S} \oslash \rho)\fatsemi\mathfrak{Q} = (\mathbb{A} \oslash \rho)^{\mathsf{T}}\fatsemi\mathfrak{Q}$

$\mathtt{Q}\fatsemi\mathbb{S}^{\mathsf{T}} = (\pi \oslash \mathtt{S})\fatsemi\mathtt{Q} = (\pi \oslash \mathbb{A})^{\mathsf{T}}\fatsemi\mathtt{Q}$

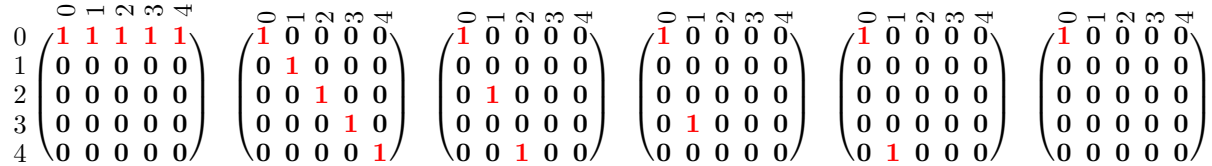


Fig. 7.2.2 Illustrating (ii.a) with transposed k_n. Cf. also $\mathfrak{Q}_{table}$ in Fig. 6.3.10

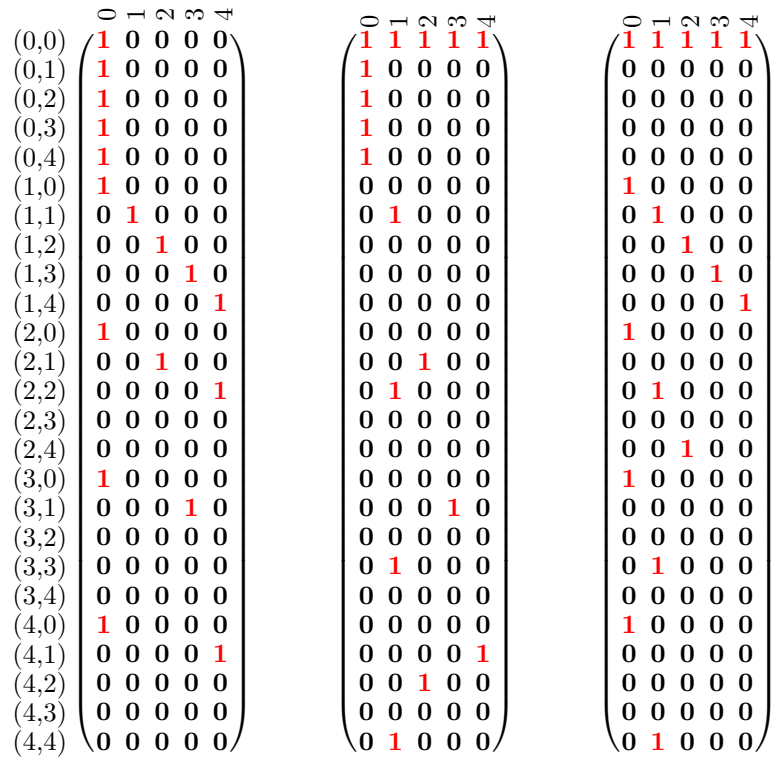


Fig. 7.2.3 When regressively finite: $\mathbb{M} = N$ $\mathbb{M}^{\hookleftarrow} = \mathfrak{Q} = {}^{\hookrightarrow}\mathfrak{Q}$ ${}^{\hookrightarrow}\mathbb{M} = \mathtt{Q} = \mathtt{Q}^{\hookleftarrow}$

Proof: We observe in advance that Lemma 4.2.1 may be applied. The claim (i.a) collects the results of Lemma 4.2.1.i.

---

i.b) Looking at Fig. 7.2.1, $\mathfrak{Q}$ will probably not be univalent.

$(\mathfrak{Q} \cap \overline{(z;\mathbb{T} \oslash z;\mathbb{T})})^{\mathsf{T}};(\mathfrak{Q} \cap \overline{(z;\mathbb{T} \oslash z;\mathbb{T})}) = \mathfrak{Q}^{\mathsf{T}};(\mathfrak{Q} \cap \overline{(z;\mathbb{T} \oslash z;\mathbb{T})})$ mask shifting
$= \left[\mathtt{sup}_{n\geq 0}\ (\mathfrak{S} \oslash \rho)^n;\mathfrak{Q}_0;\mathbb{S}^n\right]^{\mathsf{T}};\mathtt{sup}_{k\geq 0}\left[(\mathfrak{S} \oslash \rho)^k;\mathfrak{Q}_0;\mathbb{S}^k \cap \overline{(z;\mathbb{T} \oslash z;\mathbb{T})}\right]$
$= \mathtt{sup}_{n\geq 0}\,\mathtt{sup}_{k\geq 0}\,\mathbb{S}^{\mathsf{T}n};\ \mathfrak{Q}_0^{\mathsf{T}};\ (\mathfrak{S}^{\mathsf{T}} \oslash \rho^{\mathsf{T}})^n;\left[(\mathfrak{S} \oslash \rho)^k;\mathfrak{Q}_0;\mathbb{S}^k \cap \overline{(z;\mathbb{T} \oslash z;\mathbb{T})}\right]$
$= \mathtt{sup}_{n\geq 0}\,\mathtt{sup}_{k\geq 0}\,\mathbb{S}^{\mathsf{T}n};\ \mathfrak{Q}_0^{\mathsf{T}};\ (B \oslash \rho)^n;\left[(B^{\mathsf{T}} \oslash \rho^{\mathsf{T}})^k;\mathfrak{Q}_0;\mathbb{S}^k \cap \overline{(z;\mathbb{T} \oslash z;\mathbb{T})}\right]$ Prop. 6.3.1.i.e

Now we discuss separately the cases $n = k$ and, e.g., $n > k$. For $n = k$, we neglect $\overline{(z;\mathbb{T} \oslash z;\mathbb{T})}$ and proceed according to Prop. 6.2.1.i.h with

$(B \oslash \rho);(B^{\mathsf{T}} \oslash \rho^{\mathsf{T}}) = B;B^{\mathsf{T}} \cap \rho;\rho^{\mathsf{T}} \subseteq \mathbb{I}$ and $\mathfrak{Q}_0^{\mathsf{T}};\mathfrak{Q}_0 \subseteq \mathbb{I}$.

For $n > k$, we have to consider

$(B \oslash \rho);\mathfrak{Q}_0 = (B \oslash \rho);(z;\mathbb{T} \oslash \mathbb{T};z^{\mathsf{T}}) = B;z;\mathbb{T} \cap \rho;\mathbb{T};z^{\mathsf{T}}$
$= (z;\mathbb{T} \oslash z;\mathbb{T}) \cap \mathbb{T};z^{\mathsf{T}} \subseteq (z;\mathbb{T} \oslash z;\mathbb{T})$ Prop. 6.2.1.i.c

together with $(B \oslash \rho);(z;\mathbb{T} \oslash z;\mathbb{T}) = B;z;\mathbb{T} \cap \rho;z;\mathbb{T} = (z;\mathbb{T} \oslash z;\mathbb{T})$.

The other result follows analogously.

To prove $\mathfrak{P};\mathfrak{Q} = \mathfrak{Q}$, we show these two parts:

$\mathfrak{P};(\mathfrak{S} \oslash \rho) = (\mathfrak{P};\mathfrak{S} \oslash \mathfrak{P};\rho) = (\mathtt{S} \oslash \pi) = (\pi \oslash \mathtt{S});\mathfrak{P}$
$\mathfrak{P};\mathfrak{Q}_0 = \mathfrak{P};(z;\mathbb{T} \oslash \mathbb{T};z^{\mathsf{T}}) = (\mathbb{T};z^{\mathsf{T}} \oslash z;\mathbb{T}) = \mathfrak{Q}_0$

	0	1	2	3	4
(0,0)	1	1	1	1	0
(0,1)	0	0	0	0	0
(0,2)	0	0	0	0	0
(0,3)	0	0	0	0	0
(0,4)	0	0	0	0	0
(1,0)	0	0	0	0	0
(1,1)	1	0	0	0	0
(1,2)	0	0	0	0	0
(1,3)	0	0	0	0	0
(1,4)	0	0	0	0	0
(2,0)	0	0	0	0	0
(2,1)	0	1	0	0	0
(2,2)	1	0	0	0	0
(2,3)	0	0	0	0	0
(2,4)	0	0	0	0	0
(3,0)	0	0	0	0	0
(3,1)	0	0	1	0	0
(3,2)	0	0	0	0	0
(3,3)	1	0	0	0	0
(3,4)	0	0	0	0	0
(4,0)	0	0	0	0	0
(4,1)	0	0	0	1	0
(4,2)	0	1	0	0	0
(4,3)	0	0	0	0	0
(4,4)	1	0	0	0	0

	0	1	2	3	4
(0,0)	1	1	1	1	1
(0,1)	0	0	0	0	0
(0,2)	0	0	0	0	0
(0,3)	0	0	0	0	0
(0,4)	0	0	0	0	0
(1,0)	0	0	0	0	0
(1,1)	1	0	0	0	0
(1,2)	0	0	0	0	0
(1,3)	0	0	0	0	0
(1,4)	0	0	0	0	0
(2,0)	0	0	0	0	0
(2,1)	0	1	0	0	0
(2,2)	1	0	0	0	0
(2,3)	0	0	0	0	0
(2,4)	0	0	0	0	0
(3,0)	0	0	0	0	0
(3,1)	0	0	1	0	0
(3,2)	0	0	0	0	0
(3,3)	1	0	0	0	0
(3,4)	0	0	0	0	0
(4,0)	0	0	0	0	0
(4,1)	0	0	0	1	0
(4,2)	0	1	0	0	0
(4,3)	0	0	0	0	0
(4,4)	1	0	0	0	0

Fig. 7.2.4 Illustrating results (i.d), (iii.a), and (iv.a): $\mathfrak{Q};\mathbb{S}^{\mathsf{T}}$ versus $(\mathfrak{S} \oslash \rho);\mathfrak{Q}$

---

i.c) We focus on $\mathfrak{Q}_0$ first:

$\mathfrak{Q}_0^{\mathsf{T}};\ (A \oslash e;\mathbb{T}) = \left[\ (z;\mathbb{T} \oslash \mathbb{T};z^{\mathsf{T}})\ \right]^{\mathsf{T}};\ (A \oslash e;\mathbb{T}) = (\mathbb{T};z^{\mathsf{T}} \oslash z;\mathbb{T});\ (A \oslash e;\mathbb{T})$
$= \mathbb{T};z^{\mathsf{T}};A \cap z;\mathbb{T};e;\mathbb{T}$ since A is injective

$= \mathbb{T} ; z^\intercal ; A \cap z ; \mathbb{T} = (\mathbb{T} ; z^\intercal \cap z ; \mathbb{T}) ; A$
$= z ; \mathbb{T} ; z^\intercal ; A$ masking
$\subseteq A$ because z is injective

Now we proceed by induction with hypothesis $\left[q_H^n(\mathfrak{Q}_0)\right]^\intercal ; (A \oslash e ; \mathbb{T}) \subseteq A$ for injective A:
$\left[q_H^{n+1}(\mathfrak{Q}_0)\right]^\intercal ; (A \oslash e ; \mathbb{T}) = \mathbb{S}^\intercal ; \left[q_H^n(\mathfrak{Q}_0)\right]^\intercal ; (\mathfrak{S}^\intercal \oslash \rho^\intercal) ; (A \oslash e ; \mathbb{T})$
$= \mathbb{S}^\intercal ; \left[q_H^n(\mathfrak{Q}_0)\right]^\intercal ; (\mathfrak{S}^\intercal ; (A \oslash e ; \mathbb{T}) \oslash \rho^\intercal ; (A \oslash e ; \mathbb{T}))$ since A is injective
$= \mathbb{S}^\intercal ; \left[q_H^n(\mathfrak{Q}_0)\right]^\intercal ; (\mathbb{S} ; A \oslash \mathbb{T} ; A \cap e ; \mathbb{T})$ Prop. 6.3.1.i.c; destroy and append
$\subseteq \mathbb{S}^\intercal ; \left[q_H^n(\mathfrak{Q}_0)\right]^\intercal ; (\mathbb{S} ; A \oslash e ; \mathbb{T})$
$\subseteq \mathbb{S}^\intercal ; \mathbb{S} ; A \subseteq A$ hypothesis applied with $\mathbb{S} ; A$ injective and $\mathbb{S}$ univalent

---

i.d) $\mathfrak{Q} ; \mathbb{S}^\intercal = \mathtt{sup}_{n \geq 0}\, q_H^n(\mathfrak{Q}_0) ; \mathbb{S}^\intercal$
$= (z ; \mathbb{T} \oslash \mathbb{T} ; z^\intercal) ; \mathbb{S}^\intercal \cup \mathtt{sup}_{n \geq 1}\, (\mathfrak{S} \oslash \rho)^n ; \mathfrak{Q}_0 ; \mathbb{S}^n ; \mathbb{S}^\intercal$
$= \mathbb{\perp\!\!\!\perp} \cup \mathtt{sup}_{n \geq 1}\, (\mathfrak{S} \oslash \rho)^n ; \mathfrak{Q}_0 ; \mathbb{S}^n ; \mathbb{S}^\intercal$ since $\mathbb{S} ; z = \mathbb{\perp\!\!\!\perp}$
$= \mathtt{sup}_{n \geq 1}\, (\mathfrak{S} \oslash \rho) ; (\mathfrak{S} \oslash \rho)^{n-1} ; \mathfrak{Q}_0 ; \left[\mathbb{S}^{n-1} \cap \mathbb{T} ; \mathbb{S}^\intercal\right]$ Prop. 5.2.1.i.b
$= \mathtt{sup}_{n \geq 1}\, (\mathfrak{S} \oslash \rho) ; \left[(\mathfrak{S} \oslash \rho)^{n-1} ; \mathfrak{Q}_0 ; \mathbb{S}^{n-1} \cap \mathbb{T} ; \mathbb{S}^\intercal\right]$ masking
$= (\mathfrak{S} \oslash \rho) ; \left[\mathtt{sup}_{n \geq 1}\, (\mathfrak{S} \oslash \rho)^{n-1} ; \mathfrak{Q}_0 ; \mathbb{S}^{n-1} \cap \mathbb{T} ; \mathbb{S}^\intercal\right]$
$= (\mathfrak{S} \oslash \rho) ; \mathtt{sup}_{k \geq 0}\, (\mathfrak{S} \oslash \rho)^k ; \mathfrak{Q}_0 ; \mathbb{S}^k \cap \mathbb{T} ; \mathbb{S}^\intercal$
$= (\mathfrak{S} \oslash \rho) ; \mathfrak{Q} \cap \mathbb{T} ; \mathbb{S}^\intercal$

The other result follows analogously.

---

i.e) $\mathfrak{Q} ; z ; \mathbb{T} = \left[\mathfrak{Q}_0 \cup (\mathfrak{S} \oslash \rho) ; \mathfrak{Q} ; \mathbb{S}\right] ; z ; \mathbb{T} = \mathfrak{Q}_0 ; z ; \mathbb{T} \cup \mathbb{\perp\!\!\!\perp}$ since $\mathbb{S} ; z = \mathbb{\perp\!\!\!\perp}$
$= (z ; \mathbb{T} \oslash \mathbb{T} ; z^\intercal) ; z ; \mathbb{T} = (z ; \mathbb{T} ; z ; \mathbb{T} \oslash \mathbb{T} ; z^\intercal ; z ; \mathbb{T})$ since $z ; \mathbb{T}$ is injective
$= (z ; \mathbb{T} \oslash \mathbb{T})$ since z is a point
$= \pi ; z ; \mathbb{T}$

---

i.f) $\hookrightarrow\!\mathbb{M} = (\pi^\intercal \oslash \mathbb{M}^\intercal) ; \rho = (\pi^\intercal \oslash \left[\mathtt{sup}_{n \geq 0}\, (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash k_n)\right]^\intercal) ; \rho$
$= \mathtt{sup}_{n \geq 0}\, (\pi^\intercal \oslash (\mathbb{T} ; z^\intercal ; \mathbb{S}^n \oslash k_n^\intercal)) ; \rho$
$= \mathtt{sup}_{n \geq 0}\, ((\mathbb{I} \oslash \mathbb{T} ; z^\intercal ; \mathbb{S}^n) \oslash \rho ; k_n^\intercal) ; \rho$ projection migration according to Prop. 3.1.2.i
$= \mathtt{sup}_{n \geq 0}\, (\mathbb{I} \oslash \mathbb{T} ; z^\intercal ; \mathbb{S}^n) ; \mathbb{T} \cap \rho ; k_n^\intercal$ destroy and append
$= \mathtt{sup}_{n \geq 0} (\pi \cap \mathbb{T} ; z^\intercal ; \mathbb{S}^n) ; \mathbb{T} \cap \rho ; k_n^\intercal$ since ρ is total
$= \mathtt{sup}_{n \geq 0}\, \pi ; (\mathbb{T} \cap \mathbb{S}^{\intercal n} ; z ; \mathbb{T}) \cap \rho ; k_n^\intercal$ mask shifting
$= \mathtt{sup}_{n \geq 0}\, \pi ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \rho ; k_n^\intercal$
$= \mathtt{sup}_{n \geq 0}\, (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \oslash k_n^\intercal)$ only later, with regressive finiteness, $\ldots = \mathtt{Q}$

$\hookrightarrow\!\mathbb{N} = (\pi^\intercal \oslash N^\intercal) ; \rho = (\pi^\intercal \oslash (\mathfrak{P} ; \mathbb{M})^\intercal) ; \rho$ due to Prop. 7.1.1.i.h
$= (\pi^\intercal \oslash \mathbb{M}^\intercal ; \mathfrak{P}) ; \rho = (\rho^\intercal ; \mathfrak{P} \oslash \mathbb{M}^\intercal ; \mathfrak{P}) ; \rho$
$= (\rho^\intercal \oslash \mathbb{M}^\intercal) ; \mathfrak{P} ; \rho = \mathfrak{P} ; \mathfrak{P} ; (\rho^\intercal \oslash \mathbb{M}^\intercal) ; \pi$
$= \mathfrak{P} ; (\mathbb{M}^\intercal \oslash \rho^\intercal) ; \pi = \mathfrak{P} ; \mathbb{M}^\hookleftarrow$

$$\mathfrak{Q}_0 = \begin{array}{c} \\ (0,0)\\(0,1)\\(0,2)\\(0,3)\\(0,4)\\(0,5)\\(1,0)\\(1,1)\\(1,2)\\(1,3)\\(1,4)\\(1,5)\\(2,0)\\(2,1)\\(2,2)\\(2,3)\\(2,4)\\(2,5)\\(3,0)\\(3,1)\\(3,2)\\(3,3)\\(3,4)\\(3,5)\\(4,0)\\(4,1)\\(4,2)\\(4,3)\\(4,4)\\(4,5)\\(5,0)\\(5,1)\\(5,2)\\(5,3)\\(5,4)\\(5,5) \end{array} \begin{array}{c} \begin{array}{cccccc} 0&1&2&3&4&5 \end{array} \\ \begin{pmatrix} \mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\

$\mathbb{M}^{\hookleftarrow} = (\mathbb{M}^{\intercal} \circledcirc \rho^{\intercal}) ; \pi = (\big[\mathtt{sup}_{n\geq 0} (\mathbb{S}^{\intercal^n} ; z ; \mathbb{T} \circledcirc k_n) \big]^{\intercal} \circledcirc \rho^{\intercal}) ; \pi$
$= \mathtt{sup}_{n\geq 0} ((\mathbb{T} ; z^{\intercal} ; \mathbb{S}^n \circledcirc k_n^{\intercal}) \circledcirc \rho^{\intercal}) ; \pi$
$= \mathtt{sup}_{n\geq 0} \big[\pi ; (\mathbb{T} ; z^{\intercal} ; \mathbb{S}^n \circledcirc k_n^{\intercal}) \cap \rho ; \rho^{\intercal}\big] ; \pi$
$= \mathtt{sup}_{n\geq 0} \big[\pi ; (\mathbb{T} ; z^{\intercal} ; \mathbb{S}^n ; \pi^{\intercal} \cap k_n^{\intercal} ; \rho^{\intercal}) \cap \rho ; \rho^{\intercal}\big] ; \pi$
$= \mathtt{sup}_{n\geq 0} \big[\mathbb{T} ; z^{\intercal} ; \mathbb{S}^n ; \pi^{\intercal} \cap \pi ; k_n^{\intercal} ; \rho^{\intercal} \cap \rho ; \rho^{\intercal}\big] ; \pi$ masking
$= \mathtt{sup}_{n\geq 0} \big[\mathbb{T} ; z^{\intercal} ; \mathbb{S}^n ; \pi^{\intercal} \cap (\pi ; k_n^{\intercal} \cap \rho) ; \rho^{\intercal}\big] ; \pi$
$= \mathtt{sup}_{n\geq 0} \mathbb{T} ; z^{\intercal} ; \mathbb{S}^n \cap (\pi ; k_n^{\intercal} \cap \rho) ; \mathbb{T}$ destroy and append
$= \mathtt{sup}_{n\geq 0} \mathbb{T} ; z^{\intercal} ; \mathbb{S}^n \cap (k_n^{\intercal} \circledcirc \mathbb{I}) ; \mathbb{T}$
$= \mathtt{sup}_{n\geq 0} (k_n^{\intercal} \circledcirc \mathbb{I}) ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^n$ masking
$= \mathtt{sup}_{n\geq 0} \mathtt{vec}(k_n^{\intercal}) ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^n$ due to Prop. 2.2.5.i

---

Variant version with projection migration:

$\mathbb{M}^{\hookleftarrow} = (\mathbb{M}^{\intercal} \circledcirc \rho^{\intercal}) ; \pi = \mathtt{sup}_{n\geq 0} ((\mathbb{T} ; z^{\intercal} ; \mathbb{S}^n \circledcirc k_n^{\intercal}) \circledcirc \rho^{\intercal}) ; \pi$
$= \mathtt{sup}_{n\geq 0} (\pi ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^n \circledcirc (k_n^{\intercal} \circledcirc \mathbb{I})) ; \pi$ projection migration according to Prop. 3.1.2.ii
$= \mathtt{sup}_{n\geq 0} \mathbb{T} ; z^{\intercal} ; \mathbb{S}^n \cap (k_n^{\intercal} \circledcirc \mathbb{I}) ; \mathbb{T}$ destroy and append; π total
$= \mathtt{sup}_{n\geq 0} \mathtt{vec}(k_n^{\intercal}) ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^n$ due to Prop. 2.2.5.i and masking

---

Herefrom follows $\hookrightarrow \mathcal{N} = \mathfrak{P} ; \mathbb{M}^{\hookleftarrow} = \mathtt{sup}_{n\geq 0} \mathtt{vec}(k_n) ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^n$ with Prop. 2.2.6.ii.

---

ii.a) Before starting the proof proper, we show by induction

$(\mathfrak{S} \circledcirc \rho)^n ; \pi ; z ; \mathbb{T} = \mathtt{vec}(k_n^{\intercal})$,

started with

$\pi ; z ; \mathbb{T} = \pi ; z ; \mathbb{T} \cap \rho ; \mathbb{T} = (\pi ; z ; \mathbb{T} \cap \rho) ; \mathbb{T}$ since ρ is total
$= (k_0^{\intercal} \circledcirc \mathbb{I}) ; \mathbb{T} = \mathtt{vec}(k_0^{\intercal})$ using Prop. 2.2.5

and completed by the step $n \mapsto n+1$:

$(\mathfrak{S} \circledcirc \rho)^{n+1} ; \pi ; z ; \mathbb{T} = (\mathfrak{S} \circledcirc \rho) ; (\mathfrak{S} \circledcirc \rho)^n ; \pi ; z ; \mathbb{T}$
$= (\mathfrak{S} \circledcirc \rho) ; \mathtt{vec}(k_n^{\intercal})$ hypothesis
$= \mathtt{vec}(k_{n+1}^{\intercal})$ Lemma 6.4.2.ii.g

Now we prove that

$\mathfrak{Q} = \mathtt{sup}_{n\geq 0} (\mathfrak{S} \circledcirc \rho)^n ; \mathfrak{Q}_0 ; \mathbb{S}^n = \mathtt{sup}_{n\geq 0} (\mathfrak{S} \circledcirc \rho)^n ; (z ; \mathbb{T} \circledcirc \mathbb{T} ; z^{\intercal}) ; \mathbb{S}^n$
$= \mathtt{sup}_{n\geq 0} (\mathfrak{S} \circledcirc \rho)^n ; [\pi ; z ; \mathbb{T} \cap \rho ; \mathbb{T} ; z^{\intercal}] ; \mathbb{S}^n = \mathtt{sup}_{n\geq 0} (\mathfrak{S} \circledcirc \rho)^n ; [\pi ; z ; \mathbb{T} \cap \mathbb{T} ; z^{\intercal} ; \mathbb{S}^n]$ ρ total
$= \mathtt{sup}_{n\geq 0} \big[(\mathfrak{S} \circledcirc \rho)^n ; \pi ; z ; \mathbb{T}\big] \cap \mathbb{T} ; z^{\intercal} ; \mathbb{S}^n$ masking
$= \mathtt{sup}_{n\geq 0} \mathtt{vec}(k_n^{\intercal}) \cap \mathbb{T} ; z^{\intercal} ; \mathbb{S}^n$ see above
$= \mathtt{sup}_{n\geq 0} \mathtt{vec}(k_n^{\intercal}) ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^n$ masking
$= \mathbb{M}^{\hookleftarrow}$ using (i.f),

The proof of the other version is analogous; it needs however 2.2.6.ii in addition. The vec-version is the one obtained with Lemma 6.4.2.ii.h.

We now prove $\mathbb{M}^{\hookleftarrow} = \mathfrak{Q}$:

$\mathbb{M}^{\hookleftarrow} = \big[\mathtt{sup}_{n\geq 0} (\mathbb{S}^{\intercal^n} ; z ; \mathbb{T} \circledcirc k_n)\big]^{\hookleftarrow}$
$= \mathtt{sup}_{n\geq 0} (\mathbb{S}^{\intercal^n} ; z ; \mathbb{T} \circledcirc k_n)^{\hookleftarrow}$
$= \mathtt{sup}_{n\geq 0} \mathtt{vec}(k_n^{\intercal}) ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^n$ following Prop. 3.2.3.vi
$= \mathfrak{Q}$ using the result above

This allows us proceed as follows to obtain the final version:

$\mathfrak{Q} = \mathbb{M}^{\hookleftarrow}$
$= \big[\, \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\ (\mathbb{S}^{\tau n}; z; \mathbb{T} \circledcirc \mathbb{S}^{\tau i}; z; \mathbb{T}); \mathbb{T}; z^{\mathsf{T}}; \mathbb{S}^{n*i} \big]^{\hookleftarrow}$ Prop. 7.1.1.ii.g with regressive finite
$= \big(\big[\, \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\ (\mathbb{S}^{\tau n}; z; \mathbb{T} \circledcirc \mathbb{S}^{\tau i}; z; \mathbb{T}); \mathbb{T}; z^{\mathsf{T}}; \mathbb{S}^{n*i} \big]^{\mathsf{T}} \circledcirc \rho^{\mathsf{T}} \big); \pi$
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0} \big[\, \big((\mathbb{S}^{\tau n}; z; \mathbb{T} \circledcirc \mathbb{S}^{\tau i}; z; \mathbb{T}); \mathbb{T}; z^{\mathsf{T}}; \mathbb{S}^{n*i} \circledcirc \rho \big) \big]^{\mathsf{T}}; \pi$
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0} \big[\, \big((\mathbb{S}^{\tau n}; z; \mathbb{T} \circledcirc \mathbb{S}^{\tau i}; z; \mathbb{T}; z^{\mathsf{T}}; \mathbb{S}^{n*i}) \ \circledcirc \rho; \mathbb{I} \big) \big]^{\mathsf{T}}; \pi$ masking
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0} \big[\, (\mathbb{S}^{\tau n}; z; \mathbb{T}; \pi^{\mathsf{T}} \circledcirc (\mathbb{S}^{\tau i}; z; \mathbb{T}; z^{\mathsf{T}}; \mathbb{S}^{n*i} \circledcirc \mathbb{I})) \big]^{\mathsf{T}}; \pi$ projection migration
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0} (\mathbb{T}; z^{\mathsf{T}}; \mathbb{S}^{n} \circledcirc\ (\mathbb{S}^{\tau n*i}; z; \mathbb{T}; z^{\mathsf{T}}; \mathbb{S}^{i} \circledcirc \mathbb{I})); \pi$ with π total
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\, \mathbb{T}; z^{\mathsf{T}}; \mathbb{S}^{n} \cap (\mathbb{S}^{\tau n*i}; z; \mathbb{T}; z^{\mathsf{T}}; \mathbb{S}^{i} \circledcirc \mathbb{I}); \mathbb{T}$ destroy and append
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\ (\mathbb{S}^{\tau n*i}; z; \mathbb{T}; z^{\mathsf{T}}; \mathbb{S}^{i} \circledcirc \mathbb{I}); \mathbb{T}; z^{\mathsf{T}}; \mathbb{S}^{n}$ masking
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0} (\pi; \mathbb{S}^{\tau n*i}; z; \mathbb{T}; z^{\mathsf{T}}; \mathbb{S}^{i} \cap \rho); \mathbb{T}; z^{\mathsf{T}}; \mathbb{S}^{n}$
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0} (\pi; \mathbb{S}^{\tau n*i}; z; \mathbb{T}; z^{\mathsf{T}}; \mathbb{S}^{i}; \rho^{\mathsf{T}} \cap \mathbb{I}); \rho; \mathbb{T}; z^{\mathsf{T}}; \mathbb{S}^{n}$ destroy and append backwards
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0} (\pi; \mathbb{S}^{\tau n*i}; z; \mathbb{T} \cap \mathbb{T}; z^{\mathsf{T}}; \mathbb{S}^{i}; \rho^{\mathsf{T}} \cap \mathbb{I}); \mathbb{T}; z^{\mathsf{T}}; \mathbb{S}^{n}$ since ρ is total; masking
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0} \big[\pi; \mathbb{S}^{\tau n*i}; z; \mathbb{T} \cap (\mathbb{T}; z^{\mathsf{T}}; \mathbb{S}^{i}; \rho^{\mathsf{T}} \cap \mathbb{I})\big]; \mathbb{T}; z^{\mathsf{T}}; \mathbb{S}^{n}$
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0} \big[\pi; \mathbb{S}^{\tau n*i}; z; \mathbb{T} \cap (\rho; \mathbb{S}^{\tau i}; z; \mathbb{T} \cap \mathbb{I})\big]; \mathbb{T}; z^{\mathsf{T}}; \mathbb{S}^{n}$ transposing a partial identity
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0} \big[\pi; \mathbb{S}^{\tau n*i}; z; \mathbb{T} \cap \rho; \mathbb{S}^{\tau i}; z; \mathbb{T}\big] \cap \mathbb{I}; \mathbb{T}; z^{\mathsf{T}}; \mathbb{S}^{n}$ masking
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\ (\mathbb{S}^{\tau n*i}; z; \mathbb{T} \circledcirc \mathbb{S}^{\tau i}; z; \mathbb{T}); \mathbb{T}; z^{\mathsf{T}}; \mathbb{S}^{n}$

$\mathtt{vec}(k_n) = \mathfrak{P}; \mathtt{vec}(k_n^{\mathsf{T}})$ Prop. 2.2.6.ii
$= \mathfrak{P}; (k_n^{\mathsf{T}} \circledcirc \mathbb{I}); \mathbb{T} = (\mathbb{I} \circledcirc k_n^{\mathsf{T}}); \mathbb{T}$

---

ii.b) $\mathfrak{Q}^{\mathsf{T}}; (\mathbb{S}^{\tau k}; z; \mathbb{T} \circledcirc \mathbb{S}^{\tau l}; z; \mathbb{T})$
$= \big[\, \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\ (\mathbb{S}^{\tau n*i}; z; \mathbb{T} \circledcirc \mathbb{S}^{\tau i}; z; \mathbb{T}); \mathbb{T}; z^{\mathsf{T}}; \mathbb{S}^{n} \big]^{\mathsf{T}}; (\mathbb{S}^{\tau k}; z; \mathbb{T} \circledcirc \mathbb{S}^{\tau l}; z; \mathbb{T})$ due to (ii.a)
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\, \mathbb{S}^{\tau n}; z; \mathbb{T}; (\mathbb{T}; z^{\mathsf{T}}; \mathbb{S}^{n*i} \circledcirc \mathbb{T}; z^{\mathsf{T}}; \mathbb{S}^{i}); (\mathbb{S}^{\tau k}; z; \mathbb{T} \circledcirc \mathbb{S}^{\tau l}; z; \mathbb{T})$
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\, \mathbb{S}^{\tau n}; z; \mathbb{T}; \big[\mathbb{T}; z^{\mathsf{T}}; \mathbb{S}^{n*i}; \mathbb{S}^{\tau k}; z; \mathbb{T} \cap \mathbb{T}; z^{\mathsf{T}}; \mathbb{S}^{i}; \mathbb{S}^{\tau l}; z; \mathbb{T}\big]$ with $i := l$ to be $\neq \mathbb{\perp}$
$= \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\tau n}; z; \mathbb{T}; z^{\mathsf{T}}; \mathbb{S}^{n*l}; \mathbb{S}^{\tau k}; z; \mathbb{T}$ since $\mathbb{S}^{\tau l}; z; \mathbb{T}$ was assumed to be a point

Now the term is $\ldots = \mathbb{\perp}$ unless $n * l = k$. In the first case, namely $l = 0 \implies k = 0$, implies
$\ldots = \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\tau n}; z; \mathbb{T}; z^{\mathsf{T}}; z; \mathbb{T} = \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\tau n}; z; \mathbb{T} = \mathbb{T}$

It remains the case with $l \neq 0$ and thus, due to $n * l = k$, obviously $l \,|\, k$; then with $q := \frac{k}{l}$
$\ldots = \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\tau n}; z; \mathbb{T}; z^{\mathsf{T}}; \mathbb{S}^{n*l}; \mathbb{S}^{\tau k}; z; \mathbb{T} = \mathbb{S}^{\tau q}; z; \mathbb{T}; z^{\mathsf{T}}; \mathbb{S}^{k}; \mathbb{S}^{\tau k}; z; \mathbb{T} = \mathbb{S}^{\tau q}; z; \mathbb{T}$

We show the result $(\mathbb{I} \circledcirc \mathbb{I}); z; \mathbb{T} \subseteq \mathfrak{Q}$ "columnwise", proving for $n \geq 0$ that
$(\mathbb{I} \circledcirc \mathbb{I}); z; \mathbb{T} \cap \mathbb{T}; z^{\mathsf{T}}; \mathbb{S}^{n} = (\mathbb{I} \circledcirc \mathbb{I}); z; \mathbb{T}; z^{\mathsf{T}}; \mathbb{S}^{n} \subseteq (\mathfrak{S} \circledcirc \rho)^{n}; \mathfrak{Q}_0; \mathbb{S}^{n}$.

Either $\mathbb{S}^{\tau n}; z; \mathbb{T} = \mathbb{\perp}$ and we are done; or $\mathbb{S}^{\tau n}; z; \mathbb{T}$ is a point — shunted so as to obtain
$(\mathbb{I} \circledcirc \mathbb{I}); z; \mathbb{T} \subseteq (\mathfrak{S} \circledcirc \rho)^{n}; \mathfrak{Q}_0; \mathbb{S}^{n}; \mathbb{S}^{\tau n}; z; \mathbb{T} = (\mathfrak{S} \circledcirc \rho)^{n}; \mathfrak{Q}_0; z; \mathbb{T}$ since $\mathbb{S}$ is injective

This can indeed be shown: $(\mathbb{I} \circledcirc \mathbb{I}); z; \mathbb{T} = (z; \mathbb{T}; z^{\mathsf{T}} \circledcirc \mathbb{I}); z; \mathbb{T}$ Prop. 5.2.1.i.j
$\subseteq (k_n^{\mathsf{T}} \circledcirc \mathbb{I}); z; \mathbb{T}$ Lemma 6.4.2.i.b
$= (\mathfrak{S} \circledcirc \rho)^{n-1}; (\mathbb{I} \circledcirc \mathbb{I}); z; \mathbb{T} \subseteq (\mathfrak{S} \circledcirc \rho)^{n-1}; (\mathbb{I} \circledcirc \mathbb{I}); \mathbb{T}$ Lemma 6.4.2.ii.f
$= (\mathfrak{S} \circledcirc \rho)^{n-1}; \mathfrak{S}; z; \mathbb{T}$ Prop. 6.3.1.ii.a
$= (\mathfrak{S} \circledcirc \rho)^{n-1}; (\mathfrak{S} \circledcirc \rho); \pi; z; \mathbb{T}$ destroy and append in reverse direction
$= (\mathfrak{S} \circledcirc \rho)^{n}; \pi; z; \mathbb{T}$
$= (\mathfrak{S} \circledcirc \rho)^{n}; \mathfrak{Q}_0; z; \mathbb{T}$

$\mathbb{T}; \mathfrak{Q} \supseteq \mathbb{T}; (z; \mathbb{T} \circledcirc z; \mathbb{T})$ see above

$= \mathbb{T} ; \pi^\intercal ; (\pi ; z ; \mathbb{T} \cap \rho ; z ; \mathbb{T}) = \mathbb{T} ; (z ; \mathbb{T} \cap \mathbb{T} ; z ; \mathbb{T})$ destroy and append
$= \mathbb{T} ; (z ; \mathbb{T} \cap \mathbb{T}) = \mathbb{T}$ since z is a point

---

ii.c) $\hookrightarrow\!\mathfrak{V} = \hookrightarrow\!\big[\, \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\ (\mathbb{S}^{\intercal n*i} ; z ; \mathbb{T} \between \mathbb{S}^{\intercal i} ; z ; \mathbb{T}) ; \mathbb{T} ; z^\intercal ; \mathbb{S}^n \big]$ see (ii.a) above
$= (\pi^\intercal \between \big[\, \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\ (\mathbb{S}^{\intercal n*i} ; z ; \mathbb{T} \between \mathbb{S}^{\intercal i} ; z ; \mathbb{T}) ; \mathbb{T} ; z^\intercal ; \mathbb{S}^n \big]^\intercal) ; \rho$
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\ (\pi^\intercal \between \mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; (\mathbb{T} ; z^\intercal ; \mathbb{S}^{n*i} \between \mathbb{T} ; z^\intercal ; \mathbb{S}^i)) ; \rho$
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\ (\pi^\intercal \between \mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; (\mathbb{T} ; z^\intercal ; \mathbb{S}^{n*i} ; \pi^\intercal \cap \mathbb{T} ; z^\intercal ; \mathbb{S}^i ; \rho^\intercal)) ; \rho$
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\ (\pi^\intercal \between \mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{n*i} ; \pi^\intercal \cap \mathbb{T} ; z^\intercal ; \mathbb{S}^i ; \rho^\intercal) ; \rho$ masking
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0} (\pi ; \pi^\intercal \cap \rho ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{n*i} ; \pi^\intercal \cap \rho ; \mathbb{T} ; z^\intercal ; \mathbb{S}^i ; \rho^\intercal) ; \rho$
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0} (\pi ; \pi^\intercal \cap \rho ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \mathbb{T} ; z^\intercal ; \mathbb{S}^{n*i} ; \pi^\intercal \cap \mathbb{T} ; z^\intercal ; \mathbb{S}^i ; \rho^\intercal) ; \rho$ masking, ρ is total
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0} ([\pi \cap \mathbb{T} ; z^\intercal ; \mathbb{S}^{n*i}] ; \pi^\intercal \cap \mathbb{T} ; z^\intercal ; \mathbb{S}^i ; \rho^\intercal) ; \rho \cap \rho ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T}$ masking and rearranging
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0} ([\pi \cap \mathbb{T} ; z^\intercal ; \mathbb{S}^{n*i}] ; \mathbb{T} \cap \mathbb{T} ; z^\intercal ; \mathbb{S}^i) \cap \rho ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T}$ destroy and append
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\, \pi ; [\mathbb{T} \cap \mathbb{S}^{\intercal n*i} ; z ; \mathbb{T}] \cap \mathbb{T} ; z^\intercal ; \mathbb{S}^i \cap \rho ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T}$ mask shifting
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\, \pi ; \mathbb{S}^{\intercal n*i} ; z ; \mathbb{T} \cap \rho ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \mathbb{T} ; z^\intercal ; \mathbb{S}^i$
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\ (\mathbb{S}^{\intercal n*i} ; z ; \mathbb{T} \between \mathbb{S}^{\intercal n} ; z ; \mathbb{T}) \cap \mathbb{T} ; z^\intercal ; \mathbb{S}^i$
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\ (\mathbb{S}^{\intercal n*i} ; z ; \mathbb{T} \between \mathbb{S}^{\intercal n} ; z ; \mathbb{T}) ; \mathbb{T} ; z^\intercal ; \mathbb{S}^i$

This is again $\mathfrak{Q}$ as in (ii.a), however, with index positions exchanged.

New version with projection migration:

$\hookrightarrow\!\mathfrak{V} = \hookrightarrow\!\big[\, \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\ (\mathbb{S}^{\intercal n*i} ; z ; \mathbb{T} \between \mathbb{S}^{\intercal i} ; z ; \mathbb{T}) ; \mathbb{T} ; z^\intercal ; \mathbb{S}^n \big]$ see (ii.a) above
$= (\pi^\intercal \between \big[\, \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\ (\mathbb{S}^{\intercal n*i} ; z ; \mathbb{T} \between \mathbb{S}^{\intercal i} ; z ; \mathbb{T}) ; \mathbb{T} ; z^\intercal ; \mathbb{S}^n \big]^\intercal) ; \rho$
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\ (\pi^\intercal \between \mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; (\mathbb{T} ; z^\intercal ; \mathbb{S}^{n*i} \between \mathbb{T} ; z^\intercal ; \mathbb{S}^i)) ; \rho$
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\ (\pi^\intercal \between (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{n*i} \between \mathbb{T} ; z^\intercal ; \mathbb{S}^i)) ; \rho$ masking
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\ ((\mathbb{I} \between \mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{n*i}) \between \mathbb{T} ; z^\intercal ; \mathbb{S}^i) ; \rho$ proj. migr. Prop. 3.1.2.i; ρ total
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\ (\mathbb{I} \between \mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{n*i}) ; \mathbb{T} \cap \mathbb{T} ; z^\intercal ; \mathbb{S}^i$ destroy and append
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\ (\mathbb{S}^{\intercal n*i} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^n \between \mathbb{I}) ; \mathbb{T} \cap \mathbb{T} ; z^\intercal ; \mathbb{S}^i$ due to Prop. 2.1.1.iii
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\ (\mathbb{S}^{\intercal n*i} ; z ; \mathbb{T} \cap \mathbb{T} ; z^\intercal ; \mathbb{S}^n \between \mathbb{I}) ; \mathbb{T} \cap \mathbb{T} ; z^\intercal ; \mathbb{S}^i$ masking
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\ \big[\pi ; \mathbb{S}^{\intercal n*i} ; z ; \mathbb{T} \cap \mathbb{T} ; z^\intercal ; \mathbb{S}^n \cap \rho\big] ; \mathbb{T} \cap \mathbb{T} ; z^\intercal ; \mathbb{S}^i$ expanded, π total
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\ \big[\pi ; \mathbb{S}^{\intercal n*i} ; z ; \mathbb{T} \cap \rho\big] ; \big[\mathbb{T} \cap \mathbb{S}^{\intercal n} ; z ; \mathbb{T}\big] \cap \mathbb{T} ; z^\intercal ; \mathbb{S}^i$ mask shifting
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\ \big[\pi ; \mathbb{S}^{\intercal n*i} ; z ; \mathbb{T} \cap \rho\big] ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \mathbb{T} ; z^\intercal ; \mathbb{S}^i$
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\ \big[\pi ; \mathbb{S}^{\intercal n*i} ; z ; \mathbb{T} \cap \rho ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T}\big] \cap \mathbb{T} ; z^\intercal ; \mathbb{S}^i$ masking
$= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\ (\mathbb{S}^{\intercal n*i} ; z ; \mathbb{T} \between \mathbb{S}^{\intercal n} ; z ; \mathbb{T}) ; \mathbb{T} ; z^\intercal ; \mathbb{S}^i$

$\mathfrak{Q}_0^{\hookleftarrow} = (\mathfrak{Q}_0^\intercal \between \rho^\intercal) ; \pi$
$= \big[\pi ; (\mathbb{T} ; z^\intercal \between z ; \mathbb{T}) \cap \rho ; \rho^\intercal\big] ; \pi$
$= \big[\mathbb{T} ; z^\intercal ; \pi^\intercal \cap \pi ; z ; \mathbb{T} \cap \rho ; \rho^\intercal\big] ; \pi$ expanding again; π, ρ are maps
$= \big[\mathbb{T} ; z^\intercal ; \pi^\intercal \cap \rho ; \rho^\intercal\big] ; \pi \cap \pi ; z ; \mathbb{T}$ masking; shuffled
$= \mathbb{T} ; z^\intercal \cap \rho ; \rho^\intercal ; \pi \cap \pi ; z ; \mathbb{T}$ destroy and append
$= \rho ; \mathbb{T} ; z^\intercal \cap \mathbb{T} \cap \pi ; z ; \mathbb{T}$ since ρ is a map
$= (z ; \mathbb{T} \between \mathbb{T} ; z^\intercal) = \mathfrak{Q}_0$

We prove $\hookrightarrow\!\mathfrak{V}_0 = \mathfrak{Q}_0^{\hookleftarrow}$:

$\hookrightarrow\!\mathfrak{V}_0 = (\pi^\intercal \between \mathfrak{Q}_0^\intercal) ; \rho$ by definition of left inverse
$= \big[\pi ; \pi^\intercal \cap \rho ; (\mathbb{T} ; z^\intercal \between z ; \mathbb{T})\ \big] ; \rho$
$= \big[\pi ; \pi^\intercal \cap \mathbb{T} ; z^\intercal ; \pi^\intercal \cap \rho ; z ; \mathbb{T}\big] ; \rho$ expanding the fork; ρ is a map

$= \big[\pi \cap \mathbb{T}\fatsemi z^\intercal\big]\fatsemi\pi^\intercal\fatsemi\rho \cap \rho\fatsemi z\fatsemi\mathbb{T} = \big[\pi \cap \mathbb{T}\fatsemi z^\intercal\big]\fatsemi\mathbb{T} \cap \rho\fatsemi z\fatsemi\mathbb{T}$ masking
$= \pi\fatsemi(\mathbb{T} \cap z\fatsemi\mathbb{T}) \cap \rho\fatsemi z\fatsemi\mathbb{T}$ mask shifting
$= (z\fatsemi\mathbb{T} \oslash z\fatsemi\mathbb{T})$
$= \pi\fatsemi z\fatsemi\mathbb{T} \cap \rho\fatsemi(\mathbb{T} \cap z\fatsemi\mathbb{T}) = \pi\fatsemi z\fatsemi\mathbb{T} \cap \big[\rho \cap \mathbb{T}\fatsemi z^\intercal\big]\fatsemi\rho^\intercal\fatsemi\pi$ mask shifting
$= \big[\pi\fatsemi z\fatsemi\mathbb{T} \cap \rho \cap \mathbb{T}\fatsemi z^\intercal\big]\fatsemi\rho^\intercal\fatsemi\pi$ masking
$= \big[\pi\fatsemi z\fatsemi\mathbb{T} \cap \mathbb{T}\fatsemi z^\intercal\fatsemi\rho^\intercal \cap \rho\fatsemi\rho^\intercal\big]\fatsemi\pi$ since ρ is a map; rearranged
$= \big[\pi\fatsemi(z\fatsemi\mathbb{T}\fatsemi\pi^\intercal \cap \mathbb{T}\fatsemi z^\intercal\fatsemi\rho^\intercal) \cap \rho\fatsemi\rho^\intercal\big]\fatsemi\pi$ since π is a map
$= \big[\pi\fatsemi (z\fatsemi\mathbb{T} \oslash \mathbb{T}\fatsemi z^\intercal) \cap \rho\fatsemi\rho^\intercal\big]\fatsemi\pi$
$= \big((z\fatsemi\mathbb{T} \oslash \mathbb{T}\fatsemi z^\intercal) \oslash \rho^\intercal\big)\fatsemi\pi$
$= (\mathbb{Q}_0^\intercal \oslash \rho^\intercal)\fatsemi\pi = \mathbb{Q}_0^{\hookleftarrow}$

---

ii.d) $\mathfrak{Q}\fatsemi e\fatsemi\mathbb{T} = \big[\mathfrak{Q}_0 \cup (\mathfrak{S} \oslash \rho)\fatsemi\mathfrak{Q}\fatsemi\mathbb{S}\big]\fatsemi e\fatsemi\mathbb{T}$
$= \mathfrak{Q}_0\fatsemi e\fatsemi\mathbb{T} \cup (\mathfrak{S} \oslash \rho)\fatsemi\mathfrak{Q}\fatsemi\mathbb{S}\fatsemi e\fatsemi\mathbb{T}$
$= (z\fatsemi\mathbb{T} \oslash \mathbb{T}\fatsemi z^\intercal)\fatsemi e\fatsemi\mathbb{T} \cup (\mathfrak{S} \oslash \rho)\fatsemi\mathfrak{Q}\fatsemi z\fatsemi\mathbb{T}$
$= (z\fatsemi\mathbb{T}\fatsemi e\fatsemi\mathbb{T} \oslash \mathbb{T}\fatsemi z^\intercal\fatsemi e\fatsemi\mathbb{T}) \cup (\mathfrak{S} \oslash \rho)\fatsemi\pi\fatsemi z\fatsemi\mathbb{T}$ since $e\fatsemi\mathbb{T}$ is injective; (i.e)
$= \perp\!\!\!\perp \cup \mathfrak{S}\fatsemi z\fatsemi\mathbb{T}$
$= (\mathbb{I} \oslash \mathbb{I})\fatsemi\mathbb{T}$ Prop. 6.3.1.ii.a

---

ii.e) We assume that $\mathbb{S}^{\intercal l}\fatsemi z\fatsemi\mathbb{T} \neq \perp\!\!\!\perp$, i.e. that it is a point.
$\mathtt{rel}(\mathfrak{Q}\fatsemi\mathbb{S}^{\intercal l}\fatsemi z\fatsemi\mathbb{T}) = \pi^\intercal\fatsemi\big(\mathfrak{Q}\fatsemi\mathbb{S}^{\intercal l}\fatsemi z\fatsemi\mathbb{T} \cap \rho\big)$
$= \pi^\intercal\fatsemi\big(\big[\,\mathtt{sup}_{n\geq 0}\,\mathtt{vec}(k_n^\intercal)\fatsemi\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^n\big]\fatsemi\mathbb{S}^{\intercal l}\fatsemi z\fatsemi\mathbb{T} \cap \rho\big)$ (ii.a)
$= \mathtt{sup}_{n\geq 0}\,\pi^\intercal\fatsemi\big(\mathtt{vec}(k_n^\intercal)\fatsemi\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^n\fatsemi\mathbb{S}^{\intercal l}\fatsemi z\fatsemi\mathbb{T} \cap \rho\big)$ certainly $\perp\!\!\!\perp$ in case $n \neq l$
$= \pi^\intercal\fatsemi\big(\mathtt{vec}(k_l^\intercal)\fatsemi\mathbb{T}\fatsemi z^\intercal\fatsemi\mathbb{S}^l\fatsemi\mathbb{S}^{\intercal l}\fatsemi z\fatsemi\mathbb{T} \cap \rho\big)$
$= \pi^\intercal\fatsemi\big(\mathtt{vec}(k_l^\intercal)\fatsemi\mathbb{T} \cap \rho\big)$ since $\mathbb{S}^{\intercal l}\fatsemi z\fatsemi\mathbb{T}$ is a point
$= \mathtt{rel}(\mathtt{vec}(k_l^\intercal)) = k_l^\intercal$ Prop. 2.2.5.iii

---

iii.a) $(\mathfrak{S} \oslash \rho)\fatsemi\mathfrak{Q} = \big[\,(\mathfrak{S} \oslash \rho)\fatsemi\mathfrak{Q} \cap \mathbb{T}\fatsemi\mathbb{S}^\intercal\big] \cup \big[\,(\mathfrak{S} \oslash \rho)\fatsemi\mathfrak{Q} \cap \overline{\mathbb{T}\fatsemi\mathbb{S}^\intercal}\big]$
$= \mathfrak{Q}\fatsemi\mathbb{S}^\intercal \cup \big[\,(\mathfrak{S} \oslash \rho)\fatsemi\mathfrak{Q} \cap \mathbb{T}\fatsemi g_{\mathbb{S}}^\intercal\big]$ due to (i.d) and definition of $g_{\mathbb{S}}$ in Def. 5.2.1.iii.a
$= \mathfrak{Q}\fatsemi\mathbb{S}^\intercal \cup \big[\,(\mathfrak{S} \oslash \rho)\fatsemi\big(\mathtt{sup}_{n\geq 0}\,(\mathfrak{S} \oslash \rho)^n\fatsemi\mathfrak{Q}_0\fatsemi\mathbb{S}^n\big) \cap \mathbb{T}\fatsemi g_{\mathbb{S}}^\intercal\big]$ expanded
$= \mathfrak{Q}\fatsemi\mathbb{S}^\intercal \cup \mathtt{sup}_{n\geq 0}\big[\,(\mathfrak{S} \oslash \rho)^n\fatsemi\,(\mathfrak{S} \oslash \rho)\fatsemi\,(z\fatsemi\mathbb{T} \oslash \mathbb{T}\fatsemi z^\intercal)\fatsemi\mathbb{S}^n \cap \mathbb{T}\fatsemi g_{\mathbb{S}}^\intercal\big]$
$= \mathfrak{Q}\fatsemi\mathbb{S}^\intercal \cup \mathtt{sup}_{n\geq 0}\big[\,(\mathfrak{S} \oslash \rho)^n\fatsemi[\mathfrak{S}\fatsemi z\fatsemi\mathbb{T} \cap \mathbb{T}\fatsemi z^\intercal]\fatsemi\,\mathbb{S}^n \cap \mathbb{T}\fatsemi g_{\mathbb{S}}^\intercal\big]$ with ρ total
$= \mathfrak{Q}\fatsemi\mathbb{S}^\intercal \cup \mathtt{sup}_{n\geq 0}\,(\mathfrak{S} \oslash \rho)^n\fatsemi\big[\mathfrak{S}\fatsemi\,z\fatsemi\mathbb{T} \cap \mathbb{T}\fatsemi z^\intercal\fatsemi\,\mathbb{S}^n \cap \mathbb{T}\fatsemi g_{\mathbb{S}}^\intercal\big]$ implying $n := n_{n_{\mathbb{S}}}$ to be $\neq \perp\!\!\!\perp$
$= \mathfrak{Q}\fatsemi\mathbb{S}^\intercal \cup (\mathfrak{S} \oslash \rho)^{n_{\mathbb{S}}}\fatsemi\big[\,(\mathbb{I} \oslash \mathbb{I})\fatsemi z\fatsemi\mathbb{T} \cap \mathbb{T}\fatsemi g_{\mathbb{S}}^\intercal\big]$ Prop. 6.3.1.ii.a
$= \mathfrak{Q}\fatsemi\mathbb{S}^\intercal \cup \big[\,(\mathfrak{S} \oslash \rho)^{n_{\mathbb{S}}}\fatsemi\,(\mathbb{I} \oslash \mathbb{I})\fatsemi z\fatsemi\mathbb{T} \cap \mathbb{T}\fatsemi g_{\mathbb{S}}^\intercal\big]$ masking
$= \mathfrak{Q}\fatsemi\mathbb{S}^\intercal \cup \big[\,(k_{n_{\mathbb{S}}+1}^\intercal \oslash \mathbb{I})\fatsemi z\fatsemi\mathbb{T} \cap \mathbb{T}\fatsemi g_{\mathbb{S}}^\intercal\big]$ using Lemma 6.4.2.ii.f with $k_1 = \mathbb{I}$
$= \mathfrak{Q}\fatsemi\mathbb{S}^\intercal \cup \big[\,(k_{n_{\mathbb{S}}+1}^\intercal\fatsemi z\fatsemi\mathbb{T} \oslash z\fatsemi\mathbb{T}) \cap \mathbb{T}\fatsemi g_{\mathbb{S}}^\intercal\big]$ since $z\fatsemi\mathbb{T}$ is injective
$= \mathfrak{Q}\fatsemi\mathbb{S}^\intercal \cup \big[\,(z\fatsemi\mathbb{T} \oslash z\fatsemi\mathbb{T}) \cap \mathbb{T}\fatsemi g_{\mathbb{S}}^\intercal\big] = \mathfrak{Q}\fatsemi\mathbb{S}^\intercal \cup \big[\pi\fatsemi z\fatsemi\mathbb{T} \cap \rho\fatsemi z\fatsemi\mathbb{T} \cap \mathbb{T}\fatsemi g_{\mathbb{S}}^\intercal\big]$ Lemma 6.4.2.i.b

---

iv.a) uses the proof of (i.d) with $\mathbb{S}$ total together with Prop.6.3.2.iv. □

To provide yet another example of such restricted naturals, we show Fig. 7.2.6 and Fig. 7.2.7.

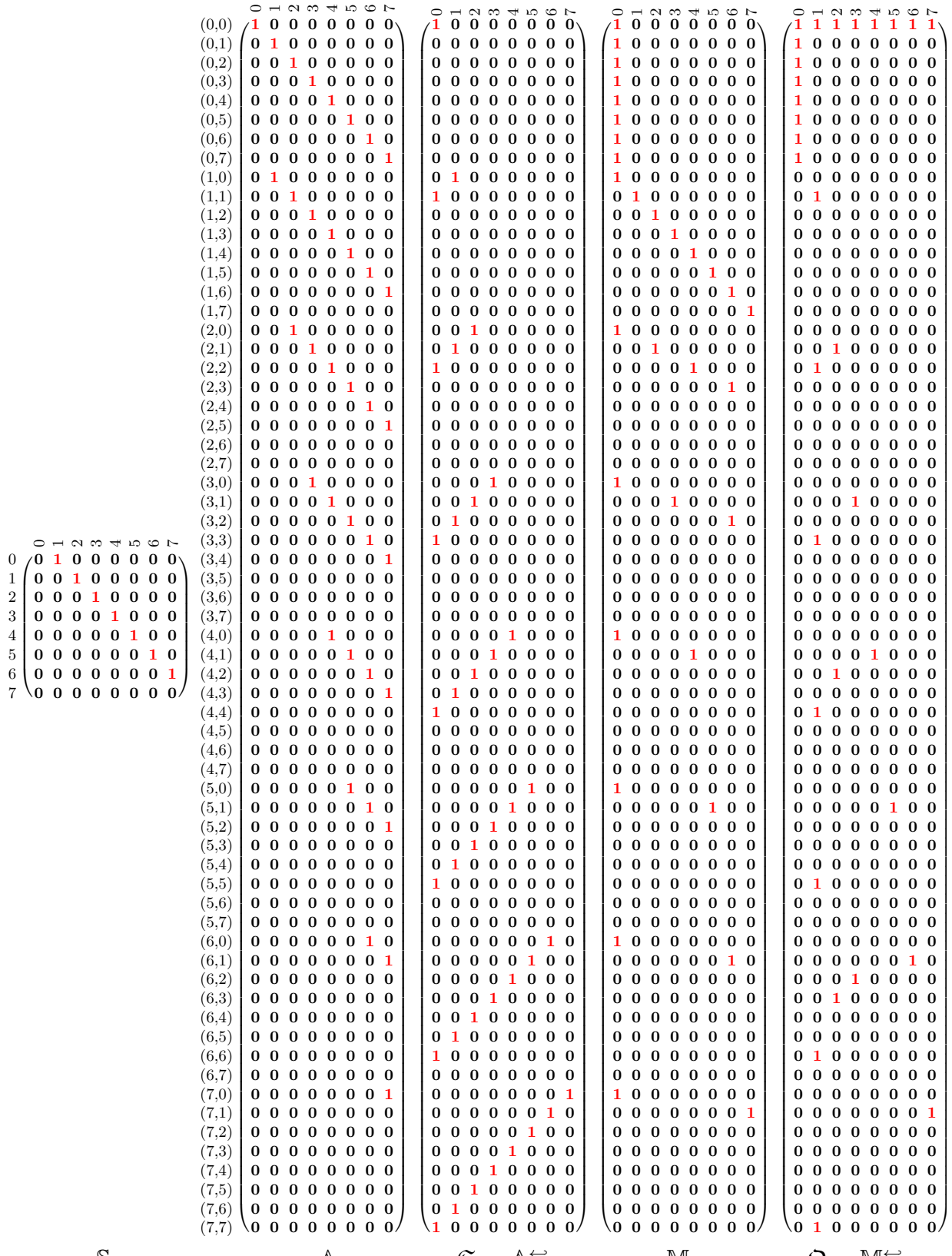


Fig. 7.2.6 Successor, addition, subtraction, multiplication, and quotients for limited naturals

We see here, that it is some sort of an overreaction when one immediately excludes 0 from any further discussion, as most mathematicians do. Looking at $\mathfrak{Q}$, it is perfectly ok that to pairs like $(n, 0)$ with $n > 0$ no result is attached, but all conceivable ones to $(0, 0)$.

	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	
2	2	3	4	5	6	7		
3	3	4	5	6	7			
4	4	5	6	7				
5	5	6	7					
6	6	7						
7	7							

$\mathbb{A}_{table}$

	0	1	2	3	4	5	6	7
0	0							
1	1	0						
2	2	1	0					
3	3	2	1	0				
4	4	3	2	1	0			
5	5	4	3	2	1	0		
6	6	5	4	3	2	1	0	
7	7	6	5	4	3	2	1	0

$\mathfrak{S}_{table}$

	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6				
3	0	3	6					
4	0	4						
5	0	5						
6	0	6						
7	0	7						

$\mathbb{M}_{table}$

	0	1	2	3	4	5	6	7
0	•	0	0	0	0	0	0	0
1		1						
2		2	1					
3		3		1				
4		4	2		1			
5		5				1		
6		6	3	2			1	
7		7						1

$\mathfrak{Q}_{table}$

Fig. 7.2.7

A remark is necessary here: Normally when working with binary maps, they are indeed mappings. Here we also had non-total binary relational operations, for which an empty space in the table is an adequate representation when undefined. It turned out that also binary relational operations that violate univalency in at most one argument may be handled in a similar form. This applies to the quotient $\mathfrak{Q}_{table}$, that can also be seen in Fig. 7.2.6 on the right: Right-inverting $\mathbb{M}$ brings $\mathfrak{Q}$, which has a full row of **1** s at just one position; in the table presentation this is visualized by a bullet. At most one may occur in such a table.

	0	1	2	3	4	5	6	7	8	9
0	•	0	0	0	0	0	0	0	0	0
1		1								
2		2	1							
3		3		1						
4		4	2		1					
5		5				1				
6		6	3	2			1			
7		7						1		
8		8	4		2				1	
9		9		3						1

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	•	0	0	0	0	0	0	0	0	0	0	0	0	0
1		1												
2		2	1											
3		3		1										
4		4	2		1									
5		5				1								
6		6	3	2			1							
7		7						1						
8		8	4		2				1					
9		9		3						1				
10		10	5			2					1			
11		11										1		
12		12	6	4	3		2						1	
13		13												1

Fig. 7.2.8 Relations $\mathfrak{Q}$ represented as tables

Looking at $\mathfrak{Q}_{table}$, the tables for $\mathfrak{Q}$, one will find out easily that all the column entries unequal to "undefined" are in sequence $0, 1, 2, \ldots$. This indicates that the n-fold sums appear in the respective column — or else the vectorizations of $k_n^\intercal$. This in turn shows the "shape" of $\mathcal{D}^\intercal$ to be discussed later in Fig. 7.4.1. Another observation: One may consider the entry $(0, 0)$ to be the pivot or hub when turning straight lines of 1s, 2s, 3s etc. from the -45 degree direction.

We have obviously that $k \,|\, n \implies \mathtt{syq}(\varepsilon, k_n^\intercal, \mathbb{T}) \mathtt{;} \mathtt{syq}(\varepsilon, k_k^\intercal, \mathbb{T})^\intercal \subseteq \Omega$.

Closely related with $\mathfrak{Q}$ are the relations `quot` and `rem`. They correspond to forming the truncated fractional part of quotient and remainder. When discussing `quot` and `rem`, we must be careful. Considering `quot` relationally as opposed to looking at it, e.g. in the HASKELL programming language, will show a difference. It is acceptable that we get in HASKELL

```
quot 3 0        *** Exception:  divide by zero
rem  3 0        *** Exception:  divide by zero
```

It is not acceptable — thinking in terms of relations — to get

```
quot 0 0        *** Exception:  divide by zero.
rem  0 0        *** Exception:  divide by zero
```

Considered as a relational question it should be handled as in Fig. 7.2.10. The relation `quot` violates univalency — `rem` is univalent — but would be relationally correct as for $\mathfrak{Q}$.

A decision had to be made according to the choice between these two conceivable versions

$\mathtt{quot}_0 = (z;\mathbb{T};z^\intercal \circledcirc \mathbb{I}) \cup (\mathbb{S}^+ \circledcirc \mathbb{I});\mathbb{T};z^\intercal$ and $\mathtt{quot}_0 = (\mathbb{S}^+ \circledcirc \mathbb{I});\mathbb{T};z^\intercal$

The difference is made up by consequences of the upper-most left-most **1** alone. It seems a good choice to take the first option because it offers the better fit in formulae to come.

7.2.2 Definition and Proposition. We define as **quotient and remainder forming** belonging to $\mathbb{S}$

$\mathtt{quot} := \mathtt{quot}_{\mathbb{S}} := \mathtt{inf}\,\{\,X \mid qu(X) \subseteq X\}$,

$\mathtt{rem} := \mathtt{rem}_{\mathbb{S}} := \mathtt{inf}\,\{\,X \mid re(X) \subseteq X\}$,

where we have in view of Lemma 4.2.1

$qu(X) := \mathtt{quot}_0 \cup qu_H(X)$
with $\mathtt{quot}_0 = (z;\mathbb{T} \circledcirc z;\mathbb{T});\mathbb{T};z^\intercal \cup (\mathbb{S}^+ \circledcirc \mathbb{I});\mathbb{T};z^\intercal$ and $qu_H(X) = (\mathfrak{S} \circledleft \rho);X;\mathbb{S}$,

$re(X) := \mathtt{rem}_0 \cup re_H(X)$
with $\mathtt{rem}_0 = (z;\mathbb{T} \circledcirc z;\mathbb{T});\mathbb{T};z^\intercal \cup \big[(\mathbb{S}^+ \circledcirc \mathbb{I});\mathbb{T} \cap \pi\big]$ and $re_H(X) = (\mathfrak{S} \circledleft \rho);X$.

Respecting possible further properties of $\mathbb{S}$, the constructs `quot` and `rem` satisfy

i) in general:

a) $\mathtt{quot} = qu(\mathtt{quot}) = \mathtt{quot}_0 \cup (\mathfrak{S} \circledleft \rho);\mathtt{quot};\mathbb{S} = \mathtt{sup}_{n\geq 0}\, qu_H^n(\mathtt{quot}_0)$

$= \mathtt{sup}_{n\geq 0}\, (\mathfrak{S} \circledleft \rho)^{n};\mathtt{quot}_0;\mathbb{S}^n \quad =: \mathtt{sup}_{n\geq 0}\, \mathtt{quot}_n$

b) $\mathtt{rem} = re(\mathtt{rem}) = \mathtt{rem}_0 \cup (\mathfrak{S} \circledleft \rho);\mathtt{rem} = \mathtt{sup}_{n\geq 0}\, re_H^n(\mathtt{rem}_0)$

$= \mathtt{sup}_{n\geq 0}\, (\mathfrak{S} \circledleft \rho)^{n};\mathtt{rem}_0 = (\mathfrak{S} \circledleft \rho)^{*};\mathtt{rem}_0 \quad =: \mathtt{sup}_{n\geq 0}\, \mathtt{rem}_n$

c) $\mathtt{quot} \cap \overline{(z;\mathbb{T} \circledcirc z;\mathbb{T})}$ as well as `rem` are univalent.

d) $((\pi \circledleft \mathtt{rem});\mathfrak{S} \circledleft \rho);\mathfrak{Q} = \mathtt{quot}$

e) $((\mathtt{quot} \circledleft \rho);\mathbb{M} \circledleft \mathtt{rem});\mathbb{A} \subseteq \pi$

$((\rho \circledleft \mathtt{quot});\mathbb{M} \circledleft \mathtt{rem});\mathbb{A} \subseteq \rho$

ii) additionally, in case $\mathbb{S}$ is regressively finite:

a) $\mathtt{quot} = (z ; \mathbb{T} \oslash z ; \mathbb{T}) \cup \mathtt{sup}_{n\geq 0}\, (\mathfrak{S} \oslash \rho)^{n ;} (\mathbb{S}^{+} \oslash \mathbb{I}) ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n}$

b) $\mathtt{rem} = (z ; \mathbb{T} \oslash z ; \mathbb{T}) ; \mathbb{T} ; z^{\intercal} \cup (\mathfrak{S} \oslash \rho)^{*} ; \big[(\mathbb{S}^{+} \oslash \mathbb{I}) \cap \pi \big]$

iii) If $\mathbb{S}$ happens to be regressively finite but not total, i.e. $\mathbb{S} ; \mathbb{T} \neq \mathbb{T}$,

a)

iv) In case $\mathbb{S}$ is regressively finite and also total, i.e. $\mathbb{S} ; \mathbb{T} = \mathbb{T}$,

a)

Proof: i.a) and (i.b) follow directly the mechanism of Prop. 4.2.1 since qu_H and re_H are isotonic, additive and $\cap$-hereditary.

---

i.c) In view of the figures presented, $(z ; \mathbb{T} \oslash z ; \mathbb{T})$ is precisely the first row of $\mathtt{quot}$. So we will abbreviate $Q_n := \mathtt{quot}_n \cap \overline{(z ; \mathbb{T} \oslash z ; \mathbb{T})}$. Obviously

$Q_0 = \mathtt{quot}_0 \cap \overline{(z ; \mathbb{T} \oslash z ; \mathbb{T})} = (\mathbb{S}^{+} \oslash \mathbb{I}) ; \mathbb{T} ; z^{\intercal}$

since $(\mathbb{S}^{+} \oslash \mathbb{I}) ; \mathbb{T} ; z^{\intercal} \subseteq (\mathbb{S}^{+} \oslash \mathbb{I}) ; \mathbb{T} \subseteq \overline{(z ; \mathbb{T} \oslash z ; \mathbb{T})}$. This follows with the Schröder rule from

$(\mathbb{S}^{\intercal +} \oslash \mathbb{I}) ; (z ; \mathbb{T} \oslash z ; \mathbb{T}) = \mathbb{S}^{\intercal +} ; z ; \mathbb{T} \cap z ; \mathbb{T} = \perp\!\!\!\perp$.

Furthermore

$\mathtt{quot} \cap \overline{(z ; \mathbb{T} \oslash z ; \mathbb{T})} = \mathtt{sup}_{n\geq 0}\, \mathtt{quot}_n \cap \overline{(z ; \mathbb{T} \oslash z ; \mathbb{T})} = \mathtt{sup}_{n\geq 0}\, Q_n$

and thus

$$
\begin{aligned}
&Q_n = \mathtt{quot}_n \cap \overline{(z ; \mathbb{T} \oslash z ; \mathbb{T})} = (\mathfrak{S} \oslash \rho)^{n ;} \mathtt{quot}_0 ; \mathbb{S}^{n} \cap \overline{(z ; \mathbb{T} \oslash z ; \mathbb{T})} \\
&= (\mathfrak{S} \oslash \rho)^{n ;} \big[(\mathbb{S}^{+} \oslash \mathbb{I}) \cup (z ; \mathbb{T} \oslash z ; \mathbb{T}) \big] ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n} \cap \overline{(z ; \mathbb{T} \oslash z ; \mathbb{T})} \\
&= \Big[(\mathfrak{S} \oslash \rho)^{n ;} (\mathbb{S}^{+} \oslash \mathbb{I}) ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n} \cup (\mathfrak{S} \oslash \rho)^{n ;} (z ; \mathbb{T} \oslash z ; \mathbb{T}) ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n} \Big] \cap \overline{(z ; \mathbb{T} \oslash z ; \mathbb{T})} \\
&= \Big[(\mathfrak{S} \oslash \rho)^{n ;} (\mathbb{S}^{+} \oslash \mathbb{I}) ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n} \cup (z ; \mathbb{T} \oslash z ; \mathbb{T}) ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n} \Big] \cap \overline{(z ; \mathbb{T} \oslash z ; \mathbb{T})} \quad \text{Prop. 6.3.2.i} \\
&= (\mathfrak{S} \oslash \rho)^{n ;} (\mathbb{S}^{+} \oslash \mathbb{I}) ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n} \cap \overline{(z ; \mathbb{T} \oslash z ; \mathbb{T})} \quad \text{since } (z ; \mathbb{T} \oslash z ; \mathbb{T}) ; R \subseteq (z ; \mathbb{T} \oslash z ; \mathbb{T}) \\
&= (\mathfrak{S} \oslash \rho)^{n ;} (\mathbb{S}^{+} \oslash \mathbb{I}) ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n}
\end{aligned}
$$

The final step is justified because

$$
\begin{aligned}
&(\mathfrak{S} \oslash \rho)^{n ;} (\mathbb{S}^{+} \oslash \mathbb{I}) ; \mathbb{T} \subseteq \overline{(z ; \mathbb{T} \oslash z ; \mathbb{T})} \\
\iff\quad &(\mathbb{S}^{\intercal +} \oslash \mathbb{I}) ; (\mathfrak{S}^{\intercal} \oslash \rho^{\intercal})^{n ;} (z ; \mathbb{T} \oslash z ; \mathbb{T}) \subseteq \perp\!\!\!\perp
\end{aligned}
$$

Here indeed

$$
\begin{aligned}
&(\mathbb{S}^{\intercal +} \oslash \mathbb{I}) ; (\mathfrak{S}^{\intercal} \oslash \rho^{\intercal})^{n ;} (z ; \mathbb{T} \oslash z ; \mathbb{T}) = (\mathbb{S}^{\intercal +} \oslash \mathbb{I}) ; (z ; \mathbb{T} \oslash z ; \mathbb{T}) \quad \text{due to Prop. 6.3.2.viii} \\
&= \mathbb{S}^{\intercal +} z ; \mathbb{T} \cap z ; \mathbb{T} = \perp\!\!\!\perp
\end{aligned}
$$

Univalency is now proved as follows:

$$
\begin{aligned}
&[\mathtt{quot} \cap \overline{(z ; \mathbb{T} \oslash z ; \mathbb{T})}]^{\intercal} ; [\mathtt{quot} \cap \overline{(z ; \mathbb{T} \oslash z ; \mathbb{T})}] \\
&= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\, Q_n^{\intercal} ; Q_i \\
&= \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\, \mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; (\mathbb{S}^{\intercal +} \oslash \mathbb{I}) ; (\mathfrak{S}^{\intercal} \oslash \rho^{\intercal})^{n ;} (\mathfrak{S} \oslash \rho)^{i ;} (\mathbb{S}^{+} \oslash \mathbb{I}) ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{i}
\end{aligned}
$$

Should now be $n = i$, we are done because $\mathbb{S}^{\intercal i} ; z ; \mathbb{T} ; \ldots ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{i} \subseteq \mathbb{I}$.

The case $n > i$, e.g., is with $(\mathfrak{S}^\intercal \oslash \rho^\intercal) ; (\mathfrak{S} \oslash \rho) \subseteq (\mathfrak{S}^\intercal ; \mathfrak{S} \otimes \rho^\intercal ; \rho) \subseteq (\mathbb{I} \otimes \mathbb{I}) = \mathbb{I}$ — justified by Prop. 2.1.1.i — handled as follows:

$\mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; (\mathbb{S}^{\intercal +} \oslash \mathbb{I}) ; (\mathfrak{S}^\intercal \oslash \rho^\intercal)^{n} ; (\mathfrak{S} \oslash \rho)^{i} ; (\mathbb{S}^{+} \oslash \mathbb{I}) ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{i}$
$\subseteq \mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; (\mathbb{S}^{\intercal +} \oslash \mathbb{I}) ; (\mathfrak{S}^\intercal \oslash \rho^\intercal)^{n-i} ; (\mathbb{S}^{+} \oslash \mathbb{I}) ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{i}$

This holds because we have $n - i \geq 1$ and we may use that according to Prop. 6.3.2.ii

$(\mathfrak{S}^\intercal \oslash \rho^\intercal) ; (\mathbb{S}^{+} \oslash \mathbb{I}) = \mathbb{\perp}$

The start of an induction for univalency of `rem` is this:

$\mathtt{rem}_0^\intercal ; \mathtt{rem}_0 =: (A^\intercal \cup B^\intercal) ; (A \cup B)$

This is treated separately:

$A^\intercal ; A = z ; \mathbb{T} ; (\mathbb{T} ; z^\intercal \oslash \mathbb{T} ; z^\intercal) ; (z ; \mathbb{T} \oslash z ; \mathbb{T}) ; \mathbb{T} ; z^\intercal \subseteq z ; \mathbb{T} ; z^\intercal \subseteq \mathbb{I}$

$B^\intercal ; A = \Big[\mathbb{T} ; (\mathbb{S}^{\intercal +} \oslash \mathbb{I}) \cap \pi^\intercal\Big] ; (z ; \mathbb{T} \oslash z ; \mathbb{T}) ; \mathbb{T} ; z^\intercal$
$\subseteq \mathbb{T} ; (\mathbb{S}^{\intercal +} \oslash \mathbb{I}) ; (z ; \mathbb{T} \oslash z ; \mathbb{T}) ; \mathbb{T} ; z^\intercal = \mathbb{T} ; (\mathbb{S}^{\intercal +} ; z ; \mathbb{T} ; z^\intercal \cap z ; \mathbb{T}) ; \mathbb{T} ; z^\intercal = \mathbb{\perp}$

$B^\intercal ; B = \Big[\mathbb{T} ; (\mathbb{S}^{\intercal +} \oslash \mathbb{I}) \cap \pi^\intercal\Big] ; \Big[(\mathbb{S}^{+} \oslash \mathbb{I}) ; \mathbb{T} \cap \pi\Big] \subseteq \pi^\intercal ; \pi \subseteq \mathbb{I}$

$\mathtt{rem}^\intercal ; \mathtt{rem} = \big[\mathtt{sup}_{n \geq 0}\, \mathtt{rem}_0^\intercal ; (\mathfrak{S}^\intercal \oslash \rho^\intercal)^{n}\big] ; \big[\mathtt{sup}_{i \geq 0}\, (\mathfrak{S} \oslash \rho)^{i} ; \mathtt{rem}_0\big]$
$= \mathtt{sup}_{n \geq 0}\, \mathtt{sup}_{i \geq 0}\, \mathtt{rem}_0^\intercal ; (\mathfrak{S}^\intercal \oslash \rho^\intercal)^{n} ; (\mathfrak{S} \oslash \rho)^{i} ; \mathtt{rem}_0$

Should now be $n = i$, we are done because

$\mathtt{rem}_0^\intercal ; (\mathfrak{S}^\intercal \oslash \rho^\intercal)^{n} ; (\mathfrak{S} \oslash \rho)^{n} ; \mathtt{rem}_0 \subseteq \mathtt{rem}_0^\intercal ; \mathbb{I} ; \mathtt{rem}_0 \subseteq \mathbb{I}$

because due to Prop. 2.1.1.i

$(\mathfrak{S}^\intercal \oslash \rho^\intercal) ; (\mathfrak{S} \oslash \rho) \subseteq (\mathfrak{S}^\intercal ; \mathfrak{S} \otimes \rho^\intercal ; \rho) \subseteq (\mathbb{I} \otimes \mathbb{I}) \subseteq \mathbb{I}$

Further we assume without loss of generality $n > i$ and obtain $n - i \geq 1$

$\mathtt{rem}_0^\intercal ; (\mathfrak{S}^\intercal \oslash \rho^\intercal)^{n-i} ; \mathtt{rem}_0$
$= \mathtt{rem}_0^\intercal ; (\mathfrak{S}^\intercal \oslash \rho^\intercal)^{n-i} ; \Big\{\big[(\mathbb{S}^{+} \oslash \mathbb{I}) ; \mathbb{T} \cap \pi\big] \cup (z ; \mathbb{T} \oslash z ; \mathbb{T}) ; \mathbb{T} ; z^\intercal\Big\}$
$= \mathtt{rem}_0^\intercal ; \Big\{(\mathfrak{S}^\intercal \oslash \rho^\intercal)^{n-i} ; \big[(\mathbb{S}^{+} \oslash \mathbb{I}) ; \mathbb{T} \cap \pi\big] \cup (\mathfrak{S}^\intercal \oslash \rho^\intercal)^{n-i} ; (z ; \mathbb{T} \oslash z ; \mathbb{T}) ; \mathbb{T} ; z^\intercal\Big\}$
$= \mathtt{rem}_0^\intercal ; \Big\{\mathbb{\perp} \cup (\mathfrak{S}^\intercal \oslash \rho^\intercal)^{n-i} ; (z ; \mathbb{T} \oslash z ; \mathbb{T}) ; \mathbb{T} ; z^\intercal\Big\}$ again with Prop. 6.3.2.ii
$= \mathtt{rem}_0^\intercal ; (z ; \mathbb{T} \oslash z ; \mathbb{T}) ; \mathbb{T} ; z^\intercal$ with Prop. 6.3.2.viii
$= \Big\{z ; \mathbb{T} ; (\mathbb{T} ; z^\intercal \oslash \mathbb{T} ; z^\intercal) \cup \big[\mathbb{T} ; (\mathbb{S}^{\intercal +} \oslash \mathbb{I}) \cap \pi^\intercal\big]\Big\} ; (z ; \mathbb{T} \oslash z ; \mathbb{T}) ; \mathbb{T} ; z^\intercal$
$= z ; \mathbb{T} ; (\mathbb{T} ; z^\intercal \oslash \mathbb{T} ; z^\intercal) ; (z ; \mathbb{T} \oslash z ; \mathbb{T}) ; \mathbb{T} ; z^\intercal \cup \big[\mathbb{T} ; (\mathbb{S}^{\intercal +} \oslash \mathbb{I}) \cap \pi^\intercal\big] ; (z ; \mathbb{T} \oslash z ; \mathbb{T}) ; \mathbb{T} ; z^\intercal$
$\subseteq z ; \mathbb{T} ; z^\intercal \cup \pi^\intercal ; (z ; \mathbb{T} \oslash z ; \mathbb{T}) ; \mathbb{T} ; z^\intercal$
$= \mathbb{I} \cup [z ; \mathbb{T} ; z^\intercal \cap \mathbb{T}] \subseteq \mathbb{I}$

---

i.d) $((\pi \oslash \mathtt{rem}) ; \mathfrak{S} \oslash \rho) ; \mathfrak{Q} = \mathtt{quot}$ examples ok — not proved

---

i.e) $((\mathtt{quot} \oslash \rho) ; \mathbb{M} \oslash \mathtt{rem}) ; \mathbb{A} = [\pi \cap \overline{\rho ; z ; \mathbb{T}}] \cup (z ; \mathbb{T} \oslash z ; \mathbb{T}) ; \mathbb{T} ; z^\intercal$ examples ok — not proved

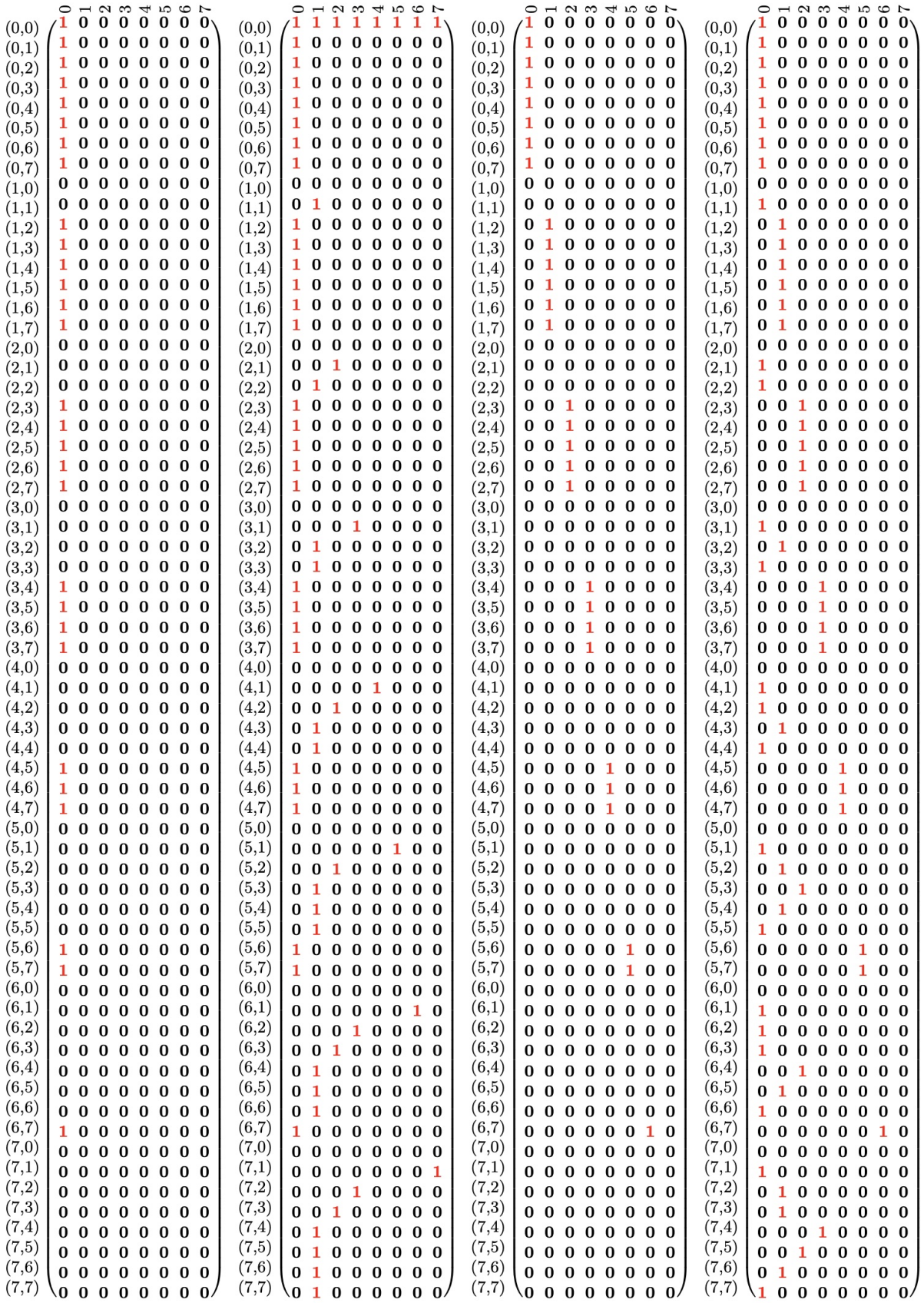


Fig. 7.2.9 $\mathtt{quot}_0$ $\mathtt{quot}$ $\mathtt{rem}_0$ $\mathtt{rem}$

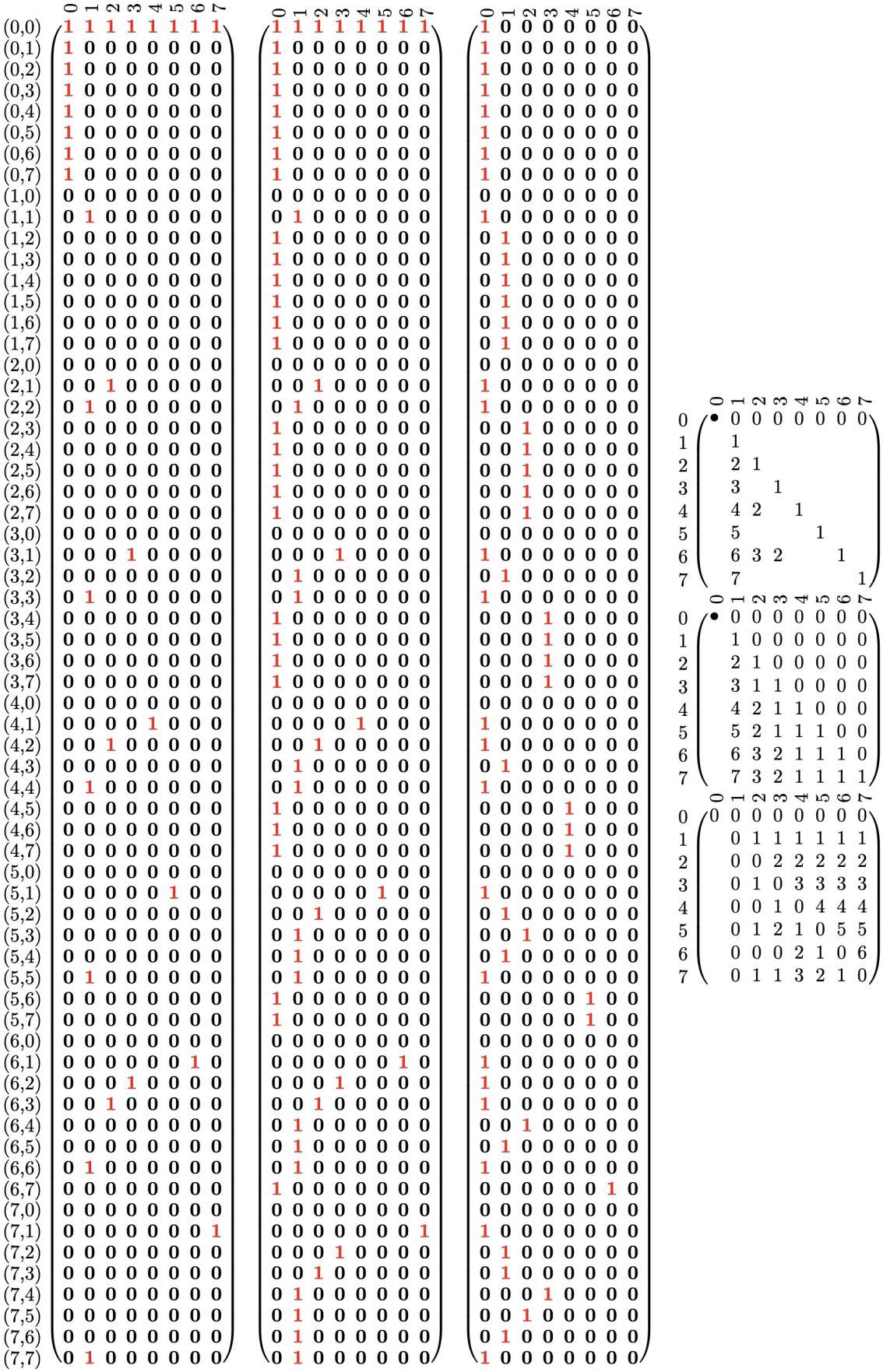


Fig. 7.2.10 Ω and then `quot` and `rem` as relations and as tables for natural numbers ≤ 7

---

ii.a) $\mathtt{quot} = \mathtt{sup}_{n\geq 0}\ (\mathfrak{S} \oslash \rho)^{n};\mathtt{quot}_0;\mathbb{S}^n$
$= \mathtt{sup}_{n\geq 0}\ (\mathfrak{S} \oslash \rho)^{n};\big[\,(z;\mathbb{T} \oslash z;\mathbb{T});\mathbb{T};z^{\mathsf{T}} \cup\ (\mathbb{S}^+ \oslash \mathbb{I});\mathbb{T};z^{\mathsf{T}}\big];\mathbb{S}^n$
$= \big[\,\mathtt{sup}_{i\geq 0}\ (\mathfrak{S} \oslash \rho)^{i};(z;\mathbb{T} \oslash z;\mathbb{T});\mathbb{T};z^{\mathsf{T}};\ \mathbb{S}^i\big] \cup \big[\,\mathtt{sup}_{n\geq 0}\ (\mathfrak{S} \oslash \rho)^{n};\ (\mathbb{S}^+ \oslash \mathbb{I});\mathbb{T};z^{\mathsf{T}};\mathbb{S}^n\big]$
We focus, thus, on just the first term.
$\mathtt{sup}_{i\geq 0}\ (\mathfrak{S} \oslash \rho)^{i};(z;\mathbb{T} \oslash z;\mathbb{T});\mathbb{T};z^{\mathsf{T}};\ \mathbb{S}^i$
$= \mathtt{sup}_{i\geq 0}\ (z;\mathbb{T} \oslash z;\mathbb{T});\mathbb{T};z^{\mathsf{T}};\mathbb{S}^i$ Prop. 6.3.2.i
$= (z;\mathbb{T} \oslash z;\mathbb{T});\mathtt{sup}_{i\geq 0}\ \mathbb{T};z^{\mathsf{T}};\mathbb{S}^i$
$= (z;\mathbb{T} \oslash z;\mathbb{T});\mathbb{T} = (z;\mathbb{T} \oslash z;\mathbb{T})$ Prop. 5.2.1.ii.a

---

ii.b) $\mathtt{rem} = (\mathfrak{S} \oslash \rho)^{*};\mathtt{rem}_0 = (\mathfrak{S} \oslash \rho)^{*};\big\{\,(z;\mathbb{T} \oslash z;\mathbb{T});\mathbb{T};z^{\mathsf{T}} \cup\ [\,(\mathbb{S}^+ \oslash \mathbb{I});\mathbb{T} \cap \pi]\big\}$
$= (\mathfrak{S} \oslash \rho)^{*};(z;\mathbb{T} \oslash z;\mathbb{T});\mathbb{T};z^{\mathsf{T}} \cup\ (\mathfrak{S} \oslash \rho)^{*};[\,(\mathbb{S}^+ \oslash \mathbb{I});\mathbb{T} \cap \pi]$
$= (z;\mathbb{T} \oslash z;\mathbb{T});\mathbb{T};z^{\mathsf{T}} \cup\ (\mathfrak{S} \oslash \rho)^{*};[\,(\mathbb{S}^+ \oslash \mathbb{I});\mathbb{T} \cap \pi]$ Prop. 6.3.2.i □

7.3 Euclidian algorithm

The Euclidian algorithm started on numbers i, k, leads to $i = k * q + r$ with $0 \leq r < k$ and always delivers a univalent relation from (i, k) to (q, r). There are two ways to introduce the Euclidean algorithm. The first is to simply write down this term:

$\mathtt{Euclid} := \mathtt{rel}\big(\big[\ (\pi \otimes \rho);\mathfrak{S} \cap\ (\rho \otimes \pi);\mathbb{M} \cap\ (\rho \oslash \rho;\mathbb{S}^+);\mathbb{T}\big];\ \mathbb{T}\big)$

Starting from $((i, k), (q, r)) \in (\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{N})$, one thus proceeds to $i - r \in \mathbb{N}$, to $k * q \in \mathbb{N}$ with $i - r = k * q$, thereby observing that k is strictly above r. This gives the result as a vector that is afterwards transformed to the relation desired. The result as shown in Fig. 7.3.1 looks quite appealing.[3]

[3]The first row is different from that in Fig. 7.3.4. The difference is just $(z;\mathbb{T} \oslash z;\mathbb{T});\mathbb{T};z^{\mathsf{T}};\rho^{\mathsf{T}}$.

	(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(1,0)	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(2,0)	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(3,0)	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(4,0)	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(5,0)	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)
(0,0)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(0,1)	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(0,2)	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(0,3)	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(0,4)	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(0,5)	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(1,0)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(1,1)	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(1,2)	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(1,3)	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(1,4)	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(1,5)	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(2,0)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(2,1)	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(2,2)	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(2,3)	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(2,4)	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(2,5)	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(3,0)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(3,1)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(3,2)	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(3,3)	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(3,4)	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(3,5)	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(4,0)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(4,1)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
(4,2)	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(4,3)	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(4,4)	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(4,5)	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(5,0)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(5,1)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
(5,2)	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(5,3)	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(5,4)	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(5,5)	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Fig. 7.3.1 First approach to Euclid's algorithm

There is also this term with precisely the same result; still without recursion:

$$\texttt{Euclid} := (\mathbb{A}^{\mathsf{T}}; (\mathbb{M}^{\mathsf{T}} \otimes \mathbb{I}) \oslash \rho^{\mathsf{T}}; \pi'^{\mathsf{T}}); ((\mathbb{I} \otimes \mathbb{S}^{\mathsf{T}+}) \oslash \rho'^{\mathsf{T}})$$

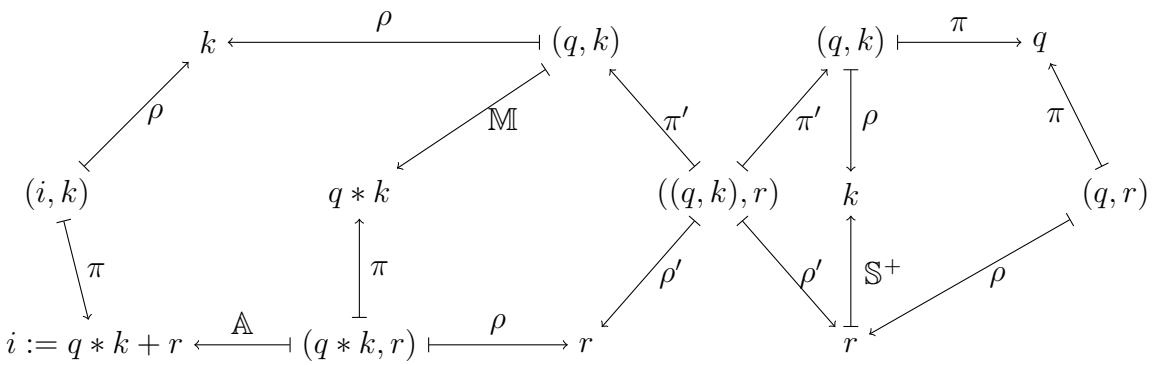


Fig. 7.3.2 Other term for Euclid's algorithm — projections from $((q, k), r)$ twice for visibility

In the term, a product can be seen with center at $((q,k),r)$: Starting from the result (q,r), first the pair is made to a triple introducing a k strictly bigger than the second component. In the second step, $i := q * k + r$ is formed.

Apart from these direct formulae, we follow our well-known track using iteration. Regrettably, I have to confess that I haven't yet undertaken the technicalities to formally prove that these three attempts lead to — essentially — the same result.

The iteration is based on the following typing diagram. Prop. 3.2.3.v is used in order to see that $(\mathbb{A}^{\mathsf{T}} \oslash \rho^{\mathsf{T}}) = (\mathfrak{S} \oslash \rho)$.

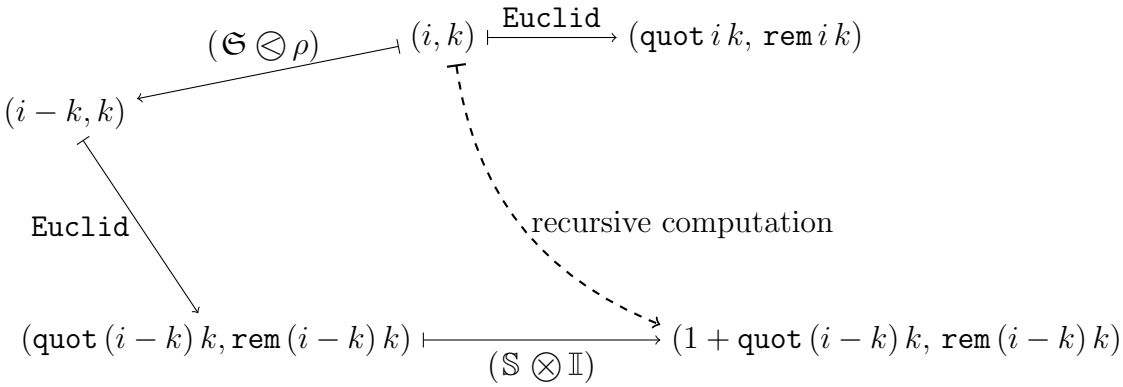


Fig. 7.3.3 Euclid's algorithm on a possibly finite initial segment of the naturals

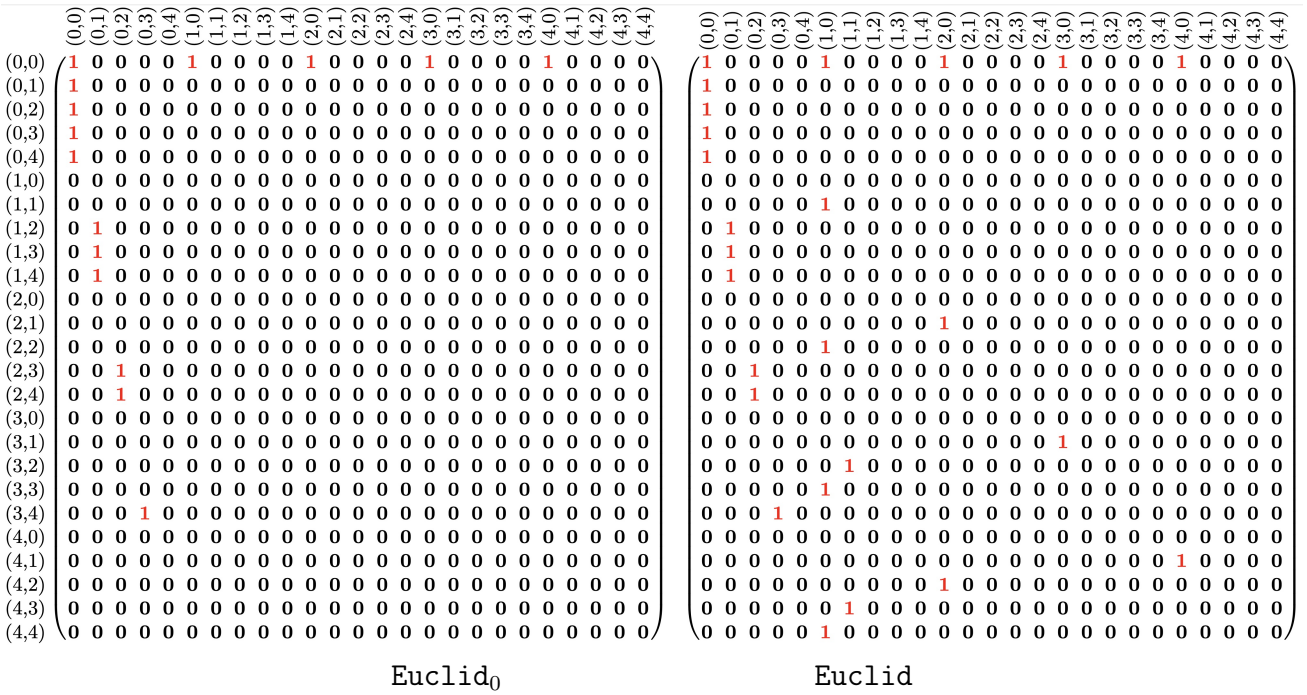


Fig. 7.3.4 Start $\mathtt{Euclid}_0$ and result $\mathtt{Euclid}$ of Euclids algorithm on natural numbers ≤ 4

One has to be very careful in view of Fig. 7.3.4 to configure the starting

$$\mathtt{Euclid}_0 = (z ; \mathbb{T} \oslash \mathbb{T} ; z^{\mathsf{T}}) ; \mathbb{T} ; z^{\mathsf{T}} ; \rho^{\mathsf{T}} \cup \big[(\mathbb{T} ; z^{\mathsf{T}} \oslash \pi) \cap (\mathbb{S}^{+} \oslash \mathbb{I}) ; \mathbb{T} \big].$$

Two concepts will be identified: The first term produces the first row in the figure, while the second term is responsible for results of shape $(0, s)$. Here, the second component s is the first component of the argument, but only if it is strictly less than the second component of the argument.

7.3.1 Definition and Proposition. We define as **Euclidean relation** for $\mathbb{S}$ the construct

$\mathtt{Euclid} := \mathtt{Euclid}_{\mathbb{S}} := \mathtt{inf}\,\{\, X \mid e(X) \subseteq X \,\}$,

where we have in view of Lemma 4.2.1

$e(X) := \mathtt{Euclid}_0 \cup e_H(X)$ with

$\mathtt{Euclid}_0 = (z;\mathbb{T} \between z;\mathbb{T});\mathbb{T};z^{\intercal};\rho^{\intercal} \cup \big[(\mathbb{T};z^{\intercal} \syq \pi) \cap (\mathbb{S}^+ \between \mathbb{I});\mathbb{T}\big]$ and

$e_H(X) = (\mathfrak{S} \syq \rho);X;(\mathbb{S} \otimes \mathbb{I})$.

Respecting possible further properties of $\mathbb{S}$, the relation $\mathtt{Euclid}$ satisfies

i) in general:

a) $\mathtt{Euclid} = e(\mathtt{Euclid}) = \mathtt{Euclid}_0 \cup (\mathfrak{S} \syq \rho);\mathtt{Euclid};(\mathbb{S} \otimes \mathbb{I})$

$= \mathtt{sup}_{n\geq 0}\, e_H^n(\mathtt{Euclid}_0) = \mathtt{sup}_{n\geq 0}\, (\mathfrak{S} \syq \rho)^{n};\mathtt{Euclid}_0;(\mathbb{S}^n \otimes \mathbb{I})$,

b) $\mathtt{Euclid}^{\intercal};\pi;z;\mathbb{T} = \mathtt{Euclid}^{\intercal};\rho;z;\mathbb{T} = \rho;z;\mathbb{T}$

c) $\mathtt{Euclid} \cap \overline{(z;\mathbb{T} \between z;\mathbb{T})}$ is univalent

ii) additionally, if $\mathbb{S}$ is regressively finite:

a) $\mathtt{Euclid};(e;\mathbb{T} \between z;\mathbb{T}) = (\mathbb{I} \between \mathbb{I});\mathbb{T}$ see column (1,0) in Fig. 7.3.4

b) $\mathtt{Euclid}_n;\pi = (z;\mathbb{T} \between z;\mathbb{T});\mathbb{S}^n \cup (\mathfrak{S} \syq \rho)^{n};(\mathbb{S}^+ \between \mathbb{I});\mathbb{T};z^{\intercal};\mathbb{S}^n$ for $n \geq 0$

$\mathtt{Euclid};\pi = (z;\mathbb{T} \between z;\mathbb{T}) \cup \mathtt{sup}_{n\geq 0}\, (\mathfrak{S} \syq \rho)^{n};(\mathbb{S}^+ \between \mathbb{I});\mathbb{T};z^{\intercal};\mathbb{S}^n = \mathtt{quot}$

$\mathtt{Euclid};\rho = (z;\mathbb{T} \between z;\mathbb{T});\mathbb{T};z^{\intercal} \cup \big[\pi \cap (\mathbb{S}^+ \between \mathbb{I});\mathbb{T}\big]$

$\cup\, (\mathfrak{S} \syq \rho)^{*};\big[\mathfrak{S} \cap (\mathfrak{S};\mathbb{S}^+ \cap \rho);\mathbb{T}\big] = \mathtt{rem}$

$\mathtt{Euclid} = (\mathtt{quot} \syq \mathtt{rem})$

$\mathtt{Euclid};\mathbb{T} = (z;\mathbb{T} \between z;\mathbb{T}) \cup \rho;\overline{z;\mathbb{T}}$

$\mathfrak{Q} = (\mathtt{Euclid} \cap \mathbb{T};z^{\intercal};\rho^{\intercal});\pi$

c) $((\mathtt{Euclid};\pi \syq \rho);\mathbb{M} \syq \mathtt{Euclid};\rho);\mathbb{A} = \pi \cap \rho;\overline{z;\mathbb{T}}$

$(\mathtt{Euclid} \syq \rho);((\pi \otimes \mathbb{I});\mathbb{M} \syq \pi';\rho);\mathbb{A} = \pi \cap \rho;\overline{z;\mathbb{T}}$

$\pi^{\intercal};\mathtt{Euclid};(\mathbb{I} \between z;\mathbb{T}) \cup z;\mathbb{T} = \mathtt{rel}(\mathfrak{Q};\mathbb{T})$

$\mathfrak{Q} = \mathtt{quot} \cap \mathtt{Euclid};\rho;z;\mathbb{T} = \mathtt{quot} \cap \mathtt{rem};z;\mathbb{T}$

iii) If $\mathbb{S}$ happens to be regressively finite but not total, i.e. $\mathbb{S};\mathbb{T} \neq \mathbb{T}$,

a)

iv) In case $\mathbb{S}$ is regressively finite and also total, i.e. $\mathbb{S};\mathbb{T}=\mathbb{T}$,

a)

Proof: We observe in advance that Lemma 4.2.1 may be applied since the functional e_H is
isotonic, i.e. $A\subseteq B \quad\Longrightarrow\quad e_H(A)\subseteq e_H(B)$,
additive, i.e.
$e_H(X\cup Y)=(\mathfrak{S}\circledcirc\rho);(X\cup Y);(\mathbb{S}\otimes\mathbb{I})$
$=(\mathfrak{S}\circledcirc\rho);X;(\mathbb{S}\otimes\mathbb{I})\cup(\mathfrak{S}\circledcirc\rho);Y;(\mathbb{S}\otimes\mathbb{I})=e_H(X)\cup e_H(Y)$,
$\cap$-hereditary, i.e.
$e_H(X\cap Y)=(\mathfrak{S}\circledcirc\rho);(X\cap Y);(\mathbb{S}\otimes\mathbb{I})$
$=(\mathfrak{S}\circledcirc\rho);X;(\mathbb{S}\otimes\mathbb{I})\cap(\mathfrak{S}\circledcirc\rho);Y;(\mathbb{S}\otimes\mathbb{I})=e_H(X)\cap e_H(Y)$.

---

i.a) The claim collects the results of Lemma 4.2.1.i,iv.

---

i.b) $\mathtt{Euclid}_0^\mathsf{T};\rho;z;\mathbb{T}=\Big\{(z;\mathbb{T}\circledrightarrow z;\mathbb{T});\mathbb{T};z^\mathsf{T};\rho^\mathsf{T}\cup\big[(\mathbb{T};z^\mathsf{T}\circledcirc\pi)\cap(\mathbb{S}^+\circledrightarrow\mathbb{I});\mathbb{T}\big]\Big\}^\mathsf{T};\rho;z;\mathbb{T}$
$=\Big\{\rho;z;\mathbb{T};(\mathbb{T};z^\mathsf{T}\circledcirc\mathbb{T};z^\mathsf{T})\cup\big[(\mathbb{T};z^\mathsf{T}\circledcirc\pi)\cap(\mathbb{S}^+\circledrightarrow\mathbb{I});\mathbb{T}\big]^\mathsf{T}\Big\};\rho;z;\mathbb{T}$
$=\rho;z;\mathbb{T};(\mathbb{T};z^\mathsf{T}\circledcirc\mathbb{T};z^\mathsf{T});\rho;z;\mathbb{T}\cup\big[(z;\mathbb{T}\circledrightarrow\pi^\mathsf{T})\cap\mathbb{T};(\mathbb{S}^{\mathsf{T}+}\circledcirc\mathbb{I})\big];\rho;z;\mathbb{T}$
$=\rho;z;\mathbb{T};(\mathbb{T};z^\mathsf{T};\mathbb{T}\cap\mathbb{T};z^\mathsf{T});z;\mathbb{T}\cup\bot\!\!\!\bot$ destroy and append; see below
$=\rho;z;\mathbb{T};z^\mathsf{T};z;\mathbb{T}=\rho;z;\mathbb{T}$

$\big[(z;\mathbb{T}\circledrightarrow\pi^\mathsf{T})\cap\mathbb{T};(\mathbb{S}^{\mathsf{T}+}\circledcirc\mathbb{I})\big];\rho;z;\mathbb{T}$
$=(z;\mathbb{T}\circledrightarrow\pi^\mathsf{T});\big[\rho;z;\mathbb{T}\cap(\mathbb{S}^+\circledrightarrow\mathbb{I});\mathbb{T}\big]$ mask shifting
$\subseteq\rho;\pi^\mathsf{T};\big[\rho;z;\mathbb{T}\cap(\pi;\mathbb{S}^+\cap\rho);\mathbb{T}\big]$
$=\rho;\pi^\mathsf{T};(\pi;\mathbb{S}^+\cap\rho\cap\rho;z;\mathbb{T});\mathbb{T}$ masking
$=\rho;(\mathbb{S}^+\cap\pi^\mathsf{T};\rho;(\mathbb{I}\cap z;\mathbb{T}));\mathbb{T}$ destroy and append
$=\rho;(\mathbb{S}^+\cap\mathbb{T};(\mathbb{I}\cap z;\mathbb{T}));\mathbb{T}$ standard property of the direct product π,ρ
$=\rho;\mathbb{S}^+;[\mathbb{T}\cap(\mathbb{I}\cap\mathbb{T};z^\mathsf{T});\mathbb{T}]=\rho;\mathbb{S}^+;(\mathbb{I}\cap\mathbb{T};z^\mathsf{T});\mathbb{T}$ mask shifting
$=\rho;\mathbb{S}^+;\mathbb{I};[\mathbb{T}\cap z;\mathbb{T}]=\rho;\mathbb{S}^+;z;\mathbb{T}=\bot\!\!\!\bot$ mask shifting

---

For the rest of the term, we prove in advance the claim $(\mathfrak{S}^\mathsf{T}\circledcirc\rho^\mathsf{T});\rho;z;\mathbb{T}\subseteq\rho;z;\mathbb{T}$ which holds since $(\mathfrak{S}^\mathsf{T}\circledcirc\rho^\mathsf{T});\rho=\pi;\mathfrak{S}^\mathsf{T};\rho\cap\rho$. This enables us to proceed with
$\mathtt{Euclid}^\mathsf{T};\rho;z;\mathbb{T}=\big[\mathtt{sup}_{n\geq0}\,(\mathfrak{S}\circledcirc\rho)^n;\mathtt{Euclid}_0;(\mathbb{S}^n\otimes\mathbb{I})\big]^\mathsf{T};\rho;z;\mathbb{T}$
$=\mathtt{sup}_{n\geq0}\,(\mathbb{S}^{\mathsf{T}n}\otimes\mathbb{I});\mathtt{Euclid}_0^\mathsf{T};(\mathfrak{S}^\mathsf{T}\circledcirc\rho^\mathsf{T})^n;\rho;z;\mathbb{T}$
$\subseteq\mathtt{sup}_{n\geq0}\,(\mathbb{S}^{\mathsf{T}n}\otimes\mathbb{I});\mathtt{Euclid}_0^\mathsf{T};\rho;z;\mathbb{T}$
$=\mathtt{sup}_{n\geq0}\,(\mathbb{S}^{\mathsf{T}n}\otimes\mathbb{I});\rho;z;\mathbb{T}=\mathtt{sup}_{n\geq0}\,\big[\pi;\mathbb{S}^{\mathsf{T}n};\mathbb{T}\cap\rho\big];z;\mathbb{T}\subseteq\rho;z;\mathbb{T}$
with equality because $\mathtt{Euclid}^\mathsf{T};\rho;z;\mathbb{T}\supseteq\mathtt{Euclid}_0^\mathsf{T};\rho;z;\mathbb{T}=\rho;z;\mathbb{T}$

---

i.c) We start proving
$E_0:=\mathtt{Euclid}_0\cap\overline{(z;\mathbb{T}\circledrightarrow z;\mathbb{T})}$

$= \Big\{ (z;\mathbb{T} \oslash z;\mathbb{T});\mathbb{T};z^\intercal;\rho^\intercal \cup \big[(\mathbb{T};z^\intercal \oslash \pi) \cap (\mathbb{S}^+ \oslash \mathbb{I});\mathbb{T} \big] \Big\} \cap \overline{(z;\mathbb{T} \oslash z;\mathbb{T})}$
$= (\mathbb{T};z^\intercal \oslash \pi) \cap (\mathbb{S}^+ \oslash \mathbb{I});\mathbb{T} \cap \overline{(z;\mathbb{T} \oslash z;\mathbb{T})}$ since $(z;\mathbb{T} \oslash z;\mathbb{T});\mathbb{T};z^\intercal;\rho^\intercal \subseteq (z;\mathbb{T} \oslash z;\mathbb{T})$
$= (\mathbb{T};z^\intercal \oslash \pi) \cap (\mathbb{S}^+ \oslash \mathbb{I});\mathbb{T}$ (**)

The latter holds because the Schröder rule may be applied:

$(\mathbb{S}^{\intercal+} \oslash \mathbb{I});(z;\mathbb{T} \oslash z;\mathbb{T}) = \mathbb{S}^{\intercal+};z;\mathbb{T} \cap z;\mathbb{T} \subseteq \mathbb{\perp}$

Therefore

$E_0^\intercal;E_0 \subseteq (\mathbb{T};z^\intercal \oslash \pi)^\intercal;(\mathbb{T};z^\intercal \oslash \pi) = (z;\mathbb{T} \oslash \pi^\intercal);(\mathbb{T};z^\intercal \oslash \pi)$
$\subseteq (z;\mathbb{T};z^\intercal \otimes \pi^\intercal;\pi) \subseteq \mathbb{I}$ due to Prop. 2.1.1.ii since z is injective and π univalent

---

$\big[\overline{(z;\mathbb{T} \oslash z;\mathbb{T})} \cap \mathtt{Euclid}\big]^\intercal;\big[\overline{(z;\mathbb{T} \oslash z;\mathbb{T})} \cap \mathtt{Euclid}\big]$
$= \big[\overline{(z;\mathbb{T} \oslash z;\mathbb{T})} \cap \mathtt{sup}_{i\geq 0}\,(\mathfrak{S} \oslash \rho)^i;\mathtt{Euclid}_0;(\mathbb{S}^i \otimes \mathbb{I})\big]^\intercal;$
$\big[\overline{(z;\mathbb{T} \oslash z;\mathbb{T})} \cap \mathtt{sup}_{n\geq 0}\,(\mathfrak{S} \oslash \rho)^n;\mathtt{Euclid}_0;(\mathbb{S}^n \otimes \mathbb{I})\big]$
$= \mathtt{sup}_{i\geq 0}\,\mathtt{sup}_{n\geq 0}\big[\overline{(\mathbb{T};z^\intercal \oslash \mathbb{T};z^\intercal)} \cap (\mathbb{S}^{\intercal i} \otimes \mathbb{I});\mathtt{Euclid}_0^\intercal;(\mathfrak{S}^\intercal \oslash \rho^\intercal)^i\big];$
$\big[\overline{(z;\mathbb{T} \oslash z;\mathbb{T})} \cap (\mathfrak{S} \oslash \rho)^n;\mathtt{Euclid}_0;(\mathbb{S}^n \otimes \mathbb{I})\big]$

For the product $\mathtt{Euclid}^\intercal;\mathtt{Euclid}$, we may focus on equal indices. If exponents disagree, we will namely find

$E_0^\intercal;(\mathfrak{S} \oslash \rho) \subseteq E_0^\intercal;\mathfrak{S};\pi^\intercal \subseteq \mathbb{T};(\mathbb{S}^{\intercal+} \oslash \mathbb{I});\mathfrak{S};\pi^\intercal = \mathbb{\perp}$ Prop. 6.3.1.i.d using (**) above

So the constituent products are

$E_{n+1}^\intercal;E_{n+1} = \big[(\mathfrak{S} \oslash \rho)^n;E_0;(\mathbb{S}^n \otimes \mathbb{I})\big]^\intercal;(\mathfrak{S} \oslash \rho)^n;E_0;(\mathbb{S}^n \otimes \mathbb{I})$
$= (\mathbb{S}^{\intercal n} \otimes \mathbb{I});E_0^\intercal;(\mathfrak{S} \oslash \rho)^{\intercal n};(\mathfrak{S} \oslash \rho)^n;E_0;(\mathbb{S}^n \otimes \mathbb{I})$
$\subseteq (\mathbb{S}^{\intercal n} \otimes \mathbb{I});E_0^\intercal;E_0;(\mathbb{S}^n \otimes \mathbb{I})$ since $(\mathfrak{S} \oslash \rho)$ is univalent
$\subseteq (\mathbb{S}^{\intercal n} \otimes \mathbb{I});(\mathbb{S}^n \otimes \mathbb{I})$ see before
$\subseteq \mathbb{I}$ since $\mathbb{S}$ is univalent

---

ii.a) $\mathtt{Euclid};(e;\mathbb{T} \oslash z;\mathbb{T}) = \mathtt{sup}_{n\geq 0}\,(\mathfrak{S} \oslash \rho)^n;\mathtt{Euclid}_0;(\mathbb{S}^n \otimes \mathbb{I});(e;\mathbb{T} \oslash z;\mathbb{T})$
$= (\mathfrak{S} \oslash \rho)^0;\mathtt{Euclid}_0;(\mathbb{S}^0 \otimes \mathbb{I});(e;\mathbb{T} \oslash z;\mathbb{T})$
$\cup\, (\mathfrak{S} \oslash \rho)^1;\mathtt{Euclid}_0;(\mathbb{S}^1 \otimes \mathbb{I});(e;\mathbb{T} \oslash z;\mathbb{T})$ since $\mathbb{S}^n;e;\mathbb{T} = \mathbb{\perp}$ for $n > 1$
$= \mathtt{Euclid}_0;(e;\mathbb{T} \oslash z;\mathbb{T}) \cup (\mathfrak{S} \oslash \rho);\mathtt{Euclid}_0;(\mathbb{S};e;\mathbb{T} \oslash z;\mathbb{T})$

We split the proof in two parts:

$\mathtt{Euclid}_0;(e;\mathbb{T} \oslash z;\mathbb{T}) = \Big\{(z;\mathbb{T} \oslash z;\mathbb{T});\mathbb{T};z^\intercal;\rho^\intercal \cup \big[(\mathbb{T};z^\intercal \oslash \pi) \cap (\mathbb{S}^+ \oslash \mathbb{I});\mathbb{T}\big]\Big\};(e;\mathbb{T} \oslash z;\mathbb{T})$
$= (z;\mathbb{T} \oslash z;\mathbb{T});\mathbb{T};z^\intercal;\rho^\intercal;(e;\mathbb{T} \oslash z;\mathbb{T}) \cup \big[(\mathbb{T};z^\intercal \oslash \pi) \cap (\mathbb{S}^+ \oslash \mathbb{I});\mathbb{T}\big];(e;\mathbb{T} \oslash z;\mathbb{T})$
$\subseteq (z;\mathbb{T} \oslash z;\mathbb{T});\mathbb{T};z^\intercal;(\mathbb{T};e;\mathbb{T} \cap z;\mathbb{T}) \cup (\mathbb{T};z^\intercal \oslash \pi);(e;\mathbb{T} \oslash z;\mathbb{T})$ destroy and append
$= (z;\mathbb{T} \oslash z;\mathbb{T});\mathbb{T};z^\intercal;z;\mathbb{T} \cup (\mathbb{T};z^\intercal;e;\mathbb{T} \cap \pi;z;\mathbb{T})$
$= (z;\mathbb{T} \oslash z;\mathbb{T});\mathbb{T} \cup \mathbb{\perp} = (z;\mathbb{T} \oslash z;\mathbb{T})$

$(\mathfrak{S} \oslash \rho);\mathtt{Euclid}_0;(\mathbb{S};e;\mathbb{T} \oslash z;\mathbb{T})$
$= (\mathfrak{S} \oslash \rho);\big\{(z;\mathbb{T} \oslash z;\mathbb{T});\mathbb{T};z^\intercal;\rho^\intercal \cup \big[(\mathbb{T};z^\intercal \oslash \pi) \cap (\mathbb{S}^+ \oslash \mathbb{I});\mathbb{T}\big]\big\};(z;\mathbb{T} \oslash z;\mathbb{T})$
$= (\mathfrak{S} \oslash \rho);\big\{(z;\mathbb{T} \oslash z;\mathbb{T});\mathbb{T};z^\intercal;\rho^\intercal;(z;\mathbb{T} \oslash z;\mathbb{T}) \cup \big[(\mathbb{T};z^\intercal \oslash \pi) \cap (\mathbb{S}^+ \oslash \mathbb{I});\mathbb{T}\big];(z;\mathbb{T} \oslash z;\mathbb{T})\big\}$
$= (\mathfrak{S} \oslash \rho);\big\{(z;\mathbb{T} \oslash z;\mathbb{T});\mathbb{T};z^\intercal;z;\mathbb{T} \cup \big[(\mathbb{T};z^\intercal \oslash \pi);(z;\mathbb{T} \oslash z;\mathbb{T}) \cap (\mathbb{S}^+ \oslash \mathbb{I});\mathbb{T}\big]\big\}$
$= (\mathfrak{S} \oslash \rho);\big\{(z;\mathbb{T} \oslash z;\mathbb{T}) \cup \big[\mathbb{T};z^\intercal;z;\mathbb{T} \cap \pi;z;\mathbb{T} \cap (\mathbb{S}^+ \oslash \mathbb{I});\mathbb{T}\big]\big\}$
$= (\mathfrak{S} \oslash \rho);\big\{(z;\mathbb{T} \oslash z;\mathbb{T}) \cup \big[\pi;z;\mathbb{T} \cap (\mathbb{S}^+ \oslash \mathbb{I});\mathbb{T}\big]\big\}$
$= (\mathfrak{S} \oslash \rho);\big\{(z;\mathbb{T} \oslash z;\mathbb{T}) \cup (z;\mathbb{T} \cap \mathbb{S}^+ \oslash \mathbb{I});\mathbb{T}\big\}$
$= (\mathfrak{S} \oslash \rho);\big\{(z;\mathbb{T} \oslash z;\mathbb{T}) \cup (z;\mathbb{T};\overline{\mathbb{T};z^\intercal} \oslash \mathbb{I});\mathbb{T}\big\}$ since $z;\mathbb{T} \cap \mathbb{S}^+ = z;\mathbb{T};\overline{\mathbb{T};z^\intercal}$
using $\overline{z;\mathbb{T}} = \mathbb{S}^{\intercal+};z;\mathbb{T}$ from Prop. 5.2.1.ii.a

$= (z;\mathbb{T} \circledcirc z;\mathbb{T}) \cup (\mathfrak{S} \circledcirc \rho);(z;\mathbb{T};\overline{\mathbb{T};z^\intercal} \circledcirc \mathbb{I});\mathbb{T}$ Prop. 6.3.2.i
$= (z;\mathbb{T} \circledcirc z;\mathbb{T}) \cup (\mathfrak{S};z;\mathbb{T};\overline{\mathbb{T};z^\intercal} \cap \rho);\mathbb{T}$
$= (z;\mathbb{T} \circledcirc z;\mathbb{T}) \cup ((\mathbb{I} \circledcirc \mathbb{I});\mathbb{T};\overline{\mathbb{T};z^\intercal} \cap \rho);\mathbb{T}$ Prop. 6.3.1.ii.a
$= (z;\mathbb{T} \circledcirc z;\mathbb{T}) \cup ((\mathbb{I} \circledcirc \mathbb{I});\mathbb{T} \cap \overline{\mathbb{T};z^\intercal} \cap \rho);\mathbb{T}$
$= (z;\mathbb{T} \circledcirc z;\mathbb{T}) \cup ((\mathbb{I} \circledcirc \mathbb{I});\mathbb{T} \cap \rho);\left[\mathbb{T} \cap \overline{z;\mathbb{T}}\right]$ mask shifting
$= (z;\mathbb{T} \circledcirc z;\mathbb{T}) \cup ((\mathbb{I} \circledcirc \mathbb{I});\mathbb{T} \cap \rho);\overline{z;\mathbb{T}}$
$= (z;\mathbb{T} \circledcirc z;\mathbb{T}) \cup ((\mathbb{I} \circledcirc \mathbb{I});\mathbb{T} \cap \rho;\overline{z;\mathbb{T}}) = (\mathbb{I} \circledcirc \mathbb{I});\mathbb{T}$

While "$\subseteq$" is obvious, we add as proof of "$\supseteq$" this Dedekind estimation:

$(\mathbb{I} \circledcirc \mathbb{I});\mathbb{T} \cap \rho;z;\mathbb{T} \subseteq ((\mathbb{I} \circledcirc \mathbb{I}) \cap \rho;z;\mathbb{T};\mathbb{T}^\intercal);(\mathbb{T} \cap (\mathbb{I} \circledcirc \mathbb{I})^\intercal;\rho;z;\mathbb{T})$
$= ((\mathbb{I} \circledcirc \mathbb{I}) \cap \rho;z;\mathbb{T});(\mathbb{I} \circledcirc \mathbb{I});\rho;z;\mathbb{T} = ((\mathbb{I} \circledcirc \mathbb{I}) \cap \rho;z;\mathbb{T});\mathbb{I};z;\mathbb{T}$
$\subseteq (\mathbb{I} \circledcirc \mathbb{I});z;\mathbb{T} = (z;\mathbb{T} \circledcirc z;\mathbb{T})$

---

ii.b) The question concerns

$\mathtt{Euclid}_0;\pi = \left\{ (z;\mathbb{T} \circledcirc z;\mathbb{T});\mathbb{T};z^\intercal;\rho^\intercal \cup \left[(\mathbb{T};z^\intercal \circledcirc \pi) \cap (\mathbb{S}^+ \circledcirc \mathbb{I});\mathbb{T}\right] \right\};\pi$
$= (z;\mathbb{T} \circledcirc z;\mathbb{T});\mathbb{T};z^\intercal;\rho^\intercal;\pi \cup \left[(\mathbb{T};z^\intercal \circledcirc \pi) \cap (\mathbb{S}^+ \circledcirc \mathbb{I});\mathbb{T}\right];\pi$
$= (z;\mathbb{T} \circledcirc z;\mathbb{T}) \cup \left[(\mathbb{T};z^\intercal \circledcirc \pi);\pi \cap (\mathbb{S}^+ \circledcirc \mathbb{I});\mathbb{T}\right]$ masking
$= (z;\mathbb{T} \circledcirc z;\mathbb{T}) \cup [(\mathbb{T};z^\intercal \cap \mathbb{T}) \cap (\mathbb{S}^+ \circledcirc \mathbb{I});\mathbb{T}]$ destroy and append
$= (z;\mathbb{T} \circledcirc z;\mathbb{T}) \cup [\mathbb{T};z^\intercal \cap (\mathbb{S}^+ \circledcirc \mathbb{I});\mathbb{T}]$
$= (z;\mathbb{T} \circledcirc z;\mathbb{T}) \cup (\mathbb{S}^+ \circledcirc \mathbb{I});\mathbb{T};z^\intercal$ masking

---

This provides us with the induction start $n = 0$ for the claim intended:

$\mathtt{Euclid}_0;\pi = (z;\mathbb{T} \circledcirc z;\mathbb{T});\mathbb{S}^0 \cup (\mathfrak{S} \circledcirc \rho)^0;(\mathbb{S}^+ \circledcirc \mathbb{I});\mathbb{T};z^\intercal;\mathbb{S}^0$

Inductive proof

$\mathtt{Euclid}_{n+1};\pi = (\mathfrak{S} \circledcirc \rho);\mathtt{Euclid}_n;(\mathbb{S} \otimes \mathbb{I});\pi$
$= (\mathfrak{S} \circledcirc \rho);\mathtt{Euclid}_n;\pi;\mathbb{S}$ destroy and append, ρ total
$= (\mathfrak{S} \circledcirc \rho);\left[(z;\mathbb{T} \circledcirc z;\mathbb{T});\mathbb{S}^n \cup (\mathfrak{S} \circledcirc \rho)^n;(\mathbb{S}^+ \circledcirc \mathbb{I});\mathbb{T};z^\intercal;\mathbb{S}^n\right];\mathbb{S}$ hypothesis
$= (\mathfrak{S} \circledcirc \rho);(z;\mathbb{T} \circledcirc z;\mathbb{T});\mathbb{S}^{n+1} \cup (\mathfrak{S} \circledcirc \rho)^{n+1};(\mathbb{S}^+ \circledcirc \mathbb{I});\mathbb{T};z^\intercal;\mathbb{S}^{n+1}$
$= (z;\mathbb{T} \circledcirc z;\mathbb{T});\mathbb{S}^{n+1} \cup (\mathfrak{S} \circledcirc \rho)^{n+1};(\mathbb{S}^+ \circledcirc \mathbb{I});\mathbb{T};z^\intercal;\mathbb{S}^{n+1}$ Prop. 6.3.2.i

This conforms with the former result Prop. 7.2.2.ii.a.

The formula concerning $\mathtt{Euclid};\pi$ is then simply obtained by summation thereby observing that $\mathbb{T} = \mathbb{T};\mathbb{S}^0 \supseteq \mathbb{T};\mathbb{S}^1 \supseteq \mathbb{T};\mathbb{S}^2 \supseteq \ldots$

Before we treat $\mathtt{Euclid}_n;\rho$, it is helpful to state that the first term results always in the same. This is true for $n = 0$

$\mathtt{Euclid}_0;\rho = \left\{(z;\mathbb{T} \circledcirc z;\mathbb{T});\mathbb{T};z^\intercal;\rho^\intercal \cup \left[(\mathbb{T};z^\intercal \circledcirc \pi) \cap (\mathbb{S}^+ \circledcirc \mathbb{I});\mathbb{T}\right]\right\};\rho$
$= (z;\mathbb{T} \circledcirc z;\mathbb{T});\mathbb{T};z^\intercal;\rho^\intercal;\rho \cup \left[(\mathbb{T};z^\intercal \circledcirc \pi) \cap (\mathbb{S}^+ \circledcirc \mathbb{I});\mathbb{T}\right];\rho$
$= (z;\mathbb{T} \circledcirc z;\mathbb{T});\mathbb{T};z^\intercal \cup \left[(\mathbb{T};z^\intercal \circledcirc \pi);\rho \cap (\mathbb{S}^+ \circledcirc \mathbb{I});\mathbb{T}\right]$ masking; ρ is a projection
$= (z;\mathbb{T} \circledcirc z;\mathbb{T});\mathbb{T};z^\intercal \cup \left[\pi \cap (\mathbb{S}^+ \circledcirc \mathbb{I});\mathbb{T}\right] = (z;\mathbb{T} \circledcirc z;\mathbb{T});\mathbb{T};z^\intercal \cup \ldots$

and may then be proved by induction:

$\mathtt{Euclid}_{n+1};\rho = (\mathfrak{S} \circledcirc \rho)^{n+1};\mathtt{Euclid}_0;(\mathbb{S} \otimes \mathbb{I})^{n+1};\rho$
$= (\mathfrak{S} \circledcirc \rho);\left\{(z;\mathbb{T} \circledcirc z;\mathbb{T});\mathbb{T};z^\intercal;\rho^\intercal \cup \ldots\right\};(\mathbb{S} \otimes \mathbb{I});\rho$
$= (\mathfrak{S} \circledcirc \rho);(z;\mathbb{T} \circledcirc z;\mathbb{T});\mathbb{T};z^\intercal;\rho^\intercal;(\mathbb{S};\mathbb{T} \circledcirc \mathbb{I}) \cup \ldots$
$= (z;\mathbb{T} \circledcirc z;\mathbb{T});\mathbb{T};z^\intercal;(\mathbb{T};\mathbb{S};\mathbb{T} \cap \mathbb{I}) \cup \ldots$ Prop. 6.3.2.i; destroy and append
$= (z;\mathbb{T} \circledcirc z;\mathbb{T});\mathbb{T};z^\intercal \cup \ldots$

We therefore focus mainly on the second term starting with $\pi \cap (\mathbb{S}^+ \circledcirc \mathbb{I}) ; \mathbb{T}$ as already obtained in case $\mathtt{Euclid}_0 ; \rho$. For $\mathtt{Euclid}_1 ; \rho$ we get as second term

$$
\begin{array}{l}
(\mathfrak{S} \circledcirc \rho) ; \mathtt{Euclid}_0 ; (\mathbb{S} \otimes \mathbb{I}) ; \rho \\
= (\mathfrak{S} \circledcirc \rho) ; \big[(\mathbb{T} ; z^\intercal \circledcirc \pi) \cap (\mathbb{S}^+ \circledcirc \mathbb{I}) ; \mathbb{T} \big] ; (\mathbb{S} ; \mathbb{T} \circledcirc \mathbb{I}) \\
= (\mathfrak{S} \circledcirc \rho) ; \big[(\mathbb{T} ; z^\intercal \circledcirc \pi) ; (\mathbb{S} ; \mathbb{T} \circledcirc \mathbb{I}) \cap (\mathbb{S}^+ \circledcirc \mathbb{I}) ; \mathbb{T} \big] \quad \text{masking} \\
= (\mathfrak{S} \circledcirc \rho) ; \big[\mathbb{T} ; z^\intercal ; \mathbb{S} ; \mathbb{T} \cap \pi \cap (\mathbb{S}^+ \circledcirc \mathbb{I}) ; \mathbb{T} \big] = (\mathfrak{S} \circledcirc \rho) ; \big[\pi \cap (\mathbb{S}^+ \circledcirc \mathbb{I}) ; \mathbb{T} \big] \\
= (\mathfrak{S} \circledcirc \rho) ; \pi \cap (\mathfrak{S} \circledcirc \rho) ; (\mathbb{S}^+ \circledcirc \mathbb{I}) ; \mathbb{T} \\
= \mathfrak{S} \cap (\mathfrak{S} ; \mathbb{S}^+ \cap \rho) ; \mathbb{T}
\end{array}
$$

The conjecture is then $\mathtt{Euclid}_n ; \rho = (\mathfrak{S} \circledcirc \rho)^{n-1} ; \big[\mathfrak{S} \cap (\mathfrak{S} ; \mathbb{S}^+ \cap \rho) ; \mathbb{T} \big]$ for $n \geq 1$, which we prove without recursion:

$$
\begin{array}{l}
\text{2nd term of } \mathtt{Euclid}_n ; \rho \\
= \text{2nd term of } (\mathfrak{S} \circledcirc \rho)^n ; \mathtt{Euclid}_0 ; (\mathbb{S} \otimes \mathbb{I})^n ; \rho \\
= (\mathfrak{S} \circledcirc \rho)^n ; \big[(\mathbb{T} ; z^\intercal \circledcirc \pi) \cap (\mathbb{S}^+ \circledcirc \mathbb{I}) ; \mathbb{T} \big] ; (\mathbb{S} \otimes \mathbb{I})^n ; \rho \\
= (\mathfrak{S} \circledcirc \rho)^n ; \big[(\mathbb{T} ; z^\intercal \circledcirc \pi) ; (\mathbb{S} \otimes \mathbb{I})^n ; \rho \cap (\mathbb{S}^+ \circledcirc \mathbb{I}) ; \mathbb{T} \big] \quad \text{masking} \\
= (\mathfrak{S} \circledcirc \rho)^n ; \big[(\mathbb{T} ; z^\intercal \circledcirc \pi) ; (\mathbb{S} \otimes \mathbb{I}) ; (\mathbb{S}^{n-1} \otimes \mathbb{I}) ; \rho \cap (\mathbb{S}^+ \circledcirc \mathbb{I}) ; \mathbb{T} \big] \\
= (\mathfrak{S} \circledcirc \rho)^n ; \big[(\mathbb{T} ; z^\intercal ; \mathbb{S} \circledcirc \pi) ; (\mathbb{S}^{n-1} ; \mathbb{T} \circledcirc \mathbb{I}) \cap (\mathbb{S}^+ \circledcirc \mathbb{I}) ; \mathbb{T} \big] \\
= (\mathfrak{S} \circledcirc \rho)^n ; \big[\mathbb{T} ; z^\intercal ; \mathbb{S}^n ; \mathbb{T} \cap \pi \cap (\mathbb{S}^+ \circledcirc \mathbb{I}) ; \mathbb{T} \big] \\
= (\mathfrak{S} \circledcirc \rho)^n ; \big[\pi \cap (\mathbb{S}^+ \circledcirc \mathbb{I}) ; \mathbb{T} \big] \\
= (\mathfrak{S} \circledcirc \rho)^{n-1} ; \big[(\mathfrak{S} \circledcirc \rho) ; \pi \cap (\mathfrak{S} \circledcirc \rho) ; (\mathbb{S}^+ \circledcirc \mathbb{I}) ; \mathbb{T} \big] \\
= (\mathfrak{S} \circledcirc \rho)^{n-1} ; \big[\mathfrak{S} \cap (\mathfrak{S} ; \mathbb{S}^+ \cap \rho) ; \mathbb{T} \big]
\end{array}
$$

In total therefore:

$$
\begin{array}{l}
\mathtt{Euclid} ; \rho = (z ; \mathbb{T} \circledcirc z ; \mathbb{T}) ; \mathbb{T} ; z^\intercal \cup \big[\pi \cap (\mathbb{S}^+ \circledcirc \mathbb{I}) ; \mathbb{T} \big] \\
\qquad\qquad \cup \, \mathtt{sup}_{n \geq 1} \, (\mathfrak{S} \circledcirc \rho)^{n-1} ; \big[\mathfrak{S} \cap (\mathfrak{S} ; \mathbb{S}^+ \cap \rho) ; \mathbb{T} \big] \\
= (z ; \mathbb{T} \circledcirc z ; \mathbb{T}) ; \mathbb{T} ; z^\intercal \cup \big[\pi \cap (\mathbb{S}^+ \circledcirc \mathbb{I}) ; \mathbb{T} \big] \cup (\mathfrak{S} \circledcirc \rho)^* ; \big[\mathfrak{S} \cap (\mathfrak{S} ; \mathbb{S}^+ \cap \rho) ; \mathbb{T} \big]
\end{array}
$$

This conforms with the former result Prop. 7.2.2.ii.b.

Since $\mathtt{Euclid}$ is "more or less" univalent, the main part of the proof follows with

$$
\begin{array}{l}
(\mathtt{quot} \circledcirc \mathtt{rem}) = (\mathtt{Euclid} ; \pi \circledcirc \mathtt{Euclid} ; \rho) = \mathtt{Euclid} ; \pi ; \pi^\intercal \cap \mathtt{Euclid} ; \rho ; \rho^\intercal \\
= \mathtt{Euclid} ; (\pi ; \pi^\intercal \cap \rho ; \rho^\intercal) = \mathtt{Euclid} ; \mathbb{I} = \mathtt{Euclid}
\end{array}
$$

Only the first row of $\mathtt{Euclid}$ needs a separate treatment:

$$
\begin{array}{l}
\big((z ; \mathbb{T} \circledcirc z ; \mathbb{T}) ; \mathbb{T} ; z^\intercal ; \rho^\intercal ; \pi \circledcirc (z ; \mathbb{T} \circledcirc z ; \mathbb{T}) ; \mathbb{T} ; z^\intercal ; \rho^\intercal ; \rho \big) \\
= \big((z ; \mathbb{T} \circledcirc z ; \mathbb{T}) \circledcirc (z ; \mathbb{T} \circledcirc z ; \mathbb{T}) ; \mathbb{T} ; z^\intercal \big) \\
= (z ; \mathbb{T} \circledcirc z ; \mathbb{T}) ; \pi^\intercal \cap (z ; \mathbb{T} \circledcirc z ; \mathbb{T}) ; \mathbb{T} ; z^\intercal ; \rho^\intercal = (z ; \mathbb{T} \circledcirc z ; \mathbb{T}) \cap (z ; \mathbb{T} \circledcirc z ; \mathbb{T}) ; \mathbb{T} ; z^\intercal ; \rho^\intercal \\
= (z ; \mathbb{T} \circledcirc z ; \mathbb{T}) ; \mathbb{T} ; z^\intercal ; \rho^\intercal \subseteq \mathtt{Euclid}_0 \subseteq \mathtt{Euclid}
\end{array}
$$

$$
\begin{array}{l}
\mathtt{Euclid} ; \mathbb{T} = \mathtt{Euclid} ; \pi ; \mathbb{T} \quad \text{see before} \\
= \Big[(z ; \mathbb{T} \circledcirc z ; \mathbb{T}) \cup \mathtt{sup}_{n \geq 0} \, (\mathfrak{S} \circledcirc \rho)^n ; (\mathbb{S}^+ \circledcirc \mathbb{I}) ; \mathbb{T} ; z^\intercal ; \mathbb{S}^n \Big] ; \mathbb{T} \\
= (z ; \mathbb{T} \circledcirc z ; \mathbb{T}) \cup \mathtt{sup}_{n \geq 0} \, (\mathfrak{S} \circledcirc \rho)^n ; (\mathbb{S}^+ \circledcirc \mathbb{I}) ; \mathbb{T} ; z^\intercal ; \mathbb{S}^n ; \mathbb{T} \\
= (z ; \mathbb{T} \circledcirc z ; \mathbb{T}) \cup \mathtt{sup}_{n \geq 0} \, (\mathfrak{S} \circledcirc \rho)^n ; (\mathbb{S}^+ \circledcirc \mathbb{I}) ; \mathbb{T} \quad \text{or} = \mathbb{\perp} \text{ when beyond limit} \\
= (z ; \mathbb{T} \circledcirc z ; \mathbb{T}) \cup \big[\mathtt{sup}_{n \geq 0} \, (\mathfrak{S} \circledcirc \rho)^n \big] ; (\mathbb{S}^+ \circledcirc \mathbb{I}) ; \mathbb{T} \\
= (z ; \mathbb{T} \circledcirc z ; \mathbb{T}) \cup (\mathfrak{S} \circledcirc \rho)^* ; (\mathbb{S}^+ \circledcirc \mathbb{I}) ; \mathbb{T} \\
= (z ; \mathbb{T} \circledcirc z ; \mathbb{T}) \cup (\rho \cap \mathbb{T} ; \mathbb{S}) ; \mathbb{T} \quad \text{Prop. 6.3.2.vi} \\
= (z ; \mathbb{T} \circledcirc z ; \mathbb{T}) \cup \rho ; (\mathbb{T} \cap \mathbb{S}^\intercal ; \mathbb{T}) \quad \text{mask shifting} \\
= (z ; \mathbb{T} \circledcirc z ; \mathbb{T}) \cup \rho ; \overline{z ; \mathbb{T}}
\end{array}
$$

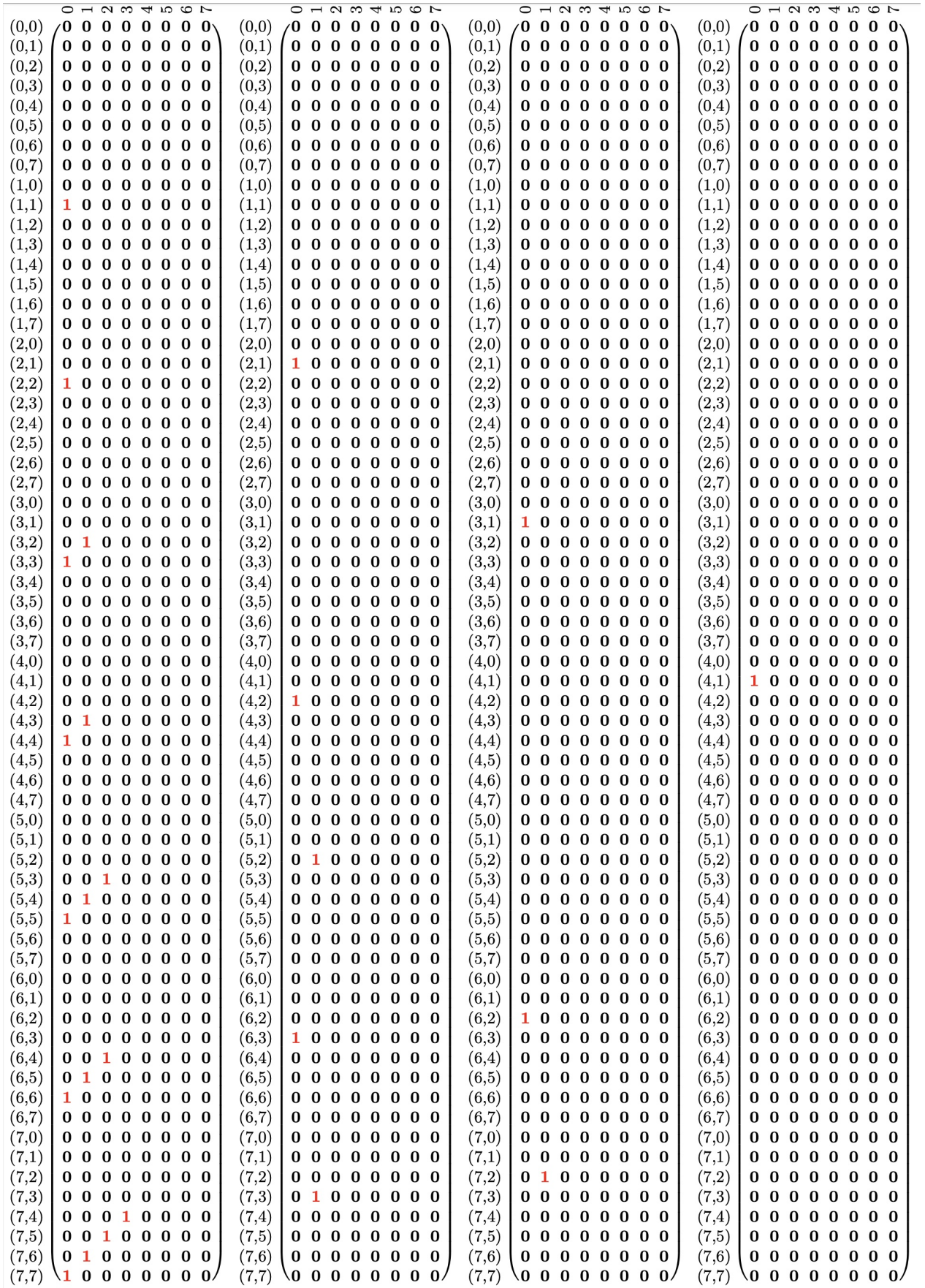


Fig. 7.3.5 $(\mathfrak{S} \circledcirc \rho)^{n;}\big[\mathfrak{S} \cap (\mathfrak{S};\mathbb{S}^{+} \cap \rho);\mathbb{T}\big]$ for $n = 0, 1, 2, 3$

$$\pi \cap (\mathbb{S}^{+} \oslash \mathbb{I});\mathbb{T} = \begin{array}{c} \\ (0,0)\\(0,1)\\(0,2)\\(0,3)\\(0,4)\\(1,0)\\(1,1)\\(1,2)\\(1,3)\\(1,4)\\(2,0)\\(2,1)\\(2,2)\\(2,3)\\(2,4)\\(3,0)\\(3,1)\\(3,2)\\(3,3)\\(3,4)\\(4,0)\\(4,1)\\(4,2)\\(4,3)\\(4,4) \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0&0&0&0&0\\ 1&0&0&0&0\\ 1&0&0&0&0\\ 1&0&0&0&0\\ 1&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&1&0&0&0\\ 0&1&0&0&0\\ 0&1&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&1&0&0\\ 0&0&1&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&1&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix} \end{array} \quad \begin{array}{c} \\ (0,0)\\(0,1)\\(0,2)\\(0,3)\\(0,4)\\(1,0)\\(1,1)\\(1,2)\\(1,3)\\(1,4)\\(2,0)\\(2,1)\\(2,2)\\(2,3)\\(2,4)\\(3,0)\\(3,1)\\(3,2)\\(3,3)\\(3,4)\\(4,0)\\(4,1)\\(4,2)\\(4,3)\\(4,4) \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix} \end{array} = (z;\mathbb{T} \oslash z;\mathbb{T});\mathbb{T};z^{\intercal}$$

With $\mathfrak{Q}$ in combination with `Euclid`, division is related to division and reduction to the integer part. To prove this means yet another induction:

The induction start

$\mathfrak{Q}_0 \subseteq \texttt{Euclid}_0;\pi$

is proved in two parts, intersected with $(z;\mathbb{T} \oslash z;\mathbb{T})$ as well as with $\overline{(z;\mathbb{T} \oslash z;\mathbb{T})}$. Looking at the first result in (ii.b), the first claim is trivial. As to the second:

$\mathfrak{Q}_0 \cap \overline{(z;\mathbb{T} \oslash z;\mathbb{T})} = (z;\mathbb{T} \oslash \mathbb{T};z^{\intercal}) \cap \overline{(z;\mathbb{T} \oslash z;\mathbb{T})}$
$= \pi;z;\mathbb{T} \cap \mathbb{T};z^{\intercal} \cap (\overline{\pi;z;\mathbb{T}} \cup \overline{\rho;z;\mathbb{T}})$
$= \mathbb{\bot} \cup \left[\pi;z;\mathbb{T} \cap \mathbb{T};z^{\intercal} \cap \rho;\overline{z;\mathbb{T}}\right] = \pi;z;\mathbb{T} \cap \mathbb{T};z^{\intercal} \cap \rho;\mathbb{S}^{\intercal+};z;\mathbb{T}$ Prop. 5.2.1.ii.a
$= (z;\mathbb{T} \oslash \mathbb{S}^{\intercal+};z;\mathbb{T}) \cap \mathbb{T};z^{\intercal}$
$= (\mathbb{I} \oslash \mathbb{S}^{\intercal+});z;\mathbb{T} \cap \mathbb{T};z^{\intercal} \subseteq (\mathbb{I} \oslash \mathbb{S}^{\intercal+});\mathbb{T} \cap \mathbb{T};z^{\intercal}$
$= (\mathbb{S}^{+} \oslash \mathbb{I});\mathbb{T} \cap \mathbb{T};z^{\intercal}$ Prop. 2.1.1.iii
$= (\mathbb{S}^{+} \oslash \mathbb{I});\mathbb{T};z^{\intercal} \subseteq \texttt{Euclid}_0;\pi$ see (ii.b) with masking

Once this is settled, we may proceed as follows

$\mathfrak{Q}_{n+1} = (\mathfrak{S} \oslash \rho);\mathfrak{Q}_n;\mathbb{S}$
$\subseteq (\mathfrak{S} \oslash \rho);\texttt{Euclid}_n;\pi;\mathbb{S}$ hypothesis
$= (\mathfrak{S} \oslash \rho);\texttt{Euclid}_n;(\mathbb{S} \otimes \mathbb{I});\pi$ since $\pi;\mathbb{S} = (\mathbb{S} \otimes \mathbb{I});\pi$
$= \texttt{Euclid}_{n+1};\pi$

---

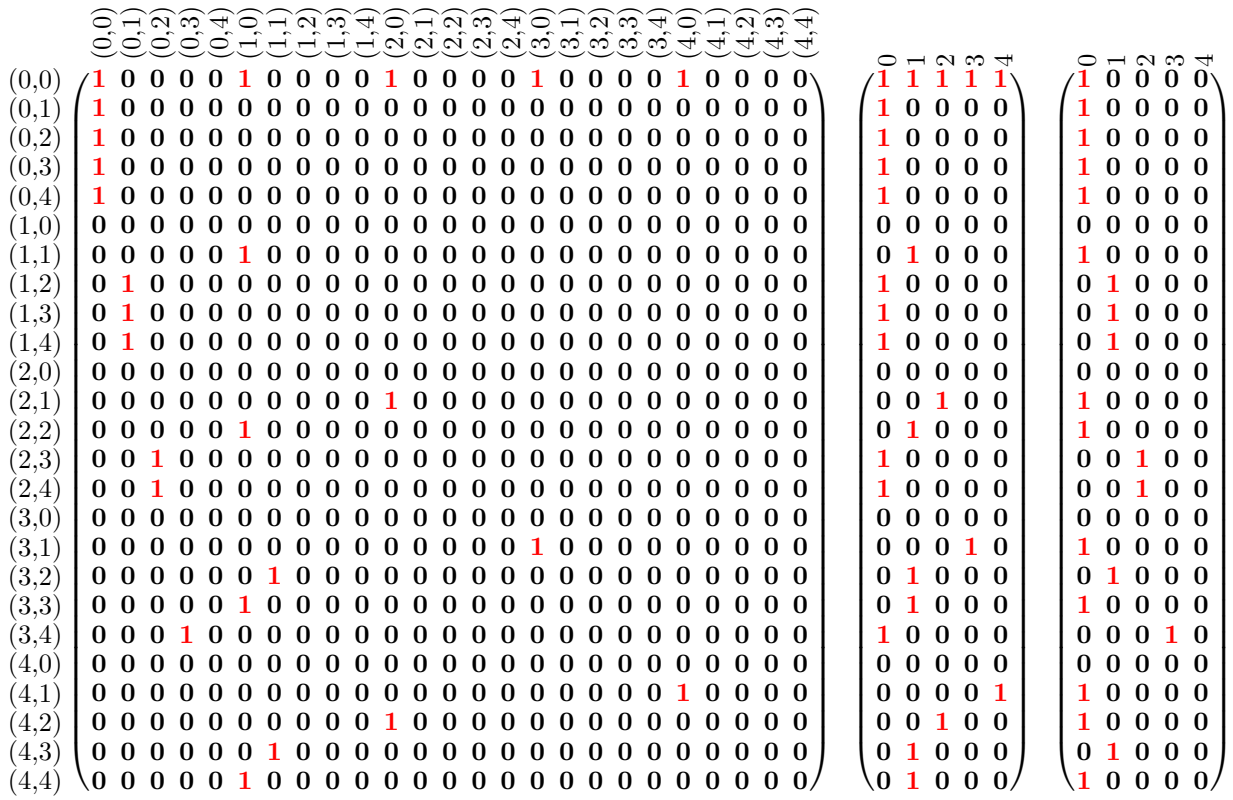
ii.c)

---

iii.a)

---

iv.a) □



Yet another possibility to express "has as divisors" is as follows

$$\mathtt{rel}((\mathtt{Euclid} \cap \mathbb{T} ; z^{\mathsf{T}} ; \rho^{\mathsf{T}}) ; \mathbb{T}) = \mathtt{rel}(\mathtt{Euclid} ; \rho ; z ; \mathbb{T}). \text{ up to } ((0,0),(0,0))!$$

We have therewith cut out just the columns of `Euclid` belonging to a second component 0 and thus divisible without residue. This has then been converted from a vector to the corresponding relation. Fig. 7.3.6 illustrates the idea. In the next we will find out that this is $\mathcal{D}^{\mathsf{T}}$.

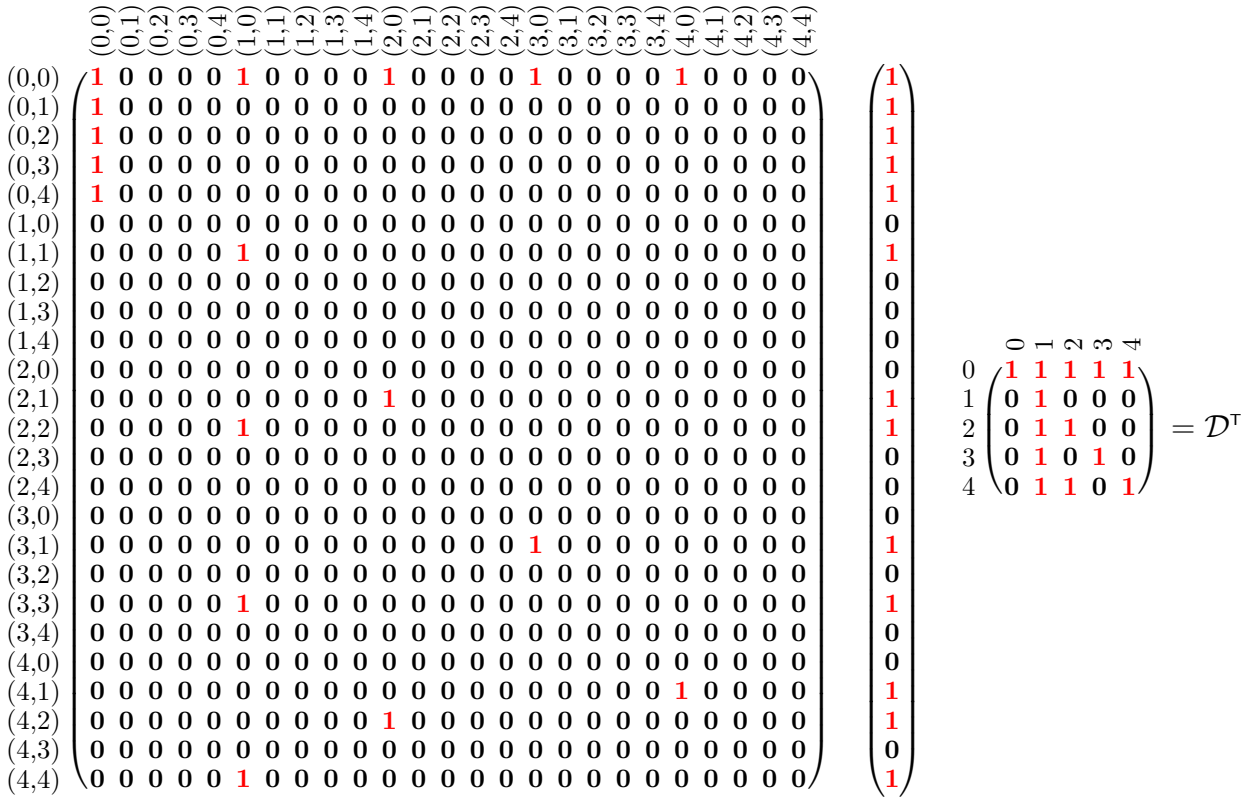


Fig. 7.3.6 "Has as divisors" obtained from `Euclid` on numbers limited by 4

The vector above is immediately identified as $\mathfrak{Q}; \mathbb{T}$; it is then automatically converted to the corresponding relation as in Prop. 2.2.5. This observation leads more or less directly to the concepts of the following section.

7.4 The divides-lattice

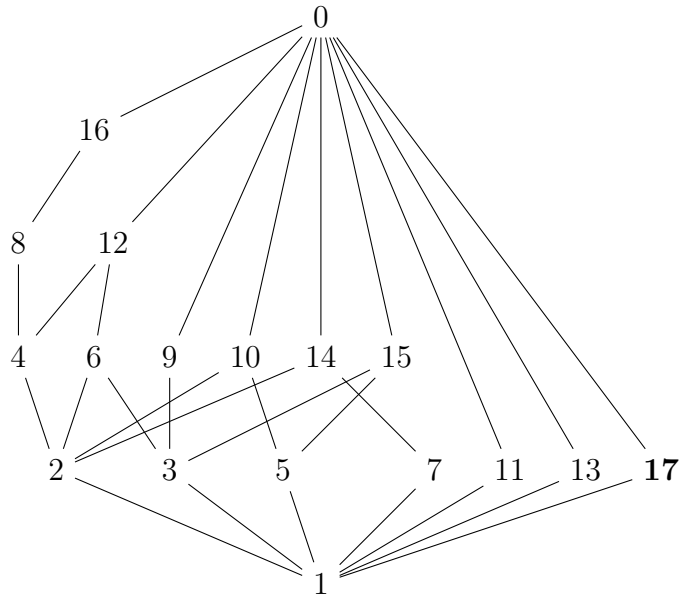
Given the multiplication operation $\mathbb{M} : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$, we have also looked for the division operation $\mathfrak{Q} : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ defined in Def. 7.2.1. Now we focus on the divisibility relation $\mathcal{D} : \mathbb{N} \longrightarrow \mathbb{N}$.

7.4.1 Definition. The relation **divides** is defined as $\mathcal{D} := \pi^{\mathsf{T}}; \mathbb{M}$. Sometimes we will also need the corresponding strict version and the Hasse relation thereof:

$$\mathcal{D}_{\mathrm{s}} := \mathcal{D} \cap \overline{\mathbb{I}} \qquad \mathcal{H} := \mathcal{D}_{\mathrm{s}} \cap \overline{\mathcal{D}_{\mathrm{s}}; \mathcal{D}_{\mathrm{s}}}$$ □

$\mathcal{D} =$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
3	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0
4	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0
5	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0
6	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0
7	1	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0
8	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0
9	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
10	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
11	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
12	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
13	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
14	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
15	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
16	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
17	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1



$\mathcal{D}_s$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	0	0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
3	1	0	0	0	0	0	1	0	0	1	0	0	1	0	0	1	0	0
4	1	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	1	0
5	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0
6	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
7	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
8	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
9	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
17	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$\mathcal{H}$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	1	1	0	1	0	1	0	0	0	1	0	1	0	0	0	1
2	0	0	0	0	1	0	1	0	0	0	1	0	0	0	1	0	0	0
3	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	1	0	0
4	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
9	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
17	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$\mathcal{H}^{\mathsf{T}}e$

0
0
1
0
0
1
0
0
1
0
0
0
0
1
0
0
0
1

e

0
1
0
0
0
0
0
0
0
0
0
0
0
0
0
0
0
0

Fig. 7.4.1 The complete lattice ordering "divides" $\mathcal{D}$ with Hasse relation on numbers $0 \ldots 17$

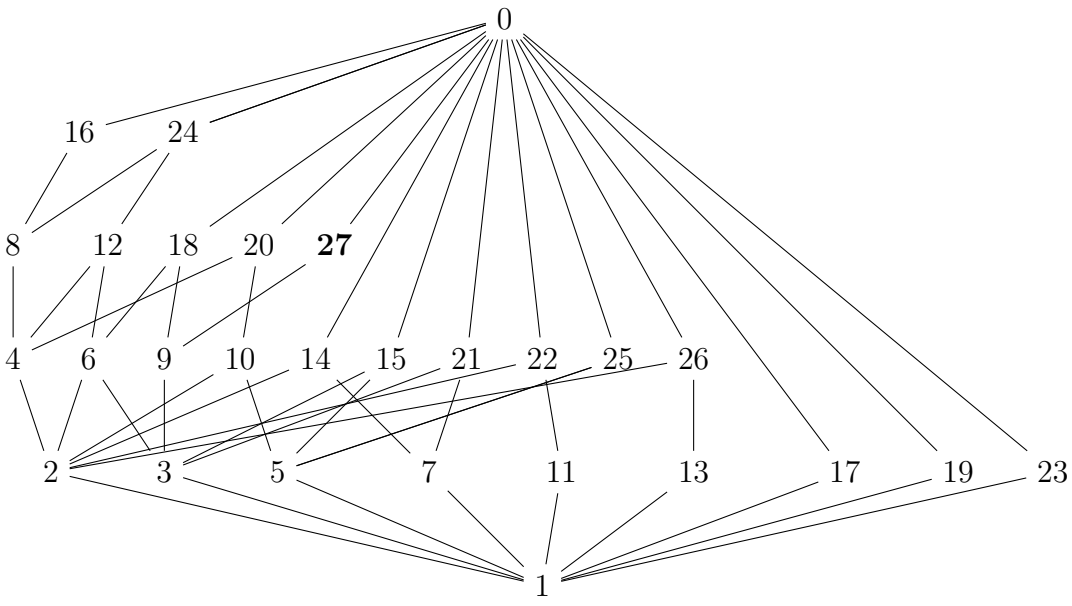


Fig. 7.4.2 The complete lattice $\mathcal{D}$ on numbers $0 \dots 27$

Now we will prove a lot of tiny relational properties. Several slightly more difficult order- and lattice-theoretic ones will come later.

7.4.2 Proposition. Assume $\mathbb{S}$ to be regressively finite. Then

i) $\mathcal{D} = \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\mathsf{T}^n} ; z ; \mathbb{T} \cap \mathbb{T} ; k_n = \mathtt{sup}_{n\geq 0}\, k_n = \mathtt{sup}_{i\geq 0}\, \mathbb{S}^{\mathsf{T}^i} ; z ; \mathbb{T} ; z^{\mathsf{T}} ; (\mathbb{S}^i)^*$

ii) $\mathcal{D} \cap \overline{\mathbb{T} ; z^{\mathsf{T}}} \subseteq \mathbb{S}^* \cap \overline{z ; \mathbb{T}}$

iii) $\mathbb{T} = \mathcal{D} ; z \qquad e ; \mathbb{T} = \mathbb{S}^{\mathsf{T}} ; z ; \mathbb{T} \subseteq \mathcal{D} \qquad \mathcal{D}^{\mathsf{T}} ; e ; \mathbb{T} = \mathbb{T} \qquad \mathcal{D} ; e ; \mathbb{T} = e ; \mathbb{T}$

$$\mathcal{D}^{\mathsf{T}} ; \mathbb{S}^{\mathsf{T}^n} ; z ; \mathbb{T} = \begin{cases} (\mathbb{S}^{\mathsf{T}^n})^* ; z ; \mathbb{T} & \text{if } \mathbb{S}^{\mathsf{T}^n} ; z ; \mathbb{T} \neq \mathbb{\perp}\!\!\!\perp, \text{ i.e. a point} \\ \mathbb{\perp}\!\!\!\perp & \text{if } \mathbb{S}^{\mathsf{T}^n} ; z ; \mathbb{T} = \mathbb{\perp}\!\!\!\perp \end{cases}$$

$\mathcal{D}_{\mathrm{s}} ; z = \overline{z} \qquad z ; \mathbb{T} \subseteq \mathcal{D}_{\mathrm{s}}^{\mathsf{T}} ; \overline{\mathbb{I}} \qquad \mathcal{D}_{\mathrm{s}} ; e ; \mathbb{T} = \mathbb{\perp}\!\!\!\perp \qquad \mathcal{D}_{\mathrm{s}}^{\mathsf{T}} ; e ; \mathbb{T} = \overline{e ; \mathbb{T}} = \mathcal{D}_{\mathrm{s}}^{\mathsf{T}} ; \mathbb{T}$

iv) $\mathcal{D} ; k_n = k_n ; \mathcal{D} \cup k_0$

v) $\mathcal{D} ; \mathbb{S} \cap \mathcal{D} = e ; \mathbb{T} ; e^{\mathsf{T}} ; \mathbb{S}^* = e ; \mathbb{T} ; \overline{\mathbb{T} ; z^{\mathsf{T}}}$

vi) $e ; \mathbb{T} ; e^{\mathsf{T}} = \overline{\mathcal{D}^{\mathsf{T}} ; \overline{\mathbb{I}}}$

vii) $\mathcal{D}_{\mathrm{s}} = \pi^{\mathsf{T}} ; (\mathbb{M} \cap \overline{\pi})$

viii) $\overline{\mathcal{D}_{\mathrm{s}}^{\mathsf{T}} ; \overline{\mathbb{I}}} = \overline{z ; \mathbb{T} ; z^{\mathsf{T}}} \cap \mathtt{inf}_{i\geq 2}\, \overline{k_i^{\mathsf{T}} ; \overline{\mathbb{I}}} = \overline{z ; \mathbb{T}} \cap \mathtt{inf}_{i\geq 2}\, \overline{k_i^{\mathsf{T}} ; \overline{\mathbb{I}}} \subseteq e ; \mathbb{T} \cup \mathbb{T} ; e^{\mathsf{T}}$

$$k_n^{\mathsf{T}} \cap \overline{\mathcal{D}_{\mathrm{s}}^{\mathsf{T}} ; \overline{\mathbb{I}}} = \begin{cases} \mathbb{S}^{\mathsf{T}^n} ; z ; \mathbb{T} ; e^{\mathsf{T}} & \text{in case } n \text{ is prime 1,2,3,5 } \dots \\ \mathbb{\perp}\!\!\!\perp & \text{otherwise} \end{cases}$$

ix) $\mathtt{upa}(\mathcal{D}_{\mathrm{s}}^{\mathsf{T}}) = \mathcal{D}_{\mathrm{s}}^{\mathsf{T}} \cap \overline{\mathcal{D}_{\mathrm{s}}^{\mathsf{T}} ; \overline{\mathbb{I}}} = \mathtt{sup}_{n\geq 2,\, n \text{ prime}}\, \mathbb{S}^{\mathsf{T}^n} ; z ; \mathbb{T} ; e^{\mathsf{T}}$

x) $\mathtt{vec}(\mathcal{D}^\intercal) = \mathfrak{Q} ; \mathbb{T}$

xi) $\mathtt{vec}(\mathcal{D}^\intercal) = (\mathfrak{S} \circledcirc \rho)^* ; (z ; \mathbb{T} \circledcirc \mathbb{I}) ; \mathbb{T}$

xii) $(\mathcal{D}^\intercal \circledcirc \mathbb{I}) = \mathtt{sup}_{n\geq 0}\, (k_n^\intercal \circledcirc \mathbb{I}) = (\mathfrak{S} \circledcirc \rho)^* ; (k_0^\intercal \circledcirc \mathbb{I})$

xiii) $\mathbb{S}^{\intercal i*k} ; z ; \mathbb{T} \neq \mathbb{\perp} \implies \mathcal{D} ; \mathbb{S}^{\intercal i*k} ; z ; \mathbb{T} \supseteq \mathbb{S}^{\intercal i} ; z ; \mathbb{T} \cup \mathbb{S}^{\intercal k} ; z ; \mathbb{T}$

xiv) $\mathtt{syq}(\mathcal{D}_\mathrm{s}, \mathbb{I}) \subseteq \mathcal{H}^\intercal$

xv) $(\mathfrak{S} \circledcirc \rho)^* ; (\mathbb{I} \circledcirc \mathbb{I}) = (\mathcal{D}^\intercal \circledcirc \mathbb{I}) \cap (\mathbb{S}^{\intercal *} \circledcirc \mathbb{I})$

Proof: i) All the variants seem to be covered by the following:

$\mathcal{D} = \pi^\intercal ; \mathbb{M} = \pi^\intercal ; \mathtt{sup}_{n\geq 0}\, (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \circledcirc k_n)$ Prop. 7.1.1.i.a
$= \mathtt{sup}_{n\geq 0}\, \pi^\intercal ; \big[\pi ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \rho ; k_n\big]$ expanded
$= \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \mathbb{T} ; k_n$ destroy and append
$= \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \mathbb{T} ; z^\intercal ; (\mathbb{S}^n)^*$ Prop. 6.4.2.ii.b
$= \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; z^\intercal ; (\mathbb{S}^n)^*$ masking
$= \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; z^\intercal ; \big(\mathtt{sup}_{i\geq 0} (\mathbb{S}^n)^i\big)$
$= \mathtt{sup}_{i\geq 0}\, \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{n*i}$
$= \mathtt{sup}_{i\geq 0}\, k_i$ Lemma 6.4.2.ii.a, that uses regressive finiteness

---

ii) $\mathcal{D} \cap \overline{\mathbb{T} ; z^\intercal} = \big[\mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \mathbb{T} ; k_n\big] \cap \overline{\mathbb{T} ; z^\intercal}$ see (i)
$= \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \mathbb{T} ; k_n \cap \overline{\mathbb{T} ; z^\intercal}$
$= \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \mathbb{T} ; z^\intercal ; (\mathbb{S}^n)^* \cap \overline{\mathbb{T} ; z^\intercal}$ Lemma 6.4.2.ii.b with $\mathbb{S}$ regressively finite
$= \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \mathbb{T} ; z^\intercal ; (\mathbb{I} \cup \mathbb{S}^{n+}) \cap \overline{\mathbb{T} ; z^\intercal}$ standard for transitive closures
$= \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap (\mathbb{T} ; z^\intercal \cup \mathbb{T} ; z^\intercal ; \mathbb{S}^{n+}) \cap \overline{\mathbb{T} ; z^\intercal}$
$= \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \mathbb{T} ; z^\intercal ; \mathbb{S}^{n+} \cap \overline{\mathbb{T} ; z^\intercal}$
$\subseteq \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \mathbb{T} ; z^\intercal ; \mathbb{S}^{n+}$
$= \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{n+}$ masking
$\subseteq \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\intercal n} ; \mathbb{S}^{n+}$ injectivity of z
$= \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\intercal n} ; \mathbb{S}^n ; \mathbb{S}^{n*}$
$\subseteq \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{n*}$ since $\mathbb{S}$ is univalent
$= \mathbb{S}^*$

Now the second containment, which we will split into two parts according to Prop. 7.1.1.i:

$\mathcal{D} \cap \overline{\mathbb{T} ; z^\intercal} = \pi^\intercal ; \mathbb{M} \cap \overline{\mathbb{T} ; z^\intercal} = \pi^\intercal ; \big[\mathbb{M}_0 \cup (\mathbb{S}^\intercal \otimes \mathbb{I}) ; (\mathbb{M} \circledcirc \rho) ; \mathbb{A}\big] \cap \overline{\mathbb{T} ; z^\intercal} \subseteq \overline{z ; \mathbb{T}}$

$\pi^\intercal ; \mathbb{M}_0 \cap \overline{\mathbb{T} ; z^\intercal} = \pi^\intercal ; (z ; \mathbb{T} \circledcirc \mathbb{T} ; z^\intercal) \cap \overline{\mathbb{T} ; z^\intercal}$
$= \pi^\intercal ; (\pi ; z ; \mathbb{T} \cap \rho ; \mathbb{T} ; z^\intercal) \cap \overline{\mathbb{T} ; z^\intercal}$
$= z ; \mathbb{T} \cap \mathbb{T} ; z^\intercal \cap \overline{\mathbb{T} ; z^\intercal} = \mathbb{\perp}$ destroy and append

$\pi^\intercal ; (\mathbb{S}^\intercal \otimes \mathbb{I}) ; (\mathbb{M} \circledcirc \rho) ; \mathbb{A}$
$= \mathbb{S}^\intercal ; \pi^\intercal ; (\mathbb{M} \circledcirc \rho) ; \mathbb{A} \subseteq \mathbb{S}^\intercal ; \mathbb{T} = \overline{z ; \mathbb{T}}$ by definition of z

---

iii) $\mathcal{D} ; z ; \mathbb{T} = \pi^\intercal ; \mathbb{M} ; z ; \mathbb{T} = \pi^\intercal ; \big[\, (z ; \mathbb{T} \circledcirc \mathbb{T}) \cup (\mathbb{T} \circledcirc z ; \mathbb{T}) \,\big]$ Prop. 7.1.1.ii.c
$= \pi^\intercal ; (z ; \mathbb{T} \circledcirc \mathbb{T}) \cup \pi^\intercal ; (\mathbb{T} \circledcirc z ; \mathbb{T})$

$= (z;\mathbb{T} \cap \pi^{\mathsf{T}};\rho;\mathbb{T}) \cup (\mathbb{T} \cap \pi^{\mathsf{T}};\rho;z;\mathbb{T})$ destroy and append
$= z;\mathbb{T} \cup \mathbb{T};z;\mathbb{T} = \mathbb{T}$

$\mathcal{D} = \pi^{\mathsf{T}};\mathbb{M}$ by definition
$\supseteq \pi^{\mathsf{T}};m_H(\mathbb{M}_0)$
$= \pi^{\mathsf{T}};(\mathbb{S}^{\mathsf{T}} \otimes \mathbb{I});(\mathbb{M}_0 \oslash \rho);\mathbb{A} = \mathbb{S}^{\mathsf{T}};\pi^{\mathsf{T}};(\mathbb{M}_0 \oslash \rho);\mathbb{A}$ destroy and append
$= \mathbb{S}^{\mathsf{T}};\pi^{\mathsf{T}};(\mathbb{M}_0 \oslash \rho);\mathtt{sup}_{n\geq 0}\,(\mathbb{S}^{\mathsf{T}n};z;\mathbb{T} \oslash \mathbb{S}^n)$
$= \mathtt{sup}_{n\geq 0}\,\mathbb{S}^{\mathsf{T}};\pi^{\mathsf{T}};\left[\mathbb{M}_0;\mathbb{S}^{\mathsf{T}n};z;\mathbb{T} \cap \rho;\mathbb{S}^n\right]$
$= \mathtt{sup}_{n\geq 0}\,\mathbb{S}^{\mathsf{T}};\pi^{\mathsf{T}};\left[(z;\mathbb{T} \oslash \mathbb{T};z^{\mathsf{T}});\mathbb{S}^{\mathsf{T}n};z;\mathbb{T} \cap \rho;\mathbb{S}^n\right]$ expanded
$= \mathtt{sup}_{n\geq 0}\,\mathbb{S}^{\mathsf{T}};\pi^{\mathsf{T}};\left[(z;\mathbb{T};\mathbb{S}^{\mathsf{T}n};z;\mathbb{T} \oslash \mathbb{T};z^{\mathsf{T}};\mathbb{S}^{\mathsf{T}n};z;\mathbb{T}) \cap \rho;\mathbb{S}^n\right]$ since $\mathbb{S}^{\mathsf{T}n};z;\mathbb{T}$ is a point
$= \mathbb{S}^{\mathsf{T}};\pi^{\mathsf{T}};\left[(z;\mathbb{T};z;\mathbb{T} \oslash \mathbb{T};z^{\mathsf{T}};z;\mathbb{T}) \cap \rho\right]$ because for $n > 0$ always occurs $\mathbb{S};z;\mathbb{T} = \mathbb{\perp}$
$= \mathbb{S}^{\mathsf{T}};\pi^{\mathsf{T}};\left[(z;\mathbb{T} \oslash \mathbb{T}) \cap \rho\right] = \mathbb{S}^{\mathsf{T}};\pi^{\mathsf{T}};\left[\pi;z;\mathbb{T} \cap \rho;\mathbb{T} \cap \rho\right] = \mathbb{S}^{\mathsf{T}};\pi^{\mathsf{T}};\left[\pi;z;\mathbb{T} \cap \rho\right]$ with ρ total
$= \mathbb{S}^{\mathsf{T}};\left[z;\mathbb{T} \cap \mathbb{T}\right] = \mathbb{S}^{\mathsf{T}};z;\mathbb{T} = e;\mathbb{T}$ destroy and append

---

$\mathcal{D}^{\mathsf{T}};e;\mathbb{T} \supseteq \mathbb{T};e^{\mathsf{T}};e;\mathbb{T} = \mathbb{T}$ due to the preceding result and, since $e;\mathbb{T} = \mathbb{S}^{\mathsf{T}};z;\mathbb{T}$ is a point

---

$\mathcal{D};e;\mathbb{T} = \pi^{\mathsf{T}};\mathbb{M};e;\mathbb{T} = \pi^{\mathsf{T}};(e;\mathbb{T} \oslash e;\mathbb{T}) = e;\mathbb{T}$ using Prop. 7.1.1.i.e

---

$\mathcal{D}^{\mathsf{T}};\mathbb{S}^{\mathsf{T}n};z;\mathbb{T} = \left[\mathtt{sup}_{i\geq 0}\,\mathbb{S}^{\mathsf{T}i};z;\mathbb{T};z^{\mathsf{T}};(\mathbb{S}^i)^*\right]^{\mathsf{T}};\mathbb{S}^{\mathsf{T}n};z;\mathbb{T}$ following (i)
$= \mathtt{sup}_{i\geq 0}(\mathbb{S}^{\mathsf{T}i})^*;z;\mathbb{T};z^{\mathsf{T}};\mathbb{S}^i;\mathbb{S}^{\mathsf{T}n};z;\mathbb{T}$ with $\mathbb{T};z^{\mathsf{T}};\mathbb{S}^i;\mathbb{S}^{\mathsf{T}n};z;\mathbb{T} = \mathbb{\perp}$ unless $n = i$
$= \begin{cases}(\mathbb{S}^{\mathsf{T}n})^*;z;\mathbb{T} & \text{if } \mathbb{S}^{\mathsf{T}n};z;\mathbb{T} \neq \mathbb{\perp} \\ \mathbb{\perp} & \text{if } \mathbb{S}^{\mathsf{T}n};z;\mathbb{T} = \mathbb{\perp}\end{cases}$

$\mathcal{D}_{\mathrm{s}};z;\mathbb{T} = \mathcal{D};z;\mathbb{T} \cap \overline{\mathbb{I}};z;\mathbb{T} = \mathbb{T} \cap \overline{z;\overline{\mathbb{T}}} = \overline{z;\overline{\mathbb{T}}}$

$z;\mathbb{T} \subseteq \mathcal{D}_{\mathrm{s}}{}^{\mathsf{T}};\overline{\mathbb{I}} \quad\Longleftrightarrow\quad \mathbb{T} \subseteq \mathbb{T};z^{\mathsf{T}};\mathcal{D}_{\mathrm{s}}{}^{\mathsf{T}};\overline{\mathbb{I}} = \overline{\mathbb{T};z^{\mathsf{T}}};\overline{\mathbb{I}} = \mathbb{T};z^{\mathsf{T}};\overline{\mathbb{I}};\overline{\mathbb{I}} = \mathbb{T};z^{\mathsf{T}};\mathbb{T}$ shunting $z;\mathbb{T}$

$\mathcal{D}_{\mathrm{s}};e;\mathbb{T} = (\mathcal{D} \cap \overline{\mathbb{I}});e;\mathbb{T} = \mathcal{D};e;\mathbb{T} \cap \overline{e;\mathbb{T}} = e;\mathbb{T} \cap \overline{e;\mathbb{T}} = \mathbb{\perp}$

$\overline{e;\mathbb{T}} = \overline{\mathbb{I}};e;\mathbb{T} = \mathbb{T} \cap \overline{\mathbb{I}};e;\mathbb{T} = \mathcal{D}^{\mathsf{T}};e;\mathbb{T} \cap \overline{\mathbb{I}};e;\mathbb{T} = (\mathcal{D}^{\mathsf{T}} \cap \overline{\mathbb{I}});e;\mathbb{T} = \mathcal{D}_{\mathrm{s}}{}^{\mathsf{T}};e;\mathbb{T} \subseteq \mathcal{D}_{\mathrm{s}}{}^{\mathsf{T}};\mathbb{T} \subseteq \overline{e;\mathbb{T}}$,
where the last step is justified by the above $\mathcal{D}_{\mathrm{s}};e;\mathbb{T} = \mathbb{\perp}$

---

iv) This proof uses Lemma 6.4.2.ii.i several times.
$k_0 \cup k_n;\mathcal{D} = k_0 \cup k_n;\mathtt{sup}_{i\geq 0}\,k_i$ due to (i)
$= k_0 \cup \left[k_n;k_0 \cup k_n;\mathtt{sup}_{i\geq 1}\,k_i\right]$
$= k_0 \cup (k_0 \cap k_n;\mathbb{T}) \cup \mathtt{sup}_{i\geq 1}\,k_n;k_i$
$= k_0 \cup \mathtt{sup}_{i\geq 1}\,k_{n*i}$
$= k_0;k_n \cup \mathtt{sup}_{i\geq 1}\,k_{i*n}$
$= k_0;k_n \cup \mathtt{sup}_{i\geq 1}\,k_i;k_n$ (*) see below
$= k_0;k_n \cup \left[\mathtt{sup}_{i\geq 1}\,k_i\right];k_n$
$= \left[\mathtt{sup}_{i\geq 0}\,k_i\right];k_n = \mathcal{D};k_n$

Concerning (*): $k_0;k_0 \cup \mathtt{sup}_{i\geq 1}\,k_i;k_0 = k_0 \cup (k_0 \cap k_i;\mathbb{T}) = k_0$

---

v) $\mathcal{D};\mathbb{S} \cap \mathcal{D} = \left[\mathtt{sup}_{n\geq 0}\,k_n\right];\mathbb{S} \cap \left[\mathtt{sup}_{i\geq 0}\,k_i\right]$
$= \mathtt{sup}_{n\geq 0}\,\mathtt{sup}_{i\geq 0}\,k_n;\mathbb{S} \cap k_i$
$= \mathtt{sup}_{n\geq 0}\,e;\mathbb{T};e^{\mathsf{T}};\mathbb{S}^n$ Lemma 6.4.2.ii.j
$= e;\mathbb{T};e^{\mathsf{T}};\mathtt{sup}_{n\geq 0}\,\mathbb{S}^n = e;\mathbb{T};e^{\mathsf{T}};\mathbb{S}^* = e;\mathbb{T};z^{\mathsf{T}};\mathbb{S};\mathbb{S}^* = e;\mathbb{T};z^{\mathsf{T}};\mathbb{S}^+ = e;\mathbb{T};\overline{\mathbb{T};z^{\mathsf{T}}}$

The last step uses Prop. 5.2.1.ii.a.

---

vi) $e ; \mathbb{T} ; e^\intercal \subseteq \overline{\mathcal{D}^\intercal ; \overline{\mathbb{I}}} \iff e ; \mathbb{T} \subseteq \overline{\mathcal{D}^\intercal ; \overline{e ; \mathbb{T}}} \iff \mathcal{D}^\intercal ; \overline{e ; \mathbb{T}} \subseteq \overline{e ; \mathbb{T}} \iff \mathcal{D} ; e ; \mathbb{T} \subseteq e ; \mathbb{T}$ see (iii)

It remains to show the other direction $\overline{\mathcal{D}^\intercal ; \overline{\mathbb{I}}} \subseteq e ; \mathbb{T} ; e^\intercal = \mathbb{I} \cap e ; \mathbb{T} = \mathbb{I} \cap \mathbb{T} ; e^\intercal$. With $\mathcal{D}$ reflexive $\overline{\mathcal{D}^\intercal ; \overline{\mathbb{I}}} \subseteq \mathbb{I} \iff \overline{\mathbb{I}} \subseteq \mathcal{D}^\intercal ; \overline{\mathbb{I}}$.

Furthermore using (iii) and shunting the point $e ; \mathbb{T}$

$\mathbb{T} ; e^\intercal \subseteq \mathcal{D}^\intercal \implies \mathbb{T} ; e^\intercal ; \overline{\mathbb{I}} \subseteq \mathcal{D}^\intercal ; \overline{\mathbb{I}} \iff e ; \mathbb{T} ; \overline{\mathcal{D}^\intercal ; \overline{\mathbb{I}}} \subseteq \mathbb{I} \iff \overline{\mathcal{D}^\intercal ; \overline{\mathbb{I}}} \subseteq \mathbb{T} ; e^\intercal ; \mathbb{I}$

---

vii) $(\mathbb{M} \cap \overline{\pi})^\intercal ; \pi = (\mathbb{M}^\intercal \cap \overline{\mathbb{I}} ; \pi^\intercal) ; \pi = \mathbb{M}^\intercal ; \pi \cap \overline{\mathbb{I}} = \mathcal{D}^\intercal \cap \overline{\mathbb{I}}$ destroy and append

---

viii) $\mathcal{D}_\mathrm{s}^\intercal ; \overline{\mathbb{I}} = \big[\mathcal{D}^\intercal \cap \overline{\mathbb{I}}\big] ; \overline{\mathbb{I}} = \big[(\mathtt{sup}_{i\geq 0}\, k_i^\intercal) \cap \overline{\mathbb{I}}\big] ; \overline{\mathbb{I}} = \big[\mathtt{sup}_{i\geq 0}(k_i^\intercal \cap \overline{\mathbb{I}})\big] ; \overline{\mathbb{I}}$

$= \big[(k_0^\intercal \cap \overline{\mathbb{I}}) \cup (k_1^\intercal \cap \overline{\mathbb{I}}) \cup \mathtt{sup}_{i\geq 2}(k_i^\intercal \cap \overline{\mathbb{I}})\big] ; \overline{\mathbb{I}}$

$= \big[(z ; \mathbb{T} \cap \overline{\mathbb{I}}) \cup \mathbb{\perp} \cup \mathtt{sup}_{i\geq 2}(k_i^\intercal \cap \overline{z ; \mathbb{T}})\big] ; \overline{\mathbb{I}}$ Lemma 6.4.2.i.f

$= (z ; \mathbb{T} \cap \overline{\mathbb{I}}) ; \overline{\mathbb{I}} \cup \mathtt{sup}_{i\geq 2}(k_i^\intercal \cap \overline{z ; \mathbb{T}}) ; \overline{\mathbb{I}}$

$= (z ; \mathbb{T} \cap \overline{\mathbb{I}} ; \overline{\mathbb{I}}) \cup (\overline{z ; \mathbb{T}} \cap \mathtt{sup}_{i\geq 2}\, k_i^\intercal ; \overline{\mathbb{I}})$ masking everywhere

$= z ; \mathbb{T} \cup \big(\overline{z ; \mathbb{T}} \cap \mathtt{sup}_{i\geq 2}\, k_i^\intercal ; \overline{\mathbb{I}}\big)$ since $\overline{\mathbb{I}} ; \overline{\mathbb{I}} = \mathbb{T}$

$= (z ; \mathbb{T} \cup \overline{z ; \mathbb{T}}) \cap \big(z ; \mathbb{T} \cup \mathtt{sup}_{i\geq 2}\, k_i^\intercal ; \overline{\mathbb{I}}\big) = z ; \mathbb{T} \cup \mathtt{sup}_{i\geq 2}\, k_i^\intercal ; \overline{\mathbb{I}}$

$= z ; \mathbb{T} ; z^\intercal \cup \mathtt{sup}_{i\geq 2}\, k_i^\intercal ; \overline{\mathbb{I}}$

The last step requires the decomposition $z ; \mathbb{T} ; \mathbb{T} = z ; \mathbb{T} ; (\mathbb{T} ; z^\intercal \cup \overline{\mathbb{T} ; z^\intercal})$ and is justified since $\overline{\mathbb{T} ; z^\intercal} = \mathbb{T} ; z^\intercal ; \overline{\mathbb{I}} = \mathbb{T} ; z^\intercal ; k_2^\intercal ; \overline{\mathbb{I}}$, Prop. 6.4.2.i.c

from which we use only "$\subseteq$", shunt the point $z ; \mathbb{T}$, and obtain

$z ; \mathbb{T} ; \overline{\mathbb{T} ; z^\intercal} \subseteq k_2^\intercal ; \overline{\mathbb{I}}$.

Now the estimation $\overline{\mathcal{D}_\mathrm{s}^\intercal ; \overline{\mathbb{I}}} \subseteq e ; \mathbb{T} \cup \mathbb{T} ; e^\intercal$, proved via $\iff \overline{e ; \mathbb{T}} \cap \overline{\mathcal{D}_\mathrm{s}^\intercal ; \overline{\mathbb{I}}} \subseteq \mathbb{T} ; e^\intercal$ and

$\overline{e ; \mathbb{T}} \cap \overline{\mathcal{D}_\mathrm{s}^\intercal ; \overline{\mathbb{I}}} = \mathcal{D}_\mathrm{s}^\intercal ; e ; \mathbb{T} \cap \overline{\mathcal{D}_\mathrm{s}^\intercal ; \overline{\mathbb{I}}}$ due to (iii)

$\subseteq (\mathcal{D}_\mathrm{s}^\intercal \cap \overline{\mathcal{D}_\mathrm{s}^\intercal ; \overline{\mathbb{I}} ; \mathbb{T} ; e^\intercal}) ; (e ; \mathbb{T} \cap \mathcal{D}_\mathrm{s} ; \overline{\mathcal{D}_\mathrm{s}^\intercal ; \overline{\mathbb{I}}})$ Dedekind rule

$\subseteq \mathbb{T} ; e^\intercal ; (\ldots \cap \mathbb{I}) = \mathbb{T} ; e^\intercal$ Schröder rule

---

For $n, i \geq 2$ we are going to proceed as follows:

$k_n^\intercal \cap \overline{\mathcal{D}_\mathrm{s}^\intercal ; \overline{\mathbb{I}}} = k_n^\intercal \cap \overline{z ; \mathbb{T} ; z^\intercal} \cap \mathtt{inf}_{i\geq 2}\, \overline{k_i^\intercal ; \overline{\mathbb{I}}}$ see above

$= \mathtt{inf}_{i\geq 2}\big[k_n^\intercal \cap \overline{k_i^\intercal ; \overline{\mathbb{I}}} \cap \overline{z ; \mathbb{T} ; z^\intercal}\big]$ rearranged; now separate treatment for $i = n$ and $i \neq n$

$= \big[k_n^\intercal \cap \overline{k_n^\intercal ; \overline{\mathbb{I}}}\big] \cap \mathtt{inf}_{n\neq i\geq 2}\big[k_n^\intercal \cap \overline{k_i^\intercal ; \overline{\mathbb{I}}} \cap \overline{z ; \mathbb{T} ; z^\intercal}\big]$ see obvious fact

$= k_n^\intercal \cap \mathtt{inf}_{n\neq i\geq 2}\big[(z ; \mathbb{T} ; z^\intercal \cup \big[k_n^\intercal \cap \overline{k_i^\intercal ; \mathbb{T}}\big]) \cap \overline{z ; \mathbb{T} ; z^\intercal}\big]$ Lemma 6.4.2.ii.a,b

$= k_n^\intercal \cap \mathtt{inf}_{n\neq i\geq 2}\big[(z ; \mathbb{T} ; z^\intercal \cap \overline{z ; \mathbb{T} ; z^\intercal}) \cup (k_n^\intercal \cap \overline{k_i^\intercal ; \mathbb{T}} \cap \overline{z ; \mathbb{T} ; z^\intercal})\big]$

$= k_n^\intercal \cap \mathtt{inf}_{n\neq i\geq 2}\big[k_n^\intercal \cap \overline{k_i^\intercal ; \mathbb{T}} \cap \overline{z ; \mathbb{T} ; z^\intercal}\big] = \mathtt{inf}_{n\neq i\geq 2}\big[k_n^\intercal \cap \overline{k_i^\intercal ; \mathbb{T}} \cap \overline{z ; \mathbb{T} ; z^\intercal}\big]$

$= \mathtt{inf}_{n\neq i\geq 2}\big[k_n^\intercal \cap \overline{k_i^\intercal ; \mathbb{T}}\big]$ because $\overline{k_i^\intercal ; \mathbb{T}} \subseteq \overline{z ; \mathbb{T} ; z^\intercal}$

$= \mathtt{inf}_{n\neq i\geq 2}\big[(\mathtt{sup}_{l\geq 0}\, \mathbb{S}^{\intercal l * n} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^l) \cap \overline{(\mathbb{S}^{\intercal i})^* ; z ; \mathbb{T}})\big]$ Lemma 6.4.2.ii.a,b

$= \mathtt{inf}_{n\neq i\geq 2}\big[(\mathtt{sup}_{l\geq 0}\, \mathbb{S}^{\intercal l * n} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^l) \cap \overline{\mathtt{sup}_{m\geq 0}\, \mathbb{S}^{\intercal i * m} ; z ; \mathbb{T}})\big]$ expanded

Let us discuss what now may happen. There is a fixed index n together with three running indices i, l, m. While $l, m \geq 0$ are unrestricted, the i is restricted by $n \neq i, \quad i \geq 2$. The term

$\mathbb{S}^{\intercal l*n}{;}\, z{;}\mathbb{T}$ will be annihilated by the term $\overline{\mathbb{S}^{\intercal i*m}{;}\, z{;}\mathbb{T}}$ provided $i*m = l*n$. Should be $l = 0$, we might choose $m = 0$. If $l > 1$, meaning $l \geq 2$, it may well contain factors n. Given $l = n^k * l'$ with $n \nmid l'$, we take $m := n^{k+1}$ and $i := l'$.

We are left with the case $l = 1$: If n is a proper product $n = a * b$ with $a > 1$ and $b > 1$, we choose $i := a$ and $m := b$ to annihilate. This possibility fails in case the $n \geq 2$ is a prime, and then indeed

$k_n^\intercal \cap \overline{\mathcal{D}_\mathrm{s}^\intercal{;}\overline{\mathbb{I}}} = \mathbb{S}^{\intercal 1*n}{;}\, z{;}\mathbb{T}{;}z^\intercal{;}\mathbb{S}^1 = \mathbb{S}^{\intercal n}{;}\, z{;}\mathbb{T}{;}e^\intercal$

---

ix) $\mathcal{D}_\mathrm{s}{}^\intercal \cap \overline{\mathcal{D}_\mathrm{s}^\intercal{;}\overline{\mathbb{I}}} = \left[\overline{\mathbb{I}} \cap \mathtt{sup}_{n\geq 0}\, k_n^\intercal\right] \cap \overline{\mathcal{D}_\mathrm{s}^\intercal{;}\overline{\mathbb{I}}}$ see (i)
$= \mathtt{sup}_{n\geq 0}\, k_n^\intercal \cap \overline{\mathbb{I}} \cap \overline{\mathcal{D}_\mathrm{s}^\intercal{;}\overline{\mathbb{I}}}$
$= (k_0^\intercal \cap \overline{\mathbb{I}} \cap \overline{\mathcal{D}_\mathrm{s}^\intercal{;}\overline{\mathbb{I}}}) \cup (k_1^\intercal \cap \overline{\mathbb{I}} \cap \overline{\mathcal{D}_\mathrm{s}^\intercal{;}\overline{\mathbb{I}}}) \cup \left[\mathtt{sup}_{n\geq 2}\, k_n^\intercal \cap \overline{\mathbb{I}} \cap \overline{\mathcal{D}_\mathrm{s}^\intercal{;}\overline{\mathbb{I}}}\right]$
$= (z{;}\mathbb{T} \cap \overline{\mathbb{I}} \cap \overline{\mathcal{D}_\mathrm{s}^\intercal{;}\overline{\mathbb{I}}}) \cup \mathbb{\perp} \cup \left[\mathtt{sup}_{n\geq 2}\, k_n^\intercal \cap \overline{z{;}\mathbb{T}} \cap \overline{\mathcal{D}_\mathrm{s}^\intercal{;}\overline{\mathbb{I}}}\right]$ due to $k_1 = \mathbb{I}$; Lemma 6.4.2.i.f
$= \mathbb{\perp} \cup \mathbb{\perp} \cup \left[\mathtt{sup}_{n\geq 2}\, k_n^\intercal \cap \overline{\mathcal{D}_\mathrm{s}^\intercal{;}\overline{\mathbb{I}}}\right]$ see (iii): $z{;}\mathbb{T} \subseteq \mathcal{D}_\mathrm{s}^\intercal{;}\overline{\mathbb{I}}$
$= \mathtt{sup}_{n\geq 2}\, k_n^\intercal \cap \overline{\mathcal{D}_\mathrm{s}^\intercal{;}\overline{\mathbb{I}}} = \mathtt{sup}_{n\geq 2,\, n \text{ prime}}\, \mathbb{S}^{\intercal n}{;}\, z{;}\mathbb{T}{;}e^\intercal$ using (viii)

---

x) $\mathtt{vec}(\mathcal{D}^\intercal) = (\pi{;}\mathcal{D}^\intercal \cap \rho){;}\mathbb{T} = (\pi{;}\,\mathbb{M}^\intercal{;}\pi \cap \rho){;}\mathbb{T}$
$= (\pi{;}\,\mathbb{M}^\intercal \cap \rho{;}\pi^\intercal){;}\pi{;}\mathbb{T}$ destroy and append
$= (\pi{;}\,\mathbb{M}^\intercal{;}\,\mathfrak{P} \cap \rho{;}\pi^\intercal){;}\mathbb{T}$ π is total and $\mathbb{M}$ is commutative
$= (\pi{;}\,\mathbb{M}^\intercal \cap \rho{;}\pi^\intercal{;}\,\mathfrak{P}\,){;}\,\mathfrak{P}{;}\mathbb{T}$
$= (\pi{;}\,\mathbb{M}^\intercal \cap \rho{;}\rho^\intercal){;}\mathbb{T}$ since $\mathfrak{P}$ is total and $\mathfrak{P}{;}\pi = \rho$
$= (\,\mathbb{M}^\intercal \circledcirc \rho^\intercal)\,{;}\,\pi{;}\mathbb{T} = \mathbb{M}^{\hookleftarrow}{;}\mathbb{T}$ with π total
$= \mathfrak{Q}{;}\mathbb{T}$ Prop. 7.2.1.ii.a

---

xi) $(\mathfrak{S} \circledcirc \rho)^*{;}\,(z{;}\mathbb{T} \circledcirc \mathbb{I}){;}\mathbb{T} = \mathtt{sup}_{n\geq 0}\,(\mathfrak{S} \circledcirc \rho)^{n}{;}\,(k_0^\intercal \circledcirc \mathbb{I}){;}\mathbb{T}$
$= \mathtt{sup}_{n\geq 0}\,\mathtt{vec}(k_n^\intercal)$ Lemma 6.4.2.ii.g
$= \mathtt{vec}(\mathtt{sup}_{n\geq 0}\, k_n^\intercal) = \mathtt{vec}(\mathcal{D}^\intercal)$

---

xii) $(\mathcal{D}^\intercal \circledcirc \mathbb{I}) = (\mathtt{sup}_{n\geq 0}\, k_n^\intercal \circledcirc \mathbb{I})$ due to (i)
$= \mathtt{sup}_{n\geq 0}\,(k_n^\intercal \circledcirc \mathbb{I}) = \mathtt{sup}_{n\geq 0}\,(\mathfrak{S} \circledcirc \rho)^{n}{;}\,(k_0^\intercal \circledcirc \mathbb{I})$ Lemma 6.4.2.ii.f
$= (\mathfrak{S} \circledcirc \rho)^*{;}\,(k_0^\intercal \circledcirc \mathbb{I})$

---

xiii) The cases $i * k = 0$ and $i * k = 1$ are covered with (iii). We may choose either i or k to reason as follows:

$\mathcal{D}{;}\,\mathbb{S}^{\intercal i*k}{;}\, z{;}\mathbb{T} = \mathtt{sup}_{n\geq 0}\, k_n{;}\,\mathbb{S}^{\intercal i*k}{;}\, z{;}\mathbb{T} \supseteq k_i{;}\,\mathbb{S}^{\intercal i*k}{;}\, z{;}\mathbb{T} = \mathbb{S}^{\intercal k}{;}\, z{;}\mathbb{T}$ Lemma 6.4.2.i.c

---

xiv) Expanding the terms together with irreflexivity of $\mathcal{D}_\mathrm{s}$ makes this evident:

$\mathtt{syq}(\mathcal{D}_\mathrm{s}, \mathbb{I}) = \overline{\overline{\mathcal{D}_\mathrm{s}^\intercal}{;}\mathbb{I}} \cap \overline{\mathcal{D}_\mathrm{s}^\intercal{;}\overline{\mathbb{I}}} \subseteq \mathcal{D}_\mathrm{s}^\intercal \cap \overline{\mathcal{D}_\mathrm{s}^\intercal{;}\mathcal{D}_\mathrm{s}^\intercal} = \mathcal{H}^\intercal$

---

xv) $(\mathfrak{S} \circledcirc \rho)^*{;}\,(\mathbb{I} \circledcirc \mathbb{I}) = \mathtt{sup}_{n\geq 0}\,(\mathfrak{S} \circledcirc \rho)^{n}{;}\,(\mathbb{I} \circledcirc \mathbb{I}) = \mathtt{sup}_{n\geq 0}\,(k_{n+1}^\intercal \circledcirc \mathbb{I})$ Prop. 6.4.2.ii.f
$= (\mathtt{sup}_{n\geq 0}\, k_{n+1}^\intercal \circledcirc \mathbb{I}) = (\mathcal{D}^\intercal \cap \mathbb{S}^{\intercal *} \circledcirc \mathbb{I})$ with $k_0^\intercal \cap \mathbb{S}^{\intercal *} = z{;}\mathbb{T}{;}z^\intercal \subseteq \mathcal{D}^\intercal$

$= (\mathcal{D}^\intercal \circledcirc \mathbb{I}) \cap (\mathbb{S}^{\intercal *} \circledcirc \mathbb{I})$ □

The interpretation of (ii) is relatively easy when looking at Fig. 7.4.1: The first column prohibits $\mathcal{D}$ to be contained in the upper right triangle $\mathbb{S}^*$ — except for the first row.

The results (viii) allow an easy interpretation remembering that the right residual shows containment of columns and the symmetric quotient equality of these: In Figs. 7.4.1 and 7.4.2, indeed, the second column of the identity is equal to the columns corresponding to primes.

Now we proceed to the order- and lattice-theoretic results. This requires standard knowledge in the relational treatment of orders; cf. e.g. [11, 12, Sect. 3.3]. Correspondingly also for lattices as e.g. in a binary meet in Def. 4.2.1 in [15].

To start with, we recall that

$\mathcal{D} = \pi^\intercal ; \mathbb{M}$ expresses "divides",
$\mathcal{D}^\intercal = \pi^\intercal ; \mathfrak{Q}$ expresses "has as divisors",
$(\mathcal{D} \circledcirc \mathcal{D})$ expresses "is a common divisor of the pair on the right".

The binary meets in the following lattice are commonly known as "greatest common divisors" and get consequently the specific denotation `gcd`. The binary joins on the other hand are commonly known as "least common multiple" and get the specific denotation `lcm`.

7.4.3 Proposition. Assume $\mathbb{S}$ to be regressively finite. Then

i) $\mathcal{D}^\intercal = \pi^\intercal ; \mathfrak{Q}$

ii) $\mathcal{D} = \pi^\intercal ; \mathbb{M} = \rho^\intercal ; \mathbb{M}$

iii) $\mathcal{D}$ is an ordering and is, thus, a lattice.

iv) This lattice is furthermore a so-called bounded lattice because it contains a least as well as a greatest element as follows

$e = \mathtt{least}_{\mathcal{D}}(\mathbb{T}) = \overline{\overline{\mathcal{D}} ; \mathbb{T}}$ $\qquad$ $z = \mathtt{greatest}_{\mathcal{D}}(\mathbb{T}) = \overline{\overline{\mathcal{D}^\intercal} ; \mathbb{T}}$

v) $\overline{\mathcal{D}^\intercal ; \overline{\mathcal{D}}} = \mathcal{D}$ $\qquad$ $\mathtt{syq}(\mathcal{D}, \mathcal{D}) = \mathbb{I}$ $\qquad$ $\overline{\mathcal{D}^\intercal ; \overline{(\mathcal{D} \circledcirc \mathcal{D})}} = (\mathcal{D} \circledcirc \mathcal{D})$

vi) The following two constructs of *greatest common divisor* resp. *least common multiple*

binary meets $\mathfrak{M}_2 = \mathtt{syq}(\,(\mathcal{D} \circledcirc \mathcal{D})\,, \mathcal{D}) =: \mathtt{gcd}$
binary joins $\mathfrak{J}_2 = \mathtt{syq}(\,(\mathcal{D}^\intercal \circledcirc \mathcal{D}^\intercal)\,, \mathcal{D}^\intercal) =: \mathtt{lcm}$

are mappings.

Proof: i) $\pi^\intercal ; \mathfrak{Q} = \pi^\intercal ; \mathbb{M}^{\hookleftarrow}$ Prop. 7.2.1.ii.a
$= \pi^\intercal ; (\,\mathbb{M}^\intercal \circledcirc \rho^\intercal) ; \pi = \mathbb{M}^\intercal ; \pi = \mathcal{D}^\intercal$

---

ii) $\pi^\intercal ; \mathbb{M} = \pi^\intercal ; \mathfrak{P} ; \mathbb{M} = \rho^\intercal ; \mathbb{M}$ since $\mathbb{M}$ is commutative due to Prop. 7.1.1.ii.b

---

iii) transitive:

$\mathcal{D} ; \mathcal{D} = \mathtt{sup}_{n\geq 0}\, k_n ; \mathtt{sup}_{m\geq 0}\, k_m = \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{m\geq 0}\, k_n ; k_m$ Prop. 7.4.2.i
$\subseteq \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{m\geq 0}\, k_{n*m} \subseteq \mathcal{D}$ Lemma 6.4.2.ii.i, including the case $k_n ; k_0$

reflexive: We recall the definitions and Prop. 7.1.1.i.a:

$\mathbb{M}_0 := (z ; \mathbb{T} \circledcirc \mathbb{T} ; z^\intercal) \qquad m_H^n(M_0) := (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \circledcirc k_n)$
$\mathbb{A} = \mathtt{sup}_{n\geq 0}\ (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \circledcirc \mathbb{S}^n)$

and may — due to distributivity — proceed piecewise:

$\mathcal{D} = \pi^\intercal ; \mathbb{M} = \pi^\intercal ; \mathtt{sup}_{n\geq 0}\ (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \circledcirc k_n)$
$= \mathtt{sup}_{n\geq 0}\ \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \mathbb{T} ; k_n$ destroy and append
$= \mathtt{sup}_{n\geq 0}\ \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \mathbb{T} ; z^\intercal ; (\mathbb{S}^n)^*$ Lemma 6.4.2.ii.b
$= \mathtt{sup}_{n\geq 0}\ \mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; z^\intercal ; (\mathbb{S}^n)^*$ masking
$\supseteq \mathtt{sup}_{n\geq 0}\ \mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^n$
$= D$ Prop. 5.2.2.i.a for D as opposed to $\mathcal{D}$
$= \mathbb{I}$ Prop. 5.2.2.ii.a if regressively finite

antisymmetric: Due to Prop. 7.4.2.ii, we have $\mathcal{D} \subseteq \mathbb{T} ; z^\intercal \cup (\mathbb{S}^* \cap \overline{z ; \mathbb{T}})$ with which simply

$\mathcal{D} \cap \mathcal{D}^\intercal \subseteq (\mathbb{T} ; z^\intercal \cup (\mathbb{S}^* \cap \overline{z ; \mathbb{T}})) \cap (z ; \mathbb{T} \cup (\mathbb{S}^{\intercal *} \cap \overline{z ; \mathbb{T}}^\intercal))$
$= (\mathbb{T} ; z^\intercal \cap z ; \mathbb{T}) \cup (\mathbb{T} ; z^\intercal \cap \mathbb{S}^{\intercal *} \cap \overline{z ; \mathbb{T}}^\intercal) \cup (\mathbb{S}^* \cap \overline{z ; \mathbb{T}} \cap z ; \mathbb{T}) \cup (\mathbb{S}^* \cap \overline{z ; \mathbb{T}} \cap \mathbb{S}^{\intercal *} \cap \overline{z ; \mathbb{T}}^\intercal)$
$\subseteq \mathbb{I} \cup \mathbb{1} \cup \mathbb{1} \cup (\mathbb{S}^* \cap \mathbb{S}^{\intercal *}) \subseteq \mathbb{I} \cup \mathbb{I} = \mathbb{I}$

---

iv) Considering 0 and 1, $\mathcal{D}$ is obviously what is called a bounded lattice. The two terms for e and z need standard knowledge in the relational treatment of orderings; cf. e.g. [11, 12, Sect. 3.3].

---

v) $\overline{\mathcal{D}^\intercal ; \overline{\mathcal{D}}} = \mathcal{D} \quad \Longleftrightarrow \quad \mathcal{D}^\intercal ; \overline{\mathcal{D}} = \overline{\mathcal{D}}$

Now follows "$\subseteq$" via Schröder rule and transitivity of $\mathcal{D}$. "$\supseteq$" holds because $\mathbb{I} \subseteq \mathcal{D}$

$\mathtt{syq}(\mathcal{D}, \mathcal{D}) = \overline{\overline{\mathcal{D}}^\intercal ; \mathcal{D}} \cap \overline{\mathcal{D}^\intercal ; \overline{\mathcal{D}}} = \mathcal{D} \cap \mathcal{D}^\intercal$ see before
$= \mathbb{I}$ since $\mathcal{D}$ is reflexive and antisymmetric

---

$\overline{\mathcal{D}^\intercal ; \overline{(\mathcal{D} \circledcirc \mathcal{D})}} = \overline{\mathcal{D}^\intercal ; \overline{\mathcal{D} ; \pi^\intercal} \cap \overline{\mathcal{D} ; \rho^\intercal}} = \overline{\mathcal{D}^\intercal ; \left[\overline{\mathcal{D} ; \pi^\intercal} \cup \overline{\mathcal{D} ; \rho^\intercal} \right]}$
$= \overline{\mathcal{D}^\intercal ; \overline{\mathcal{D} ; \pi^\intercal}} \cup \overline{\mathcal{D}^\intercal ; \overline{\mathcal{D} ; \rho^\intercal}} = \overline{\mathcal{D}^\intercal ; \overline{\mathcal{D} ; \pi^\intercal}} \cap \overline{\mathcal{D}^\intercal ; \overline{\mathcal{D} ; \rho^\intercal}}$
$= \overline{\mathcal{D}^\intercal ; \overline{\mathcal{D}}} ; \pi^\intercal \cap \overline{\mathcal{D}^\intercal ; \overline{\mathcal{D}}} ; \rho^\intercal = \mathcal{D} ; \pi^\intercal \cap \mathcal{D} ; \rho^\intercal$ see before
$= (\mathcal{D} \circledcirc \mathcal{D})$

---

vi) We show that `gcd` is indeed the greatest common divisor — taken row-wise. To this end, we first refer to the definition of `gre` in [9, Def. 9.6] or earlier already in [12, Def. 3.3.5] or [11, Def. 3.3.5] and convince us that

$\mathtt{syq}(\,(\mathcal{D} \circledcirc \mathcal{D})\,, \mathcal{D}) = \overline{\overline{(\mathcal{D} \circledcirc \mathcal{D})}^\intercal ; \mathcal{D}} \cap \overline{(\mathcal{D} \circledcirc \mathcal{D})^\intercal ; \overline{\mathcal{D}}}$
$= (\mathcal{D} \circledcirc \mathcal{D})^\intercal \cap \overline{(\mathcal{D} \circledcirc \mathcal{D})^\intercal ; \overline{\mathcal{D}}}$ due to (v)
$= \left[(\mathcal{D} \circledcirc \mathcal{D}) \cap \overline{\overline{\mathcal{D}}^\intercal ; (\mathcal{D} \circledcirc \mathcal{D})} \right]^\intercal$
$= \left[\mathtt{gre}_{\mathcal{D}}(\,(\mathcal{D} \circledcirc \mathcal{D})\,) \right]^\intercal$ □

$\mathtt{gcd} =$
$\mathtt{syq}\big(\,(\mathcal{D} \circledcirc \mathcal{D})\,, \mathcal{D}\big) =$

$\|$

	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1	1
2	2	1	2	1	2	1	2	1
3	3	1	1	3	1	1	3	1
4	4	1	2	1	4	1	2	1
5	5	1	1	1	1	5	1	1
6	6	1	2	3	2	1	6	1
7	7	1	1	1	1	1	1	7

as table ↑
as relation →

	0	1	2	3	4	5	6	7
(0,0)	1	0	0	0	0	0	0	0
(0,1)	0	1	0	0	0	0	0	0
(0,2)	0	0	1	0	0	0	0	0
(0,3)	0	0	0	1	0	0	0	0
(0,4)	0	0	0	0	1	0	0	0
(0,5)	0	0	0	0	0	1	0	0
(0,6)	0	0	0	0	0	0	1	0
(0,7)	0	0	0	0	0	0	0	1
(1,0)	0	1	0	0	0	0	0	0
(1,1)	0	1	0	0	0	0	0	0
(1,2)	0	1	0	0	0	0	0	0
(1,3)	0	1	0	0	0	0	0	0
(1,4)	0	1	0	0	0	0	0	0
(1,5)	0	1	0	0	0	0	0	0
(1,6)	0	1	0	0	0	0	0	0
(1,7)	0	1	0	0	0	0	0	0
(2,0)	0	0	1	0	0	0	0	0
(2,1)	0	1	0	0	0	0	0	0
(2,2)	0	0	1	0	0	0	0	0
(2,3)	0	1	0	0	0	0	0	0
(2,4)	0	0	1	0	0	0	0	0
(2,5)	0	1	0	0	0	0	0	0
(2,6)	0	0	1	0	0	0	0	0
(2,7)	0	1	0	0	0	0	0	0
(3,0)	0	0	0	1	0	0	0	0
(3,1)	0	1	0	0	0	0	0	0
(3,2)	0	1	0	0	0	0	0	0
(3,3)	0	0	0	1	0	0	0	0
(3,4)	0	1	0	0	0	0	0	0
(3,5)	0	1	0	0	0	0	0	0
(3,6)	0	0	0	1	0	0	0	0
(3,7)	0	1	0	0	0	0	0	0
(4,0)	0	0	0	0	1	0	0	0
(4,1)	0	1	0	0	0	0	0	0
(4,2)	0	0	1	0	0	0	0	0
(4,3)	0	1	0	0	0	0	0	0
(4,4)	0	0	0	0	1	0	0	0
(4,5)	0	1	0	0	0	0	0	0
(4,6)	0	0	1	0	0	0	0	0
(4,7)	0	1	0	0	0	0	0	0
(5,0)	0	0	0	0	0	1	0	0
(5,1)	0	1	0	0	0	0	0	0
(5,2)	0	1	0	0	0	0	0	0
(5,3)	0	1	0	0	0	0	0	0
(5,4)	0	1	0	0	0	0	0	0
(5,5)	0	0	0	0	0	1	0	0
(5,6)	0	1	0	0	0	0	0	0
(5,7)	0	1	0	0	0	0	0	0
(6,0)	0	0	0	0	0	0	1	0
(6,1)	0	1	0	0	0	0	0	0
(6,2)	0	0	1	0	0	0	0	0
(6,3)	0	0	0	1	0	0	0	0
(6,4)	0	0	1	0	0	0	0	0
(6,5)	0	1	0	0	0	0	0	0
(6,6)	0	0	0	0	0	0	1	0
(6,7)	0	1	0	0	0	0	0	0
(7,0)	0	0	0	0	0	0	0	1
(7,1)	0	1	0	0	0	0	0	0
(7,2)	0	1	0	0	0	0	0	0
(7,3)	0	1	0	0	0	0	0	0
(7,4)	0	1	0	0	0	0	0	0
(7,5)	0	1	0	0	0	0	0	0
(7,6)	0	1	0	0	0	0	0	0
(7,7)	0	0	0	0	0	0	0	1

	0	1	2	3	4	5	6	7
(0,0)	1	0	0	0	0	0	0	0
(0,1)	1	0	0	0	0	0	0	0
(0,2)	1	0	0	0	0	0	0	0
(0,3)	1	0	0	0	0	0	0	0
(0,4)	1	0	0	0	0	0	0	0
(0,5)	1	0	0	0	0	0	0	0
(0,6)	1	0	0	0	0	0	0	0
(0,7)	1	0	0	0	0	0	0	0
(1,0)	1	0	0	0	0	0	0	0
(1,1)	0	1	0	0	0	0	0	0
(1,2)	0	0	1	0	0	0	0	0
(1,3)	0	0	0	1	0	0	0	0
(1,4)	0	0	0	0	1	0	0	0
(1,5)	0	0	0	0	0	1	0	0
(1,6)	0	0	0	0	0	0	1	0
(1,7)	0	0	0	0	0	0	0	1
(2,0)	1	0	0	0	0	0	0	0
(2,1)	0	0	1	0	0	0	0	0
(2,2)	0	0	1	0	0	0	0	0
(2,3)	0	0	0	0	0	0	1	0
(2,4)	0	0	0	0	1	0	0	0
(2,5)	1	0	0	0	0	0	0	0
(2,6)	0	0	0	0	0	0	1	0
(2,7)	1	0	0	0	0	0	0	0
(3,0)	1	0	0	0	0	0	0	0
(3,1)	0	0	0	1	0	0	0	0
(3,2)	0	0	0	0	0	0	1	0
(3,3)	0	0	0	1	0	0	0	0
(3,4)	1	0	0	0	0	0	0	0
(3,5)	1	0	0	0	0	0	0	0
(3,6)	0	0	0	0	0	0	1	0
(3,7)	1	0	0	0	0	0	0	0
(4,0)	1	0	0	0	0	0	0	0
(4,1)	0	0	0	0	1	0	0	0
(4,2)	0	0	0	0	1	0	0	0
(4,3)	1	0	0	0	0	0	0	0
(4,4)	0	0	0	0	1	0	0	0
(4,5)	1	0	0	0	0	0	0	0
(4,6)	1	0	0	0	0	0	0	0
(4,7)	1	0	0	0	0	0	0	0
(5,0)	1	0	0	0	0	0	0	0
(5,1)	0	0	0	0	0	1	0	0
(5,2)	1	0	0	0	0	0	0	0
(5,3)	1	0	0	0	0	0	0	0
(5,4)	1	0	0	0	0	0	0	0
(5,5)	0	0	0	0	0	1	0	0
(5,6)	1	0	0	0	0	0	0	0
(5,7)	1	0	0	0	0	0	0	0
(6,0)	1	0	0	0	0	0	0	0
(6,1)	0	0	0	0	0	0	1	0
(6,2)	0	0	0	0	0	0	1	0
(6,3)	0	0	0	0	0	0	1	0
(6,4)	1	0	0	0	0	0	0	0
(6,5)	1	0	0	0	0	0	0	0
(6,6)	0	0	0	0	0	0	1	0
(6,7)	1	0	0	0	0	0	0	0
(7,0)	1	0	0	0	0	0	0	0
(7,1)	0	0	0	0	0	0	0	1
(7,2)	1	0	0	0	0	0	0	0
(7,3)	1	0	0	0	0	0	0	0
(7,4)	1	0	0	0	0	0	0	0
(7,5)	1	0	0	0	0	0	0	0
(7,6)	1	0	0	0	0	0	0	0
(7,7)	0	0	0	0	0	0	0	1

$= \mathtt{lcm}$
$= \mathtt{syq}\big(\,(\mathcal{D}^{\mathsf{T}} \circledcirc \mathcal{D}^{\mathsf{T}})\,, \mathcal{D}^{\mathsf{T}}\big)$

$\|$

	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	2	6	4	0	6	0
3	0	3	6	3	0	0	6	0
4	0	4	4	0	4	0	0	0
5	0	5	0	0	0	5	0	0
6	0	6	6	6	0	0	6	0
7	0	7	0	0	0	0	0	7

as table ↑
← as relation

Fig. 7.4.3 Greatest common divisor $\mathtt{syq}\big(\,(\mathcal{D} \circledcirc \mathcal{D})\,, \mathcal{D}\big)$, least common multiple $\mathtt{syq}\big(\,(\mathcal{D}^{\mathsf{T}} \circledcirc \mathcal{D}^{\mathsf{T}})\,, \mathcal{D}^{\mathsf{T}}\big)$

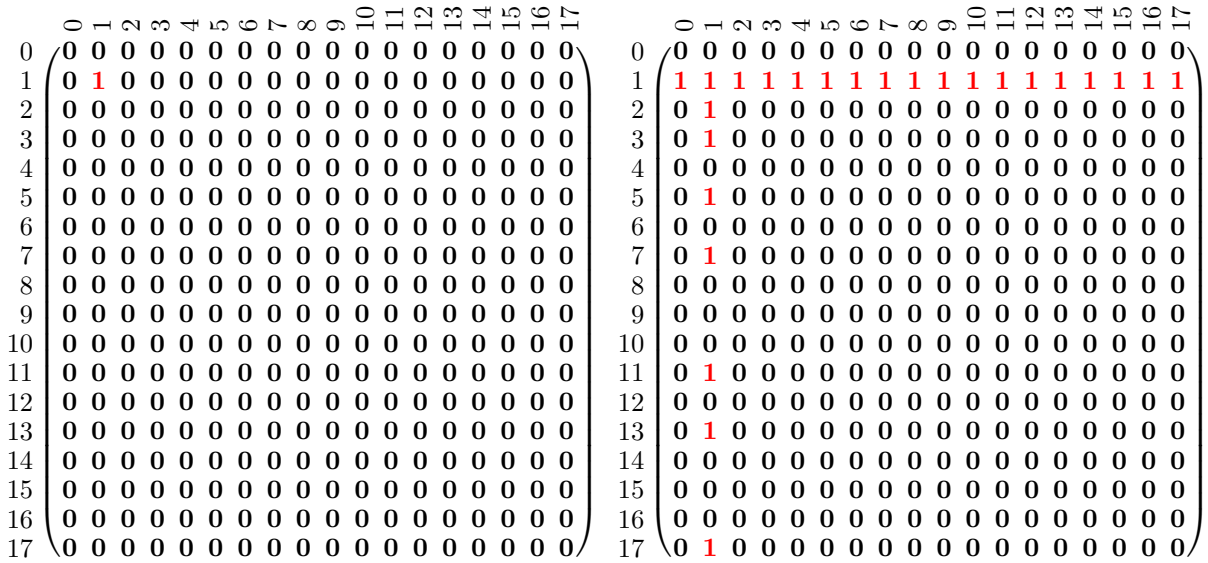


Fig. 7.4.4 $\overline{\mathcal{D}^{\mathsf{T}};\overline{\mathbb{I}}}$ and $\overline{\mathcal{D}_{\mathrm{s}}{}^{\mathsf{T}};\overline{\mathbb{I}}}$ — a hint as to primes

A result to get grip on the matrix patterns of Fig. 7.4.5:

7.4.4 Proposition. i) $e;\mathbb{T};e^{\mathsf{T}};\overline{\mathbb{I}} \cup \overline{\mathbb{I}};z;\mathbb{T};z^{\mathsf{T}} \subseteq \mathcal{D}_{\mathrm{s}}$

ii) $e;\mathbb{T} \cup \overline{\mathbb{I}};z;\mathbb{T};e^{\mathsf{T}} \supseteq \overline{\mathcal{D}_{\mathrm{s}}{}^{\mathsf{T}};\overline{\mathbb{I}}} \supseteq \overline{\mathcal{D}^{\mathsf{T}};\overline{\mathbb{I}}} = e;\mathbb{T};e^{\mathsf{T}}$

iii) $\overline{e;\mathbb{T} \cup z;\mathbb{T}};\mathbb{T};e^{\mathsf{T}} \supseteq \mathcal{D}_{\mathrm{s}}{}^{\mathsf{T}} \cap \overline{\mathcal{D}_{\mathrm{s}}{}^{\mathsf{T}};\overline{\mathbb{I}}} = \mathtt{syq}(\mathcal{D}_{\mathrm{s}},\mathbb{I})$

Proof: i) The first statement is relatively simple:

$e;\mathbb{T};\overline{\mathbb{T};e^{\mathsf{T}}} \subseteq \mathcal{D}_{\mathrm{s}} \iff \overline{\mathbb{T};e^{\mathsf{T}}} \subseteq \mathbb{T};e^{\mathsf{T}};\mathcal{D}_{\mathrm{s}}$ which holds since $\mathcal{D}_{\mathrm{s}}{}^{\mathsf{T}};e;\mathbb{T} = \overline{e;\mathbb{T}}$ due to 7.4.2.iii

$\overline{z;\mathbb{T}};\mathbb{T};z^{\mathsf{T}} \subseteq \mathcal{D}_{\mathrm{s}} \iff \overline{z;\mathbb{T}} \subseteq \mathcal{D}_{\mathrm{s}};z;\mathbb{T}$ which holds since $\mathcal{D}_{\mathrm{s}};z;\mathbb{T} = \overline{z;\mathbb{T}}$ due to 7.4.2.iii

---

ii) The rightmost equality simply recalls Prop. 7.4.2.vi. Furthermore:

$\mathcal{D}_{\mathrm{s}}{}^{\mathsf{T}};\overline{\mathbb{I}} \supseteq \left[e;\mathbb{T};e^{\mathsf{T}};\overline{\mathbb{I}} \cup \overline{\mathbb{I}};z;\mathbb{T};z^{\mathsf{T}}\right]^{\mathsf{T}};\overline{\mathbb{I}}$ following (i)

$= \left[\overline{\mathbb{I}};e;\mathbb{T};e^{\mathsf{T}} \cup z;\mathbb{T};z^{\mathsf{T}};\overline{\mathbb{I}}\right];\overline{\mathbb{I}}$

$= \overline{\mathbb{I}};e;\mathbb{T};e^{\mathsf{T}};\overline{\mathbb{I}} \cup z;\mathbb{T};z^{\mathsf{T}};\overline{\mathbb{I}};\overline{\mathbb{I}}$

$= \overline{\mathbb{I}};e;\mathbb{T};e^{\mathsf{T}};\overline{\mathbb{I}} \cup z;\mathbb{T}$ since $\mathbb{T};z^{\mathsf{T}};\overline{\mathbb{I}};\overline{\mathbb{I}} = \mathbb{T};z^{\mathsf{T}};\mathbb{T} = \mathbb{T}$

$= \overline{e;\mathbb{T}};\overline{\mathbb{T};e^{\mathsf{T}}} \cup z;\mathbb{T}$

$= \left[\overline{e;\mathbb{T}} \cap \overline{\mathbb{T};e^{\mathsf{T}}}\right] \cup \left[\overline{e;\mathbb{T}} \cap z;\mathbb{T}\right]$ because $z;\mathbb{T} \subseteq \overline{e;\mathbb{T}}$

$= \overline{e;\mathbb{T}} \cap \left[\overline{\mathbb{T};e^{\mathsf{T}}} \cup z;\mathbb{T}\right] = \overline{e;\mathbb{T} \cup \left[\mathbb{T};e^{\mathsf{T}} \cap \overline{z;\mathbb{T}}\right]}$

$= \overline{e;\mathbb{T} \cup \overline{z;\mathbb{T}};\mathbb{T};e^{\mathsf{T}}} = \overline{e;\mathbb{T} \cup \overline{\mathbb{I}};z;\mathbb{T};e^{\mathsf{T}}}$

---

iii) $\mathtt{syq}(\mathcal{D}_{\mathrm{s}},\mathbb{I}) = \mathcal{D}_{\mathrm{s}}{}^{\mathsf{T}} \cap \overline{\mathcal{D}_{\mathrm{s}}{}^{\mathsf{T}};\overline{\mathbb{I}}} \subseteq \mathcal{D}_{\mathrm{s}}{}^{\mathsf{T}} \cap \left[e;\mathbb{T} \cup \overline{z;\mathbb{T}};\mathbb{T};e^{\mathsf{T}}\right]$ preceeding result (ii)

$\subseteq \overline{e;\mathbb{T}} \cap \left[e;\mathbb{T} \cup \overline{z;\mathbb{T}};\mathbb{T};e^{\mathsf{T}}\right]$ since $\mathcal{D}_{\mathrm{s}};e;\mathbb{T} = \mathbb{L}$

$= \left[e;\mathbb{T} \cap e;\mathbb{T}\right] \cup \left[e;\mathbb{T} \cap \overline{z;\mathbb{T}};\mathbb{T};e^{\mathsf{T}}\right] = \overline{e;\mathbb{T}} \cap \overline{z;\mathbb{T}};\mathbb{T};e^{\mathsf{T}}$

$= \overline{e;\mathbb{T} \cup z;\mathbb{T}};\mathbb{T};e^{\mathsf{T}}$ masking □

Fig. 7.4.5 shows these sub-patterns of $\mathcal{D}_\mathrm{s}$ and forms a right residual that is remarkably similar when just using the sub-pattern. On the lower right, one will already recognize primes — a fact that justifies our interest.

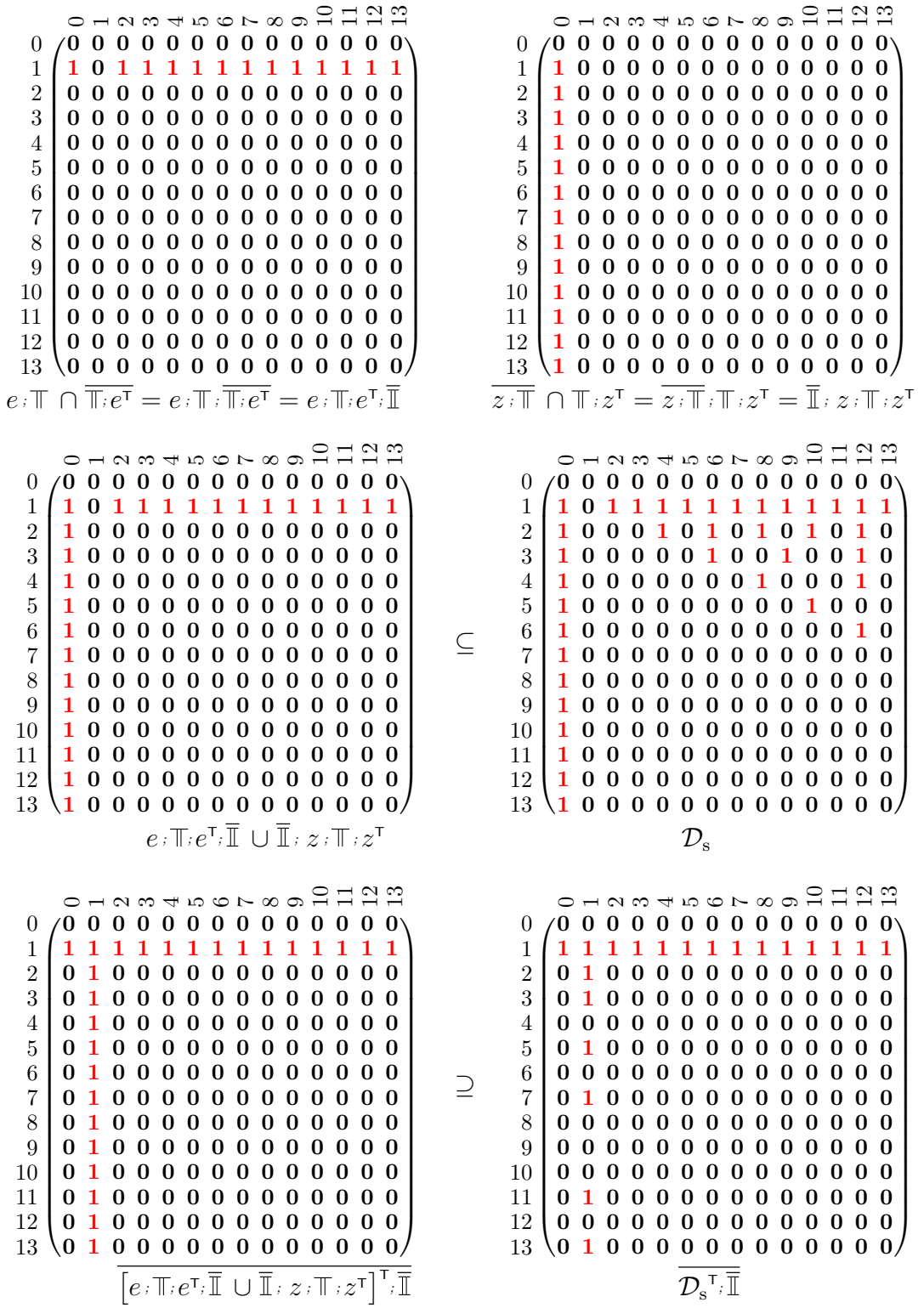


Fig. 7.4.5 Sub-patterns of $\mathcal{D}_\mathrm{s}$ with residuals

7.4.5 Proposition. Given a divides lattice $\mathcal{D}$ and number $\mathbb{S}\,{}^{\intercal n};z;\mathbb{T}$ therein, we define successively

$$\begin{array}{rll} v_n;\mathbb{T} := \mathcal{D}^{\intercal};\mathbb{S}\,{}^{\intercal n};z;\mathbb{T}, & & \text{the non-empty vector denoting the set of numbers divided by } n \\ N_n := \texttt{Extrude}\ v_n;\mathbb{T}, & & \text{this number set extruded} \\ \iota_n := \texttt{Inject}\ v_n, & & \text{the corresponding natural injection satisfying } \iota_n^{\intercal};\mathbb{T} = v_n;\mathbb{T} \\ \mathcal{D}_n := \iota_n;\mathcal{D};\iota_n^{\intercal}. & & \end{array}$$

Then $\mathcal{D}_n;\iota_n = \iota_n;\mathcal{D}$.

Proof: The containment "$\subseteq$" is obvious by construction, because $\mathcal{D}_n;\iota_n = \iota_n;\mathcal{D};\iota_n^{\intercal};\iota_n \subseteq \iota_n;\mathcal{D}$, since ι_n is univalent.

The relation $\mathcal{D}$ is a complete lattice and, thus, reflexive and transitive, so that

$$\mathcal{D}^{\intercal};\iota_n^{\intercal};\mathbb{T} = \mathcal{D}^{\intercal};v_n;\mathbb{T} = \mathcal{D}^{\intercal};\mathcal{D}^{\intercal};\mathbb{S}\,{}^{\intercal n};z;\mathbb{T} = \mathcal{D}^{\intercal};\mathbb{S}\,{}^{\intercal n};z;\mathbb{T} = v_n;\mathbb{T} = \iota_n^{\intercal};\mathbb{T}.$$

Now follows

$$\begin{array}{ll} \mathcal{D}_n;\iota_n = \iota_n;\mathcal{D};\iota_n^{\intercal};\iota_n = \iota_n;\mathcal{D};(\mathbb{I} \cap \iota_n^{\intercal};\mathbb{T}) = \iota_n;\mathcal{D};(\mathbb{I} \cap \mathbb{T};\iota_n) & \text{transposing a partial identity} \\ = \iota_n;\mathcal{D} \cap \mathbb{T};\iota_n & \text{masking} \\ = \iota_n;\mathcal{D} \cap \mathbb{T};\iota_n;\mathcal{D} & \text{see above} \\ \supseteq (\iota_n \cap \mathbb{T};\iota_n);\mathcal{D} = \iota_n;\mathcal{D} & \end{array}$$

□

$\mathcal{D}$ is a complementary lattice: complements of n are all primes not dividing n. These complements are, thus, not uniquely determined.

Whenever one takes some number $\mathbb{S}\,{}^{\intercal n};z;\mathbb{T}$, looks for all numbers it divides and extrudes this set, there comes up a situation as depicted for $n = 4$:

$\iota_4 =$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
0→	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4→	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8→	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12→	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
20→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
24→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
28→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
32→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
36→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
40→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

$\mathcal{D}_4 =$

	0→	4→	8→	12→	16→	20→	24→	28→	32→	36→	40→
0→	1	0	0	0	0	0	0	0	0	0	0
4→	1	1	1	1	1	1	1	1	1	1	1
8→	1	0	1	0	1	0	1	0	1	0	1
12→	1	0	0	1	0	0	1	0	0	1	0
16→	1	0	0	0	1	0	0	0	1	0	0
20→	1	0	0	0	0	1	0	0	0	0	1
24→	1	0	0	0	0	0	1	0	0	0	0
28→	1	0	0	0	0	0	0	1	0	0	0
32→	1	0	0	0	0	0	0	0	1	0	0
36→	1	0	0	0	0	0	0	0	0	1	0
40→	1	0	0	0	0	0	0	0	0	0	1

compare to $\mathcal{D} =$

	0	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	1
2	1	0	1	0	1	0	1	0	1	0	1
3	1	0	0	1	0	0	1	0	0	1	0
4	1	0	0	0	1	0	0	0	1	0	0
5	1	0	0	0	0	1	0	0	0	0	1
6	1	0	0	0	0	0	1	0	0	0	0
7	1	0	0	0	0	0	0	1	0	0	0
8	1	0	0	0	0	0	0	0	1	0	0
9	1	0	0	0	0	0	0	0	0	1	0
10	1	0	0	0	0	0	0	0	0	0	1

on $\mathbb{N}_{10}$

$\mathcal{D}_4 ; \iota_4 = \iota_4 ; \mathcal{D} =$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
0→	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4→	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1
8→	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1
12→	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
16→	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
20→	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
24→	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
28→	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
32→	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
36→	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
40→	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Fig. 7.4.6 Selfsimilarity of $\mathcal{D}$

When given multiplication $\mathbb{M}$, we may also ask for being relatively prime.

7.4.6 Definition. i) Using $\mathcal{D}_1 := \mathcal{D} \cap \overline{e ; \mathbb{T}}$, thus annihilating the full row of 1s at 1, the relation

$\overline{\mathtt{rel}((\mathcal{D}_1^\intercal \otimes \mathcal{D}_1^\intercal) ; \mathbb{T})}$. or better: $\overline{\mathcal{D}^\intercal ; \mathcal{D}_1}$

expresses whether a pair of $\mathbb{N}$ is **relatively prime**.

ii) The $\mathbb{N}$-valued number-theoretic function $f : \mathbb{N} \longrightarrow \mathbb{N}$ is called **$\cap$-hereditary** when $\mathbb{M} ; f \subseteq (f \otimes f) ; \mathbb{M}$ when relatively prime.

iii) The $\mathbb{N}$-valued number-theoretic function $f : \mathbb{N} \longrightarrow \mathbb{N}$ is called **multiplicative** when $\mathbb{M} ; f \subseteq (f \otimes f) ; \mathbb{M}$. □

One should observe that (iii,iv) mean equality when f, $\mathbb{M}$ are mappings; they are, however not when only restricted naturals are considered.

One will get the relation of being relatively prime also when forming

$$\overline{\mathtt{rel}((\mathtt{gcd} \cap \overline{\mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}}) ; \mathbb{T})},$$

i.e. eliminating the 1-column $e = \mathbb{S}^{\mathsf{T}} ; z ; \mathbb{T}$ from gcd in Fig. 7.4.3 and looking for vanishing rows afterwards.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
3	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1
4	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
5	0	1	1	1	1	0	1	1	1	1	0	1	1	1	1	0	1	1	1	1	0	1	1	1	1	0	1	1	1	1
6	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1
7	0	1	1	1	1	1	1	0	1	1	1	1	1	1	0	1	1	1	1	1	1	0	1	1	1	1	1	1	0	1
8	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
9	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1
10	0	1	0	1	0	0	0	1	0	1	0	1	0	1	0	0	0	1	0	1	0	1	0	1	0	0	0	1	0	1
11	0	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1
12	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1
13	0	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1
14	0	1	0	1	0	1	0	0	0	1	0	1	0	1	0	1	0	1	0	1	0	0	0	1	0	1	0	1	0	1
15	0	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0	1	1	0	1	0	0	1	1	0	0	1	0	1	1
16	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
17	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1
18	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1
19	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1
20	0	1	0	1	0	0	0	1	0	1	0	1	0	1	0	0	0	1	0	1	0	1	0	1	0	0	0	1	0	1
21	0	1	1	0	1	1	0	0	1	0	1	1	0	1	0	0	1	1	0	1	1	0	1	1	0	1	1	0	0	1
22	0	1	0	1	0	1	0	1	0	1	0	0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
23	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1
24	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1
25	0	1	1	1	1	0	1	1	1	1	0	1	1	1	1	0	1	1	1	1	0	1	1	1	1	0	1	1	1	1
26	0	1	0	1	0	1	0	1	0	1	0	1	0	0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
27	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1
28	0	1	0	1	0	1	0	0	0	1	0	1	0	1	0	1	0	1	0	1	0	0	0	1	0	1	0	1	0	1
29	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0

Fig. 7.4.7 The symmetric relation $\overline{\mathcal{D}^{\mathsf{T}} ; \mathcal{D}_1}$ of being relatively prime on numbers $0 \ldots 29$

We mention in passing that the lower left (or upper right) half of the above relation has a direct correspondence to the Euler function which is tabulated as:

$n =$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24...
$\varphi(n) =$	1	1	2	2	4	2	6	4	6	4	10	4	12	6	8	8	16	6	18	8	12	10	22	8...

and shows the number of **1** 's in the row of the lower left triangle.

7.4.7 Definition. $\mathtt{Congr} := (\mathbb{I} \cup \mathfrak{P}) ; \mathfrak{S} ; \mathcal{D}^{\mathsf{T}} = \mathbb{X} ; \mathcal{D}^{\mathsf{T}}$ gives the relation **are congruent modulo**. Closely related is a ternary one $(x, y) \mapsto z$, expressing $x \equiv y \bmod z$, or else $z \mid |x - y|$. □

$$\texttt{Congr} = \begin{array}{c} \\ (0,0)\\(0,1)\\(0,2)\\(0,3)\\(0,4)\\(0,5)\\(0,6)\\(0,7)\\(1,0)\\(1,1)\\(1,2)\\(1,3)\\(1,4)\\(1,5)\\(1,6)\\(1,7)\\(2,0)\\(2,1)\\(2,2)\\(2,3)\\(2,4)\\(2,5)\\(2,6)\\(2,7)\\(3,0)\\(3,1)\\(3,2)\\(3,3)\\(3,4)\\(3,5)\\(3,6)\\(3,7)\\(4,0)\\(4,1)\\(4,2)\\(4,3)\\(4,4)\\(4,5)\\(4,6)\\(4,7)\\(5,0)\\(5,1)\\(5,2)\\(5,3)\\(5,4)\\(5,5)\\(5,6)\\(5,7)\\(6,0)\\(6,1)\\(6,2)\\(6,3)\\(6,4)\\(6,5)\\(6,6)\\(6,7)\\(7,0)\\(7,1)\\(7,2)\\(7,3)\\(7,4)\\(7,5)\\(7,6)\\(7,7) \end{array} \begin{array}{c} \begin{array}{cccccccc} 0&1&2&3&4&5&6&7 \end{array} \\ \begin{pmatrix} 1&1&1&1&1&1&1&1\\ 0&1&0&0&0&0&0&0\\ 0&1&1&0&0&0&0&0\\ 0&1&0&1&0&0&0&0\\ 0&1&1&0&1&0&0&0\\ 0&1&0&0&0&1&0&0\\ 0&1&1&1&0&0&1&0\\ 0&1&0&0&0&0&0&1\\ 0&1&0&0&0&0&0&0\\ 1&1&1&1&1&1&1&1\\ 0&1&0&0&0&0&0&0\\ 0&1&1&0&0&0&0&0\\ 0&1&0&1&0&0&0&0\\ 0&1&1&0&1&0&0&0\\ 0&1&0&0&0&1&0&0\\ 0&1&1&1&0&0&1&0\\ 0&1&1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 1&1&1&1&1&1&1&1\\ 0&1&0&0&0&0&0&0\\ 0&1&1&0&0&0&0&0\\ 0&1&0&1&0&0&0&0\\ 0&1&1&0&1&0&0&0\\ 0&1&0&0&0&1&0&0\\ 0&1&1&1&0&0&1&0\\ 0&1&1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 1&1&1&1&1&1&1&1\\ 0&1&0&0&0&0&0&0\\ 0&1&1&0&0&0&0&0\\ 0&1&0&1&0&0&0&0\\ 0&1&1&0&1&0&0&0\\ 0&1&0&0&0&1&0&0\\ 0&1&1&1&0&0&1&0\\ 0&1&1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 1&1&1&1&1&1&1&1\\ 0&1&0&0&0&0&0&0\\ 0&1&1&0&0&0&0&0\\ 0&1&0&1&0&0&0&0\\ 0&1&1&0&1&0&0&0\\ 0&1&0&0&0&1&0&0\\ 0&1&1&1&0&0&1&0\\ 0&1&1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 1&1&1&1&1&1&1&1 \end{pmatrix} \end{array}$$

$$\mathbb{S}^{\tau 2};z = \begin{array}{c}0\\1\\2\\3\\4\\5\\6\\7\end{array}\begin{pmatrix}0\\0\\1\\0\\0\\0\\0\\0\end{pmatrix} \quad \mathbb{S}^{\tau 3};z = \begin{pmatrix}0\\0\\0\\1\\0\\0\\0\\0\end{pmatrix} \quad \mathbb{S}^{\tau 4};z = \begin{pmatrix}0\\0\\0\\0\\1\\0\\0\\0\end{pmatrix}$$

$$\mathbb{I} = \texttt{rel}(\,\texttt{Congr}\,;\mathbb{S}^{\tau 0};z)$$
$$\mathbb{T} = \texttt{rel}(\,\texttt{Congr}\,;\mathbb{S}^{\tau 1};z)$$

$$\begin{array}{c}\begin{array}{cccccccc}0&1&2&3&4&5&6&7\end{array}\\ \begin{pmatrix}1&0&1&0&1&0&1&0\\0&1&0&1&0&1&0&1\\1&0&1&0&1&0&1&0\\0&1&0&1&0&1&0&1\\1&0&1&0&1&0&1&0\\0&1&0&1&0&1&0&1\\1&0&1&0&1&0&1&0\\0&1&0&1&0&1&0&1\end{pmatrix}\end{array} = \texttt{rel}(\,\texttt{Congr}\,;\mathbb{S}^{\tau 2};z)$$

$$\begin{array}{c}\begin{array}{cccccccc}0&1&2&3&4&5&6&7\end{array}\\ \begin{pmatrix}1&0&0&1&0&0&1&0\\0&1&0&0&1&0&0&1\\0&0&1&0&0&1&0&0\\1&0&0&1&0&0&1&0\\0&1&0&0&1&0&0&1\\0&0&1&0&0&1&0&0\\1&0&0&1&0&0&1&0\\0&1&0&0&1&0&0&1\end{pmatrix}\end{array} = \texttt{rel}(\,\texttt{Congr}\,;\mathbb{S}^{\tau 3};z)$$

$$\begin{array}{c}\begin{array}{cccccccc}0&1&2&3&4&5&6&7\end{array}\\ \begin{pmatrix}1&0&0&0&1&0&0&0\\0&1&0&0&0&1&0&0\\0&0&1&0&0&0&1&0\\0&0&0&1&0&0&0&1\\1&0&0&0&1&0&0&0\\0&1&0&0&0&1&0&0\\0&0&1&0&0&0&1&0\\0&0&0&1&0&0&0&1\end{pmatrix}\end{array} = \texttt{rel}(\,\texttt{Congr}\,;\mathbb{S}^{\tau 4};z)$$

$$\begin{array}{c}\begin{array}{cccccccc}0&1&2&3&4&5&6&7\end{array}\\ \begin{pmatrix}1&0&0&0&0&1&0&0\\0&1&0&0&0&0&1&0\\0&0&1&0&0&0&0&1\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\1&0&0&0&0&1&0&0\\0&1&0&0&0&0&1&0\\0&0&1&0&0&0&0&1\end{pmatrix}\end{array} = \texttt{rel}(\,\texttt{Congr}\,;\mathbb{S}^{\tau 5};z)$$

$$\begin{array}{c}\begin{array}{cccccccc}0&1&2&3&4&5&6&7\end{array}\\ \begin{pmatrix}1&0&0&0&0&0&1&0\\0&1&0&0&0&0&0&1\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\1&0&0&0&0&0&1&0\\0&1&0&0&0&0&0&1\end{pmatrix}\end{array} = \texttt{rel}(\,\texttt{Congr}\,;\mathbb{S}^{\tau 6};z)$$

$$\begin{array}{c}\begin{array}{cccccccc}0&1&2&3&4&5&6&7\end{array}\\ \begin{pmatrix}1&0&0&0&0&0&0&1\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\1&0&0&0&0&0&0&1\end{pmatrix}\end{array} = \texttt{rel}(\,\texttt{Congr}\,;\mathbb{S}^{\tau 7};z)$$

Fig. 7.4.8 Congruence relation `Congr`; its columns determine, e.g, the congruences modulo 2, 3, ...

The absolute value of the difference is given by the map $\mathbb{X} = (\mathbb{I} \cup \mathfrak{P}) ; \mathfrak{S}$. Then divisors are looked for.

An example is provided by Fig. 7.4.8. It rightly shows rows constantly filled with **1**s in case both components are equal. Also $(3, 7)$, e.g., turn out to be congruent modulo the numbers $1, 2, 4$.

In addition to the three cases of congruence modulo $2, 3, 4$ visualized in Fig. 7.4.8, we may state that — in the sense of `Congr` — congruence modulo 0 is identity and congruence modulo 1 is the universal relation.

7.4.8 Proposition. We assume $\mathbb{S}$ to be regressively finite.

i) $\mathbb{T} = (\mathbb{I} \oslash \mathbb{I}) ; \mathtt{Congr}$

ii) $\mathtt{Congr} ; e ; \mathbb{T} = \mathbb{T}$

iii) $\mathfrak{P} ; \mathtt{Congr} = \mathtt{Congr}$

iv) $\mathtt{FlipPairsTD} ; \mathtt{Congr} = \mathtt{Congr}$

v) If $\mathbb{S}^{\mathsf{T}^n} ; z ; \mathbb{T}$ is a point, $\Xi := \mathtt{rel}(\mathtt{Congr} ; \mathbb{S}^{\mathsf{T}^n} ; z ; \mathbb{T}) = (\mathbb{S}^{\mathsf{T}^n})^* \cup (\mathbb{S}^n)^*$ is an equivalence.

Proof: i) $(\mathbb{I} \oslash \mathbb{I}) ; \mathtt{Congr} = (\mathbb{I} \oslash \mathbb{I}) ; \mathbb{X} ; \mathcal{D}^{\mathsf{T}} = \mathbb{T} ; z^{\mathsf{T}} ; \mathcal{D}^{\mathsf{T}} = \mathbb{T}$ Prop. 6.3.1.ii.b and Prop. 7.4.2.iii

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ii) $\mathtt{Congr} ; e ; \mathbb{T} = \mathbb{X} ; \mathcal{D}^{\mathsf{T}} ; e ; \mathbb{T}$ by definition
$= \mathbb{X} ; \mathbb{T}$ Prop. 7.4.2.iii
$= \mathbb{T}$ since $\mathbb{X}$ is a map, Prop. 6.3.1.ii.f

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iii) $\mathfrak{P} ; \mathtt{Congr} = \mathfrak{P} ; \mathbb{X} ; \mathcal{D}^{\mathsf{T}} = \mathbb{X} ; \mathcal{D}^{\mathsf{T}} = \mathtt{Congr}$ Prop. 6.3.1.i.h

---

iv) $\mathtt{FlipPairsTD} ; \mathtt{Congr} = (\mathtt{FlipTD} \otimes \mathtt{FlipTD}) ; (\mathbb{I} \cup \mathfrak{P}) ; \mathfrak{S} ; \mathcal{D}^{\mathsf{T}}$
$= (\mathbb{I} \cup \mathfrak{P}) ; (\mathtt{FlipTD} \otimes \mathtt{FlipTD}) ; \mathfrak{S} ; \mathcal{D}^{\mathsf{T}}$ Prop. 5.2.1.iii.j
$= (\mathbb{I} \cup \mathfrak{P}) ; \mathfrak{P} ; \mathfrak{S} ; \mathcal{D}^{\mathsf{T}}$ Prop. 6.3.1.iii.c
$= (\mathfrak{P} \cup \mathbb{I}) ; \mathfrak{S} ; \mathcal{D}^{\mathsf{T}} = \mathtt{Congr}$

---

v) $\Xi = \mathtt{rel}(\mathtt{Congr} ; \mathbb{S}^{\mathsf{T}^n} ; z ; \mathbb{T}) = \mathtt{rel}(\mathbb{X} ; \mathcal{D}^{\mathsf{T}} ; \mathbb{S}^{\mathsf{T}^n} ; z ; \mathbb{T})$
$= \mathtt{rel}(\mathbb{X} ; (\mathbb{S}^{\mathsf{T}^n})^* ; z ; \mathbb{T})$ Prop. 7.4.2.iii
$= \mathtt{rel}(\mathtt{sup}_{r \geq 0}\, \mathbb{X} ; (\mathbb{S}^{\mathsf{T}^n})^r ; z ; \mathbb{T}) = \mathtt{sup}_{r \geq 0}\, \mathtt{rel}(\mathbb{X} ; \mathbb{S}^{\mathsf{T}^{n*r}} ; z ; \mathbb{T})$
$= \mathtt{sup}_{r \geq 0}\, \mathtt{rel}((\mathbb{I} \cup \mathfrak{P}) ; \mathfrak{S} ; \mathbb{S}^{\mathsf{T}^{n*r}} ; z ; \mathbb{T})$
$= \mathtt{sup}_{r \geq 0}\, \mathtt{rel}((\mathbb{I} \cup \mathfrak{P}) ; \mathtt{sup}_{l \geq 0}\, (\mathbb{S}^{\mathsf{T}^l} \oslash \mathbb{S}^{\mathsf{T}^l} ; z ; \mathbb{T}) ; \mathbb{S}^{\mathsf{T}^{n*r}} ; z ; \mathbb{T})$
$= \mathtt{sup}_{r \geq 0}\, \mathtt{sup}_{l \geq 0}\, \mathtt{rel}((\mathbb{I} \cup \mathfrak{P}) ; (\mathbb{S}^{\mathsf{T}^l} \oslash \mathbb{S}^{\mathsf{T}^l} ; z ; \mathbb{T}) ; \mathbb{S}^{\mathsf{T}^{n*r}} ; z ; \mathbb{T})$
$= \mathtt{sup}_{r \geq 0}\, \mathtt{sup}_{l \geq 0}\, \mathtt{rel}((\mathbb{I} \cup \mathfrak{P}) ; (\mathbb{S}^{\mathsf{T}^l} ; \mathbb{S}^{\mathsf{T}^{n*r}} ; z ; \mathbb{T} \oslash \mathbb{S}^{\mathsf{T}^l} ; z ; \mathbb{T}))$ masking a join
$= \mathtt{sup}_{r \geq 0}\, \mathtt{sup}_{l \geq 0}\, \big[\mathtt{rel}((\mathbb{S}^{\mathsf{T}^{l+n*r}} ; z ; \mathbb{T} \oslash \mathbb{S}^{\mathsf{T}^l} ; z ; \mathbb{T})) \cup \mathtt{rel}((\mathbb{S}^{\mathsf{T}^l} ; z ; \mathbb{T} \oslash \mathbb{S}^{\mathsf{T}^{l+n*r}} ; z ; \mathbb{T}))\big]$
$= \mathtt{sup}_{l \geq 0}\, \mathtt{sup}_{r \geq 0}\, \big[\mathbb{S}^{\mathsf{T}^{l+n*r}} ; z ; \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^l \cup \mathbb{S}^{\mathsf{T}^l} ; z ; \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{l+n*r}\big]$ Prop. 2.2.6.v
$= \mathtt{sup}_{l \geq 0}\, \mathtt{sup}_{r \geq 0}\, \mathbb{S}^{\mathsf{T}^l} ; \mathbb{S}^{\mathsf{T}^{n*r}} ; z ; \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^l \cup \mathbb{S}^{\mathsf{T}^l} ; z ; \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{n*r} ; \mathbb{S}^l$

$$
\begin{aligned}
&= \left\{ \mathtt{sup}_{l\geq 0} \left[\mathtt{sup}_{r\geq 0} \, \mathbb{S}^{\intercal l} \fatsemi \mathbb{S}^{\intercal n*r} \fatsemi z \fatsemi \mathbb{T} \fatsemi z^{\intercal} \fatsemi \mathbb{S}^{l} \right] \right\} \cup \left\{ \mathtt{sup}_{i\geq 0} \left[\mathtt{sup}_{k\geq 0} \, \mathbb{S}^{\intercal i} \fatsemi z \fatsemi \mathbb{T} \fatsemi z^{\intercal} \fatsemi \mathbb{S}^{n*k} \fatsemi \mathbb{S}^{i} \right] \right\} \\
&= \left\{ \mathtt{sup}_{r\geq 0} \left[\mathtt{sup}_{l\geq 0} \, \mathbb{S}^{\intercal n*r} \fatsemi \mathbb{S}^{\intercal l} \fatsemi z \fatsemi \mathbb{T} \fatsemi z^{\intercal} \fatsemi \mathbb{S}^{l} \right] \right\} \cup \left\{ \mathtt{sup}_{i\geq 0} \left[\mathtt{sup}_{k\geq 0} \, \mathbb{S}^{\intercal i} \fatsemi z \fatsemi \mathbb{T} \fatsemi z^{\intercal} \fatsemi \mathbb{S}^{i} \fatsemi \mathbb{S}^{n*k} \right] \right\} \\
&= \left\{ \mathtt{sup}_{r\geq 0} \, \mathbb{S}^{\intercal n*r} \fatsemi \left[\mathtt{sup}_{l\geq 0} \, \mathbb{S}^{\intercal l} \fatsemi z \fatsemi \mathbb{T} \fatsemi z^{\intercal} \fatsemi \mathbb{S}^{l} \right] \right\} \cup \left\{ \left[\mathtt{sup}_{i\geq 0} \, \mathbb{S}^{\intercal i} \fatsemi z \fatsemi \mathbb{T} \fatsemi z^{\intercal} \fatsemi \mathbb{S}^{i} \right] \fatsemi \mathtt{sup}_{k\geq 0} \, \mathbb{S}^{n*k} \right\} \\
&= \left\{ \mathtt{sup}_{r\geq 0} \, \mathbb{S}^{\intercal n*r} \fatsemi \mathbb{I} \right\} \cup \left\{ \mathbb{I} \fatsemi \mathtt{sup}_{k\geq 0} \, \mathbb{S}^{n*k} \right\} \quad \text{applying Prop. 5.2.2.ii.a twice} \\
&= (\mathbb{S}^{\intercal n})^{*} \cup (\mathbb{S}^{n})^{*}
\end{aligned}
$$

That Ξ is an equivalence follows from Prop. 5.2.2.ii.d. □

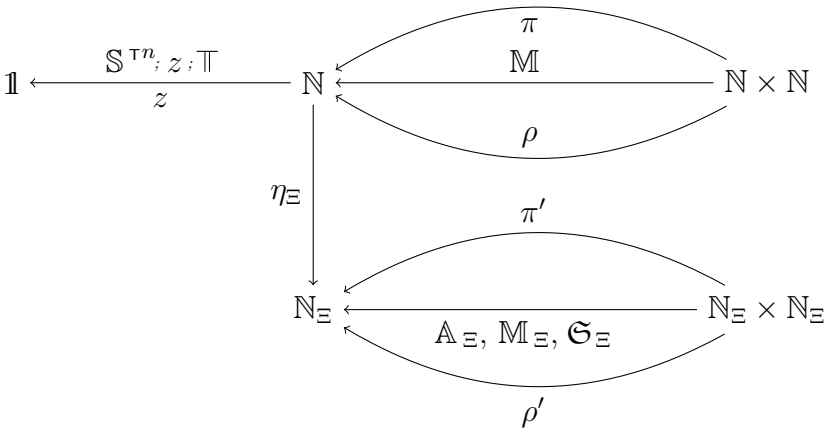


Fig. 7.4.9 Typing around the modular arithmetics

It seems that we have operations modulo $\mathbb{S}^{\intercal n} \fatsemi z \fatsemi \mathbb{T}$ fully available with

$\Xi := \Xi_n := \mathtt{rel}(\, \mathtt{Congr} \fatsemi \mathbb{S}^{\intercal n} \fatsemi z \fatsemi \mathbb{T})$
$\eta_{\Xi} := \mathtt{Project}\ \Xi$
$\mathbb{A}_{\Xi} := (\eta_{\Xi}^{\intercal} \otimes \eta_{\Xi}^{\intercal}) \fatsemi \mathbb{A} \fatsemi \eta_{\Xi}$
$\mathbb{M}_{\Xi} := (\eta_{\Xi}^{\intercal} \otimes \eta_{\Xi}^{\intercal}) \fatsemi \mathbb{M} \fatsemi \eta_{\Xi}$
$\mathfrak{S}_{\Xi} := (\eta_{\Xi}^{\intercal} \otimes \eta_{\Xi}^{\intercal}) \fatsemi \mathfrak{S} \fatsemi \eta_{\Xi}$
etc.

As one will see ahead, these attempts are insufficient. It is, however, possible to improve a binary relational operation A as follows

$A \mapsto (\mathbb{I} \oslash A^{\intercal}) \fatsemi assoc \fatsemi (A \otimes \mathbb{I}) \fatsemi A$

For the typing of the following, we refer to Fig. 7.4.8:

7.4.9 Proposition. Assuming the relation $\mathtt{Congr}$ together with a point $\mathbb{S}^{\intercal n} \fatsemi z \fatsemi \mathbb{T}$ for $n > 1$ in view of vectorization as in Prop. 7.4.8.v

$\mathtt{rel}(\, \mathtt{Congr} \fatsemi \mathbb{S}^{\intercal n} \fatsemi z \fatsemi \mathbb{T}) = (\mathbb{S}^{\intercal n})^{*} \cup (\mathbb{S}^{n})^{*}$

and denoting n-congruence and its natural projection as

$\Xi := \Xi_n := (\mathbb{S}^{\intercal n})^{*} \cup (\mathbb{S}^{n})^{*} \qquad \eta := \eta_n := \mathtt{Project}\ \Xi,$

we proceed to defining

$\lambda := \eta^{\intercal} \fatsemi \mathtt{rel}(\, (\mathbb{M} \oslash \mathbb{T} \fatsemi e^{\intercal}) \fatsemi \mathtt{Congr} \fatsemi \mathbb{S}^{\intercal n} \fatsemi z \fatsemi \mathbb{T}) \fatsemi \eta.$

If n is prime, λ will be an injective and univalent relation — a matching of classes of Ξ.

Proof: by a longer evaluation

$\lambda = \eta^\intercal ; \mathtt{rel}(\,(\,\mathbb{M} \oslash \mathbb{T} ; e^\intercal) ; \mathtt{Congr} ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T}) ; \eta$
$= \eta^\intercal ; \mathtt{rel}(\,(\,\mathbb{M} \oslash \mathbb{T} ; e^\intercal) ; \mathtt{vec}((\mathbb{S}^{\intercal n})^* \cup (\mathbb{S}^n)^*)) ; \eta$ since $\mathtt{vec}(\mathtt{rel}(v)) = v$
$= \eta^\intercal ; \mathtt{rel}(\,(\,\mathbb{M} \oslash \mathbb{T} ; e^\intercal) \,;\, ((\mathbb{S}^{\intercal n})^* \cup (\mathbb{S}^n)^* \oslash \mathbb{I}) ; \mathbb{T}) ; \eta$ Prop. 2.2.5
$= \eta^\intercal ; \mathtt{rel}(\big[\,\mathbb{M} ; \{(\mathbb{S}^{\intercal n})^* \cup (\mathbb{S}^n)^*\} \cap \mathbb{T} ; e^\intercal\big] ; \mathbb{T}) ; \eta$
$= \eta^\intercal ; \mathtt{rel}(\,\mathbb{M} ; \big[(\mathbb{S}^{\intercal n})^* \cup (\mathbb{S}^n)^*\big] ; e ; \mathbb{T}) ; \eta$ mask shift
$= \eta^\intercal ; \mathtt{rel}(\,\mathbb{M} ; (\mathbb{S}^{\intercal n})^* ; e ; \mathbb{T}) ; \eta \;\cup\; \eta^\intercal ; \mathtt{rel}(\,\mathbb{M} ; (\mathbb{S}^n)^* ; e ; \mathbb{T}) ; \eta$

The second term with $n > 1$ is rather easy to evaluate:

$\mathbb{M} ; (\mathbb{S}^n)^* ; e ; \mathbb{T} = \mathbb{M} ; e ; \mathbb{T} \cup \mathtt{sup}_{i>0}\; \mathbb{M} ; \mathbb{S}^{n*i} ; e ; \mathbb{T} = \mathbb{M} ; e ; \mathbb{T} \cup \mathbb{\perp}$

It is $\mathbb{\perp}$, since the remaining $\mathbb{M} ; e ; \mathbb{T}$ is already contained in the first.

Now the first term that obviously includes the $\mathbb{M} ; e ; \mathbb{T}$ just left out:

$\mathtt{rel}(\,\mathbb{M} ; (\mathbb{S}^{\intercal n})^* ; e ; \mathbb{T}) = \pi^\intercal ; \big[\rho \cap \mathbb{M} ; (\mathbb{S}^{\intercal n})^* ; e ; \mathbb{T}\big]$ due to Prop. 2.2.5
$= \pi^\intercal ; \big[\,\rho \cap \mathtt{sup}_{i\geq 0}\; (\mathbb{S}^{\intercal i} ; z ; \mathbb{T} \oslash k_i) ; (\mathbb{S}^{\intercal n})^* ; e ; \mathbb{T}\big]$ Prop. 7.1.1.i.a
$= \mathtt{sup}_{i\geq 0}\, \pi^\intercal ; \big[\,\rho \cap\; (\mathbb{S}^{\intercal i} ; z ; \mathbb{T} \oslash k_i ;\; (\mathbb{S}^{\intercal n})^* ; e ; \mathbb{T})\;\big]$ masking a join
$= \mathtt{sup}_{i\geq 0}\, \pi^\intercal ; \big[\,\pi ; \mathbb{S}^{\intercal i} ; z ; \mathbb{T} \cap \rho ; (\mathbb{I} \cap k_i ;\; (\mathbb{S}^{\intercal n})^* ; e ; \mathbb{T}\big]$
$= \mathtt{sup}_{i\geq 0}\, \mathbb{S}^{\intercal i} ; z ; \mathbb{T} \cap \mathbb{T} ; (\mathbb{I} \cap k_i ;\; (\mathbb{S}^{\intercal n})^* ; e ; \mathbb{T})$ destroy and append
$= \mathtt{sup}_{i\geq 0}\, \mathbb{S}^{\intercal i} ; z ; \mathbb{T} \cap \mathbb{T} ; e^\intercal ; (\mathbb{S}^n)^* ; k_i^\intercal$ mask shift
$= \mathtt{sup}_{i\geq 0}\, \mathbb{S}^{\intercal i} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S} ;\; (\mathbb{S}^n)^* ; k_i^\intercal$ masking and expansion of e
$= \mathtt{sup}_{i\geq 0}\, \mathtt{sup}_{l\geq 0}\, \mathbb{S}^{\intercal i} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S} ; \mathbb{S}^{n*l} ; k_i^\intercal$
$= \mathtt{sup}_{i\geq 1}\, \mathtt{sup}_{l\geq 0}\, \mathbb{S}^{\intercal i} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{1+n*l} ; k_i^\intercal$ observing that $\mathbb{S} ; k_0^\intercal = \mathbb{S} ; z ; \mathbb{T} = \mathbb{\perp}$
$= \mathtt{sup}_{i\geq 1}\, \mathtt{sup}_{l\geq 0,\ \text{where } i|(1+n*l)}\, \mathbb{S}^{\intercal i} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{\frac{1+n*l}{i}}$ Lemma 6.4.2.i.c

All these are entries $x ; \mathbb{T} ; y^\intercal$. For every i exists at most one l, because

$\frac{1+n*l_1}{i} = \frac{1+n*l_2}{i} \implies 1 + n * l_1 = 1 + n * l_2 \implies n * l_1 = n * l_2 \implies l_1 = l_2$;

therefore the relation λ is univalent and since symmetric also injective — also when n should happen not to be prime. Non-univalence can be introduced by quotient forming with η. □

We show examples of the relation on limited naturals in Figs. 7.4.10:

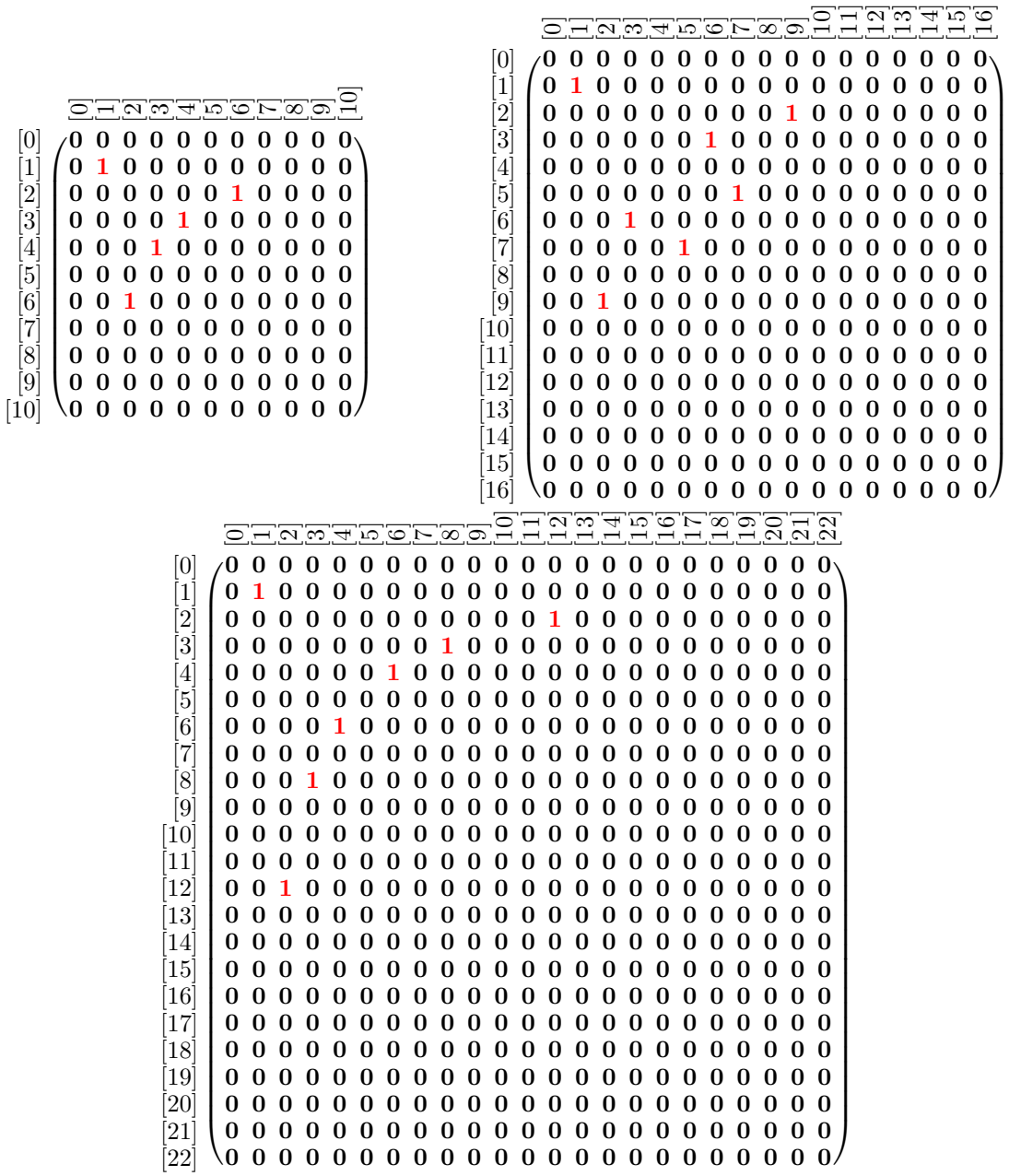


Fig. 7.4.10 Matching in connection with Wilson's Theorem with $p = 11, 17$ and 23

↑ limited by n =? □

7.5 Canceling

To understand canceling in the present context, we consider two fractions, e.g. $\frac{3}{2}$ and $\frac{6}{4}$, which obviously mean the same rational number, justified by $3*4 = 12 = 2*6$. But we must not use numbers beyond our limit — here e.g. 11 — when proceeding constructively with relations. Thus, we better switch to the quotients. These quotients will stay inside the limits.

The following matrix shows how we intend the relation `cancel` to look like. We are still indifferent as to design the first row. There should obviously be empty lines whenever the denominator is 0. Canceling may certainly be proved to be a univalent relation.

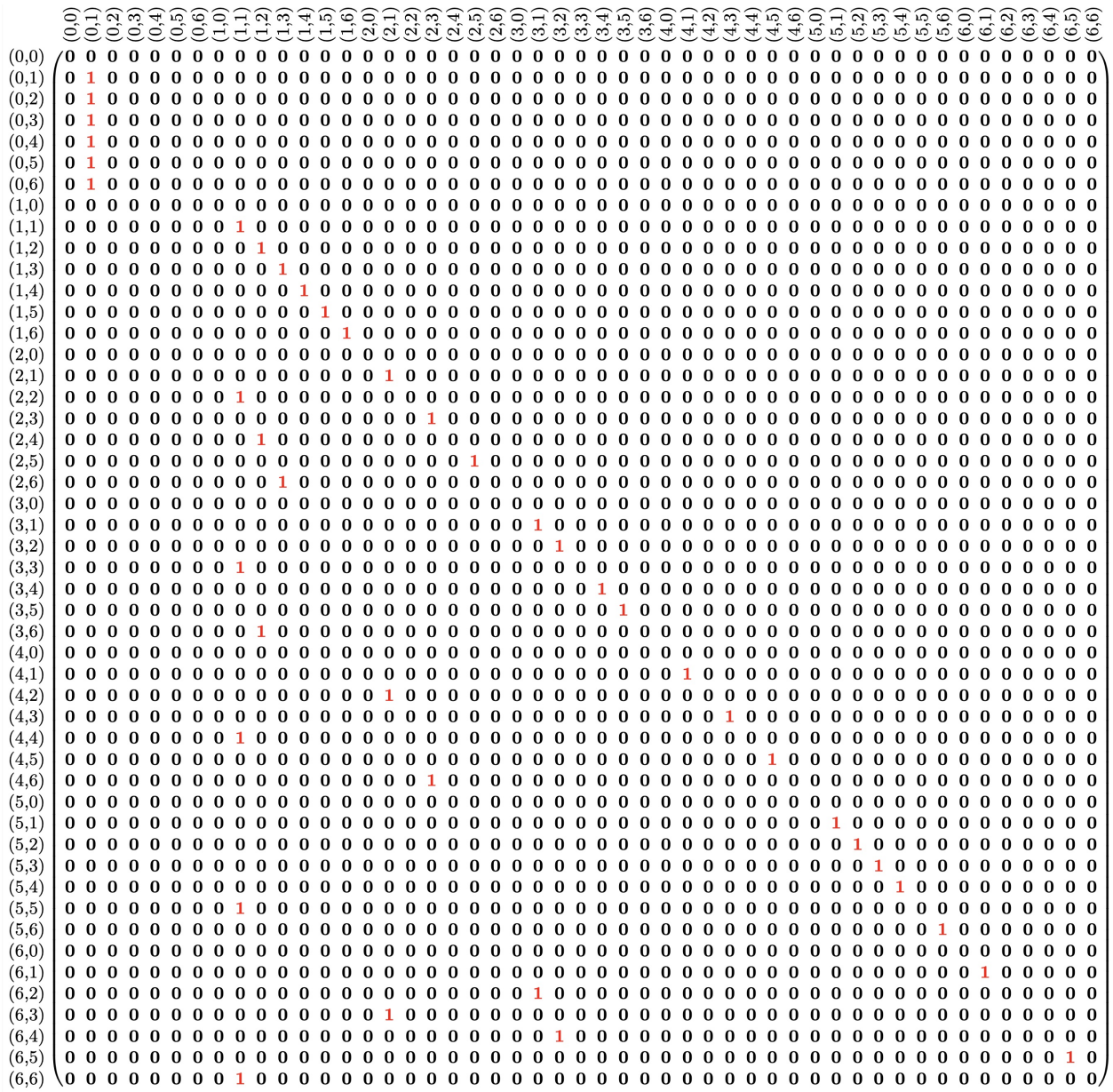


Fig. 7.5.1 Canceling `cancel` as intended

Fig. 7.5.2 shows further work with canceling, which is related to the introduction of rationals. A certain modification of the cancel relation, namely, leads finally to an equivalence, and, thus, to the projection on quotients.

$\mathcal{D}_0^\intercal$

	0	1	2	3	4
0	0	0	0	0	0
1	0	1	0	0	0
2	0	1	1	0	0
3	0	1	0	1	0
4	0	1	1	0	1

C'

	(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	(1,0)	(1,1)	(1,2)	(1,3)	(1,4)	(2,0)	(2,1)	(2,2)	(2,3)	(2,4)	(3,0)	(3,1)	(3,2)	(3,3)	(3,4)	(4,0)	(4,1)	(4,2)	(4,3)	(4,4)
(0,0)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(0,1)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(0,2)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(0,3)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(0,4)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(1,0)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(1,1)	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(1,2)	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(1,3)	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(1,4)	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(2,0)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(2,1)	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
(2,2)	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
(2,3)	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
(2,4)	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
(3,0)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(3,1)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
(3,2)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
(3,3)	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
(3,4)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
(4,0)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(4,1)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
(4,2)	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0
(4,3)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
(4,4)	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1

$\Xi := C'; C'^\intercal \cup \overline{C'; \mathbb{T}}; \overline{C'; \mathbb{T}}^\intercal$

	(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	(1,0)	(1,1)	(1,2)	(1,3)	(1,4)	(2,0)	(2,1)	(2,2)	(2,3)	(2,4)	(3,0)	(3,1)	(3,2)	(3,3)	(3,4)	(4,0)	(4,1)	(4,2)	(4,3)	(4,4)
(0,0)	1	1	1	1	1	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0
(0,1)	1	1	1	1	1	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0
(0,2)	1	1	1	1	1	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0
(0,3)	1	1	1	1	1	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0
(0,4)	1	1	1	1	1	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0
(1,0)	1	1	1	1	1	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0
(1,1)	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	1
(1,2)	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
(1,3)	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(1,4)	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(2,0)	1	1	1	1	1	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0
(2,1)	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0
(2,2)	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	1
(2,3)	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
(2,4)	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
(3,0)	1	1	1	1	1	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0
(3,1)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
(3,2)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
(3,3)	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1
(3,4)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
(4,0)	1	1	1	1	1	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0
(4,1)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
(4,2)	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0
(4,3)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
(4,4)	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1

η_Ξ

	[(0,0)]	[(1,1)]	[(1,2)]	[(1,3)]	[(1,4)]	[(2,1)]	[(2,3)]	[(3,1)]	[(3,2)]	[(3,4)]	[(4,1)]	[(4,3)]
(0,0)	1	0	0	0	0	0	0	0	0	0	0	0
(0,1)	1	0	0	0	0	0	0	0	0	0	0	0
(0,2)	1	0	0	0	0	0	0	0	0	0	0	0
(0,3)	1	0	0	0	0	0	0	0	0	0	0	0
(0,4)	1	0	0	0	0	0	0	0	0	0	0	0
(1,0)	1	0	0	0	0	0	0	0	0	0	0	0
(1,1)	0	1	0	0	0	0	0	0	0	0	0	0
(1,2)	0	0	1	0	0	0	0	0	0	0	0	0
(1,3)	0	0	0	1	0	0	0	0	0	0	0	0
(1,4)	0	0	0	0	1	0	0	0	0	0	0	0
(2,0)	1	0	0	0	0	0	0	0	0	0	0	0
(2,1)	0	0	0	0	0	1	0	0	0	0	0	0
(2,2)	0	1	0	0	0	0	0	0	0	0	0	0
(2,3)	0	0	0	0	0	0	1	0	0	0	0	0
(2,4)	0	0	1	0	0	0	0	0	0	0	0	0
(3,0)	1	0	0	0	0	0	0	0	0	0	0	0
(3,1)	0	0	0	0	0	0	0	1	0	0	0	0
(3,2)	0	0	0	0	0	0	0	0	1	0	0	0
(3,3)	0	1	0	0	0	0	0	0	0	0	0	0
(3,4)	0	0	0	0	0	0	0	0	0	1	0	0
(4,0)	1	0	0	0	0	0	0	0	0	0	0	0
(4,1)	0	0	0	0	0	0	0	0	0	0	1	0
(4,2)	0	0	0	0	0	1	0	0	0	0	0	0
(4,3)	0	0	0	0	0	0	0	0	0	0	0	1
(4,4)	0	1	0	0	0	0	0	0	0	0	0	0

Fig. 7.5.2 Relations involved in fraction canceling for naturals limited by 4

A first approach is this: We get the cancel relation C' (or `cancel'`) using the distributivity shuffle $\mathfrak{D}$ and the slightly modified divides relation $\mathcal{D}$:

$$C' := (\mathbb{I} \oslash (\mathcal{D}_0^\intercal \oslash \mathcal{D}_0^\intercal)) ; \mathfrak{D} ; (\mathfrak{Q} \otimes \mathfrak{Q}) \quad \text{with} \quad \mathcal{D}_0 := \mathcal{D} \cap \overline{z ; \mathbb{T}}^\intercal \cap \mathbb{S}^*$$

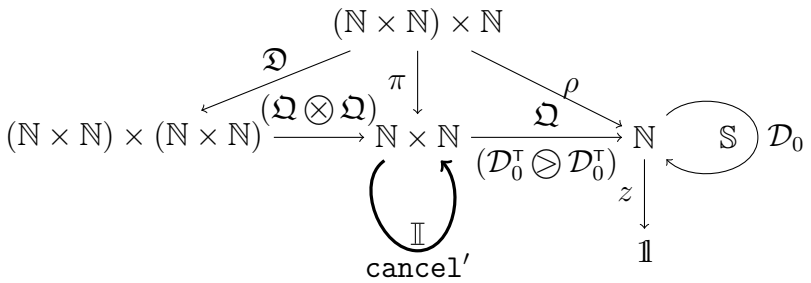


Fig. 7.5.3 Typing around canceling

$C' =$

	(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	(1,0)	(1,1)	(1,2)	(1,3)	(1,4)	(2,0)	(2,1)	(2,2)	(2,3)	(2,4)	(3,0)	(3,1)	(3,2)	(3,3)	(3,4)	(4,0)	(4,1)	(4,2)	(4,3)	(4,4)	
(0,0)	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0
(0,1)	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
(0,2)	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2
(0,3)	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3
(0,4)	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4
(1,0)	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	5
(1,1)	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	6
(1,2)	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	7
(1,3)	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	8
(1,4)	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	9
(2,0)	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	10
(2,1)	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	11
(2,2)	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	12
(2,3)	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	13
(2,4)	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	14
(3,0)	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	15
(3,1)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	16
(3,2)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	17
(3,3)	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	18
(3,4)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	19
(4,0)	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	20
(4,1)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	21
(4,2)	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	22
(4,3)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	23
(4,4)	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	24

	0	1	2	3	4
0	•	1	1	1	1
1	5	6	7	8	9
2	5	11	6	13	7
3	5	16	17	6	19
4	5	21	11	23	6

Fig. 7.5.4 Canceling in ℕ in a first attempt

check again!

The corresponding typing diagram is shown as Fig. 7.5.3. Executing this as a computation, one will obtain the result shown in Fig. 7.5.4. Up to the first row, this is indeed a mapping.

It may not be immediately evident how this works. Consider the pair $(4, 2)$ in Fig. 7.5.2 below. With $\mathbb{I}$ it is preserved and with $(\mathcal{D}_0^\intercal \oslash \mathcal{D}_0^\intercal)$ one looks for common divisors; here the number 2. This is then distributed so as to obtain $((4, 2), (2, 2))$, followed by dividing out the 2 in both cases. This results in the pair $(2, 1)$ — and $(4, 2)$ should one have chosen the common factor 1.

A far better variant, delivering the same result as in Fig. 7.5.4, uses the greatest common divisor mapping `gcd`:

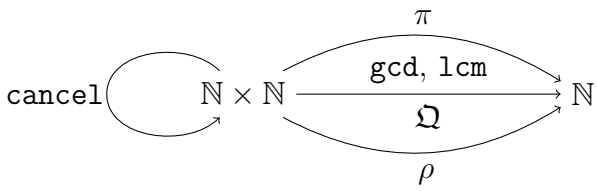
$C := ((\pi \oslash \mathtt{gcd}) ; \mathfrak{Q} \oslash (\rho \oslash \mathtt{gcd}) ; \mathfrak{Q})$

Still better is it to get rid of 0 in second position and use

$C := ((\pi \oslash \mathtt{gcd}) ; \mathfrak{Q} \oslash (\rho \oslash \mathtt{gcd}) ; \mathfrak{Q}) \cap \overline{\rho ; z ; \mathbb{T}}.$

	0	1	2	3	4
0	0	1	2	3	4
1	1	1	1	1	1
2	2	1	2	1	2
3	3	1	1	3	1
4	4	1	2	1	4

gcd



	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	2	0	4
3	0	3	0	3	0
4	0	4	4	0	4

lcm

Fig. 7.5.5 Typing around `gcd` and `lcm`

From here, we define $R := C \cup v ; v^{\mathsf{T}}$ with $v := (\pi \cup \rho) ; z ; \mathbb{T}$ as well as

$\eta := C ; \mathbb{T} \cap \mathtt{Project}(R ; R^{\mathsf{T}})$.

8 Further mathematical concepts

There are, of course further mathematical constructs that should also be considered.

8.1 Exponentiation

We here assume $\mathbb{S}$ to be regressively finite. The following concept prepares exponentiation by forming the n-fold product:

8.1.1 Definition. We define the n-**fold product** relations by

$$p_0 := \mathbb{T}; e^\mathsf{T} \qquad p_{n+1} := (p_n \otimes \mathbb{I}); \mathbb{M}.$$
$$\mathtt{p}_0 := \mathbb{T}; e^\mathsf{T} \qquad \mathtt{p}_{n+1} := (\mathbb{I} \otimes \mathtt{p}_n); N. \qquad \square$$

So p_0, considered as a mapping, will always deliver 1,

$$p_1 = (p_0 \otimes \mathbb{I}); \mathbb{M} = (\mathbb{T}; e^\mathsf{T} \otimes \mathbb{I}); \mathbb{M} = \mathbb{I} \quad \text{Prop. 7.1.1.ii.e,}$$

and one will grasp the meaning of

$$p_2 = (p_1 \otimes \mathbb{I}); \mathbb{M} = (\mathbb{I} \otimes \mathbb{I}); \mathbb{M},$$
$$p_3 = (p_2 \otimes \mathbb{I}); \mathbb{M} = (\,(\mathbb{I} \otimes \mathbb{I}); \mathbb{M} \otimes \mathbb{I}); \mathbb{M}.$$

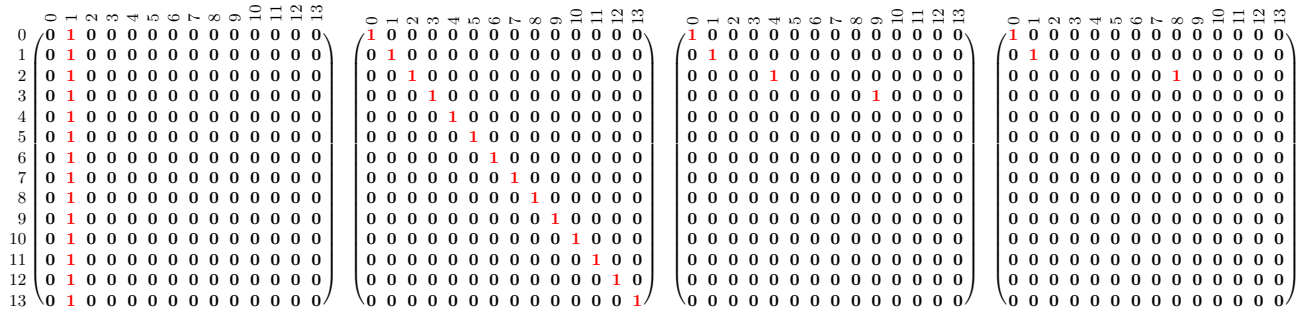


Fig. 8.1.1 Initial sequence p_0, p_1, p_2, p_3 of the n-fold product

8.1.2 Proposition. i) p_n is univalent and $p_n = \mathtt{p}_n$ for $n \geq 0$

ii) $$p_n; \mathbb{S}^{\intercal m}; z; \mathbb{T} = \begin{cases} \mathbb{T} & n = 0, \quad m = 1 \\ \mathbb{\bot} & n = 0, \quad m \neq 1 \\ \mathbb{\bot} & n \geq 1, \quad m \neq l^n \text{ for all } l > 0 \\ \mathbb{S}^{\intercal i}; z; \mathbb{T} & n \geq 1, \quad m = i^n \text{ with } i := \sqrt[n]{m} \end{cases} \quad \text{or} \quad \mathbb{S}^{\intercal m}; z; \mathbb{T} = \mathbb{\bot} \quad \text{and} \quad \mathbb{S}^{\intercal m}; z; \mathbb{T} \neq \mathbb{\bot}$$

iii) $p_0^\mathsf{T} ; \mathbb{S}^{\mathsf{T} i} ; z ; \mathbb{T} = \mathbb{S}^\mathsf{T} ; z ; \mathbb{T} ; \mathbb{S}^{\mathsf{T} i} ; z ; \mathbb{T} = \begin{cases} \mathbb{\perp\!\!\!\perp} & \text{when } \mathbb{S}^{\mathsf{T} i} ; z ; \mathbb{T} = \mathbb{\perp\!\!\!\perp} \\ \mathbb{S}^\mathsf{T} ; z ; \mathbb{T} = e ; \mathbb{T} & \text{otherwise} \end{cases}$

$p_k^\mathsf{T} ; \mathbb{S}^{\mathsf{T} i} ; z ; \mathbb{T} = \mathbb{S}^{\mathsf{T} (i^k)} ; z ; \mathbb{T}$ when $k \geq 1$

iv) $p_n = \mathtt{sup}_{i \geq 0} \ \mathbb{S}^{\mathsf{T} i} ; z ; \mathbb{T} \cap \mathbb{S}^{(i^n - i)}$ for $n \geq 1$
$= \mathtt{sup}_{i \geq 0} \ \mathbb{S}^{\mathsf{T} i} ; z ; \mathbb{T} ; z^\mathsf{T} ; \mathbb{S}^{(i^n)}$ for $n \geq 0$

v) The p_n are total if $\mathbb{S}$ is total.

vi) $(\mathbb{I} \cap e ; \mathbb{T}) ; p_n = e ; e^\mathsf{T} ; p_n = e ; e^\mathsf{T}$ for $n \geq 0$

vii) $p_0 ; p_0^\mathsf{T} = \mathbb{T}$ $p_1 ; p_1^\mathsf{T} = \mathbb{I}$ $p_n ; p_n^\mathsf{T} \subseteq \mathbb{I}$ p_n injective for $n > 0$

$p_2 ; \mathbb{S}^{\mathsf{T} 16} ; z ; \mathbb{T} =$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$=$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	1	1	1	1	1	1	1	1	1	1	1	1	1	1
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$= \mathbb{S}^{\mathsf{T} 4} ; z ; \mathbb{T}$

$p_3 ; \mathbb{S}^{\mathsf{T} 8} ; z ; \mathbb{T} =$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$=$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$= \mathbb{S}^{\mathsf{T} 2} ; z ; \mathbb{T}$

Fig. 8.1.2 Illustrating Prop. 8.1.2.ii for naturals restricted to 13

Proof: i) p_0 is certainly a mapping. Induction:

$p_{n+1}^\mathsf{T} ; p_{n+1} = \mathbb{M}^\mathsf{T} ; (p_n^\mathsf{T} \rhd \mathbb{I}) ; (p_n \lhd \mathbb{I}) ; \mathbb{M} \subseteq \mathbb{M}^\mathsf{T} ; (p_n^\mathsf{T} ; p_n \otimes \mathbb{I}) ; \mathbb{M}$ Prop. 2.1.1.i
$\subseteq \mathbb{M}^\mathsf{T} ; (\mathbb{I} \otimes \mathbb{I}) ; \mathbb{M}$ hypothesis
$\subseteq \mathbb{I}$ since $\mathbb{M}$ is univalent

It is trivial to observe that $p_0 = \mathtt{p}_0$. In addition:

$p_{n+1} = (p_n \lhd \mathbb{I}) ; \mathbb{M} = (p_n \lhd \mathbb{I}) ; \mathfrak{P} ; \mathfrak{P} ; \mathbb{M}$
$= (p_n \lhd \mathbb{I}) ; \mathfrak{P} ; N$ Prop. 7.1.1.i.h

$= (\mathbb{I} \leftthreetimes p_n) ; N$ Prop. 3.1.1
$= (\mathbb{I} \leftthreetimes \mathtt{p}_n) ; N = \mathtt{p}_{n+1}$ hypothesis

---

ii) $p_0 ; \mathbb{S}^{\intercal m} ; z ; \mathbb{T} = \mathbb{T} ; e^{\intercal} ; \mathbb{S}^{\intercal m} ; z ; \mathbb{T} = \mathbb{T} ; z^{\intercal} ; \mathbb{S} ; \mathbb{S}^{\intercal m} ; z ; \mathbb{T} = \begin{cases} \mathbb{T} & \text{in case } m = 1 \\ \perp\!\!\!\perp & \text{otherwise} \end{cases}$

This was because $\mathbb{S} ; z = \perp\!\!\!\perp$. The case $n = 1$ with $p_1 = \mathbb{I}$ and $i := m$ is trivial.[1] For $n > 1$:

$p_{n+1} ; \mathbb{S}^{\intercal m} ; z ; \mathbb{T} = (p_n \leftthreetimes \mathbb{I}) ; \mathbb{M} ; \mathbb{S}^{\intercal m} ; z ; \mathbb{T} = (p_n \leftthreetimes \mathbb{I}) ; \big[\mathtt{sup}_{l \geq 0} \, (\mathbb{S}^{\intercal l} ; z ; \mathbb{T} \leftthreetimes k_l) \, \big] ; \mathbb{S}^{\intercal m} ; z ; \mathbb{T}$
$= \mathtt{sup}_{l \geq 0} \, (p_n \leftthreetimes \mathbb{I}) ; (\mathbb{S}^{\intercal l} ; z ; \mathbb{T} \leftthreetimes k_l) ; \mathbb{S}^{\intercal m} ; z ; \mathbb{T}$
$= \mathtt{sup}_{l \geq 0} \big(p_n ; \mathbb{S}^{\intercal l} ; z ; \mathbb{T} \cap k_l \big) ; \mathbb{S}^{\intercal m} ; z ; \mathbb{T}$
$= \mathtt{sup}_{l \geq 0} \, p_n ; \mathbb{S}^{\intercal l} ; z ; \mathbb{T} \cap k_l ; \mathbb{S}^{\intercal m} ; z ; \mathbb{T}$ masking
$= \mathbb{S}^{\intercal i} ; z ; \mathbb{T} \cap k_{(i^n)} ; \mathbb{S}^{\intercal m} ; z ; \mathbb{T}$ hypothesis; $\neq \perp\!\!\!\perp$ at most when $i := \sqrt[n]{l}$ exists and $\mathbb{S}^{\intercal l} ; z ; \mathbb{T} \neq \perp\!\!\!\perp$
$= \mathbb{S}^{\intercal i} ; z ; \mathbb{T} \cap \mathbb{S}^{\intercal k} ; z ; \mathbb{T}$ Prop. 6.4.2.i.c with $m = k * i^n$ for some k; otherwise $= \perp\!\!\!\perp$
$= \mathbb{S}^{\intercal i} ; z ; \mathbb{T}$ both are points implying $i = k$ and, thus, $m = i * i^n = i^{n+1}$ or $i := \sqrt[n+1]{m}$

---

iii) Proof for $k = 0$ plus induction starting with $k = 1$:

$p_0^{\intercal} ; \mathbb{S}^{\intercal i} ; z ; \mathbb{T} = e ; \mathbb{T} ; \mathbb{S}^{\intercal i} ; z ; \mathbb{T} = \mathbb{S}^{\intercal} ; z ; \mathbb{T} ; \mathbb{S}^{\intercal i} ; z ; \mathbb{T}$, where the last part is either $\perp\!\!\!\perp$ or $\mathbb{T}$

$p_1^{\intercal} ; \mathbb{S}^{\intercal i} ; z ; \mathbb{T} = \mathbb{I} ; \mathbb{S}^{\intercal i} ; z ; \mathbb{T} = \mathbb{S}^{\intercal (i^1)} ; z ; \mathbb{T}$

$p_{k+1}^{\intercal} ; \mathbb{S}^{\intercal i} ; z ; \mathbb{T} = \mathbb{M}^{\intercal} ; (p_k^{\intercal} \leftthreetimes \mathbb{I}) ; \mathbb{S}^{\intercal i} ; z ; \mathbb{T}$ by definition
$= \mathbb{M}^{\intercal} ; (p_k^{\intercal} ; \mathbb{S}^{\intercal i} ; z ; \mathbb{T} \leftthreetimes \mathbb{S}^{\intercal i} ; z ; \mathbb{T})$
$= \mathbb{M}^{\intercal} ; (\mathbb{S}^{\intercal (i^k)} ; z ; \mathbb{T} \leftthreetimes \mathbb{S}^{\intercal i} ; z ; \mathbb{T})$ hypothesis
$= \mathbb{S}^{\intercal (i^k) * i} ; z ; \mathbb{T}$ Prop. 7.1.1.ii.h
$= \mathbb{S}^{\intercal (i^{k+1})} ; z ; \mathbb{T}$

---

iv) The case $n = 0$ needs a separate proof:

$\mathtt{sup}_{i \geq 0} \; \mathbb{S}^{\intercal i} ; z ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{\, i^0} = \big[\mathtt{sup}_{i \geq 0} \; \mathbb{S}^{\intercal i} ; z ; \mathbb{T} \big] ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}$
$= \mathbb{T} ; \mathbb{T} ; e^{\intercal} = p_0$ using regressive finiteness via Prop. 5.2.1.ii.a

The two versions of case $n = 1$:

$\mathtt{sup}_{i \geq 0} \; \mathbb{S}^{\intercal i} ; z ; \mathbb{T} \cap \mathbb{S}^{\, (i^1 - i)} = \big[\mathtt{sup}_{i \geq 0} \, \mathbb{S}^{\intercal i} ; z ; \mathbb{T} \big] \cap \mathbb{S}^{\, 0} = \mathbb{T} \cap \mathbb{I} = p_1$ Prop. 5.2.1.ii.a
$\mathtt{sup}_{i \geq 0} \; \mathbb{S}^{\intercal i} ; z ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{\, i^1} = \mathbb{I} = p_1$ Prop. 5.2.2.ii.a

Induction $n \mapsto n + 1$ starting at $n = 1$:

$p_{n+1} = (p_n \leftthreetimes \mathbb{I}) ; \mathbb{M} = (p_n \leftthreetimes \mathbb{I}) ; \mathtt{sup}_{l \geq 0} \, (\mathbb{S}^{\intercal l} ; z ; \mathbb{T} \leftthreetimes k_l)$
$= \mathtt{sup}_{l \geq 0} \; p_n ; \mathbb{S}^{\intercal l} ; z ; \mathbb{T} \cap k_l$
$= \mathtt{sup}_{l \geq 0} \; k_l \cap \begin{cases} \perp\!\!\!\perp & l \neq q^n \text{ for all } q > 0 \quad \text{or} \quad \mathbb{S}^{\intercal l} ; z ; \mathbb{T} = \perp\!\!\!\perp \\ \mathbb{S}^{\intercal i} ; z ; \mathbb{T} & l = (\sqrt[n]{l})^n \quad \text{with } i := \sqrt[n]{l} \quad \text{and} \quad \mathbb{S}^{\intercal l} ; z ; \mathbb{T} \neq \perp\!\!\!\perp \end{cases}$ following (ii)
$= \mathtt{sup}_{i \geq 0} \; k_{(i^n)} \cap \mathbb{S}^{\intercal i} ; z ; \mathbb{T}$ those with $l \neq i^n$ deleted since they contribute just $\perp\!\!\!\perp$

From now on continued in two versions according to the representation of $k_{(i^n)}$ in 6.4.2.ii.a:

$= \mathtt{sup}_{i \geq 0} \; \big[\mathtt{sup}_{a \geq 0} \, \mathbb{S}^{\intercal a} ; z ; \mathbb{T} \cap \mathbb{S}^{\, a * (i^n - 1)} \big] \cap \mathbb{S}^{\intercal i} ; z ; \mathbb{T}$ thus determining equal points, i.e. $a := i$
$= \mathtt{sup}_{i \geq 0} \; \mathbb{S}^{\intercal i} ; z ; \mathbb{T} \cap \mathbb{S}^{\, i * (i^n - 1)}$

[1] See Prop. 7.1.1.ii.d

$= \mathsf{sup}_{i\geq 0}\ \mathbb{S}^{\intercal i} ; z ; \mathbb{T} \cap \mathbb{S}^{(i^{n+1}-i)}$

or else

$= \mathsf{sup}_{i\geq 0}\ \big[\mathsf{sup}_{a\geq 0}\, \mathbb{S}^{\intercal a} ; z ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{a*i^n}\big] \cap \mathbb{S}^{\intercal i} ; z ; \mathbb{T}$

$= \mathsf{sup}_{i\geq 0}\ \big[\mathsf{sup}_{a\geq 0}\, \mathbb{S}^{\intercal a} ; z ; \mathbb{T} \cap \mathbb{S}^{\intercal i} ; z ; \mathbb{T}\big] ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{a*i^n}$ masking

$= \mathsf{sup}_{i\geq 0}\ \mathbb{S}^{\intercal i} ; z ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{i^{n+1}}$ implying $a := i$ to be $\neq \mathbb{\perp}$

---

v) is proved by induction starting with

$p_0 ; \mathbb{T} = \mathbb{T} ; e^{\intercal} ; \mathbb{T} = \mathbb{T}$ since e is a point

$p_{n+1} ; \mathbb{T} = (p_n \oslash \mathbb{I}) ; \mathbb{M} ; \mathbb{T}$

$= (p_n \oslash \mathbb{I}) ; \mathbb{T}$ since $\mathbb{S}$ is assumed to be total, which implies $\mathbb{M}$ total; Prop. 7.1.1.ii.e

$= (p_n \oslash \mathbb{I}) ; \pi ; \mathbb{T}$ because π is total

$= p_n ; \mathbb{T} = \mathbb{T}$ induction hypothesis

---

vi) Again a proof by induction — applying $\mathbb{I} \cap e ; \mathbb{T} = e ; e^{\intercal}$ several times:

$e ; e^{\intercal} ; p_0 = e ; e^{\intercal} ; \mathbb{T} ; e^{\intercal} = e ; \mathbb{T} ; e^{\intercal} = e ; e^{\intercal}$ since e is surjective

$e ; e^{\intercal} ; p_{n+1} = e ; e^{\intercal} ; (p_n \oslash \mathbb{I}) ; \mathbb{M} = (e ; e^{\intercal} ; p_n \oslash e ; e^{\intercal}) ; \mathbb{M}$

$= (e ; e^{\intercal} \oslash e ; e^{\intercal}) ; \mathbb{M}$ hypothesis

$= e ; e^{\intercal}$ due to Prop. 7.1.1.ii.d

---

vii) The cases $n = 0$ and $n = 1$ are trivial. For $n > 0$ we use (iv):

$p_n ; p_n^{\intercal} = \big[\mathsf{sup}_{i\geq 0}\ \mathbb{S}^{\intercal i} ; z ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{(i^n)}\big] ; \big[\mathsf{sup}_{k\geq 0}\ \mathbb{S}^{\intercal(k^n)} ; z ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{k}\big]$

$= \mathsf{sup}_{i\geq 0}\, \mathsf{sup}_{k\geq 0}\ \mathbb{S}^{\intercal i} ; z ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{(i^n)} ; \mathbb{S}^{\intercal(k^n)} ; z ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{k}$ which terms are $= \mathbb{\perp}$ for $i^n \neq k^n$

$= \mathsf{sup}_{i\geq 0}\ \mathbb{S}^{\intercal i} ; z ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{(i^n)} ; \mathbb{S}^{\intercal(i^n)} ; z ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{i}$

$\subseteq \mathsf{sup}_{i\geq 0}\ \mathbb{S}^{\intercal i} ; z ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{i} \subseteq \mathsf{sup}_{i\geq 0}\ \mathbb{S}^{\intercal i} ; \mathbb{S}^{i}$ since $z ; \mathbb{T}$ is injective

$\subseteq \mathbb{I}$ because $\mathbb{S}$ is univalent □

Exponentiation is a number-theoretic classic defined by $n^0 = 1$, $n^{e+1} = n^e * n$, which shall now be lifted to a relational form.

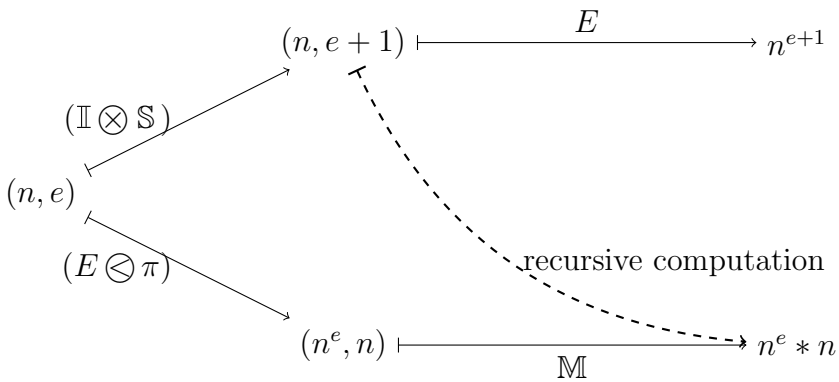


Fig. 8.1.3 Defining exponentiation of natural numbers

The version (ii.a) below expresses in a more or less directly graspable formula the assignment $(n, i) \mapsto n^i$, i.e. of the point $\mathbb{S}^{\intercal(n^i)} ; z ; \mathbb{T}$ to the pair formed by the points $\mathbb{S}^{\intercal n} ; z ; \mathbb{T}$ and $\mathbb{S}^{\intercal i} ; z ; \mathbb{T}$.

8.1.3 Definition and Proposition. Given the successor relation $\mathbb{S}$, we define as **exponentiation** belonging to $\mathbb{S}$ the construct

$E := E_{\mathbb{S}} := \mathtt{inf}\,\{\, X \mid \xi(X) \subseteq X \,\}$,

where we have in view of Lemma 4.2.1

$\xi(X) := E_0 \cup \xi_H(X)$ with $E_0 := (\mathbb{T};e^{\mathsf{T}} \bigotimes z;\mathbb{T})$ and
$\xi_H(X) := (\mathbb{I} \otimes \mathbb{S}^{\mathsf{T}});(X \bigotimes \pi);\mathbb{M}$.

Respecting possible further properties of $\mathbb{S}$, the exponentiation E satisfies

i) in general:

a) $E = \xi(E) = E_0 \cup (\mathbb{I} \otimes \mathbb{S}^{\mathsf{T}});(E \bigotimes \pi);\mathbb{M} = \mathtt{sup}_{n\geq 0}\, \xi_H^n(E_0) = \mathtt{sup}_{n\geq 0}\, (p_n \bigotimes \mathbb{S}^{\mathsf{T}n};z;\mathbb{T})$

b) E is univalent

c) Assuming $\mathbb{S}^{\mathsf{T}i};z;\mathbb{T} \neq \mathbb{\perp}$ and $\mathbb{S}^{\mathsf{T}k};z;\mathbb{T} \neq \mathbb{\perp}$:

$$E^{\mathsf{T}};(\mathbb{S}^{\mathsf{T}i};z;\mathbb{T} \bigotimes \mathbb{S}^{\mathsf{T}k};z;\mathbb{T}) = \mathbb{S}^{\mathsf{T}(i^k)};z;\mathbb{T} = \begin{cases} e;\mathbb{T} & \text{when } k = 0 \\ z;\mathbb{T} & \text{when } i = 0 \text{ and } k > 0 \\ \mathbb{S}^{\mathsf{T}(i^k)};z;\mathbb{T} & \text{when } i > 0 \text{ and } k > 0 \end{cases}$$

ii) additionally, in case $\mathbb{S}$ is regressively finite:

a) $E = \mathtt{sup}_{n\geq 0}\, \mathtt{sup}_{i\geq 0}\, (\mathbb{S}^{\mathsf{T}n};z;\mathbb{T} \bigotimes \mathbb{S}^{\mathsf{T}i};z;\mathbb{T});\mathbb{T};z^{\mathsf{T}};\mathbb{S}^{(n^i)}$

iii) If $\mathbb{S}$ happens to be regressively finite but not total, i.e. $\mathbb{S};\mathbb{T} \neq \mathbb{T}$,

...

iv) In case $\mathbb{S}$ is regressively finite and also total, i.e. $\mathbb{S};\mathbb{T} = \mathbb{T}$,

a) E is total

b) $(\mathbb{I} \bigotimes \mathbb{T};z^{\mathsf{T}};\mathbb{S}^{n});E = p_n$

Proof: We observe in advance that Lemma 4.2.1 may be applied since the functional ξ_H is
isotonic, i.e. $A \subseteq B \implies \xi_H(A) \subseteq \xi_H(B)$,
additive, i.e. $\xi_H(X \cup Y) = (\mathbb{I} \otimes \mathbb{S}^{\mathsf{T}});(X \cup Y \bigotimes \pi);\mathbb{M}$
$= (\mathbb{I} \otimes \mathbb{S}^{\mathsf{T}});\big[\, (X \bigotimes \pi) \cup (Y \bigotimes \pi) \,\big];\mathbb{M} = \xi_H(X) \cup \xi_H(Y)$,
$\cap$-sub-hereditary, i.e. $\xi_H(X \cap Y) = (\mathbb{I} \otimes \mathbb{S}^{\mathsf{T}});(X \cap Y \bigotimes \pi);\mathbb{M}$
$= (\mathbb{I} \otimes \mathbb{S}^{\mathsf{T}});\big[\, (X \bigotimes \pi) \cap (Y \bigotimes \pi) \,\big];\mathbb{M}$
$= \big[\, (\mathbb{I} \otimes \mathbb{S}^{\mathsf{T}});(X \bigotimes \pi) \cap (\mathbb{I} \otimes \mathbb{S}^{\mathsf{T}});(Y \bigotimes \pi) \,\big];\mathbb{M}$ since $\mathbb{S}$ is injective
$\subseteq (\mathbb{I} \otimes \mathbb{S}^{\mathsf{T}});(X \bigotimes \pi);\mathbb{M} \cap (\mathbb{I} \otimes \mathbb{S}^{\mathsf{T}});(Y \bigotimes \pi);\mathbb{M} = \xi_H(X) \cap \xi_H(Y)$.

---

i.a) Concerning the explicit presentation, certainly
$\xi_H^0(E_0) = E_0 = (\mathbb{T};e^{\mathsf{T}} \bigotimes z;\mathbb{T}) = (p_0 \bigotimes \mathbb{S}^{\mathsf{T}0};z;\mathbb{T})$

$\xi_H^{n+1}(E_0) = \xi_H(\xi_H^n(E_0)) = \xi_H(\,(p_n \bigotimes \mathbb{S}^{\mathsf{T}n};z;\mathbb{T})\,)$ induction hypothesis
$= (\mathbb{I} \otimes \mathbb{S}^{\mathsf{T}});(\,(p_n \bigotimes \mathbb{S}^{\mathsf{T}n};z;\mathbb{T}) \bigotimes \pi);\mathbb{M}$

$= (\mathbb{I} \otimes \mathbb{S}^{\mathsf{T}}) ; ((p_n \oslash \mathbb{I}) \oslash \mathbb{S}^{\mathsf{T}^n} ; z ; \mathbb{T}) ; \mathbb{M}$ projection migration Prop. 3.1.2.iii; π surjective
$= ((p_n \oslash \mathbb{I}) \oslash \mathbb{S}^{\mathsf{T}^{n+1}} ; z ; \mathbb{T}) ; \mathbb{M}$
$= \left[\pi ; (p_n \oslash \mathbb{I}) \cap \rho ; \mathbb{S}^{\mathsf{T}^{n+1}} ; z ; \mathbb{T}\right] ; \mathbb{M}$
$= \pi ; (p_n \oslash \mathbb{I}) ; \mathbb{M} \cap \rho ; \mathbb{S}^{\mathsf{T}^{n+1}} ; z ; \mathbb{T}$ masking
$= \pi ; p_{n+1} \cap \rho ; \mathbb{S}^{\mathsf{T}^{n+1}} ; z ; \mathbb{T}$ due to Def. 8.1.1
$= (p_{n+1} \oslash \mathbb{S}^{\mathsf{T}^{n+1}} ; z ; \mathbb{T})$

---

i.b) is proved using Prop. 8.1.2.i:

$\left[\xi_H^n(E_0)\right]^{\mathsf{T}} ; \xi_H^m(E_0) = \left[\ (p_n \oslash \mathbb{S}^{\mathsf{T}^n} ; z ; \mathbb{T})\ \right]^{\mathsf{T}} ; (p_m \oslash \mathbb{S}^{\mathsf{T}^m} ; z ; \mathbb{T})$
$= (p_n^{\mathsf{T}} \oslash \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^n) ; (p_m \oslash \mathbb{S}^{\mathsf{T}^m} ; z ; \mathbb{T})$
$= p_n^{\mathsf{T}} ; p_m \cap \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^n ; \mathbb{S}^{\mathsf{T}^m} ; z ; \mathbb{T}$ which will due to $\mathbb{S} ; z = \mathbb{\perp}$ be $\mathbb{\perp}$ when $n \neq m$ and $\mathbb{T}$ otherwise
$= p_n^{\mathsf{T}} ; p_n \subseteq \mathbb{I}$ hypothesis

---

i.c) $E^{\mathsf{T}} ; (\mathbb{S}^{\mathsf{T}^i} ; z ; \mathbb{T} \oslash \mathbb{S}^{\mathsf{T}^k} ; z ; \mathbb{T}) = \left[\ \mathtt{sup}_{n \geq 0}\ (p_n \oslash \mathbb{S}^{\mathsf{T}^n} ; z ; \mathbb{T})\ \right]^{\mathsf{T}} ; (\mathbb{S}^{\mathsf{T}^i} ; z ; \mathbb{T} \oslash \mathbb{S}^{\mathsf{T}^k} ; z ; \mathbb{T})$
$= \mathtt{sup}_{n \geq 0}\ (p_n^{\mathsf{T}} \oslash \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^n) ; (\mathbb{S}^{\mathsf{T}^i} ; z ; \mathbb{T} \oslash \mathbb{S}^{\mathsf{T}^k} ; z ; \mathbb{T})$
$= \mathtt{sup}_{n \geq 0}\, p_n^{\mathsf{T}} ; \mathbb{S}^{\mathsf{T}^i} ; z ; \mathbb{T} \cap \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^n ; \mathbb{S}^{\mathsf{T}^k} ; z ; \mathbb{T}$ implying $n := k$ to be $\neq \mathbb{\perp}$
$= p_k^{\mathsf{T}} ; \mathbb{S}^{\mathsf{T}^i} ; z ; \mathbb{T} \cap \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^k ; \mathbb{S}^{\mathsf{T}^k} ; z ; \mathbb{T}$
$= p_k^{\mathsf{T}} ; \mathbb{S}^{\mathsf{T}^i} ; z ; \mathbb{T}$ by assumption $\mathbb{S}^{\mathsf{T}^k} ; z ; \mathbb{T} \neq \mathbb{\perp}$
$= \mathbb{S}^{\mathsf{T}^{(i^k)}} ; z ; \mathbb{T}$ due to Prop. 8.1.2.iii

For the case distinction, we recall that $0^0 = 1$ and $0^n = 0$ for $n > 0$.

$$p_{12} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} \mathbf{1} & \mathbf{0} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \quad \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} = (\mathbb{I} \oslash \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{12}) ; E$$

Fig. 8.1.4 Illustrating (i.c)

---

ii.a) To prove

$\mathtt{sup}_{n \geq 0}\, \mathtt{sup}_{i \geq 0}\ (\mathbb{S}^{\mathsf{T}^n} ; z ; \mathbb{T} \oslash \mathbb{S}^{\mathsf{T}^i} ; z ; \mathbb{T}) ; \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{(n^i)} = \mathtt{sup}_{n \geq 0}\ (p_n \oslash \mathbb{S}^{\mathsf{T}^n} ; z ; \mathbb{T})\,,$

it suffices to show

$(p_n \oslash \mathbb{S}^{\mathsf{T}^n} ; z ; \mathbb{T}) = (\mathtt{sup}_{l \geq 0}\ \mathbb{S}^{\mathsf{T}^l} ; z ; \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{(l^n)} \oslash \mathbb{S}^{\mathsf{T}^n} ; z ; \mathbb{T})$ Prop. 8.1.2.iv
$= \mathtt{sup}_{l \geq 0}\ (\mathbb{S}^{\mathsf{T}^l} ; z ; \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{(l^n)} \oslash \mathbb{S}^{\mathsf{T}^n} ; z ; \mathbb{T})$
$= \mathtt{sup}_{l \geq 0}\, \pi ; \mathbb{S}^{\mathsf{T}^l} ; z ; \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{(l^n)} \cap \rho ; \mathbb{S}^{\mathsf{T}^n} ; z ; \mathbb{T}$
$= \mathtt{sup}_{l \geq 0}\, \pi ; \mathbb{S}^{\mathsf{T}^l} ; z ; \mathbb{T} \cap \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{(l^n)} \cap \rho ; \mathbb{S}^{\mathsf{T}^n} ; z ; \mathbb{T}$ masking
$= \mathtt{sup}_{l \geq 0}\, \left[\pi ; \mathbb{S}^{\mathsf{T}^l} ; z ; \mathbb{T} \cap \rho ; \mathbb{S}^{\mathsf{T}^n} ; z ; \mathbb{T}\right] ; \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{(l^n)}$ masking
$= \mathtt{sup}_{l \geq 0}\ (\mathbb{S}^{\mathsf{T}^l} ; z ; \mathbb{T} \oslash \mathbb{S}^{\mathsf{T}^n} ; z ; \mathbb{T}) ; \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{(l^n)}$

---

iii)

---

iv.a) $E ; \mathbb{T} = \mathtt{sup}_{n\geq 0}\ (p_n \circledcirc \mathbb{S}^{\intercal n} ; z ; \mathbb{T}) ; \mathbb{T} = \mathtt{sup}_{n\geq 0}(\pi ; p_n \cap \rho ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T}) ; \mathbb{T}$
$= \mathtt{sup}_{n\geq 0}\, \pi ; p_n ; \mathbb{T} \cap \rho ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T}$ masking
$= \mathtt{sup}_{n\geq 0}\, \rho ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T}$ since the p_n are then total following Prop. 8.1.2.v
$= \rho ; \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\intercal n} ; z ; \mathbb{T} = \mathbb{T}$ when regressively finite due to Prop. 5.2.1.ii.a

---

iv.b) $(\mathbb{I} \circledcirc \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n}) ; E = (\mathbb{I} \circledcirc \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n}) ; \mathtt{sup}_{k\geq 0}\ (p_k \circledcirc \mathbb{S}^{\intercal k} ; z ; \mathbb{T})$
$= \mathtt{sup}_{k\geq 0}\, p_k \cap \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n} ; \mathbb{S}^{\intercal k} ; z ; \mathbb{T}$ implying $k = n$ to be $\neq \bot\!\!\!\bot$
$= p_n \cap \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n} ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T}$
$= p_n$ since $\mathbb{S}^{\intercal n} ; z ; \mathbb{T}$ is necessarily a point for every n, the last term results in $\mathbb{T}$. □

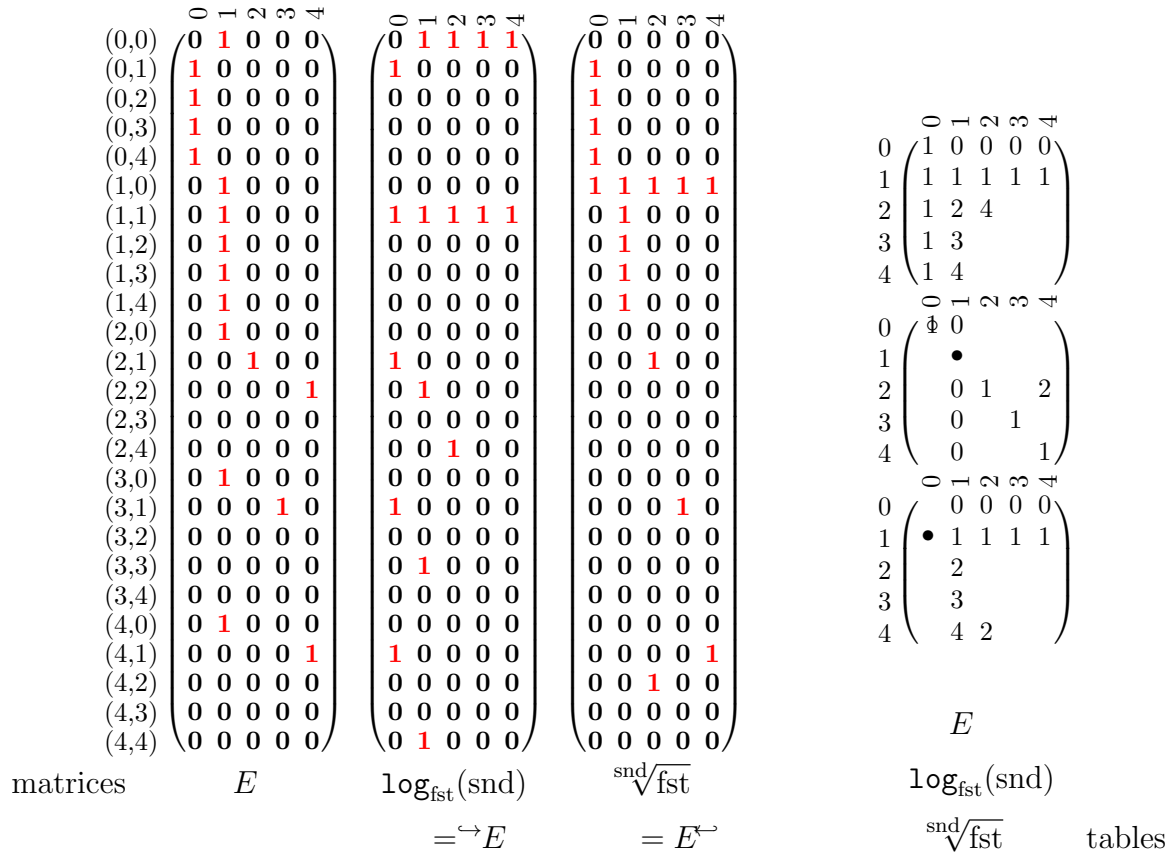


Fig. 8.1.5 Exponentiation, logarithm and root, limited to $0, 1, 2, 3, 4$ as matrix and as table

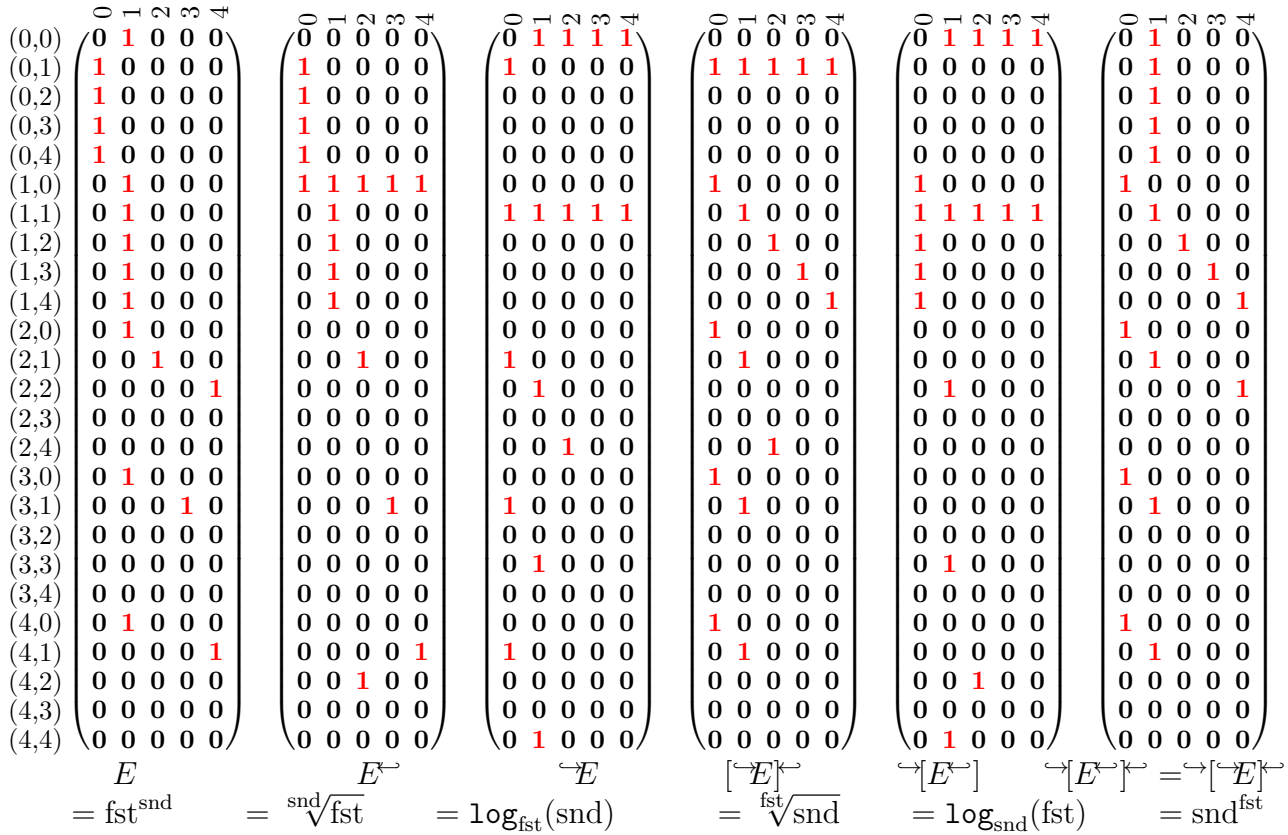
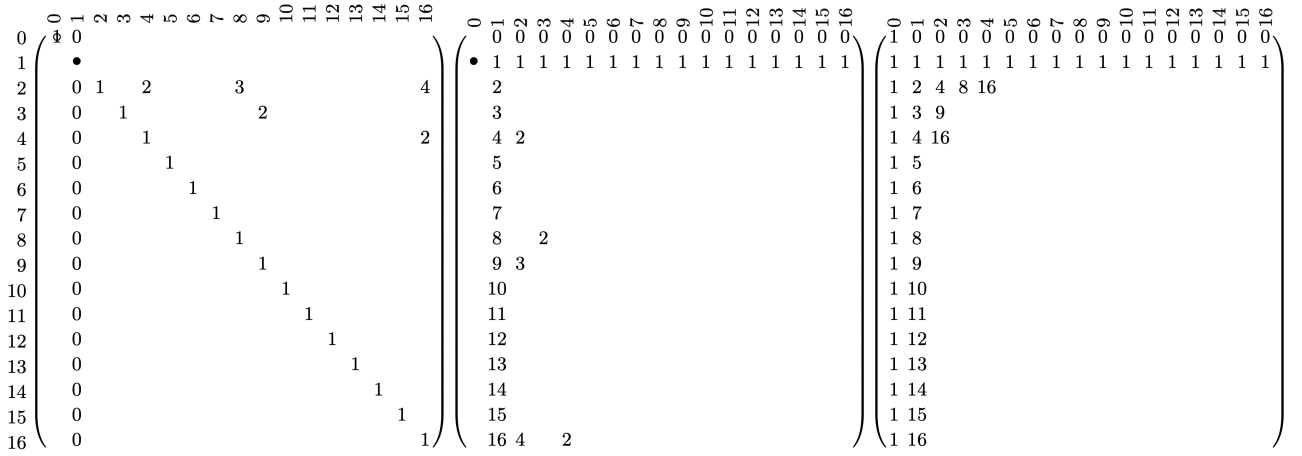


Fig. 8.1.6 Exponentiation as an example with 6 different iterated inverses

A tiny hint: the three matrices on the left all contain the same number of non-zero entries, namely 17. An explanation is necessary concerning the symbol ⧱ in the very upper left of the tables in Fig. 8.1.5 and Fig. 8.1.7. We had the bullet • to indicate a row of **1**'s, if there exists just one. This applies to row $(1,1)$ of the binary relational operation for the logarithm and the $(1,0)$ to extract the 0'th root of 1. It occurs not least in tables representing division as on page 158. Here, however, we have a row like $(0,0)$ in the middle matrix, left-inverting E so as to obtain $\mathtt{log}_{\mathrm{fst}}(\mathrm{snd})$. It is certainly a borderline case; but exponentiation together with left and right inversion demands the first row of E, i.e. $0^0 = 1$, but $0^n = 0$ for $n > 0$. The symbol ⧱ in the representation of $\hookrightarrow\!\!\!\!/\,E$ shall indicate such a row with just one 0 and otherwise 1.

The construct E computes $E(n,e) = n^e$. The right-inverse is $E^{\hookleftarrow}$, which corresponds to the computation $E^{\hookleftarrow}(m,e) = \sqrt[e]{m}$. In Fig. 8.1.7, yet another abbreviation has taken place: The entry $(0,0)$ has here been handled nearly in the same way as $(1,0)$; however, it means what is presented as a (smaller) matrix on the lower right.



$\hookrightarrow E = \mathtt{log}_{\mathrm{fst}}(\mathrm{snd})$ $E^{\hookleftarrow} = \sqrt[\mathrm{snd}]{\mathrm{fst}}$ E

Fig. 8.1.7 Exponentiation $E : (r, c) \mapsto r^c$ with inverses

8.1.4 Proposition. i) $\hookrightarrow E = \mathtt{sup}_{n\geq 0}\ (\mathbb{I} \circledcirc p_n^\mathsf{T}) ; \mathbb{T} ; z^\mathsf{T} ; \mathbb{S}^n =: \mathtt{log}_{\mathrm{fst}}(\mathrm{snd})$

ii) $E^{\hookleftarrow} = \mathtt{sup}_{n\geq 0}\ (p_n^\mathsf{T} \circledcirc \mathbb{S}^{\mathsf{T}n} ; z ; \mathbb{T}) =: \sqrt[\mathrm{snd}]{\mathrm{fst}}$

Proof: i) $\hookrightarrow E = (\pi^\mathsf{T} \circledcirc E^\mathsf{T}) ; \rho$

$= (\pi^\mathsf{T} \circledcirc \big[\, \mathtt{sup}_{n\geq 0}\ (p_n \circledcirc \mathbb{S}^{\mathsf{T}n} ; z ; \mathbb{T}) \,\big]^\mathsf{T}) ; \rho$

$= \mathtt{sup}_{n\geq 0}\ (\pi^\mathsf{T} \circledcirc (p_n^\mathsf{T} \circledcirc \mathbb{T} ; z^\mathsf{T} ; \mathbb{S}^n)\,) ; \rho$

$= \mathtt{sup}_{n\geq 0}\ \big[\pi ; \pi^\mathsf{T} \cap \rho ; p_n^\mathsf{T} ; \pi^\mathsf{T} \cap \rho ; \mathbb{T} ; z^\mathsf{T} ; \mathbb{S}^n ; \rho^\mathsf{T}\big] ; \rho$

$= \mathtt{sup}_{n\geq 0}\ \big[(\pi \cap \rho ; p_n^\mathsf{T}) ; \pi^\mathsf{T} \cap \mathbb{T} ; z^\mathsf{T} ; \mathbb{S}^n ; \rho^\mathsf{T}\big] ; \rho$ since ρ is total

$= \mathtt{sup}_{n\geq 0}\ (\mathbb{I} \circledcirc p_n^\mathsf{T}) ; \mathbb{T} \cap \mathbb{T} ; z^\mathsf{T} ; \mathbb{S}^n$ destroy and append

$= \mathtt{sup}_{n\geq 0}\ (\mathbb{I} \circledcirc p_n^\mathsf{T}) ; \mathbb{T} ; z^\mathsf{T} ; \mathbb{S}^n$ masking

---

ii) $E^{\hookleftarrow} = (E^\mathsf{T} \circledcirc \rho^\mathsf{T}) ; \pi$

$= (\big[\, \mathtt{sup}_{n\geq 0}\ (p_n \circledcirc \mathbb{S}^{\mathsf{T}n} ; z ; \mathbb{T}) \,\big]^\mathsf{T} \circledcirc \rho^\mathsf{T}) ; \pi$

$= \mathtt{sup}_{n\geq 0}\ (\,(p_n^\mathsf{T} \circledcirc \mathbb{T} ; z^\mathsf{T} ; \mathbb{S}^n) \circledcirc \rho^\mathsf{T}) ; \pi$

$= \mathtt{sup}_{n\geq 0}\ (\pi ; p_n^\mathsf{T} \circledcirc\ (\mathbb{T} ; z^\mathsf{T} ; \mathbb{S}^n \circledcirc \mathbb{I})\,) ; \pi$ projection migration according to Prop. 3.1.2.ii

$= \mathtt{sup}_{n\geq 0}\ \pi ; p_n^\mathsf{T} \cap (\mathbb{T} ; z^\mathsf{T} ; \mathbb{S}^n \circledcirc \mathbb{I}) ; \mathbb{T}$ destroy and append

$= \mathtt{sup}_{n\geq 0}\ \pi ; p_n^\mathsf{T} \cap \rho ; (\mathbb{T} \cap \mathbb{S}^{\mathsf{T}n} ; z ; \mathbb{T})$ mask shifting

$= \mathtt{sup}_{n\geq 0}\ (p_n^\mathsf{T} \circledcirc \mathbb{S}^{\mathsf{T}n} ; z ; \mathbb{T})$ □

One should compare this with Prop. 7.2.1.ii.a, where k_n occurs instead of p_n.

With the two following figures, resembling Fig. 3.2.1 and Fig. 3.2.2, we recall the general concept of inverting a binary relational operation in another special case.

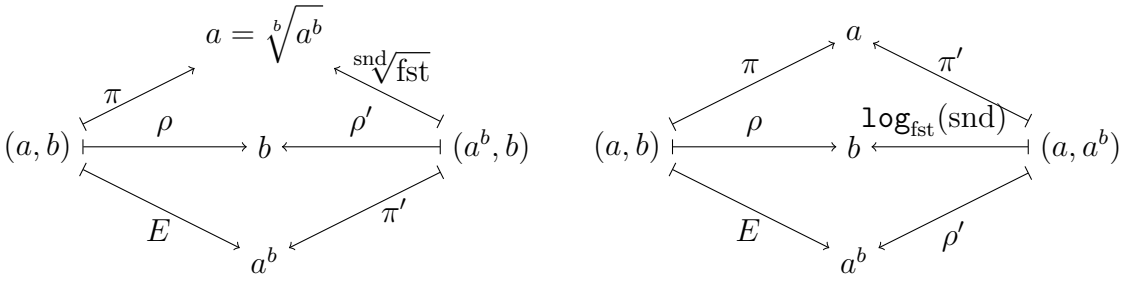


Fig. 8.1.8 Right- and left-inverse of E so as to obtain $E^{\hookleftarrow} = \sqrt[\mathrm{snd}]{\mathrm{fst}}$ and ${}^{\hookrightarrow}E = \mathtt{log}_{\mathrm{fst}}(\mathrm{snd})$

8.2 Factorial and Gamma function

The factorial is a mathematical classic characterized by

$(n+1)\,! = (n+1) * n\,!,$
$0\,! = 1.$

It shall now be converted into a relation `fact`.

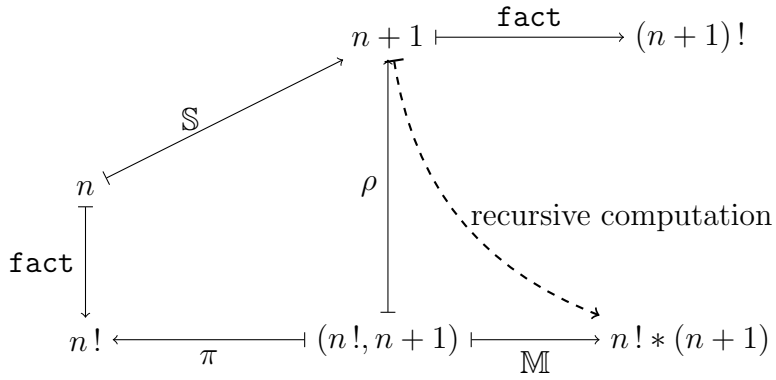


Fig. 8.2.1 Defining the factorial relation `fact` on a possibly finite initial segment of the naturals

We define this factorial relation according to Fig. 8.2.1, using $e := \mathbb{S}^{\mathsf{T}};z$.

8.2.1 Definition. We define the n-**fold downwards product** relations by

$f_0 := z;\mathbb{T};e^{\mathsf{T}} \qquad f_{n+1} := (\mathbb{S}^{\mathsf{T}};f_n \oslash \mathbb{I});\mathbb{M}\,.$ □

These are first examples:

$f_1 = (\mathbb{S}^{\mathsf{T}};f_0 \oslash \mathbb{I});\mathbb{M} = (\mathbb{S}^{\mathsf{T}};z;\mathbb{T};e^{\mathsf{T}} \oslash \mathbb{I});\mathbb{M} = (e;\mathbb{T};e^{\mathsf{T}} \oslash \mathbb{I});\mathbb{M}$
$= (e;\mathbb{T};e^{\mathsf{T}} \oslash \mathbb{I});\mathtt{sup}_{n\geq 0}\,(\mathbb{S}^{\mathsf{T}n};z;\mathbb{T} \oslash k_n)$
$= \mathtt{sup}_{n\geq 0}\, e;\mathbb{T};e^{\mathsf{T}};\mathbb{S}^{\mathsf{T}n};z;\mathbb{T} \cap k_n \quad$ implying $n := 1$ to be $\neq \mathbb{\perp}$
$= e;\mathbb{T};e^{\mathsf{T}};\mathbb{S}^{\mathsf{T}};z;\mathbb{T} \cap \mathbb{I}$
$= e;\mathbb{T} \cap \mathbb{I} = e;\mathbb{T};e^{\mathsf{T}}$

$f_2 = (\mathbb{S}^{\intercal};f_1 \circledcirc \mathbb{I});\mathbb{M} = (\mathbb{S}^{\intercal};e;\mathbb{T};e^{\intercal} \circledcirc \mathbb{I});\mathbb{M}$
$= (\mathbb{S}^{\intercal};e;\mathbb{T};e^{\intercal} \circledcirc \mathbb{I});\, \mathtt{sup}_{n\geq 0}\, (\mathbb{S}^{\intercal n};z;\mathbb{T} \circledcirc k_n)$
$= \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\intercal};e;\mathbb{T};e^{\intercal};\mathbb{S}^{\intercal n};z;\mathbb{T} \cap k_n \quad$ implying $n := 1$ to be $\neq \perp\!\!\!\perp$
$= \mathbb{S}^{\intercal};e;\mathbb{T};e^{\intercal};\mathbb{S}^{\intercal};z;\mathbb{T} \cap \mathbb{I}$
$= \mathbb{S}^{\intercal};e;\mathbb{T} \cap \mathbb{I} = \mathbb{S}^{\intercal};e;\mathbb{T};e^{\intercal};\mathbb{S}$

$f_3 = (\mathbb{S}^{\intercal};f_2 \circledcirc \mathbb{I});\mathbb{M} = (\mathbb{S}^{\intercal 2};e;\mathbb{T};e^{\intercal};\mathbb{S} \circledcirc \mathbb{I});\mathbb{M}$
$= (\mathbb{S}^{\intercal 2};e;\mathbb{T};e^{\intercal};\mathbb{S} \circledcirc \mathbb{I});\, \mathtt{sup}_{n\geq 0}\, (\mathbb{S}^{\intercal n};z;\mathbb{T} \circledcirc k_n)$
$= \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\intercal 2};e;\mathbb{T};e^{\intercal};\mathbb{S};\mathbb{S}^{\intercal n};z;\mathbb{T} \cap k_n \quad$ implying $n := 2$ to be $\neq \perp\!\!\!\perp$
$= \mathbb{S}^{\intercal 2};e;\mathbb{T} \cap k_2$
$= \mathbb{S}^{\intercal 3};z;\mathbb{T} \cap \mathtt{sup}_{i\geq 0}\, \mathbb{S}^{\intercal i};z;\mathbb{T};z^{\intercal};\mathbb{S}^{i*2} \quad$ Lemma 6.4.2.ii.a, implying $i := 3$ to be $\neq \perp\!\!\!\perp$
$= \mathbb{S}^{\intercal 3};z;\mathbb{T} \cap \mathbb{S}^{\intercal 3};z;\mathbb{T};z^{\intercal};\mathbb{S}^{3*2}$
$= \mathbb{S}^{\intercal 3};z;\mathbb{T};z^{\intercal};\mathbb{S}^{3*2}$

8.2.2 Proposition. Assuming $\mathbb{S}$ to be regressively finite, for every $n \geq 0$

i) $f_n;z = \perp\!\!\!\perp$,

ii) $f_n = \mathbb{S}^{\intercal n};z;\mathbb{T};z^{\intercal};\mathbb{S}^{n!}$.

Proof: i) by induction:

$f_0;z = z;\mathbb{T};e^{\intercal};z = \perp\!\!\!\perp$
$f_{n+1};z = (\mathbb{S}^{\intercal};f_n \circledcirc \mathbb{I});\mathbb{M};z = (\mathbb{S}^{\intercal};f_n \circledcirc \mathbb{I});\big[\,(z;\mathbb{T} \circledcirc \mathbb{T}) \cup (\mathbb{T} \circledcirc z;\mathbb{T})\,\big] \quad$ Prop. 7.1.1.ii.c
$= (\mathbb{S}^{\intercal};f_n \circledcirc \mathbb{I});(z;\mathbb{T} \circledcirc \mathbb{T}) \cup (\mathbb{S}^{\intercal};f_n \circledcirc \mathbb{I});(\mathbb{T} \circledcirc z;\mathbb{T})$
$= \mathbb{S}^{\intercal};f_n;z;\mathbb{T} \cup (\mathbb{S}^{\intercal};f_n;\mathbb{T} \cap z;\mathbb{T})$
$= \perp\!\!\!\perp \cup (\mathbb{S}^{\intercal};f_n;\mathbb{T} \cap z;\mathbb{T}) \subseteq \mathbb{S}^{\intercal};\mathbb{T} \cap z;\mathbb{T} = \overline{z;\mathbb{T}} \cap z;\mathbb{T} = \perp\!\!\!\perp \quad$ using the hypothesis

---

ii) Starting with $f_0 = z;\mathbb{T};e^{\intercal} = \mathbb{S}^{\intercal 0};z;\mathbb{T};z^{\intercal};\mathbb{S}^{0!}$, we prove by induction

$f_{n+1} = (\mathbb{S}^{\intercal};f_n \circledcirc \mathbb{I});\mathbb{M} = (\mathbb{S}^{\intercal};\mathbb{S}^{\intercal n};z;\mathbb{T};z^{\intercal};\mathbb{S}^{n!} \circledcirc \mathbb{I});\big[\,\mathtt{sup}_{j\geq 0}\, (\mathbb{S}^{\intercal j};z;\mathbb{T} \circledcirc k_j)\,\big]$
$= \mathtt{sup}_{j\geq 0}\, \mathbb{S}^{\intercal n+1};z;\mathbb{T};z^{\intercal};\mathbb{S}^{n!};\mathbb{S}^{\intercal j};z;\mathbb{T} \cap k_j \quad$ first term $= \perp\!\!\!\perp$ unless $j := n!$ and $\mathbb{S}^{\intercal n!};z;\mathbb{T} \neq \perp\!\!\!\perp$
$= \mathbb{S}^{\intercal n+1};z;\mathbb{T} \cap k_{n!}$
$= \mathbb{S}^{\intercal n+1};z;\mathbb{T} \cap \mathtt{sup}_{m\geq 0}\, \mathbb{S}^{\intercal m};z;\mathbb{T};z^{\intercal};\mathbb{S}^{n!*m} \quad$ Lemma 6.4.2.ii.a; will be $\perp\!\!\!\perp$ in case $m \neq n+1$
$= \mathbb{S}^{\intercal n+1};z;\mathbb{T} \cap \mathbb{S}^{\intercal n+1};z;\mathbb{T};z^{\intercal};\mathbb{S}^{n!*(n+1)}$
$= \mathbb{S}^{\intercal n+1};z;\mathbb{T};z^{\intercal};\mathbb{S}^{(n+1)!}$ □

These downwards products are now combined into the factorial function. The item (i.c) resembles more or less directly on the left side $(n+1)!$ and on the right one: forming the pair of $n!$ and $n+1$ which is then multiplied.

8.2.3 Definition and Proposition. Given the successor relation $\mathbb{S}$, we define as **factorial** relation belonging to $\mathbb{S}$ the construct

$\mathtt{fact} := \mathtt{fact}_{\mathbb{S}} := \mathtt{inf}\,\{\, X \mid g(X) \subseteq X \,\}$,

where we have in view of Lemma 4.2.1

$g(X) := f_0 \cup g_H(X) \quad$ with $\quad f_0 = z;\mathbb{T};e^{\intercal} \quad$ and $\quad g_H(X) = (\mathbb{S}^{\intercal};X \circledcirc \mathbb{I});\mathbb{M}$.

Respecting possible further properties of $\mathbb{S}$, the factorial relation $\mathtt{fact}$ satisfies

i) in general:

a) $\mathtt{fact} = \mathtt{fact} ; g(\mathtt{fact}) = z ; \mathbb{T} ; e^\mathsf{T} \cup (\mathbb{S}^\mathsf{T} ; \mathtt{fact} \oslash \mathbb{I}) ; \mathbb{M} = \mathtt{sup}_{n\geq 0}\, g_H^n(f_0) = \mathtt{sup}_{n\geq 0}\, f_n$

b) $\mathtt{fact}$ is univalent

c) $\mathbb{S} ; \mathtt{fact} = (\mathtt{fact} \oslash \mathbb{S}) ; \mathbb{M}$

d) $(\mathtt{fact} \oslash \mathbb{I}) ; \mathfrak{Q} \subseteq \mathbb{S}^\mathsf{T} ; \mathtt{fact}$

e) $\mathtt{fact} ; z = \mathbb{\perp} \qquad \mathtt{fact} ; e = z \cup e$

ii) additionally, in case $\mathbb{S}$ is regressively finite:

a) $\mathtt{fact} = \mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\mathsf{T}^n} ; z ; \mathbb{T} ; z^\mathsf{T} ; \mathbb{S}^{n!}$

iii) If $\mathbb{S}$ happens to be regressively finite but not total, i.e. $\mathbb{S} ; \mathbb{T} \neq \mathbb{T}$,

a)

iv) In case $\mathbb{S}$ is regressively finite and also total, i.e. $\mathbb{S} ; \mathbb{T} = \mathbb{T}$,

a) $\mathtt{fact}$ is total

Proof: We observe in advance that Lemma 4.2.1 may be applied since the functional g_H is
isotonic, i.e. $A \subseteq B \implies g_H(A) \subseteq g_H(B)$,
additive, i.e. $g_H(X \cup Y) = (\mathbb{S}^\mathsf{T} ; (X \cup Y) \oslash \mathbb{I}) ; \mathbb{M}$
$= \big[(\mathbb{S}^\mathsf{T} ; X \oslash \mathbb{I}) \cup (\mathbb{S}^\mathsf{T} ; Y \oslash \mathbb{I}) \big] ; \mathbb{M} = g_H(X) \cup g_H(Y)$,
$\cap$-sub-hereditary, i.e. $g_H(X \cap Y) = (\mathbb{S}^\mathsf{T} ; (X \cap Y) \oslash \mathbb{I}) ; \mathbb{M}$
$= \big[(\mathbb{S}^\mathsf{T} ; X \oslash \mathbb{I}) \cap (\mathbb{S}^\mathsf{T} ; Y \oslash \mathbb{I}) \big] ; \mathbb{M}$
$\subseteq (\mathbb{S}^\mathsf{T} ; X \oslash \mathbb{I}) ; \mathbb{M} \cap (\mathbb{S}^\mathsf{T} ; Y \oslash \mathbb{I}) ; \mathbb{M} = g_H(X) \cap g_H(Y)$.

---

i.a) Proof by induction:
$g_H^0(f_0) = f_0 = z ; \mathbb{T} ; e^\mathsf{T} = f_0$

$g_H^{n+1}(f_0) = g_H(g_H^n(f_0)) = g_H(f_n) = (\mathbb{S}^\mathsf{T} ; f_n \oslash \mathbb{I}) ; \mathbb{M} = f_{n+1}$

---

i.b) We have by definition
$\mathtt{fact}^\mathsf{T} ; \mathtt{fact} = \big[e ; \mathbb{T} ; z^\mathsf{T} \cup \mathbb{M}^\mathsf{T} ; (\mathtt{fact}^\mathsf{T} ; \mathbb{S} \oslash \mathbb{I}) \big] ; \big[z ; \mathbb{T} ; e^\mathsf{T} \cup (\mathbb{S}^\mathsf{T} ; \mathtt{fact} \oslash \mathbb{I}) ; \mathbb{M} \big]$
and see three parts of products:
$(z ; e^\mathsf{T})^\mathsf{T} ; z ; e^\mathsf{T} = e ; z^\mathsf{T} ; z ; e^\mathsf{T} = e ; \mathbb{T} ; e^\mathsf{T} \subseteq \mathbb{I}$
$(\mathtt{fact}^\mathsf{T} ; \mathbb{S} \oslash \mathbb{I}) ; z ; \mathbb{T} = (\mathtt{fact}^\mathsf{T} ; \mathbb{S} ; z ; \mathbb{T} \oslash z ; \mathbb{T}) = (\mathbb{\perp} \oslash z ; \mathbb{T}) = \mathbb{\perp}$
$(\mathtt{fact}^\mathsf{T} ; \mathbb{S} \oslash \mathbb{I}) ; (\mathbb{S}^\mathsf{T} ; \mathtt{fact} \oslash \mathbb{I}) \subseteq (\ldots \otimes \mathbb{I}) \subseteq \mathbb{I}$ Prop. 2.1.1.i

---

i.c) $\mathbb{S} ; \mathtt{fact} = \mathbb{S} ; \big[z ; e^\mathsf{T} \cup (\mathbb{S}^\mathsf{T} ; \mathtt{fact} \oslash \mathbb{I}) ; \mathbb{M} \big]$
$= \mathbb{S} ; z ; e^\mathsf{T} \cup \mathbb{S} ; (\mathbb{S}^\mathsf{T} ; \mathtt{fact} \oslash \mathbb{I}) ; \mathbb{M} = \mathbb{\perp} \cup (\mathbb{S} ; \mathbb{S}^\mathsf{T} ; \mathtt{fact} \oslash \mathbb{S}) ; \mathbb{M}$
$= ((\mathbb{I} \cap \mathbb{S} ; \mathbb{T}) ; \mathtt{fact} \oslash \mathbb{S}) ; \mathbb{M}$ Prop. 5.2.1.i.b
$= (\mathtt{fact} \cap \mathbb{S} ; \mathbb{T} \oslash \mathbb{S}) ; \mathbb{M}$ masking

$= (\mathtt{fact} ; \pi^\intercal \cap \mathbb{S} ; \mathbb{T} \cap \mathbb{S} ; \rho^\intercal) ; \mathbb{M} = (\mathtt{fact} ; \pi^\intercal \cap \mathbb{S} ; \rho^\intercal) ; \mathbb{M} = (\mathtt{fact} \oslash \mathbb{S}) ; \mathbb{M}$

---

i.d) $(\mathtt{fact} \oslash \mathbb{I}) ; \mathfrak{Q} = (\mathtt{fact} \oslash \mathbb{I}) ; \big[\mathtt{sup}_{n\geq 0} \, (k_n^\intercal \oslash \mathbb{I}) ; \mathbb{T} ; z^\intercal ; \mathbb{S}^n \big]$ Prop. 7.2.1.ii.a

$= \mathtt{sup}_{n\geq 0} (\mathtt{fact} ; k_n^\intercal \cap \mathbb{I}) ; \mathbb{T} ; z^\intercal ; \mathbb{S}^n$

$= \mathtt{sup}_{n\geq 0} \big([\, \mathtt{sup}_{l\geq 0} \, f_l \,] ; [\, \mathtt{sup}_{i\geq 0} \, \mathbb{S}^{\intercal i*n} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^i] \cap \mathbb{I} \big) ; \mathbb{T} ; z^\intercal ; \mathbb{S}^n$ Lemma 6.4.2.ii.a

$= \mathtt{sup}_{n\geq 0} \, \mathtt{sup}_{l\geq 0} \, \mathtt{sup}_{i\geq 0} \big(f_l ; \mathbb{S}^{\intercal i*n} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^i \cap \mathbb{I} \big) ; \mathbb{T} ; z^\intercal ; \mathbb{S}^n$

$= \mathtt{sup}_{n\geq 0} \, \mathtt{sup}_{l\geq 0} \, \mathtt{sup}_{i\geq 0} \big(\mathbb{S}^{\intercal l} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{l!} ; \mathbb{S}^{\intercal i*n} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^i \cap \mathbb{I} \big) ; \mathbb{T} ; z^\intercal ; \mathbb{S}^n$ Prop. 8.2.2.ii

this implies $i := l$ as a consequence of $\mathbb{I}$ to be $\neq \mathbb{\perp}$

$= \mathtt{sup}_{l\geq 0} \, \mathtt{sup}_{n\geq 0} \big(\mathbb{S}^{\intercal l} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{l!} ; \mathbb{S}^{\intercal l*n} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^l \cap \mathbb{I} \big) ; \mathbb{T} ; z^\intercal ; \mathbb{S}^n$

$= \mathtt{sup}_{n\geq 0} \big(z ; \mathbb{T} ; z^\intercal ; \mathbb{S} ; z ; \mathbb{T} ; z^\intercal \cap \mathbb{I} \big) ; \mathbb{T} ; z^\intercal ; \mathbb{S}^n \quad \longleftarrow$ case $l = 0$ vanishes since $\mathbb{S} ; z ; \mathbb{T} = \mathbb{\perp}$

$\cup \, \mathtt{sup}_{l\geq 1} \, \mathtt{sup}_{n\geq 0} \big(\mathbb{S}^{\intercal l} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{l!} ; \mathbb{S}^{\intercal l*n} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^l \cap \mathbb{I} \big) ; \mathbb{T} ; z^\intercal ; \mathbb{S}^n$

$= \mathtt{sup}_{l\geq 1} \, \mathtt{sup}_{n\geq 0} \big(\mathbb{S}^{\intercal l} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{l!} ; \mathbb{S}^{\intercal l*n} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^l \cap \mathbb{I} \big) ; \mathbb{T} ; z^\intercal ; \mathbb{S}^n$

this implies $l! = l * n$ to be $\neq \mathbb{\perp}$

$\subseteq \mathtt{sup}_{l\geq 1} \big(\mathbb{S}^{\intercal l} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^l \cap \mathbb{I} \big) ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{(l-1)!}$ since possibly $\mathbb{S}^{l!} = \mathbb{\perp}$

$= \mathtt{sup}_{l\geq 1} \big(\mathbb{S}^{\intercal l} ; z ; \mathbb{T} \cap \mathbb{I} \big) ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{(l-1)!}$

$= \mathtt{sup}_{l\geq 1} \, \mathbb{S}^{\intercal l} ; z ; \mathbb{T} \cap \mathbb{T} ; z^\intercal ; \mathbb{S}^{(l-1)!}$

$= \mathtt{sup}_{l\geq 1} \, \mathbb{S}^{\intercal l} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{(l-1)!}$

$= \mathbb{S}^\intercal ; \mathtt{sup}_{m\geq 0} \, \mathbb{S}^{\intercal m} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{m!}$

$= \mathbb{S}^\intercal ; \mathtt{fact}$

An example when numbers are delimited by 7 is that $(\mathtt{fact} \oslash \mathbb{I}) ; \mathfrak{Q}$ delivers $\mathbb{\perp}$ for the argument 4, but $\mathbb{S}^\intercal ; \mathtt{fact}$ brings 6.

---

i.e) $\mathtt{fact} ; z = \mathtt{sup}_{n\geq 0} \, f_n ; z = \mathbb{\perp}$ Prop. 8.2.2.i

$\mathtt{fact} ; e ; \mathbb{T} = \big[z ; \mathbb{T} ; e^\intercal \cup (\mathbb{S}^\intercal ; \mathtt{fact} \oslash \mathbb{I}) ; \mathbb{M} \big] ; e ; \mathbb{T} = z ; \mathbb{T} ; e^\intercal ; e ; \mathbb{T} \cup (\mathbb{S}^\intercal ; \mathtt{fact} \oslash \mathbb{I}) ; \mathbb{M} ; e ; \mathbb{T}$

$= z ; \mathbb{T} \cup (\mathbb{S}^\intercal ; \mathtt{fact} \oslash \mathbb{I}) ; (\mathbb{I} \oslash \mathbb{I}) \, ; e ; \mathbb{T}$ Prop. 7.1.1.i.e

$= z ; \mathbb{T} \cup (\mathbb{S}^\intercal ; \mathtt{fact} \cap \mathbb{I}) ; e ; \mathbb{T} \subseteq z ; \mathbb{T} \cup e ; \mathbb{T}$

Since $\mathbb{S}^\intercal ; \mathtt{fact} \supseteq \mathbb{S}^\intercal ; z ; \mathbb{T} ; e^\intercal = e ; \mathbb{T} ; e^\intercal$, we may shunt the point e to obtain $\mathbb{S}^\intercal ; \mathtt{fact} ; e ; \mathbb{T} \supseteq e ; \mathbb{T}$, and so the latter is indeed an equality.

---

ii.a) Due to (i.a) combined with Prop. 8.2.2.ii.

---

iii) ...

---

iv.a) We use (ii.a)

$\mathtt{fact} ; \mathbb{T} = \mathtt{sup}_{n\geq 0} \, \mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; z^\intercal ; \mathbb{S}^{n!} ; \mathbb{T} = \mathtt{sup}_{n\geq 0} \, \mathbb{S}^{\intercal n} ; z ; \mathbb{T}$ since $\mathbb{S}$ is total

$= \mathbb{T}$ Prop. 5.2.1.ii.a □

$\texttt{fact} =$

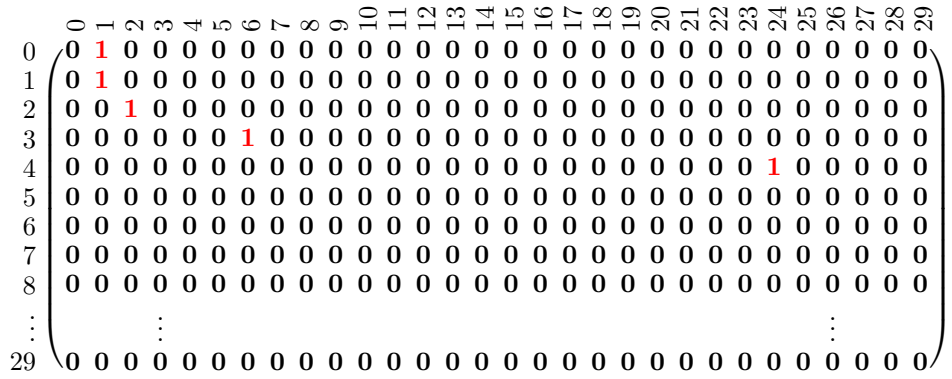


Fig. 8.2.2 The factorial function `fact` on a finite initial segment of the naturals

The thus introduced factorial relation is nearly the same as the Γ-relation, resembling the **Gamma function**, traditionally studied in number theory, defined by

$$\Gamma := \mathbb{S}^{\mathsf{T}} ; \texttt{fact},$$

to the study of which an immense literature has appeared.

8.3 Wilson's Theorem

It is a nice exercise to consider Wilson's Theorem from this point of view. A standard formulation is: Precisely for prime numbers p — not necessarily odd — will always hold

$$(p-1)\,! \equiv -1 \ \texttt{mod}\ p.$$

This means $p \mid 1 + (p-1)\,!$. The right side translated to relations is $\mathbb{S}^{\mathsf{T}} ; \texttt{fact} ; \mathbb{S}$, which is shown in Fig. 8.3.1 below. We will take the liberty to include the number 1, which does not disturb too much.

$\mathbb{S}^{\mathsf{T}} ; \texttt{fact} ; \mathbb{S} =$

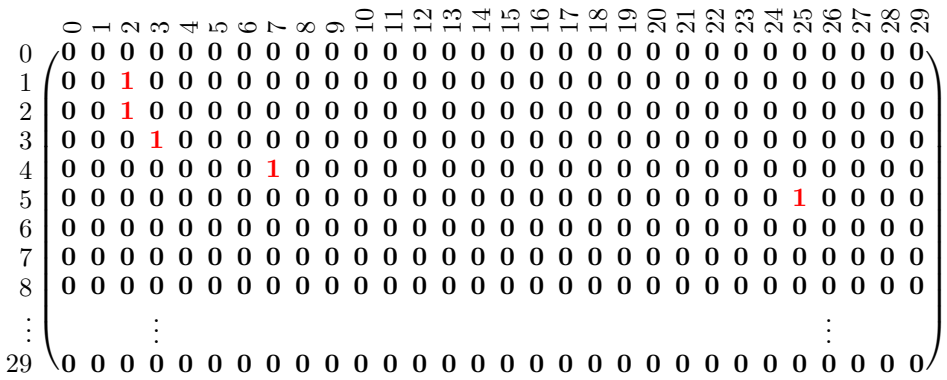


Fig. 8.3.1 Wilson's theorem for the factorial function: $1 \,|\, 2$, $2 \,|\, 2$, $3 \,|\, 3$, and $5 \,|\, 25$, but $4 \not| \, 7$

The primes — including the even 2 as well as 1 — are precisely what $(\mathcal{D} \cap \mathbb{S}^{\mathsf{T}} ; \texttt{fact} ; \mathbb{S}) ; \mathbb{T}$ presents. This will, however, need a relational proof.

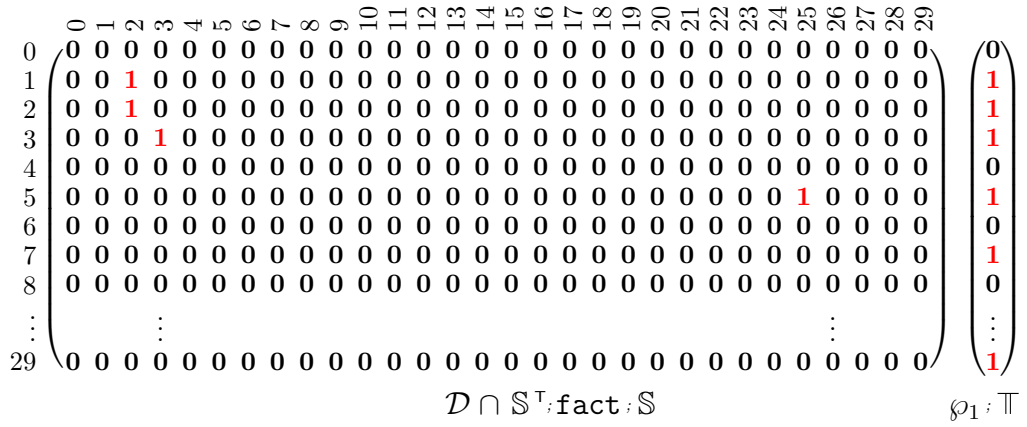


Fig. 8.3.2 A bit of evidence for Prop. 8.3.1

8.3.1 Proposition (*Wilson's Theorem*)**.** Let $\mathbb{S}$ be regressively finite. Then

i) $\mathcal{D} \cap \mathbb{S}^{\mathsf{T}} ; \mathtt{fact} ; \mathbb{S} \subseteq \wp_1 ; \mathbb{T}$

ii) $\left[\mathcal{D} \cap \mathbb{S}^{\mathsf{T}} ; \mathtt{fact} ; \mathbb{S}\right] ; \mathbb{T} = \wp_1 ; \mathbb{T}$, if $\mathbb{S}$ is totally defined.

Proof: i) We start evaluating the left side:

$$
\begin{aligned}
&\mathcal{D} \cap \mathbb{S}^{\mathsf{T}} ; \mathtt{fact} ; \mathbb{S} = \mathcal{D} \cap \mathbb{S}^{\mathsf{T}} ; [\mathtt{sup}_{n\geq 0}\, f_n] ; \mathbb{S} \quad \text{Prop. 8.2.3.i.a} \\
&= [\mathtt{sup}_{i\geq 0}\, \mathbb{S}^{\mathsf{T}i} ; z ; \mathbb{T} ; z^{\mathsf{T}} ; (\mathbb{S}^{i})^*] \cap \mathbb{S}^{\mathsf{T}} ; [\mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\mathsf{T}n} ; z ; \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{n!}] ; \mathbb{S} \quad \text{Prop. 7.4.2.i, Prop. 8.2.2} \\
&= [\mathtt{sup}_{i\geq 0}\, \mathbb{S}^{\mathsf{T}i} ; z ; \mathbb{T} ; z^{\mathsf{T}} ; (\mathbb{S}^{i})^*] \cap [\mathtt{sup}_{n\geq 0}\, \mathbb{S}^{\mathsf{T}} ; \mathbb{S}^{\mathsf{T}n} ; z ; \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{1+n!}] \\
&= \mathtt{sup}_{i\geq 1}\, \mathbb{S}^{\mathsf{T}i} ; z ; \mathbb{T} ; z^{\mathsf{T}} ; (\mathbb{S}^{i})^* \cap \mathbb{S}^{\mathsf{T}i} ; z ; \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{1+(i-1)!} \quad \text{since } \ldots = \mathbb{\bot} \text{ when not } i = n+1 \\
&= \mathtt{sup}_{i\geq 1}\, \mathbb{S}^{\mathsf{T}i} ; z ; \mathbb{T} ; z^{\mathsf{T}} ; \left[(\mathbb{S}^{i})^* \cap \mathbb{S}^{1+(i-1)!}\right] \quad \text{left factor is univalent since } z \text{ is injective} \\
&= \mathtt{sup}_{i\geq 1}\, \mathtt{sup}_{k\geq 0}\, \mathbb{S}^{\mathsf{T}i} ; z ; \mathbb{T} ; z^{\mathsf{T}} ; \left[\mathbb{S}^{i*k} \cap \mathbb{S}^{1+(i-1)!}\right] \quad \text{definition of reflexive-transitive closure} \\
&= \mathtt{sup}_{i\geq 1}\, \mathbb{S}^{\mathsf{T}i} ; z ; \mathbb{T} ; z^{\mathsf{T}} ; \begin{cases} \mathbb{S}^{1+(i-1)!} & \text{if } i*k = 1+(i-1)! \\ \mathbb{\bot} & \text{otherwise} \end{cases} \quad \text{Prop. 5.2.1.ii.f}
\end{aligned}
$$

Since certainly $\mathbb{\bot} \subseteq \wp_1 ; \mathbb{T}$, the claim of the present proposition therefore reduces to the statement that for all i

$$i \mid 1+(i-1)! \quad \Longrightarrow \quad \mathbb{S}^{\mathsf{T}i} ; z ; \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{1+(i-1)!} \subseteq \wp_1 ; \mathbb{T}.$$

To investigate whether $\mathbb{S}^{\mathsf{T}i} ; z ; \mathbb{T} \subseteq \wp_1 ; \mathbb{T}$, we choose $\wp_1 ; \mathbb{T} = \mathtt{DIV} ; e ; \mathbb{T} = \overline{\mathcal{D}_{\mathrm{s}}^{\mathsf{T}} ; \overline{\mathbb{I}}} ; e ; \mathbb{T}$ from the variants of Def. 9.3.3:

$$
\begin{aligned}
&\mathbb{S}^{\mathsf{T}i} ; z ; \mathbb{T} \subseteq \overline{\mathcal{D}_{\mathrm{s}}^{\mathsf{T}} ; \overline{\mathbb{I}}} ; e ; \mathbb{T} = \overline{\mathcal{D}_{\mathrm{s}}^{\mathsf{T}} ; \overline{e ; \mathbb{T}}} \\
&\iff \mathcal{D}_{\mathrm{s}}^{\mathsf{T}} ; \overline{e ; \mathbb{T}} \subseteq \overline{\mathbb{S}^{\mathsf{T}i} ; z ; \mathbb{T}} \quad \text{negated} \\
&\iff \mathcal{D}_{\mathrm{s}} ; \mathbb{S}^{\mathsf{T}i} ; z ; \mathbb{T} \subseteq e ; \mathbb{T} \quad \text{Schröder rule}
\end{aligned}
$$

This expresses indeed that every strict divisor of $i \geq 1$ must necessarily be 1.

---

ii) In the infinite case also $\wp_1 ; \mathbb{T} \subseteq \mathtt{sup}_{i\geq 1}\, \mathbb{S}^{\mathsf{T}i} ; z ; \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{1+(i-1)!} ; \mathbb{T}$ is required. This is, however, trivial because always $\mathbb{S}^{1+(i-1)!} \neq \mathbb{\bot}$, so that

$$\mathtt{sup}_{i\geq 1}\, \mathbb{S}^{\mathsf{T}i} ; z ; \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{1+(i-1)!} ; \mathbb{T} = \mathtt{sup}_{i\geq 1}\, \mathbb{S}^{\mathsf{T}i} ; z ; \mathbb{T}. \qquad \square$$

n	$1 + (n-1)!$	`quot` $(1 + (n-1)!)$ n	`rem` $(1 + (n-1)!)$ n
1	2	2	0
2	2	1	0
3	3	1	0
5	25	5	0
7	721	103	0
9	40321	4480	1
11	3628801	329891	0
13	479001601	36846277	0
15	87178291201	5811886080	1
17	20922789888001	1230752346353	0
19	6402373705728001	336967037143579	0
21	2432902008176640001	115852476579840000	1

The table above is an indication that also here $\wp_1$; $\mathbb{T}$ may be used.

9 Primes

A most prominent concept of number theory are prime numbers characterized by the requirement that only 1 and the number itself occur as its divisors. At a first view, thus $1, 2, 3, 5, 7, \ldots$ should be considered being prime. The number 1, however, is so omnipresent that one will soon exclude it from the discussion of primes; even the 2 behaves very special and one often restricts to odd primes. For the denotation intended see page 14, which is more or less repeated in Fig. 9.1.1. It is not immediately clear, however, which relational definition one should use. Earliest attempts may already be seen around Fig. 6.3.12, Fig. 6.3.13, the remark on page 134, Prop. 7.4.2.xi,xii and Fig. 7.4.5.

9.1 Several ideas for the introduction of primes

Denotation of the sets of primes is planned as indicated in Fig. 9.1.1 — which will soon be replaced by a formal definition.

$$\wp_1 ; \mathbb{T} = \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \\ 16 \\ 17 \end{matrix} \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{pmatrix} \qquad \wp_2 ; \mathbb{T} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{pmatrix} \qquad \wp ; \mathbb{T} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{pmatrix}$$

$$\wp = \begin{matrix} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \\ 16 \\ 17 \end{matrix} \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \end{matrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}$$

Fig. 9.1.1 Vectors and one of the three diagonals to denote primes

Several conceivable definitions shall be investigated, and their mutual properties be proved. It is thereby tried to identify a definition that offers an easily manipulable algebraic flavour.

A first possible variant is based on the multiplication relation $\mathbb{M}$. The basic requirement for primes demands that when decomposing a number multiplicatively, one factor is necessarily the number 1. So we start with the two relations

$$\mathbb{M} \quad \text{and} \quad (\mathbb{I} \otimes e ; \mathbb{T}) \cup (e ; \mathbb{T} \otimes \mathbb{I}) .$$

Several of their properties are immediate:

9.1.1 Proposition. Let us abbreviate $\alpha := (\mathbb{I} \oslash e ; \mathbb{T}) \cup (e ; \mathbb{T} \oslash \mathbb{I})$ and assume $\mathbb{S}$ to be regressively finite. Then the following hold:

i) $\mathbb{M}^{\mathsf{T}} ; \alpha = \mathbb{I}$

ii) $\alpha \subseteq \mathbb{M}$

iii) $\mathcal{D}^{\mathsf{T}} ; \overline{\mathbb{I} \cup e ; \mathbb{T}} = \overline{\mathtt{DIA} \cup e ; \mathbb{T}}$

Proof: i) $\mathbb{M}^{\mathsf{T}} ; \alpha = \mathbb{M}^{\mathsf{T}} ; \big[(\mathbb{I} \oslash e ; \mathbb{T}) \cup (e ; \mathbb{T} \oslash \mathbb{I}) \big]$
$= \mathbb{M}^{\mathsf{T}} ; (\mathbb{I} \oslash e ; \mathbb{T}) \cup \mathbb{M}^{\mathsf{T}} ; (e ; \mathbb{T} \oslash \mathbb{I}) = \mathbb{I} \cup \mathbb{I} = \mathbb{I}$ Prop. 7.1.1.i.e with $\mathbb{M} = N$ of (ii.b)

ii) Since $\mathbb{M}$ is — due to Prop. 7.1.1.ii.b — commutative, it suffices to show that
$(e ; \mathbb{T} \oslash \mathbb{I}) = (\mathbb{S}^{\mathsf{T}} ; z ; \mathbb{T} \oslash k_1) \subseteq \mathtt{sup}_{n \geq 0} (\mathbb{S}^{\mathsf{T}^n} ; z ; \mathbb{T} \oslash k_n) = \mathbb{M}$. Prop. 7.1.1.i.a

iii) Computationally ok; Exercise! □

Then we ask which columns of $\mathbb{M}$ — selected by the diagonal relation $\mathtt{DIA} \subseteq \mathbb{I}$ — are contained in some columns of the latter, i.e. $\mathbb{M} ; \mathtt{DIA} \subseteq (\mathbb{I} \oslash e ; \mathbb{T}) \cup (e ; \mathbb{T} \oslash \mathbb{I})$. We recall that the right residual models precisely the construct that models this situation and find out

$$\mathtt{DIA} := \overline{\mathbb{M}^{\mathsf{T}} ; \overline{(\mathbb{I} \oslash e ; \mathbb{T}) \cup (e ; \mathbb{T} \oslash \mathbb{I})}}.$$

$$\mathtt{DIA} = \overline{\mathbb{M}^{\mathsf{T}} ; \overline{(\mathbb{I} \oslash e ; \mathbb{T}) \cup (e ; \mathbb{T} \oslash \mathbb{I})}} =$$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Fig. 9.1.2 The primes starting at 1 obtained as a residual — limited by 17

Pictorially, this partial diagonal is obviously suited to characterize the set of primes starting with 1 as

$\wp_1 := \mathtt{DIA}$ or $\wp_1 ; \mathbb{T} := \mathtt{DIA} ; \mathbb{T}$

There are other ideas of this type that shall now be presented with Fig. 9.1.4. The column for 5 of $\mathbb{M}$, e.g. marks just the pairs $(1, 5)$ and $(5, 1)$. Since it seems to model the primes more or less directly, we define also

$\mathtt{MIQ} := \overline{\mathbb{M}^{\mathsf{T}} ; (\overline{\mathbb{I}} \oslash \overline{\mathbb{I}})}$,

thus modelling, where a product is equal to one or the other of its factors. This applies not least to 0. One would probably try to prove

$\wp_1 ; \mathbb{T} := \mathtt{MIQ} ; e ; \mathbb{T}$ or $\wp_1 := \mathtt{MIQ} \cap \mathbb{I} \cap \overline{z ; \mathbb{T}}$ or $\wp_2 ; \mathbb{T} := (\mathtt{MIQ} \cap \mathbb{I}) ; \mathbb{T}$.

$$\mathbb{M} \cap (\pi \cup \rho) \cap \overline{\mathbb{T}_{\cdot} z^{\intercal}} =$$

	0	1	2	3	4	5	6	7
(0,0)	0	0	0	0	0	0	0	0
(0,1)	0	0	0	0	0	0	0	0
(0,2)	0	0	0	0	0	0	0	0
(0,3)	0	0	0	0	0	0	0	0
(0,4)	0	0	0	0	0	0	0	0
(0,5)	0	0	0	0	0	0	0	0
(0,6)	0	0	0	0	0	0	0	0
(0,7)	0	0	0	0	0	0	0	0
(1,0)	0	0	0	0	0	0	0	0
(1,1)	0	1	0	0	0	0	0	0
(1,2)	0	0	1	0	0	0	0	0
(1,3)	0	0	0	1	0	0	0	0
(1,4)	0	0	0	0	1	0	0	0
(1,5)	0	0	0	0	0	1	0	0
(1,6)	0	0	0	0	0	0	1	0
(1,7)	0	0	0	0	0	0	0	1
(2,0)	0	0	0	0	0	0	0	0
(2,1)	0	0	1	0	0	0	0	0
(2,2)	0	0	0	0	0	0	0	0
(2,3)	0	0	0	0	0	0	0	0
(2,4)	0	0	0	0	0	0	0	0
(2,5)	0	0	0	0	0	0	0	0
(2,6)	0	0	0	0	0	0	0	0
(2,7)	0	0	0	0	0	0	0	0
(3,0)	0	0	0	0	0	0	0	0
(3,1)	0	0	0	1	0	0	0	0
(3,2)	0	0	0	0	0	0	0	0
(3,3)	0	0	0	0	0	0	0	0
(3,4)	0	0	0	0	0	0	0	0
(3,5)	0	0	0	0	0	0	0	0
(3,6)	0	0	0	0	0	0	0	0
(3,7)	0	0	0	0	0	0	0	0
(4,0)	0	0	0	0	0	0	0	0
(4,1)	0	0	0	0	1	0	0	0
(4,2)	0	0	0	0	0	0	0	0
(4,3)	0	0	0	0	0	0	0	0
(4,4)	0	0	0	0	0	0	0	0
(4,5)	0	0	0	0	0	0	0	0
(4,6)	0	0	0	0	0	0	0	0
(4,7)	0	0	0	0	0	0	0	0
(5,0)	0	0	0	0	0	0	0	0
(5,1)	0	0	0	0	0	1	0	0
(5,2)	0	0	0	0	0	0	0	0
(5,3)	0	0	0	0	0	0	0	0
(5,4)	0	0	0	0	0	0	0	0
(5,5)	0	0	0	0	0	0	0	0
(5,6)	0	0	0	0	0	0	0	0
(5,7)	0	0	0	0	0	0	0	0
(6,0)	0	0	0	0	0	0	0	0
(6,1)	0	0	0	0	0	0	1	0
(6,2)	0	0	0	0	0	0	0	0
(6,3)	0	0	0	0	0	0	0	0
(6,4)	0	0	0	0	0	0	0	0
(6,5)	0	0	0	0	0	0	0	0
(6,6)	0	0	0	0	0	0	0	0
(6,7)	0	0	0	0	0	0	0	0
(7,0)	0	0	0	0	0	0	0	0
(7,1)	0	0	0	0	0	0	0	1
(7,2)	0	0	0	0	0	0	0	0
(7,3)	0	0	0	0	0	0	0	0
(7,4)	0	0	0	0	0	0	0	0
(7,5)	0	0	0	0	0	0	0	0
(7,6)	0	0	0	0	0	0	0	0
(7,7)	0	0	0	0	0	0	0	0

Fig. 9.1.3 Pairs containing the number 1 on either side, with 0-column annihilated

$$\begin{array}{c|cccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
(0,0) & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(0,1) & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(0,2) & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(0,3) & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(0,4) & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(0,5) & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(0,6) & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(0,7) & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(1,0) & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(1,1) & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
(1,2) & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
(1,3) & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
(1,4) & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
(1,5) & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
(1,6) & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
(1,7) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
(2,0) & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(2,1) & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
(2,2) & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
(2,3) & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
(2,4) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(2,5) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(2,6) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(2,7) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(3,0) & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(3,1) & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
(3,2) & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
(3,3) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(3,4) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(3,5) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(3,6) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(3,7) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(4,0) & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(4,1) & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
(4,2) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(4,3) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(4,4) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(4,5) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(4,6) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(4,7) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(5,0) & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(5,1) & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
(5,2) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(5,3) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(5,4) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(5,5) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(5,6) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(5,7) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(6,0) & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(6,1) & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
(6,2) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(6,3) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(6,4) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(6,5) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(6,6) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(6,7) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(7,0) & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(7,1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
(7,2) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(7,3) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(7,4) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(7,5) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(7,6) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(7,7) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\qquad
\begin{array}{c|cccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
(0,0) & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(0,1) & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
(0,2) & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
(0,3) & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
(0,4) & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
(0,5) & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
(0,6) & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
(0,7) & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
(1,0) & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
(1,1) & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
(1,2) & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
(1,3) & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
(1,4) & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
(1,5) & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
(1,6) & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
(1,7) & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
(2,0) & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
(2,1) & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
(2,2) & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
(2,3) & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
(2,4) & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
(2,5) & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
(2,6) & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
(2,7) & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
(3,0) & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
(3,1) & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
(3,2) & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
(3,3) & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
(3,4) & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
(3,5) & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
(3,6) & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
(3,7) & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
(4,0) & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
(4,1) & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
(4,2) & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
(4,3) & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
(4,4) & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
(4,5) & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
(4,6) & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
(4,7) & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
(5,0) & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
(5,1) & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
(5,2) & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
(5,3) & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
(5,4) & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
(5,5) & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
(5,6) & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
(5,7) & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
(6,0) & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
(6,1) & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
(6,2) & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
(6,3) & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
(6,4) & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
(6,5) & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
(6,6) & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
(6,7) & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
(7,0) & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
(7,1) & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
(7,2) & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
(7,3) & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
(7,4) & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
(7,5) & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
(7,6) & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
(7,7) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}$$

$$\Longleftarrow \quad \Longleftarrow \mathbb{M}$$

$$= \pi \cup \rho = \overline{(\overline{\mathbb{I}} \circledcirc \overline{\mathbb{I}})}$$

$$\texttt{MIQ} = \overline{\mathbb{M}^{\intercal_{;}} (\overline{\mathbb{I}} \circledcirc \overline{\mathbb{I}})} = \begin{array}{c|cccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
7 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}$$

correspondingly for limitation by 17:

$$\begin{array}{c|cccccccccccccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
7 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
11 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
13 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
14 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
17 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}$$

Fig. 9.1.4 Multiplication $\mathbb{M}$, $\pi \cup \rho$ and MIQ in approaching a conceivable definition of primes

The relation $\overline{(\overline{\mathbb{I}} \circledcirc \overline{\mathbb{I}})}$ describes the transition from a pair to the — at most 2-element- — set of its constituent elements; see Fig. 9.1.4. In a first attempt, we might intersect it with $\mathbb{M}$, providing us nearly with the pairs desired; just the pairs containing the number 0 must be eliminated.

Fig. 9.1.4 shows that the residual $\mathtt{MIQ} = \overline{\mathbb{M}^{\intercal;} (\overline{\mathbb{I}} \circledcirc \overline{\mathbb{I}})}$ also incorporates pictorially all the information on primes without referring to a single element such as e. Multiplied with e it gives the primes $\wp_{1\,;}\mathbb{T}$ starting at 1. Deleting the diagonal it indicates the possibly even primes $\wp_{2\,;}\mathbb{T}$.

A not that immediate variant computes the residuals $\overline{k_n^{\intercal;}\overline{\mathbb{I}}}$ (see page 122) for $n \geq 2$ and intersects them as shown in the following. The final intersection will certainly trigger us, to investigate

$$\mathtt{ZPE} := \mathtt{inf}_{\,n\geq 2}\, \overline{k_n^{\intercal;}\overline{\mathbb{I}}}$$

and consider

$$\wp_{1\,;}\mathbb{T} := \mathtt{ZPE}_{;}e_{\,;}\mathbb{T}$$

as possible definition. Below follows first the final intersection and then the underlying sequence $\overline{k_2^{\intercal;}\overline{\mathbb{I}}}\ \ldots\ \overline{k_9^{\intercal;}\overline{\mathbb{I}}}$.

$$\mathtt{ZPE} = \mathtt{inf}_{\,n\geq 2}\, \overline{k_n^{\intercal;}\overline{\mathbb{I}}} =$$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
17	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Fig. 9.1.5 Final intersection of $\overline{k_2^{\intercal;}\overline{\mathbb{I}}}\ \ldots\ \overline{k_9^{\intercal;}\overline{\mathbb{I}}}$

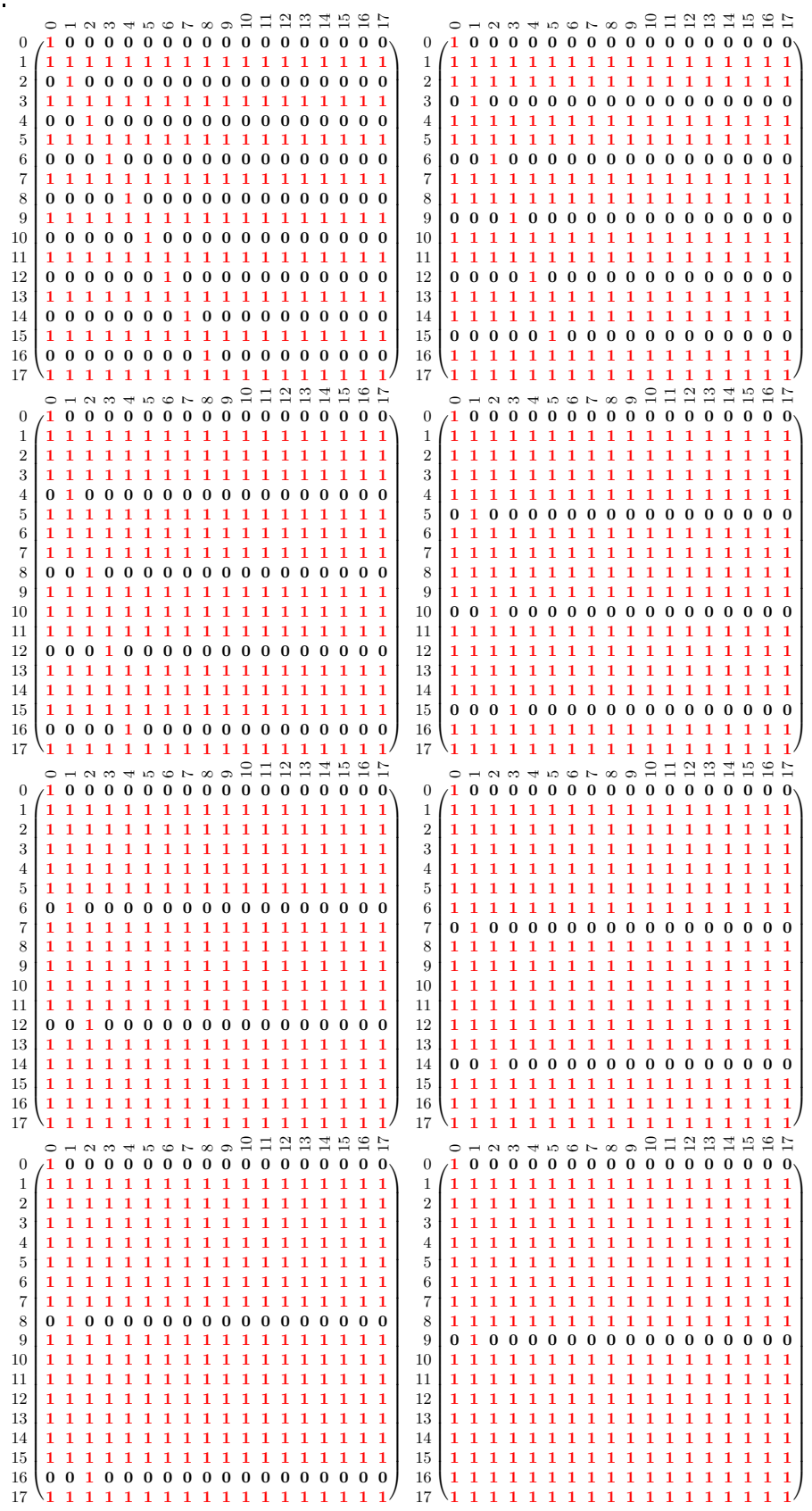


Fig. 9.1.6 $\overline{k_2^{\mathrm{I};\mathbb{I}}}$ … $\overline{k_9^{\mathrm{I};\mathbb{I}}}$

Yet another iteration that comes close to primes is this one $\texttt{MTI} := \texttt{inf}_{n\geq 2}\left[\mathbb{T}\,;z^{\mathsf{T}};\mathbb{S}^{\,n}\cup\overline{k_n^{\mathsf{T}};\mathbb{I}}\right]$, the parts $2\ldots6$ of which we show below together with the final intersection:

$$\wp_1\,;\mathbb{T} := \texttt{MTI}\,;e\,;\mathbb{T} \quad \text{or} \quad \wp_1 := \texttt{MTI}\cap\mathbb{I}\cap\overline{z\,;\mathbb{T}}\ .$$

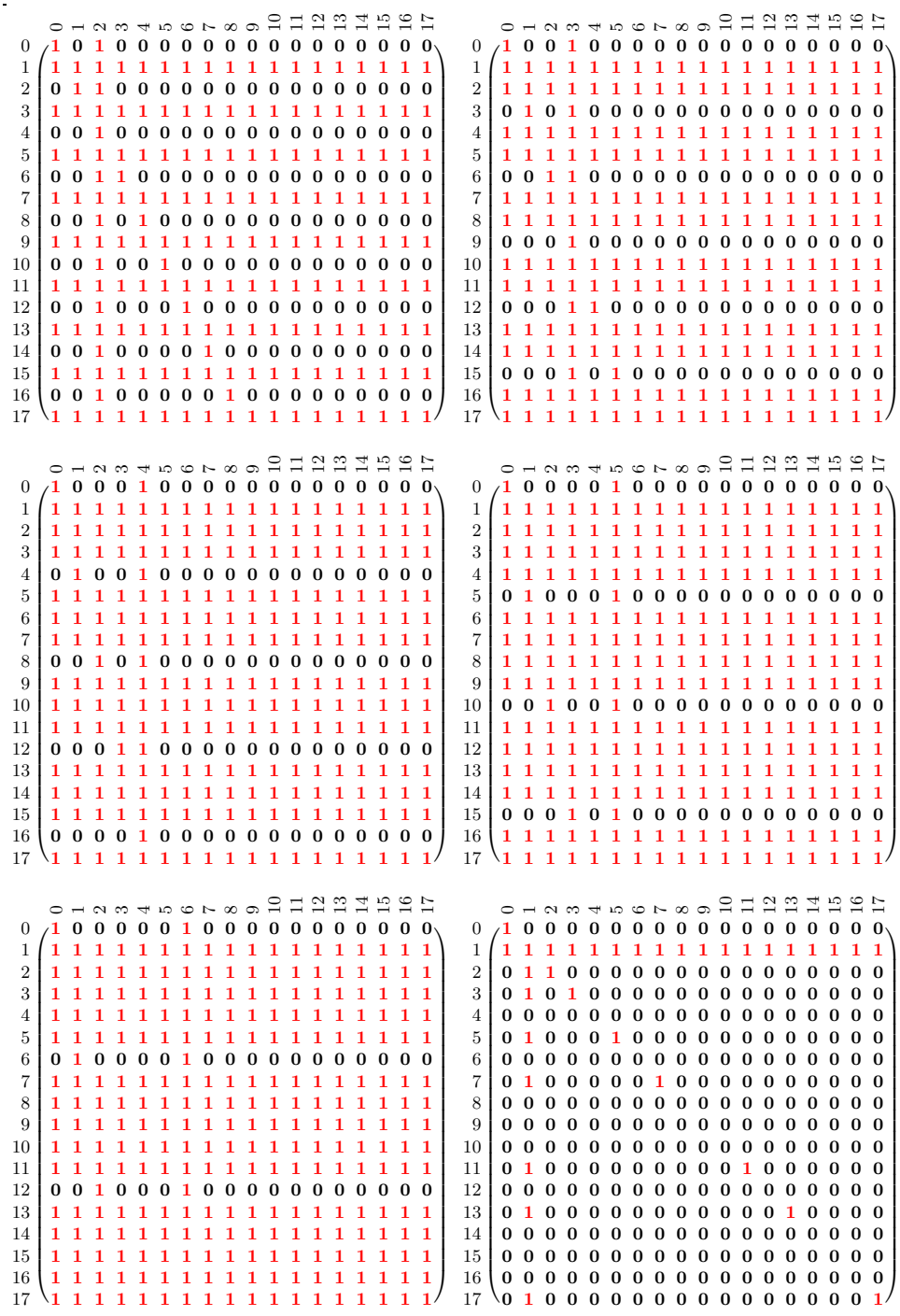


Fig. 9.1.7 Iteration steps to obtain MTI

In another attempt, we might look at the divides order $\mathcal{D}$ of Fig. 7.4.1, or even better the divides strictorder $\mathcal{D}_\mathrm{s} := \mathcal{D} \cap \overline{\mathbb{I}}$, that give some hints as to primes. Deleting the 1-row, $\mathcal{D}_\mathrm{s}$ reduces to the left relation $\mathcal{D}_\mathrm{s} \cap \overline{e \mathbin{;} \mathbb{T}}$ of Fig. 9.1.8, out of which the empty columns provide access to the primes.

$$\wp_1 \mathbin{;} \mathbb{T} := \overline{\left[\mathcal{D}_\mathrm{s} \cap \overline{e \mathbin{;} \mathbb{T}}\right]^\mathsf{T} \mathbin{;} \mathbb{T}} = \begin{pmatrix} 0\\1\\1\\1\\0\\1\\0\\1\\0\\0\\0\\1\\0\\1\\0\\0\\0\\1 \end{pmatrix}$$

Fig. 9.1.8 Intermediate step to indicate primes via $\mathcal{D}_\mathrm{s} \cap \overline{e \mathbin{;} \mathbb{T}}$

This may be modified as

$$\wp_1 \mathbin{;} \mathbb{T} := \overline{\left[\mathcal{D}_\mathrm{s} \cap \overline{e \mathbin{;} \mathbb{T}}\right]^\mathsf{T} \mathbin{;} \mathbb{T}} = \overline{\mathcal{D}_\mathrm{s}^\mathsf{T} \mathbin{;} \left[\mathbb{T} \cap \overline{e \mathbin{;} \mathbb{T}}\right]} = \overline{\mathcal{D}_\mathrm{s}^\mathsf{T} \mathbin{;} \overline{\mathbb{I}}} \mathbin{;} e \mathbin{;} \mathbb{T}$$

and we define

$$\mathtt{DIV} := \overline{\mathcal{D}_\mathrm{s}^\mathsf{T} \mathbin{;} \overline{\mathbb{I}}},$$

so that $\wp_1 \mathbin{;} \mathbb{T} := \mathtt{DIV} \mathbin{;} \mathbb{T} = \mathtt{DIV} \mathbin{;} e \mathbin{;} \mathbb{T}$ look — in view of Fig. 9.1.9 — like a definition.

With the symmetric quotient $\mathtt{SYD} := \mathtt{syq}(\mathcal{D}_\mathrm{s}, \mathbb{I})$, we are lead to $\wp_2 \mathbin{;} \mathbb{T} := \mathtt{SYD} \mathbin{;} \mathbb{T} = \mathtt{SYD} \mathbin{;} e \mathbin{;} \mathbb{T}$

$$\mathtt{DIV} = \overline{\mathcal{D}_\mathrm{s}^\mathsf{T} \mathbin{;} \overline{\mathbb{I}}} \qquad\qquad \mathtt{SYD} = \mathtt{syq}(\mathcal{D}_\mathrm{s}, \mathbb{I})$$

Fig. 9.1.9 Showing DIV and SYD pictorially

$$\mathbb{M}_{\mathtt{SYD}}\mathbb{T} = \begin{array}{c} \\ (0,0)\\(0,1)\\(0,2)\\(0,3)\\(0,4)\\(0,5)\\(0,6)\\(0,7)\\(1,0)\\(1,1)\\(1,2)\\(1,3)\\(1,4)\\(1,5)\\(1,6)\\(1,7)\\(2,0)\\(2,1)\\(2,2)\\(2,3)\\(2,4)\\(2,5)\\(2,6)\\(2,7)\\(3,0)\\(3,1)\\(3,2)\\(3,3)\\(3,4)\\(3,5)\\(3,6)\\(3,7)\\(4,0)\\(4,1)\\(4,2)\\(4,3)\\(4,4)\\(4,5)\\(4,6)\\(4,7)\\(5,0)\\(5,1)\\(5,2)\\(5,3)\\(5,4)\\(5,5)\\(5,6)\\(5,7)\\(6,0)\\(6,1)\\(6,2)\\(6,3)\\(6,4)\\(6,5)\\(6,6)\\(6,7)\\(7,0)\\(7,1)\\(7,2)\\(7,3)\\(7,4)\\(7,5)\\(7,6)\\(7,7) \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\

9.2 Further ideas on how to introduce primes

Also the relations $\mathcal{D}$ of Fig. 7.4.1 or Fig. 7.4.2 persuade us to identify 1 as the least element of the divides lattice and then call its atoms the (not necessarily odd) primes. With $\mathcal{H} := \mathcal{D}_\mathrm{s} \cap \overline{\mathcal{D}_\mathrm{s} ; \mathcal{D}_\mathrm{s}}$ the Hasse-relation of $\mathcal{D}_\mathrm{s}$, we define $\wp_2 ; \mathbb{T} := \mathtt{HSD}$ with

$\mathtt{HSD} := \mathcal{H}^\mathsf{T} ; e ; \mathbb{T}$.

While the above HSD will be discussed in addition to the preceding ones, there are further possibilities to detect primes out of the relational context. Some are presented in pictorial form, giving strong hints. Of course — these deserve to be proved, which we do not yet.

We begin with the Ferrers operator, that has according to [9] lots of formula connections, applied to the division strictorder $\mathcal{D}_\mathrm{s}$. Fig. 9.2.1 immediately triggers us to try a proof of the statement on the right of the following line:

$\mathtt{FerrOP}(R) := R ; \overline{R}^\mathsf{T} ; R \qquad\qquad \wp_1 ; \mathbb{T} = \overline{\mathbb{T} ; \mathtt{FerrOP}(\mathcal{D}_\mathrm{s})}^\mathsf{T}$

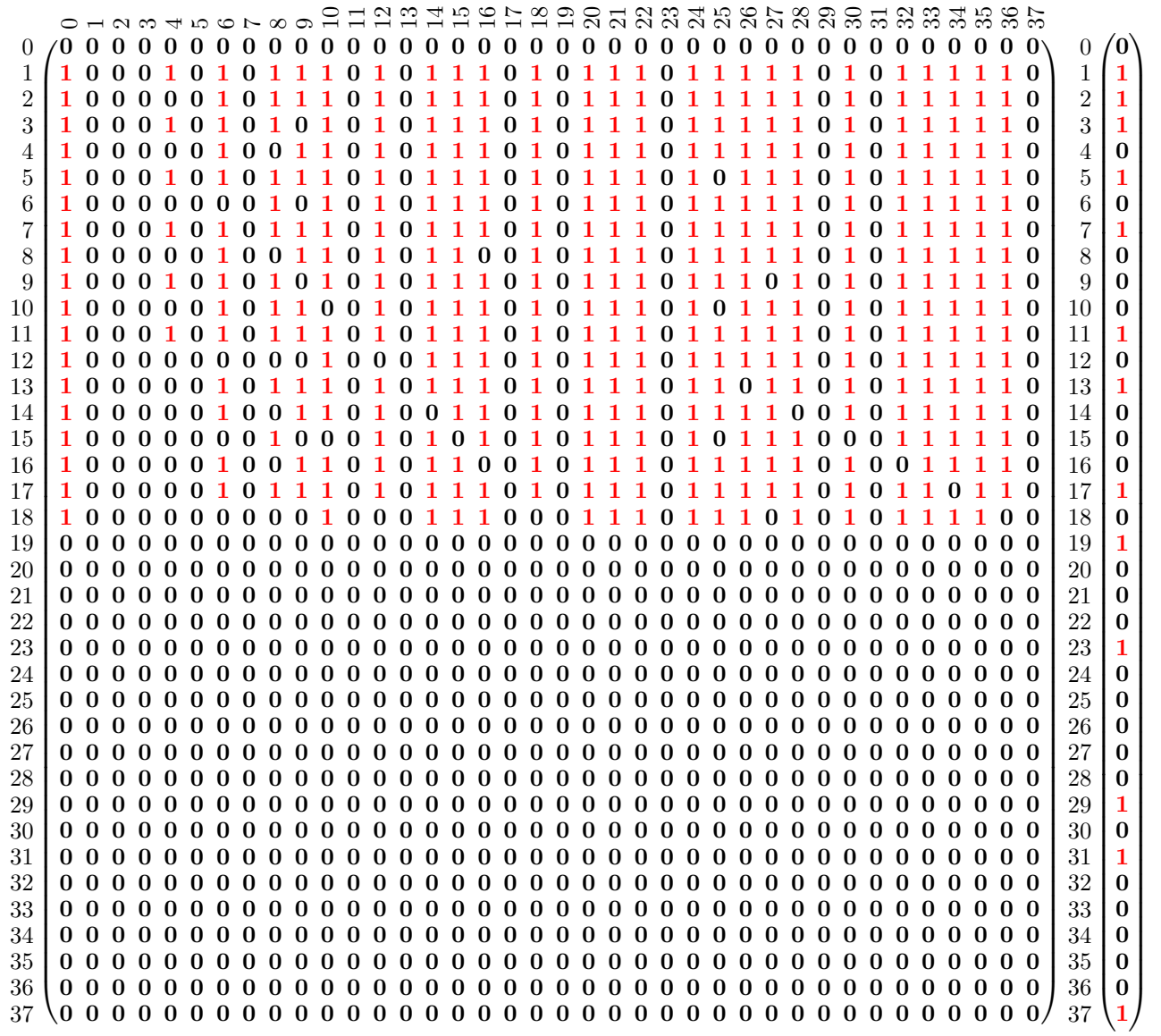


Fig. 9.2.1 Illustrating $\mathtt{FerrOP}(\mathcal{D}_\mathrm{s})$ and $\overline{\mathbb{T} ; \mathtt{FerrOP}(\mathcal{D}_\mathrm{s})}^\mathsf{T}$

Next we test the fringe operation $\nabla(R)$ as to being connected with primes. Also this is broadly discussed in [9] and here visualized with Fig. 9.2.2. It leads to the attempt shown here on the right:

$$\nabla(R) := R \cap \overline{\texttt{FerrOP}(R)} \qquad \wp_2 \, ; \mathbb{T} = \nabla(\mathcal{D}_\mathrm{s})^\intercal ; e \, ; \mathbb{T}$$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	1	1	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0	1	0
2	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
3	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
4	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
19	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
20	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
21	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
22	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
24	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
25	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
26	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
27	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
28	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
29	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
30	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
31	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
32	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
33	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
34	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
35	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
36	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Fig. 9.2.2 Illustrating $\nabla(\mathcal{D}_\mathrm{s})$ and $\nabla(\mathcal{D}_\mathrm{s})^\intercal ; e \, ; \mathbb{T}$

In this way, we get a bunch of suspected formulae and descriptions, that no serious mathematician — should he/she be interested in such topics — would ever dare to distrust. On the other side, we need here a lot of highly detailed formal proofs of seemingly obvious statements. The idea was, to execute all this hoping to find a firm substrate on which to later erect ever more complex constructions.

Yet another concept is the relation with the fringe mentioned deleted; also for Fig. 9.2.3 an attempt is made to extract the primes as shown in:

$$\texttt{FringeDel}(R) := R \cap \texttt{FerrOP}(R) \qquad \wp_1 \, ; \, \mathbb{T} = \overline{\mathbb{T} \, ; \, \texttt{FringeDel}(\mathcal{D}_\mathrm{s})}^{\mathsf{T}}$$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	0	0	1	0	1	0	1	1	1	0	1	0	1	1	1	0	1	0	1	1	1	0	1	1	1	1	1	0	1	0	1	1	1	1	1	0	1
2	1	0	0	0	0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
3	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	1
4	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0
5	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	1
6	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0
7	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	1
8	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0
9	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
10	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
11	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1
12	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
13	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
14	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1
16	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
17	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1
18	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1
20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
21	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1
24	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
25	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
26	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
27	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
28	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
29	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
30	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
31	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
32	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
33	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
34	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
35	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Fig. 9.2.3 Illustrating $\texttt{FringeDel}(\mathcal{D}_\mathrm{s})$ and $\overline{\mathbb{T} \, ; \texttt{FringeDel}(\mathcal{D}_\mathrm{s})}^{\mathsf{T}}$

There exists according to [9, Def. 10.40] also the concept of a block-transitive kernel of a relation. A connection to primes exists as to Fig. 9.2.4, using $\nabla := \nabla(R) = R \cap \overline{\mathtt{FerrOP}(R)}$ and the definition of the univalent part, but is not that immediate:

$\mathtt{BTK}(R) := \nabla ; \nabla^{\mathsf{T}} ; R ; \nabla^{\mathsf{T}} ; \nabla$ $\qquad$ $\wp_2 ; \mathbb{T} := \mathtt{upa}\big(\mathtt{BTK}(\mathcal{D}_{\mathrm{s}})^{\mathsf{T}}\big) ; \mathbb{T}$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37		
0	**0**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**0**
1	1	**0**	1	1	1	1	0	1	1	1	0	1	0	1	0	0	1	1	0	1	0	0	0	1	0	1	1	1	1	1	1	1	1	0	1	0	1	1	1	**0**
2	1	0	**0**	0	1	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	1	0	1	0	1	0	2	**1**
3	1	0	0	**0**	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	1	0	3	**1**
4	1	0	0	0	**0**	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	1	0	4	**0**
5	1	0	0	0	0	**0**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	5	**1**
6	0	0	0	0	0	0	**0**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	6	**0**
7	0	0	0	0	0	0	0	**0**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	7	**1**
8	1	0	0	0	0	0	0	0	**0**	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	8	**0**
9	1	0	0	0	0	0	0	0	0	**0**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	9	**0**
10	0	0	0	0	0	0	0	0	0	0	**0**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	10	**0**
11	0	0	0	0	0	0	0	0	0	0	0	**0**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	11	**1**
12	0	0	0	0	0	0	0	0	0	0	0	0	**0**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	12	**0**
13	1	0	0	0	0	0	0	0	0	0	0	0	0	**0**	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	13	**1**
14	1	0	0	0	0	0	0	0	0	0	0	0	0	0	**0**	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	14	**0**
15	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**0**	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	15	**0**
16	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**0**	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	16	**0**
17	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**0**	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	17	**1**
18	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**0**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	18	**0**
19	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**0**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	19	**1**
20	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**0**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	20	**0**
21	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**0**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	21	**0**
22	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**0**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	22	**0**
23	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**0**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	23	**1**
24	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**0**	0	0	0	0	0	0	0	0	0	0	0	0	0	24	**0**
25	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**0**	0	0	0	0	0	0	0	0	0	0	0	0	25	**0**
26	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**0**	0	0	0	0	0	0	0	0	0	0	0	26	**0**
27	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**0**	0	0	0	0	0	0	0	0	0	0	27	**0**
28	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**0**	0	0	0	0	0	0	0	0	0	28	**0**
29	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**0**	0	0	0	0	0	0	0	0	29	**1**
30	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**0**	0	0	0	0	0	0	0	30	**0**
31	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**0**	0	0	0	0	0	0	31	**1**
32	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**0**	0	0	0	0	0	32	**0**
33	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**0**	0	0	0	0	33	**0**
34	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**0**	0	0	0	34	**0**
35	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**0**	0	0	35	**0**
36	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**0**	0	36	**0**
37	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**0**	37	**1**

Fig. 9.2.4 Illustrating $\mathtt{BTK}(\mathcal{D}_{\mathrm{s}})$ and $\mathtt{upa}\big(\mathtt{BTK}(\mathcal{D}_{\mathrm{s}})^{\mathsf{T}}\big) ; \mathbb{T}$

Also the theory of inverses (sub-inverses, generalized inverses, Thierrin-Vagner inverses and Moore-Penrose inverses) gives some help. The relation G is called a subinverse of the relation R, provided $R ; G ; R \subseteq R$. There exists always the greatest sub-inverse as shown in Fig. 9.2.5; again this leads to an attempt to define primes:

$\mathtt{GreSub}(R) := \overline{\mathtt{FerrOP}(R^{\mathsf{T}})}$ $\qquad$ $\wp_1 ; \mathbb{T} := \mathtt{GreSub}(\mathcal{D}_{\mathrm{s}}) ; e ; \mathbb{T}$

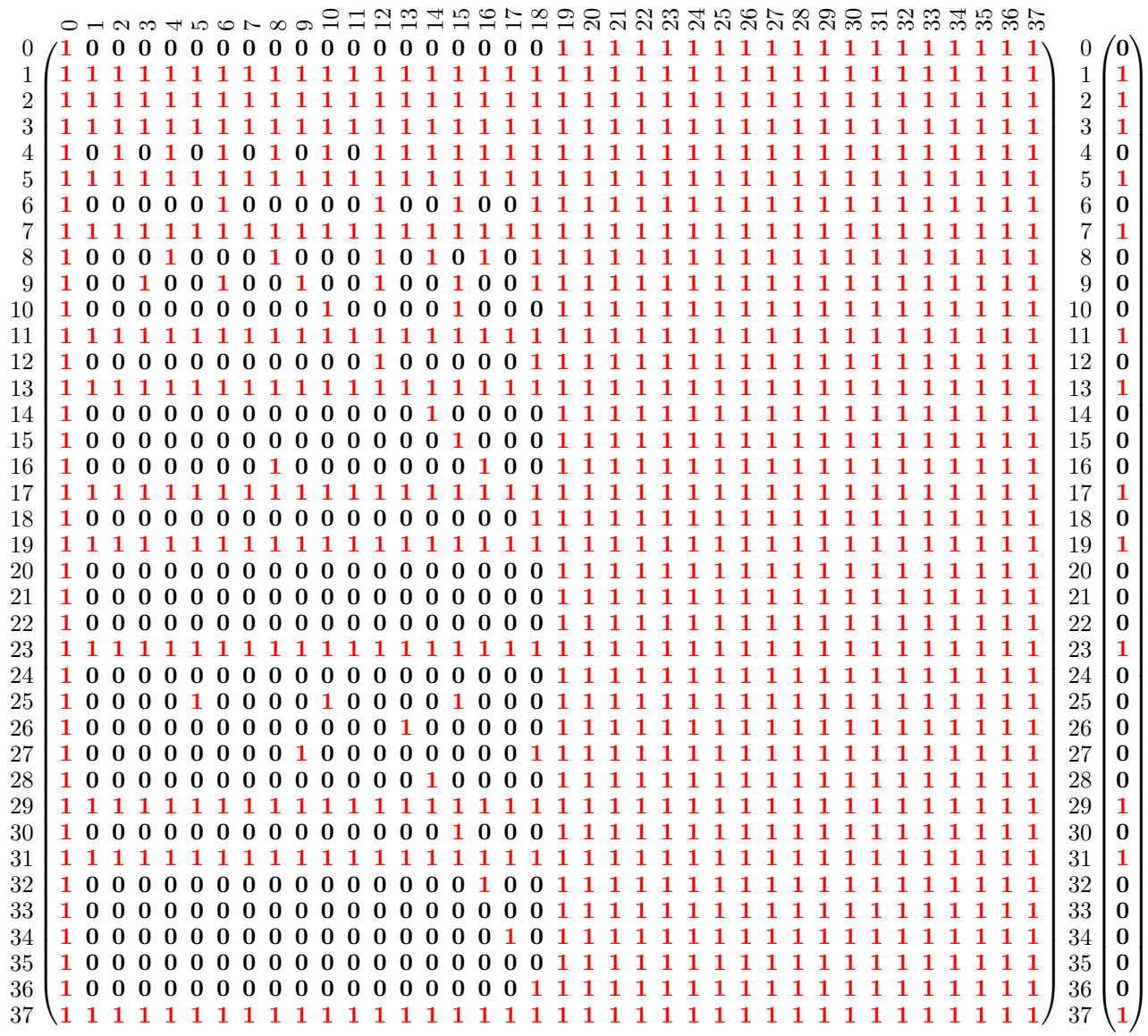


Fig. 9.2.5 Illustrating $\mathtt{GreSub}(\mathcal{D}_{\mathrm{s}})$ and $\mathtt{GreSub}(\mathcal{D}_{\mathrm{s}}) ; e ; \mathbb{T}$

Remark: In general, we have

$\mathcal{D}_{\mathrm{s}} ; \mathtt{GreSub}(\mathcal{D}_{\mathrm{s}}) ; \mathcal{D}_{\mathrm{s}} \subsetneqq \mathcal{D}_{\mathrm{s}}$

in view of entry $(6, 12)$. Also

$\mathtt{GreSub}(\mathcal{D}_{\mathrm{s}}) ; \mathcal{D}_{\mathrm{s}} ; \mathtt{GreSub}(\mathcal{D}_{\mathrm{s}}) \subsetneqq \mathtt{GreSub}(\mathcal{D}_{\mathrm{s}})$.

It must be mentioned that this inverse is unequal to the candidate for being a Thierrin-Vagner-inverse in Fig. 9.2.6; cf. [9, Sect. 10.7]. However, $\mathcal{D}_\mathrm{s}$ turns out not to be regular, i.e. posessing a generalized inverse, in which case such a Thierrin-Vagner inverse need not exist, that is characterized by $R ; G ; R = R$ as well as $G ; A ; G = G$. Nevertheless, we also obtain a hint as to the primes:

$\mathtt{ThiVag}(R) := \mathtt{GreSub}(R) ; R ; \mathtt{GreSub}(R)$ $\qquad$ $\wp_1 ; \mathbb{T} := \mathtt{ThiVag}(\mathcal{D}_\mathrm{s}) ; e ; \mathbb{T}$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37		
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	**0**
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	**1**
2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	**1**
3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	3	**1**
4	1	0	1	0	1	0	1	0	1	0	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	4	**0**
5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	5	**1**
6	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	6	**0**
7	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	7	**1**
8	1	0	0	0	1	0	0	0	1	0	0	0	1	0	1	0	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	8	**0**
9	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	9	**0**
10	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	10	**0**
11	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	11	**1**
12	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	12	**0**
13	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	13	**1**
14	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	14	**0**
15	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	15	**0**
16	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	16	**0**
17	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	17	**1**
18	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	18	**0**
19	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	19	**1**
20	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	20	**0**
21	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	21	**0**
22	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	22	**0**
23	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	23	**1**
24	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	24	**0**
25	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	25	**0**
26	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	26	**0**
27	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	27	**0**
28	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	28	**0**
29	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	29	**1**
30	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	30	**0**
31	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	31	**1**
32	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	32	**0**
33	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	33	**0**
34	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	34	**0**
35	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	35	**0**
36	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	36	**0**
37	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	37	**1**

Fig. 9.2.6 Illustrating $\mathtt{ThiVag}(\mathcal{D}_\mathrm{s})$ and $\mathtt{ThiVag}(\mathcal{D}_\mathrm{s}) ; e ; \mathbb{T}$

as well as

$$\texttt{GreSub}(\mathcal{D}_{\mathrm{s}})\texttt{;}\mathcal{D}_{\mathrm{s}}\texttt{;}\texttt{GreSub}(\mathcal{D}_{\mathrm{s}}) \subsetneqq \texttt{ThiVag}(\mathcal{D}_{\mathrm{s}}).$$

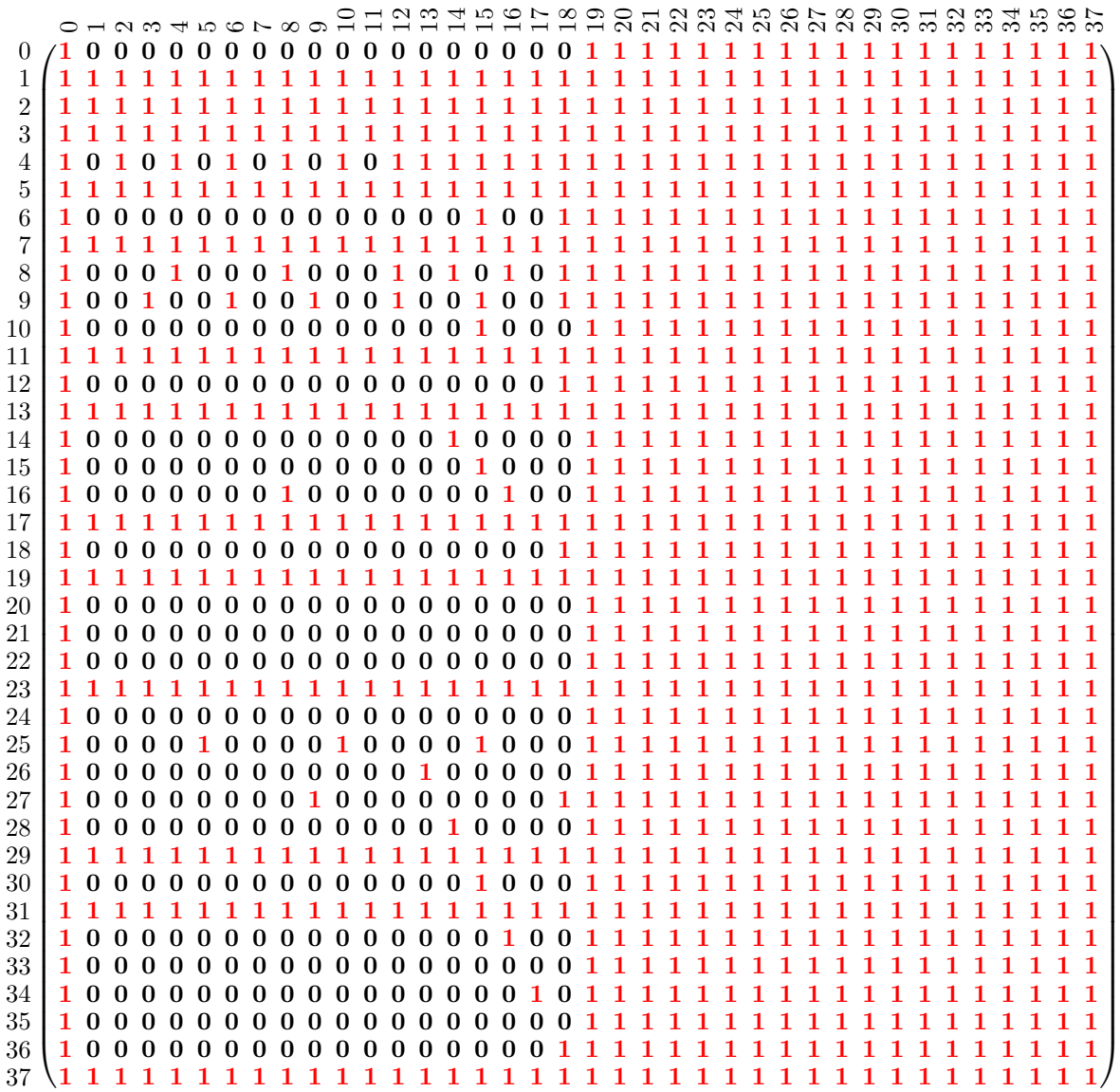


Fig. 9.2.7 Illustrating $\texttt{GreSub}(\mathcal{D}_{\mathrm{s}})\texttt{;}\mathcal{D}_{\mathrm{s}}\texttt{;}\texttt{GreSub}(\mathcal{D}_{\mathrm{s}}) \subsetneqq \texttt{ThiVag}(\mathcal{D}_{\mathrm{s}})$ with entry $(6, 12)$

Remark. As for every ordering, these inverses specialize considerably and satisfy the following identities:

$\texttt{FerrOP}(\mathcal{D}) = \overline{\mathcal{D}}^{\mathsf{T}}$ $\qquad$ $\nabla(\mathcal{D}) = \mathbb{I}$

$\texttt{FringeDel}(\mathcal{D}) = \mathcal{D}_{\mathrm{s}}$ $\qquad$ $\texttt{GreSub}(\mathcal{D}) = \mathcal{D}$

$\texttt{ThiVag}(\mathcal{D}) = \mathcal{D}$ $\qquad$ $\texttt{BTK}(\mathcal{D}) = \mathcal{D}$

9.3 Interdependency

It is obvious that all these pictorially presented ideas may lead to some definition of the concept of being prime; the necessary adaptations being modest. To prove that all these mean essentially the same, may however be tricky in some cases. The following list recalls these concepts and points to their respective visual presentation:

$$\texttt{DIA} := \overline{\mathbb{M}^{\intercal}\mathbin{;}\overline{(\mathbb{I}\oslash e\mathbin{;}\mathbb{T})\cup(e\mathbin{;}\mathbb{T}\oslash\mathbb{I})}} \qquad \text{Fig. 9.1.2}$$

$$\texttt{MIQ} := \overline{\mathbb{M}^{\intercal}\mathbin{;}(\overline{\mathbb{I}}\oslash\overline{\mathbb{I}})} \qquad \text{Fig. 9.1.4}$$

$$\texttt{MTI} := \texttt{inf}_{n\geq 2}\left[\mathbb{T}\mathbin{;}z^{\intercal}\mathbin{;}\mathbb{S}^{n}\cup\overline{k_n^{\intercal}\mathbin{;}\overline{\mathbb{I}}}\right] \qquad \text{Fig. 9.1.7}$$

$$\texttt{ZPE} := \texttt{inf}_{n\geq 2}\,\overline{k_n^{\intercal}\mathbin{;}\overline{\mathbb{I}}} \qquad \text{Fig. 9.1.5}$$

$$\texttt{DIV} := \overline{\mathcal{D}_{\mathrm{s}}{}^{\intercal}\mathbin{;}\overline{\mathbb{I}}} \qquad \text{Fig. 9.1.9}$$

$$\texttt{SYD} := \texttt{syq}(\mathcal{D}_{\mathrm{s}},\mathbb{I}) \qquad \text{Fig. 9.1.9}$$

and in a slightly different way also

$$\texttt{HSD} := \mathcal{H}^{\intercal}\mathbin{;}e\mathbin{;}\mathbb{T} \qquad \text{Fig. 7.4.1}$$

Looking at these different approaches, one will easily find out pictorially that they are all united of the following disjoint pieces

$$z\mathbin{;}\mathbb{T}\mathbin{;}z^{\intercal} \qquad \wp_1 \qquad \wp_2\mathbin{;}\mathbb{T}\mathbin{;}e^{\intercal},$$

possibly together with an additional $e\mathbin{;}\mathbb{T}$. All these are, quite obviously, heavily intertwined and also bound to the basic geometric patterns of Fig. 9.3.1. Later we will in particular find out that

$$\wp_1\subseteq\overline{z\mathbin{;}\overline{\mathbb{T}}}\cap\mathbb{I} \qquad \text{and} \qquad \wp_2\mathbin{;}\mathbb{T}\mathbin{;}e^{\intercal}\subseteq\overline{z\mathbin{;}\overline{\mathbb{T}}}\cap\mathbb{T}\mathbin{;}e^{\intercal}=\overline{z\mathbin{;}\overline{\mathbb{T}}}\mathbin{;}\mathbb{T}\mathbin{;}e^{\intercal}.$$

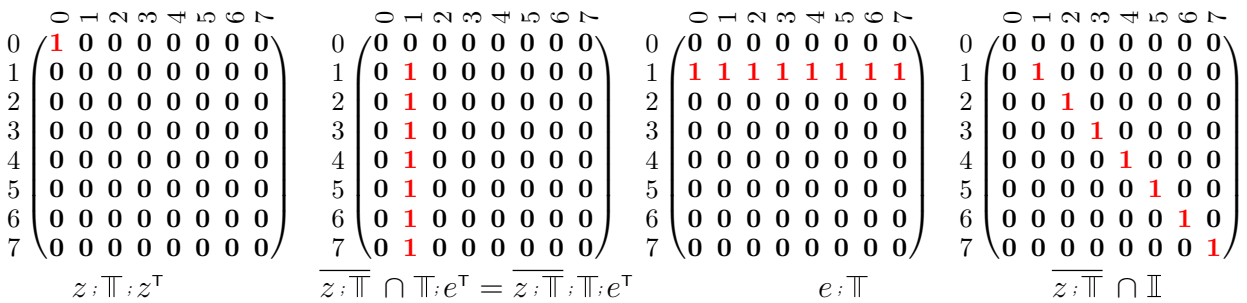


Fig. 9.3.1 Basic geometric patterns

	$\wp_1$	$z\,;\mathbb{T}\,;z^\intercal$	$\wp_2\,;\mathbb{T}\,;e^\intercal$	$e\,;\mathbb{T}$
DIA	•			
MIQ	•	•	•	
MTI	•	•	•	•
ZPE		•	•	•
DIV			•	•
SYD			•	

Fig. 9.3.2 Composition of primes by elementary pictorial elements

We will underpin this pictorial evidence formally:

9.3.1 Proposition.

i) $\mathtt{DIA} = \mathtt{MIQ} \cap \overline{z\,;\mathbb{T}\,;z^\intercal} \cap (e\,;\mathbb{T}\,;e^\intercal \cup \overline{\mathbb{T}\,;e^\intercal}) \subseteq \overline{z\,;\mathbb{T}} \cap \overline{\mathbb{T}\,;z^\intercal} \cap \mathbb{I}$

ii) $\mathtt{MIQ} = \mathtt{MTI} \cap \big[\,\overline{z\,;\mathbb{T}}\,;\mathbb{T}\,;e^\intercal \cup \mathbb{I}\big] \subseteq \overline{z\,;\mathbb{T}}\,;\mathbb{T}\,;e^\intercal \cup \mathbb{I}$

iii) $\mathtt{MTI} = \mathtt{MIQ} \cup e\,;\mathbb{T} \subseteq \overline{z\,;\mathbb{T}}\,;\mathbb{T}\,;e^\intercal \cup e\,;\mathbb{T} \cup \mathbb{I}$

iv) $\mathtt{DIV} = \mathtt{SYD} \cup e\,;\mathbb{T} = \mathtt{ZPE} \cap \overline{z\,;\mathbb{T}} \subseteq \overline{z\,;\mathbb{T}}\,;\mathbb{T}\,;e^\intercal \cup e\,;\mathbb{T}$

v) $\mathtt{ZPE} = \mathtt{DIV} \cup z\,;\mathbb{T}\,;z^\intercal \subseteq \overline{z\,;\mathbb{T}}\,;\mathbb{T}\,;e^\intercal \cup e\,;\mathbb{T} \cup z\,;\mathbb{T}\,;z^\intercal$

vi) $\mathtt{SYD} = \mathtt{DIV} \cap \overline{e\,;\mathbb{T}} \subseteq (\overline{e\,;\mathbb{T}} \cap \overline{z\,;\mathbb{T}})\,;\mathbb{T}\,;e^\intercal$

Proof: i) Pictorially obvious — yet to be demonstrated

---

ii) $\mathtt{MIQ} = \overline{\mathbb{M}^\intercal\,;(\overline{\mathbb{I}} \otimes \overline{\mathbb{I}})}$

$= \overline{\big[\,\mathtt{sup}_{n\geq 0}\,(\mathbb{T}\,;z^\intercal\,;\mathbb{S}^n \otimes k_n^\intercal)\,\big]\,;(\overline{\mathbb{I}} \otimes \overline{\mathbb{I}})}$ expanding $\mathbb{M}$ via Def. 7.1.1.i.a

$= \overline{\mathtt{sup}_{n\geq 0}\,\big[\,(\mathbb{T}\,;z^\intercal\,;\mathbb{S}^n \otimes k_n^\intercal)\,;(\overline{\mathbb{I}} \otimes \overline{\mathbb{I}})\,\big]}$

$= \overline{(\mathbb{T}\,;z^\intercal \otimes z\,;\mathbb{T})\,;(\overline{\mathbb{I}} \otimes \overline{\mathbb{I}})} \;\cap\; \overline{\mathtt{sup}_{n\geq 1}\,(\mathbb{T}\,;z^\intercal\,;\mathbb{S}^n \otimes k_n^\intercal)\,;(\overline{\mathbb{I}} \otimes \overline{\mathbb{I}})}$ as $k_0^\intercal$ is not univalent

$= \overline{(\mathbb{T}\,;z^\intercal\,;\pi^\intercal \cap z\,;\mathbb{T}\,;\rho^\intercal)\,;(\overline{\mathbb{I}} \otimes \overline{\mathbb{I}})} \;\cap\; \overline{\mathtt{sup}_{n\geq 1}\,\mathbb{T}\,;z^\intercal\,;\mathbb{S}^n\,;\overline{\mathbb{I}} \cap k_n^\intercal\,;\overline{\mathbb{I}}}$

$= \overline{\mathbb{T}\,;z^\intercal\,;\pi^\intercal\,;(\overline{\mathbb{I}} \otimes \overline{\mathbb{I}}) \cap z\,;\mathbb{T}} \cap \overline{\mathbb{T}\,;e^\intercal\,;\overline{\mathbb{I}} \cap \overline{\mathbb{I}}} \;\cap\; \mathtt{inf}_{n\geq 2}\,\overline{\mathbb{T}\,;z^\intercal\,;\mathbb{S}^n\,;\overline{\mathbb{I}} \cap k_n^\intercal\,;\overline{\mathbb{I}}}$

$= \overline{\mathbb{T}\,;z^\intercal\,;(\overline{\mathbb{I}} \cap \mathbb{T}\,;\overline{\mathbb{I}})} \cap z\,;\mathbb{T} \cap \overline{\mathbb{T}\,;e^\intercal} \cap \overline{\mathbb{I}} \;\cap\; \mathtt{inf}_{n\geq 2}\,\overline{\mathbb{T}\,;z^\intercal\,;\mathbb{S}^n} \cap k_n^\intercal\,;\overline{\mathbb{I}}$

$= \overline{\mathbb{T}\,;z^\intercal} \cap z\,;\mathbb{T} \cap (\mathbb{T}\,;e^\intercal \cup \mathbb{I}) \;\cap\; \mathtt{inf}_{n\geq 2}\,\big[\mathbb{T}\,;z^\intercal\,;\mathbb{S}^n \cup k_n^\intercal\,;\overline{\mathbb{I}}\big]$

$= (\mathbb{T}\,;z^\intercal \cup \overline{z\,;\mathbb{T}}) \cap (\mathbb{T}\,;e^\intercal \cup \mathbb{I}) \cap \mathtt{MTI}$

$= \big[(\mathbb{T}\,;z^\intercal \cap \mathbb{T}\,;e^\intercal) \cup (\mathbb{T}\,;z^\intercal \cap \mathbb{I}) \cup (\overline{z\,;\mathbb{T}} \cap \mathbb{T}\,;e^\intercal) \cup (\overline{z\,;\mathbb{T}} \cap \mathbb{I})\big] \cap \mathtt{MTI}$

$= \big[\mathbb{1} \cup z\,;\mathbb{T}\,;z^\intercal \cup (\overline{z\,;\mathbb{T}} \cap \mathbb{T}\,;e^\intercal) \cup (\overline{z\,;\mathbb{T}\,;z^\intercal} \cap \mathbb{I})\big] \cap \mathtt{MTI}$

$= \big[(\overline{z\,;\mathbb{T}} \cap \mathbb{T}\,;e^\intercal) \cup z\,;\mathbb{T}\,;z^\intercal \cup (\overline{z\,;\mathbb{T}\,;z^\intercal} \cap \mathbb{I})\big] \cap \mathtt{MTI}$ just rearranged

$= \big[(\overline{z\,;\mathbb{T}} \cap \mathbb{T}\,;e^\intercal) \cup \{(z\,;\mathbb{T}\,;z^\intercal \cup \overline{z\,;\mathbb{T}\,;z^\intercal}) \cap (z\,;\mathbb{T}\,;z^\intercal \cup \mathbb{I})\}\big] \cap \mathtt{MTI}$

$= \big[\overline{z\,;\mathbb{T}}\,;\mathbb{T}\,;e^\intercal \cup \mathbb{I}\big] \cap \mathtt{MTI}$

---

iii) $\mathtt{MIQ} \cup e ; \mathbb{T} = \overline{\mathbb{M}^{\mathsf{T}} ; (\overline{\mathbb{I}} \circledcirc \overline{\mathbb{I}})} \cup e ; \mathbb{T}$
$= \overline{\left[\mathtt{sup}_{n \geq 0} \, (\mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\, n} \circledcirc k_n^{\mathsf{T}}) \right] ; (\overline{\mathbb{I}} \circledcirc \overline{\mathbb{I}})} \cup e ; \mathbb{T}$ expanding $\mathbb{M}$ via Def. 7.1.1.i.a
$= \overline{\mathtt{sup}_{n \geq 0} \left[(\mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\, n} \circledcirc k_n^{\mathsf{T}}) ; (\overline{\mathbb{I}} \circledcirc \overline{\mathbb{I}}) \cap \overline{e ; \mathbb{T}} \right]}$
$= \overline{(\mathbb{T} ; z^{\mathsf{T}} \circledcirc z ; \mathbb{T}) ; (\overline{\mathbb{I}} \circledcirc \overline{\mathbb{I}}) \cap \overline{e ; \mathbb{T}}} \cap \overline{\mathtt{sup}_{n \geq 1} \, (\mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\, n} \circledcirc k_n^{\mathsf{T}}) ; (\overline{\mathbb{I}} \circledcirc \overline{\mathbb{I}}) \cap \overline{e ; \mathbb{T}}}$
$= \overline{(\mathbb{T} ; z^{\mathsf{T}} ; \pi^{\mathsf{T}} \cap z ; \mathbb{T} ; \rho^{\mathsf{T}}) ; (\overline{\mathbb{I}} \circledcirc \overline{\mathbb{I}}) \cap \overline{e ; \mathbb{T}}}$
$\quad \cap \overline{\mathtt{sup}_{n \geq 1} \, \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\, n} ; \overline{\mathbb{I}} \cap k_n^{\mathsf{T}} ; \overline{\mathbb{I}} \cap \overline{e ; \mathbb{T}}}$ as k_n^{T} univalent only for $n \geq 1$
$= \overline{\mathbb{T} ; z^{\mathsf{T}} ; \pi^{\mathsf{T}} ; (\overline{\mathbb{I}} \circledcirc \overline{\mathbb{I}}) \cap z ; \mathbb{T} \cap \overline{e ; \mathbb{T}}} \cap \overline{\mathbb{T} ; e^{\mathsf{T}} ; \overline{\mathbb{I}} \cap \overline{\mathbb{I}} \cap \overline{e ; \mathbb{T}}}$
$\quad \cap \mathtt{inf}_{n \geq 2} \overline{\mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\, n} ; \overline{\mathbb{I}} \cap k_n^{\mathsf{T}} ; \overline{\mathbb{I}}}$ as ρ total; $n \geq 2$ implies $k_n^{\mathsf{T}} ; \overline{\mathbb{I}} \subseteq \overline{e ; \mathbb{T}}$, Lemma 6.4.2.i.c
$= \overline{\mathbb{T} ; z^{\mathsf{T}} ; (\overline{\mathbb{I}} \cap \mathbb{T} ; \overline{\mathbb{I}}) \cap z ; \mathbb{T}} \cap \overline{\overline{\mathbb{T} ; e^{\mathsf{T}}} \cap \overline{\mathbb{I}} \cap \overline{e ; \mathbb{T}}}$
$\quad \cap \mathtt{inf}_{n \geq 2} \overline{\overline{\mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\, n}} \cap k_n^{\mathsf{T}} ; \overline{\mathbb{I}}}$ since $z ; \mathbb{T} \subseteq \overline{e ; \mathbb{T}}$
$= \overline{\overline{\mathbb{T} ; z^{\mathsf{T}}} \cap z ; \mathbb{T} \cap (\mathbb{T} ; e^{\mathsf{T}} \cup \mathbb{I} \cup e ; \mathbb{T}) \cap \mathtt{inf}_{n \geq 2} \left[\mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\, n} \cup \overline{k_n^{\mathsf{T}} ; \overline{\mathbb{I}}} \right]}$
$= (\mathbb{T} ; z^{\mathsf{T}} \cup \overline{z ; \mathbb{T}}) \cap (\mathbb{T} ; e^{\mathsf{T}} \cup \mathbb{I} \cup e ; \mathbb{T}) \cap \mathtt{MTI}$
$= \mathtt{MTI}$ because what follows now

$\mathtt{MTI} \subseteq \mathbb{T} ; z^{\mathsf{T}} \cup \overline{z ; \mathbb{T}}$ and $\mathtt{MTI} \subseteq \mathbb{T} ; e^{\mathsf{T}} \cup \mathbb{I} \cup e ; \mathbb{T}$

We prove these two items thus:

$z ; \mathbb{T} ; \mathbb{T} \subseteq \mathbb{T} ; z^{\mathsf{T}} \cup \overline{\mathtt{MTI}} = \mathbb{T} ; z^{\mathsf{T}} \cup \overline{\mathtt{inf}_{n \geq 2} \left[\mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\, n} \cup \overline{k_n^{\mathsf{T}} ; \overline{\mathbb{I}}} \right]} = \mathbb{T} ; z^{\mathsf{T}} \cup \mathtt{sup}_{n \geq 2} \left[\overline{\mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\, n}} \cap k_n^{\mathsf{T}} ; \overline{\mathbb{I}} \right]$
$\iff \quad \mathbb{T} \subseteq \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{T} ; z^{\mathsf{T}} \cup \mathtt{sup}_{n \geq 2} \left[\mathbb{T} ; z^{\mathsf{T}} ; \overline{\mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\, n}} \cap \mathbb{T} ; z^{\mathsf{T}} ; k_n^{\mathsf{T}} ; \overline{\mathbb{I}} \right]$ shunting $z ; \mathbb{T}$ with $\mathbb{T} ; z^{\mathsf{T}}$ a map
$= \mathbb{T} ; z^{\mathsf{T}} \cup \mathtt{sup}_{n \geq 2} \left[\overline{\mathbb{T} ; z^{\mathsf{T}} ; \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\, n}} \cap \mathbb{T} ; z^{\mathsf{T}} ; \overline{\mathbb{I}} \right]$ since $k_n ; z ; \mathbb{T} = z ; \mathbb{T}$ for $n \geq 2$
$= \mathbb{T} ; z^{\mathsf{T}} \cup \mathtt{sup}_{n \geq 2} \left[\overline{\mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\, n}} \cap \overline{\mathbb{T} ; z^{\mathsf{T}}} \right]$
$= \mathbb{T} ; z^{\mathsf{T}} \cup \left[\overline{\mathbb{T} ; z^{\mathsf{T}}} \cap \mathtt{sup}_{n \geq 2} \, \overline{\mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\, n}} \right]$
$= \left[\mathbb{T} ; z^{\mathsf{T}} \cup \overline{\mathbb{T} ; z^{\mathsf{T}}} \right] \cap \left[\mathbb{T} ; z^{\mathsf{T}} \cup \mathtt{sup}_{n \geq 2} \, \overline{\mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\, n}} \right]$
$= \mathbb{T} \cap \left[\mathbb{T} ; z^{\mathsf{T}} \cup \mathtt{sup}_{n \geq 2} \, \overline{\mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\, n}} \right] = \mathbb{T} ; z^{\mathsf{T}} \cup \mathtt{sup}_{n \geq 2} \, \overline{\mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\, n}}$
$= \mathbb{T} ; z^{\mathsf{T}} \cup \overline{\mathtt{inf}_{n \geq 2} \, \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\, n}} = \mathbb{T} ; z^{\mathsf{T}} \cup \overline{\mathbb{L}} = \mathbb{T} ; z^{\mathsf{T}} \cup \mathbb{T}$ what holds indeed

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The next proof uses point-wise reasoning:

$\mathtt{MTI} = \mathtt{inf}_{n \geq 2} \left[\mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\, n} \cup \overline{k_n^{\mathsf{T}} ; \overline{\mathbb{I}}} \right] \subseteq \mathbb{T} ; e^{\mathsf{T}} \cup \mathbb{I} \cup e ; \mathbb{T}$:
$\iff \quad \overline{\mathbb{T} ; e^{\mathsf{T}}} \cap \overline{\mathbb{I}} \cap \overline{e ; \mathbb{T}} \subseteq \mathtt{sup}_{n \geq 2} \left[\overline{\mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\, n}} \cap k_n^{\mathsf{T}} ; \overline{\mathbb{I}} \right]$

Assume an entry $\mathbb{S}^{\, \mathsf{T} r} ; z ; \mathbb{T} ; \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\, c} \subseteq \overline{\mathbb{T} ; e^{\mathsf{T}}} \cap \overline{\mathbb{I}} \cap \overline{e ; \mathbb{T}}$. Then certainly

$c \neq 1 \qquad r \neq c \qquad r \neq 1.$

We will try to identify an appropriate $n \geq 2$ for the right side.

$\mathbb{S}^{\, \mathsf{T} r} ; z ; \mathbb{T} ; \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\, c} \subseteq \overline{\mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\, n}} \cap k_n^{\mathsf{T}} ; \overline{\mathbb{I}}$
$\iff \quad \mathbb{T} \subseteq \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\, r} ; \overline{\mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\, n}} ; \mathbb{S}^{\, \mathsf{T} c} ; z ; \mathbb{T} \cap \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\, r} ; k_n^{\mathsf{T}} ; \overline{\mathbb{I}} ; \mathbb{S}^{\, \mathsf{T} c} ; z ; \mathbb{T}$
$= \overline{\mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\, r} ; \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\, n} ; \mathbb{S}^{\, \mathsf{T} c} ; z ; \mathbb{T}} \cap \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\, r} ; k_n^{\mathsf{T}} ; \overline{\mathbb{S}^{\, \mathsf{T} c} ; z ; \mathbb{T}}$
$= \overline{\mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\, n} ; \mathbb{S}^{\, \mathsf{T} c} ; z ; \mathbb{T}} \cap \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\, r} ; k_n^{\mathsf{T}} ; \overline{\mathbb{S}^{\, \mathsf{T} c} ; z ; \mathbb{T}}$

Taking $n = c$ would make the left term to $\mathbb{L}$; so we restrict to $n \neq c$. In case we take n to be a divisor of r, we had to apply Lemma 6.4.2.i.c and get $\mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\, r} ; k_n^{\mathsf{T}} = \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\, \frac{r}{n}}$. Then

$\mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\, r} ; k_n^{\mathsf{T}} ; \overline{\mathbb{S}^{\, \mathsf{T} c} ; z ; \mathbb{T}} = \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\, \frac{r}{n}} ; \overline{\mathbb{S}^{\, \mathsf{T} c} ; z ; \mathbb{T}} = \overline{\mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\, \frac{r}{n}} ; \mathbb{S}^{\, \mathsf{T} c} ; z ; \mathbb{T}}$

leaving us with the requirement that $\frac{r}{n} \neq c$ which is easily satisfied.

$\mathtt{MTI} = \mathtt{MIQ} \cup e;\mathbb{T} \subseteq \mathbb{I} \cup \overline{z;\mathbb{T}};\mathbb{T};e^{\mathsf{T}} \cup e;\mathbb{T}$ see (ii)

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iv) $\mathtt{DIV} = \overline{\mathtt{sup}_{n\geq 0}(k_n^{\mathsf{T}} \cap \overline{\mathbb{I}});\overline{\mathbb{I}}}$ since $\mathtt{DIV} = \overline{\mathcal{D}_{\mathrm{s}}{}^{\mathsf{T}};\overline{\mathbb{I}}} = \overline{[\mathtt{sup}_{n\geq 0}\, k_n \cap \overline{\mathbb{I}}]^{\mathsf{T}};\overline{\mathbb{I}}}$
$= \overline{(z;\mathbb{T} \cap \overline{\mathbb{I}});\overline{\mathbb{I}} \cup \mathbb{1} \cup \mathtt{sup}_{n\geq 2}(k_n^{\mathsf{T}} \cap \overline{\mathbb{I}});\overline{\mathbb{I}}}$ because $k_1 = \mathbb{I}$
$= \overline{(z;\mathbb{T} \cap \overline{\mathbb{I}});\overline{\mathbb{I}}} \cap \overline{\mathtt{sup}_{n\geq 2}(k_n^{\mathsf{T}} \cap \overline{\mathbb{I}});\overline{\mathbb{I}}}$
$= \overline{z;\mathbb{T} \cap \overline{\mathbb{I}};\overline{\mathbb{I}}} \cap \mathtt{inf}_{n\geq 2}\, \overline{(k_n^{\mathsf{T}} \cap \overline{\mathbb{I}});\overline{\mathbb{I}}}$ masking
$= \overline{z;\mathbb{T}} \cap \mathtt{inf}_{n\geq 2}\, \overline{(k_n^{\mathsf{T}} \cap \overline{z;\mathbb{T}});\overline{\mathbb{I}}}$ due to $\overline{\mathbb{I}};\overline{\mathbb{I}} = \mathbb{T}$ and Lemma 6.4.2.i.f
$= \overline{z;\mathbb{T}} \cap \mathtt{inf}_{n\geq 2}\, k_n^{\mathsf{T}};\overline{\mathbb{I}} \cap \overline{z;\mathbb{T}}$ masking
$= \overline{z;\mathbb{T}} \cap \mathtt{inf}_{n\geq 2}\, \big[k_n^{\mathsf{T}};\overline{\mathbb{I}} \cup z;\mathbb{T}\big]$
$= \overline{z;\mathbb{T}} \cap \big[z;\mathbb{T} \cup \mathtt{inf}_{n\geq 2}\, k_n^{\mathsf{T}};\overline{\mathbb{I}}\big]$ trivially
$= \overline{z;\mathbb{T}} \cap \mathtt{inf}_{n\geq 2}\, k_n^{\mathsf{T}};\overline{\mathbb{I}}$
$= \overline{z;\mathbb{T}} \cap \mathtt{ZPE}$

The estimation is precisely the result Prop. 7.4.4.ii.

$\mathtt{SYD} \cup e;\mathbb{T} = \mathtt{SYD} \cup \overline{\mathcal{D}_{\mathrm{s}}{}^{\mathsf{T}};\mathbb{T}}$ Prop. 7.4.2.iii
$= \mathtt{SYD} \cup \overline{\mathcal{D}_{\mathrm{s}}{}^{\mathsf{T}};(\overline{\mathbb{I}} \cup \mathbb{I})} = \mathtt{SYD} \cup \overline{\mathcal{D}_{\mathrm{s}}{}^{\mathsf{T}};\overline{\mathbb{I}} \cup \mathcal{D}_{\mathrm{s}}{}^{\mathsf{T}};\mathbb{I}}$
$= \mathtt{SYD} \cup \Big[\overline{\mathcal{D}_{\mathrm{s}}{}^{\mathsf{T}};\overline{\mathbb{I}}} \cap \overline{\mathcal{D}_{\mathrm{s}}{}^{\mathsf{T}};\mathbb{I}}\Big]$
$= \Big[\overline{\mathcal{D}_{\mathrm{s}}{}^{\mathsf{T}};\overline{\mathbb{I}}} \cap \mathcal{D}_{\mathrm{s}}{}^{\mathsf{T}}\Big] \cup \Big[\overline{\mathcal{D}_{\mathrm{s}}{}^{\mathsf{T}};\overline{\mathbb{I}}} \cap \overline{\mathcal{D}_{\mathrm{s}}{}^{\mathsf{T}}}\Big] = \overline{\mathcal{D}_{\mathrm{s}}{}^{\mathsf{T}};\overline{\mathbb{I}}} \cap \mathbb{T} = \mathtt{DIV}$

$\mathtt{DIV} = \mathtt{SYD} \cup e;\mathbb{T}$ (iv)
$\subseteq \overline{z;\mathbb{T}};\mathbb{T};e^{\mathsf{T}} \cup e;\mathbb{T}$ obvious

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v) We use that according to Prop. 7.4.2.i by definition $\mathtt{DIV} = \overline{\mathcal{D}_{\mathrm{s}}{}^{\mathsf{T}};\overline{\mathbb{I}}} = \overline{[\mathtt{sup}_{n\geq 0}\, k_n \cap \overline{\mathbb{I}}]^{\mathsf{T}};\overline{\mathbb{I}}}$.

$z;\mathbb{T};z^{\mathsf{T}} \cup \mathtt{DIV} = z;\mathbb{T};z^{\mathsf{T}} \cup \overline{\mathtt{sup}_{n\geq 0}(k_n^{\mathsf{T}} \cap \overline{\mathbb{I}});\overline{\mathbb{I}}}$
$= z;\mathbb{T};z^{\mathsf{T}} \cup \overline{(z;\mathbb{T} \cap \overline{\mathbb{I}});\overline{\mathbb{I}} \cup \mathbb{1} \cup \mathtt{sup}_{n\geq 2}(k_n^{\mathsf{T}} \cap \overline{\mathbb{I}});\overline{\mathbb{I}}}$ because $k_1 = \mathbb{I}$
$= z;\mathbb{T};z^{\mathsf{T}} \cup \Big[\overline{(z;\mathbb{T} \cap \overline{\mathbb{I}});\overline{\mathbb{I}}} \cap \overline{\mathtt{sup}_{n\geq 2}(k_n^{\mathsf{T}} \cap \overline{\mathbb{I}});\overline{\mathbb{I}}}\Big]$
$= \Big[z;\mathbb{T};z^{\mathsf{T}} \cup \overline{z;\mathbb{T} \cap \overline{\mathbb{I}};\overline{\mathbb{I}}}\Big] \cap \Big[z;\mathbb{T};z^{\mathsf{T}} \cup \mathtt{inf}_{n\geq 2}\, \overline{(k_n^{\mathsf{T}} \cap \overline{\mathbb{I}});\overline{\mathbb{I}}}\Big]$ masking
$= \Big[z;\mathbb{T};z^{\mathsf{T}} \cup \overline{z;\mathbb{T}}\Big] \cap \Big[z;\mathbb{T};z^{\mathsf{T}} \cup \mathtt{inf}_{n\geq 2}\, \overline{(k_n^{\mathsf{T}} \cap \overline{z;\mathbb{T}});\overline{\mathbb{I}}}\Big]$ due to $\overline{\mathbb{I}};\overline{\mathbb{I}} = \mathbb{T}$ and Lemma 6.4.2.i.f
$= \Big[z;\mathbb{T};z^{\mathsf{T}} \cup \overline{z;\mathbb{T}}\Big] \cap \Big[z;\mathbb{T};z^{\mathsf{T}} \cup \mathtt{inf}_{n\geq 2}\, k_n^{\mathsf{T}};\overline{\mathbb{I}} \cap \overline{z;\mathbb{T}}\Big]$ masking
$= \Big[z;\mathbb{T};z^{\mathsf{T}} \cup \overline{z;\mathbb{T}}\Big] \cap \Big[z;\mathbb{T};z^{\mathsf{T}} \cup z;\mathbb{T} \cup \mathtt{inf}_{n\geq 2}\, \overline{k_n^{\mathsf{T}};\overline{\mathbb{I}}}\Big]$
$= \Big[z;\mathbb{T};z^{\mathsf{T}} \cup \overline{z;\mathbb{T}}\Big] \cap \Big[z;\mathbb{T} \cup \mathtt{ZPE}\Big]$
$= \Big[(z;\mathbb{T};z^{\mathsf{T}} \cup \overline{z;\mathbb{T}}) \cap z;\mathbb{T}\Big] \cup \Big[(z;\mathbb{T};z^{\mathsf{T}} \cup \overline{z;\mathbb{T}}) \cap \mathtt{ZPE}\Big]$
$= \Big[z;\mathbb{T};z^{\mathsf{T}} \cap z;\mathbb{T}\Big] \cup \mathtt{ZPE}$ see (A) below
$= z;\mathbb{T};z^{\mathsf{T}} \cup \mathtt{ZPE} = \mathtt{ZPE}$ see (B) below

(A): For arbitrary $n \geq 2$ indeed

$\overline{k_n^{\mathsf{T}};\overline{\mathbb{I}}} \subseteq z;\mathbb{T};z^{\mathsf{T}} \cup \overline{z;\mathbb{T}} \quad \Longleftrightarrow \quad z;\mathbb{T} = z;\mathbb{T};z^{\mathsf{T}} \cup z;\mathbb{T};\overline{\mathbb{T};z^{\mathsf{T}}} \subseteq z;\mathbb{T};z^{\mathsf{T}} \cup k_n^{\mathsf{T}};\overline{\mathbb{I}}$
$\Longleftarrow \quad \overline{\mathbb{T};z^{\mathsf{T}}} \subseteq \mathbb{T};z^{\mathsf{T}};k_n^{\mathsf{T}};\overline{\mathbb{I}} = \mathbb{T};z^{\mathsf{T}};\overline{\mathbb{I}} = \overline{\mathbb{T};z^{\mathsf{T}};\mathbb{I}}$ shunting the point $z;\mathbb{T}$ and $k_n;z;\mathbb{T} = z;\mathbb{T}$

(B): Furthermore for all $n \geq 2$:

$z;\mathbb{T};z^{\mathsf{T}} \subseteq \overline{k_n^{\mathsf{T}};\overline{\mathbb{I}}} \quad \Longleftrightarrow \quad k_n^{\mathsf{T}};\overline{\mathbb{I}} \subseteq \overline{z;\mathbb{T};z^{\mathsf{T}}} \quad \Longleftrightarrow \quad k_n;z;\mathbb{T};z^{\mathsf{T}} = z;\mathbb{T};z^{\mathsf{T}} \subseteq \mathbb{I}$

Finally using (iv): $\mathtt{ZPE} = z\,;\mathbb{T}\,;z^{\mathsf{T}} \cup \mathtt{DIV} \subseteq z\,;\mathbb{T}\,;z^{\mathsf{T}} \cup \overline{z\,;\mathbb{T}}\,;\mathbb{T};e^{\mathsf{T}} \cup e\,;\mathbb{T}$ obvious

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vi) This is precisely the result Prop. 7.4.4.iii. □

We will take the figure above — obtained by pictorial evidence and also proof — as a guide for the sequence in which to prove interdependencies. It immediately suggests to prove several containments and equalities:

9.3.2 Proposition. i) $\mathtt{MTI} \supseteq \mathtt{ZPE}$

ii) $\mathtt{MIQ} \supseteq \mathtt{SYD} \subseteq \mathtt{HSD}$

Proof: i) is immediate.

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ii) Above, we have already obtained that $\mathtt{SYD} \subseteq \mathtt{DIV} \subseteq \mathtt{ZPE} \subseteq \mathtt{MTI}$ as well as $\mathtt{MIQ} \cup e\,;\mathbb{T} = \mathtt{MTI}$. The latter implies $\mathtt{MTI} \cap \overline{e\,;\mathbb{T}} \subseteq \mathtt{MIQ}$, so that it remains the task to prove $\mathtt{SYD} \subseteq \overline{e\,;\mathbb{T}}$ in order to finally obtain $\mathtt{SYD} \subseteq \mathtt{MIQ}$. To this end, we use the Dedekind rule to get

$e\,;\mathbb{T}\,;\mathbb{T} \cap \mathcal{D}_{\mathrm{s}}{}^{\mathsf{T}} \subseteq (\ldots)\,;(\mathbb{T} \cap \mathbb{T};e^{\mathsf{T}};\mathcal{D}_{\mathrm{s}}{}^{\mathsf{T}}) = (\ldots)\,;(\perp\!\!\!\perp) = \perp\!\!\!\perp \subseteq \mathcal{D}_{\mathrm{s}}{}^{\mathsf{T}};\overline{\mathbb{I}}$

This in turn leads to

$\mathtt{SYD} = \mathtt{syq}(\mathcal{D}_{\mathrm{s}}, \mathbb{I}) = \mathcal{D}_{\mathrm{s}}{}^{\mathsf{T}} \cap \overline{\mathcal{D}_{\mathrm{s}}{}^{\mathsf{T}};\overline{\mathbb{I}}} \subseteq \overline{e\,;\mathbb{T}}$.

The final $\mathtt{SYD} \subseteq \mathtt{HSD}$ follows from Prop. 7.4.2.xiv and iii. □

9.3.3 Definition and Proposition. We consider these partial diagonals, resp. vectors, and call them the **primes starting at** 1

$\wp_1 := \mathtt{DIA} = \mathtt{MIQ} \cap \mathbb{I} \cap \overline{z\,;\mathbb{T}\,;z^{\mathsf{T}}} = \mathtt{MTI} \cap \mathbb{I} \cap \overline{z\,;\mathbb{T}\,;z^{\mathsf{T}}} \subseteq \mathbb{I}$

$\wp_1\,;\mathbb{T} := \mathtt{DIA}\,;\mathbb{T} = \mathtt{DIV};e;\mathbb{T} = \mathtt{HSD} \cup e\,;\mathbb{T} = \mathtt{DIV}\,;\mathbb{T} = \mathtt{ZPE};e;\mathbb{T} = \mathtt{ZPE};\overline{z\,;\mathbb{T}} = \mathtt{MTI};e;\mathbb{T} = \mathtt{MIQ};e;\mathbb{T}$.

Additionally, the constructs

$\wp_2 := \wp_1 \cap \overline{e\,;\mathbb{T}} \subseteq \mathbb{I}$

$\wp_2\,;\mathbb{T} := \wp_1\,;\mathbb{T} \cap \overline{e\,;\mathbb{T}} = \mathtt{SYD}\,;\mathbb{T} = \mathtt{SYD}\,;e\,;\mathbb{T} = \mathtt{HSD}$,

and furthermore

$\wp := \wp_2 \cap \overline{\mathbb{S}^{\mathsf{T}};e\,;\mathbb{T}} \subseteq \mathbb{I}$

$\wp\,;\mathbb{T} := \wp_1\,;\mathbb{T} \cap \overline{e\,;\mathbb{T}} \cap \overline{\mathbb{S}^{\mathsf{T}2};z\,;\mathbb{T}} = \wp_2\,;\mathbb{T} \cap \overline{\mathbb{S}^{\mathsf{T}};e\,;\mathbb{T}}$

are said to describe **all prime numbers** (i.e. including 2) and the **odd prime numbers**.

We thereby also introduce these natural extrusions with corresponding injection mappings (assuming in addition $\mathbb{S}^{\mathsf{T}3};z\,;\mathbb{T} \neq \perp\!\!\!\perp$):

$\iota_{\wp_1;\mathbb{T}} = \mathtt{Inject}(\wp_1\,;\mathbb{T}) : \mathbb{P}_1 \longrightarrow \mathbb{N}$, thus *defining* the extruded set $\mathbb{P}_1$ of primes ≥ 1

$\iota_{\wp_2;\mathbb{T}} = \mathtt{Inject}(\wp_2\,;\mathbb{T}) : \mathbb{P}_2 \longrightarrow \mathbb{N}$, thus *defining* the extruded set $\mathbb{P}_2$ of primes ≥ 2

$\iota_{\wp;\mathbb{T}} = \mathtt{Inject}(\wp\,;\mathbb{T}) : \mathbb{P} \longrightarrow \mathbb{N}$, thus *defining* the extruded set $\mathbb{P}$ of odd primes

all three of which schematically satisfy $\wp = \iota^\intercal ; \iota$ and $\wp ; \mathbb{T} = \iota^\intercal ; \mathbb{T}$.

Proof: Although pictorially evident, this means a lot of tiny bits of proofs:

We first show that $\mathtt{ZPE} ; e ; \mathbb{T} = \mathtt{MTI} ; e ; \mathbb{T} = \mathtt{MIQ} ; e ; \mathbb{T}$:

$\mathtt{MTI} ; e ; \mathbb{T} = \left[\mathtt{inf}_{n \geq 2} \, \mathbb{T} ; z^\intercal ; \mathbb{S}^n \cup \overline{k_n^\intercal ; \overline{\mathbb{I}}} \right] ; e ; \mathbb{T}$

$= \mathtt{inf}_{n \geq 2} \left[\mathbb{T} ; z^\intercal ; \mathbb{S}^n ; e ; \mathbb{T} \cup \overline{k_n^\intercal ; \overline{\mathbb{I}}} ; e ; \mathbb{T} \right]$ since $e ; \mathbb{T}$ is injective

$= \mathtt{inf}_{n \geq 2} \left[\mathbb{1} \cup \overline{k_n^\intercal ; \overline{\mathbb{I}}} ; e ; \mathbb{T} \right]$

$= \left[\mathtt{inf}_{n \geq 2} \, \overline{k_n^\intercal ; \overline{\mathbb{I}}} \right] ; e ; \mathbb{T} = \mathtt{ZPE} ; e ; \mathbb{T}$

Now $\mathtt{MIQ} ; e ; \mathbb{T} = \ldots = \mathtt{ZPE} ; e ; \mathbb{T}$: Since $\mathtt{MIQ} \subseteq \mathtt{MTI}$ as well as $\mathtt{MTI} ; e ; \mathbb{T} = \mathtt{ZPE} ; e ; \mathbb{T}$ are already known, it suffices to prove $\mathtt{ZPE} ; e ; \mathbb{T} \subseteq \mathtt{MIQ} ; e ; \mathbb{T}$, or expanded using that $e ; \mathbb{T}$ is a point

$\mathtt{MIQ} ; e ; \mathbb{T} = \overline{\mathbb{M}^\intercal ; \left(\overline{e ; \mathbb{T}} \otimes \overline{e ; \mathbb{T}} \right)} = \overline{\mathtt{sup}_{k \geq 0} \, \left(\mathbb{T} ; z^\intercal ; \mathbb{S}^k \otimes k_k^\intercal \right) ; \left(\overline{e ; \mathbb{T}} \otimes \overline{e ; \mathbb{T}} \right)}$

$= \overline{\left(\mathbb{T} ; z^\intercal \otimes k_0^\intercal \right) ; \left(\overline{e ; \mathbb{T}} \otimes \overline{e ; \mathbb{T}} \right)} \cap \overline{\mathtt{sup}_{k \geq 1} \, \left(\mathbb{T} ; z^\intercal ; \mathbb{S}^k \otimes k_k^\intercal \right) ; \left(\overline{e ; \mathbb{T}} \otimes \overline{e ; \mathbb{T}} \right)}$

$= \overline{\left(\mathbb{T} ; z^\intercal \otimes z ; \mathbb{T} \right) ; \left(\overline{e ; \mathbb{T}} \otimes \overline{e ; \mathbb{T}} \right)} \cap \overline{\mathtt{sup}_{k \geq 1} (\mathbb{T} ; z^\intercal ; \mathbb{S}^k ; \overline{e ; \mathbb{T}} \cap k_k^\intercal ; \overline{e ; \mathbb{T}})}$ observe: $k_0^\intercal$ not univalent

$= \overline{\left(\mathbb{T} ; z^\intercal ; \pi^\intercal \cap z ; \mathbb{T} ; \rho^\intercal \right) ; \left(\overline{e ; \mathbb{T}} \otimes \overline{e ; \mathbb{T}} \right)} \cap \mathtt{inf}_{k \geq 1} \overline{\left(\mathbb{T} ; z^\intercal ; \mathbb{S}^k ; e ; \mathbb{T} \cap k_k^\intercal ; \overline{e ; \mathbb{T}} \right)}$

$= \overline{\mathbb{T} ; z^\intercal ; \pi^\intercal ; \left(\overline{e ; \mathbb{T}} \otimes \overline{e ; \mathbb{T}} \right)} \cap z ; \mathbb{T} \cap \overline{\mathbb{T} ; e^\intercal ; e ; \mathbb{T}} \cap \overline{e ; \mathbb{T}} \cap \mathtt{inf}_{k \geq 2} \, \overline{\mathbb{1}} \cap k_k^\intercal ; \overline{e ; \mathbb{T}}$

$= \overline{\mathbb{T} ; z^\intercal ; \left(\overline{e ; \mathbb{T}} \cap \mathbb{T} ; \overline{e ; \mathbb{T}} \right)} \cap z ; \mathbb{T} \cap \mathbb{T} \cap \mathtt{inf}_{k \geq 2} \, \overline{k_k^\intercal ; \overline{e ; \mathbb{T}}}$

$= \overline{\mathbb{T} ; z^\intercal ; \overline{e ; \mathbb{T}}} \cap z ; \mathbb{T} \cap \mathtt{inf}_{k \geq 2} \, \overline{k_k^\intercal ; \overline{\mathbb{I}}} ; e ; \mathbb{T}$

$= \overline{z ; \mathbb{T}} \cap \mathtt{ZPE} ; e ; \mathbb{T}$

$= \mathtt{ZPE} ; e ; \mathbb{T}$ where $\mathtt{ZPE} ; e ; \mathbb{T} \subseteq \overline{z ; \mathbb{T}}$ is proved as follows:

$\mathtt{ZPE} ; e ; \mathbb{T} = \mathtt{inf}_{n \geq 2} \, \overline{k_n^\intercal ; \overline{e ; \mathbb{T}}}$

$= \mathtt{inf}_{n \geq 2} \, \overline{k_n^\intercal ; \mathbb{T} \cap \overline{\mathbb{S}^{\intercal n} ; z ; \mathbb{T}}}$ Prop. 6.4.2.ii.a

$= \mathtt{inf}_{n \geq 2} \, \overline{k_n^\intercal ; \mathbb{T}} \cup \mathbb{S}^{\intercal n} ; z ; \mathbb{T}$

$= \mathtt{inf}_{n \geq 2} \, \overline{\left(\mathbb{S}^{n \intercal} \right)^* ; z ; \mathbb{T} \cup \mathbb{S}^{\intercal n} ; z ; \mathbb{T}}$ Prop. 6.4.2.ii.b

$\subseteq \mathtt{inf}_{n \geq 2} \, \overline{\mathbb{I} ; z ; \mathbb{T}} \cup \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \subseteq \overline{z ; \mathbb{T}}$

Also $\mathtt{SYD} ; \mathbb{T} = \mathtt{SYD} ; e ; \mathbb{T}$ is pictorially evident, but $\mathtt{SYD} ; \mathbb{T} \subseteq \mathtt{SYD} ; e ; \mathbb{T}$ requires a tricky proof:

$\iff \quad \mathtt{syq}(\mathcal{D}_\mathrm{s}, \mathbb{I}) ; \mathbb{T} \subseteq \mathtt{syq}(\mathcal{D}_\mathrm{s}, \mathbb{I}) ; e ; \mathbb{T} = \mathtt{syq}(\mathcal{D}_\mathrm{s}, e ; \mathbb{T})$ by definition with $e ; \mathbb{T}$ injective

$\iff \quad \mathtt{syq}(\mathcal{D}_\mathrm{s}, \mathbb{I})^\intercal ; \overline{\mathtt{syq}(\mathcal{D}_\mathrm{s}, e ; \mathbb{T})} = \mathtt{syq}(\mathbb{I}, \mathcal{D}_\mathrm{s}) ; \left[\overline{\mathcal{D}_\mathrm{s}^\intercal ; e ; \mathbb{T}} \cup \mathcal{D}_\mathrm{s}^\intercal ; \overline{e ; \mathbb{T}} \right] \subseteq \mathbb{1}$ Schröder rule

For the first constituent, we use that $e ; \mathbb{T}$ is a point and Prop. 7.4.2.iii

$\mathtt{syq}(\mathbb{I}, \mathcal{D}_\mathrm{s}) ; \overline{\mathcal{D}_\mathrm{s}^\intercal ; e ; \mathbb{T}} = \mathtt{syq}(\mathbb{I}, \mathcal{D}_\mathrm{s}) ; e ; \mathbb{T} = \mathtt{syq}(\mathbb{I}, \mathcal{D}_\mathrm{s} ; e ; \mathbb{T}) = \mathtt{syq}(\mathbb{I}, \mathbb{1}) = \mathbb{1}$.

In the proof for the second constituent

$\mathtt{syq}(\mathbb{I}, \mathcal{D}_\mathrm{s}) ; \mathcal{D}_\mathrm{s}^\intercal ; \overline{e ; \mathbb{T}} = \left[\mathcal{D}_\mathrm{s} ; \mathtt{SYD} \right]^\intercal ; \overline{e ; \mathbb{T}} = e ; \mathbb{T} ; e^\intercal ; \overline{e ; \mathbb{T}} = \mathbb{1}$,

the proof of the step $\mathcal{D}_\mathrm{s} ; \mathtt{SYD} = e ; \mathbb{T} ; e^\intercal = e ; \mathbb{T} \cap \mathbb{T} ; e^\intercal$ is still open:

$\mathcal{D}_\mathrm{s} ; \mathtt{SYD} = \mathcal{D}_\mathrm{s} ; \mathtt{syq}(\mathcal{D}_\mathrm{s}, \mathbb{I}) = \left[\mathcal{D}_\mathrm{s} \cap \mathbb{T} \right] ; \mathtt{syq}(\mathcal{D}_\mathrm{s}, \mathbb{I}) = \left[\mathcal{D}_\mathrm{s} \cap (\overline{\mathbb{I}} ; \mathcal{D}_\mathrm{s} \cup \overline{\mathbb{I} ; \mathcal{D}_\mathrm{s}}) \right] ; \mathtt{syq}(\mathcal{D}_\mathrm{s}, \mathbb{I})$

It now splits, with first part simple

$(\mathcal{D}_\mathrm{s} \cap \overline{\mathbb{I}} ; \mathcal{D}_\mathrm{s}) ; \mathtt{syq}(\mathcal{D}_\mathrm{s}, \mathbb{I}) \subseteq \mathcal{D}_\mathrm{s} ; \mathtt{syq}(\mathcal{D}_\mathrm{s}, \mathbb{I}) \cap \overline{\mathbb{I}} ; \mathcal{D}_\mathrm{s} ; \mathtt{syq}(\mathcal{D}_\mathrm{s}, \mathbb{I}) \subseteq \mathbb{I} \cap \overline{\mathbb{I}} ; \mathbb{I} = \mathbb{1}$

and a second part:

$(\mathcal{D}_\mathrm{s} \cap \overline{\mathbb{I} ; \mathcal{D}_\mathrm{s}}) ; \mathtt{syq}(\mathcal{D}_\mathrm{s}, \mathbb{I}) = \mathtt{SYD}^\intercal ; \mathtt{SYD} \subseteq e ; \mathbb{T} \quad \iff \quad \mathtt{SYD} ; \overline{e ; \mathbb{T}} \subseteq \overline{\mathtt{SYD}}$,

where we apply a standard rule for symmetric quotients

$\mathtt{SYD};\overline{e;\mathbb{T}} = \mathtt{syq}(\mathcal{D}_\mathrm{s},\mathbb{I});\overline{\mathbb{I}};e;\mathbb{T} = \mathtt{syq}(\overline{\mathcal{D}_\mathrm{s}},\overline{\mathbb{I}});\overline{\mathbb{I}};e;\mathbb{T} \subseteq \overline{\mathcal{D}_\mathrm{s}^{\mathsf{T}}};e;\mathbb{T} = \overline{\mathcal{D}_\mathrm{s}^{\mathsf{T}};e;\mathbb{T}}$
$= e;\mathbb{T}$ due to Prop. 7.4.2.iii
$\subseteq \overline{\mathtt{SYD}}$

The last obvious result — to be mentioned separately — is proved via the Dedekind rule:

$e;\mathbb{T} \subseteq \overline{\mathtt{SYD}} = \overline{\mathtt{syq}(\mathcal{D}_\mathrm{s},\mathbb{I})} = \overline{\mathcal{D}_\mathrm{s}^{\mathsf{T}}} \cup \mathcal{D}_\mathrm{s}^{\mathsf{T}};\overline{\mathbb{I}}$ two times expanded
$\iff \quad e;\mathbb{T};\mathbb{T} \cap \mathcal{D}_\mathrm{s}^{\mathsf{T}} \subseteq (e;\mathbb{T} \cap \mathcal{D}_\mathrm{s}^{\mathsf{T}};\mathbb{T}^{\mathsf{T}});(\mathbb{T} \cap \mathbb{T};e^{\mathsf{T}};\mathcal{D}_\mathrm{s}^{\mathsf{T}}) = (\ldots);(\perp\!\!\!\perp) = \perp\!\!\!\perp \subseteq \mathcal{D}_\mathrm{s}^{\mathsf{T}};\overline{\mathbb{I}}$

Altogether, since $\mathtt{SYD}^{\mathsf{T}};\mathtt{SYD} \subseteq \mathbb{T};e^{\mathsf{T}}$ follows simply by transposition:

$\mathcal{D}_\mathrm{s};\mathtt{SYD} \subseteq e;\mathbb{T} \cap \mathbb{T};e^{\mathsf{T}} = e;\mathbb{T};e^{\mathsf{T}}$

Since $e;\mathbb{T};e^{\mathsf{T}}$ is an entry, it suffices to show

$\mathcal{D}_\mathrm{s};\mathtt{SYD} \neq \perp\!\!\!\perp$

to establish an equality. So we assume $\mathcal{D}_\mathrm{s};\mathtt{SYD} \subseteq \perp\!\!\!\perp$ in order to find the following contradiction:

$\perp\!\!\!\perp \supseteq \mathcal{D}_\mathrm{s};\mathtt{SYD} = \mathcal{D}_\mathrm{s};\mathtt{syq}(\mathcal{D}_\mathrm{s},\mathbb{I}) = \mathcal{D}_\mathrm{s};\mathtt{upa}(\mathcal{D}_\mathrm{s}^{\mathsf{T}})$ expanded
$= \mathtt{upa}(\mathcal{D}_\mathrm{s}^{\mathsf{T}})^{\mathsf{T}};\mathtt{upa}(\mathcal{D}_\mathrm{s}^{\mathsf{T}})$ using the result [11, 4.2.9.iv]
$\implies \mathtt{upa}(\mathcal{D}_\mathrm{s}^{\mathsf{T}}) = \mathtt{SYD} = \perp\!\!\!\perp$

The very last one follows from Prop. 7.4.2.xiv and Prop. 7.4.4.iii below:

$\mathtt{SYD} \subseteq \mathcal{H}^{\mathsf{T}}$ and $\mathtt{SYD} \subseteq \mathbb{T};e^{\mathsf{T}}$,

then namely

$\mathtt{SYD} \subseteq \mathcal{H}^{\mathsf{T}} \cap \mathbb{T};e^{\mathsf{T}} \subseteq (\mathcal{H}^{\mathsf{T}} \cap \mathbb{T};e^{\mathsf{T}});\mathbb{T} = \mathcal{H}^{\mathsf{T}};(\mathbb{T} \cap e;\mathbb{T}) = \mathcal{H}^{\mathsf{T}};e;\mathbb{T} = \mathtt{HSD}$

$\mathtt{HSD} \cup e;\mathbb{T} = \left[\mathcal{D}_\mathrm{s}^{\mathsf{T}} \cap \overline{\mathcal{D}_\mathrm{s}^{\mathsf{T}};\mathcal{D}_\mathrm{s}^{\mathsf{T}}}\right];e;\mathbb{T} \cup e;\mathbb{T}$
$= \left[\mathcal{D}_\mathrm{s}^{\mathsf{T}};e;\mathbb{T} \cap \overline{\mathcal{D}_\mathrm{s}^{\mathsf{T}};\mathcal{D}_\mathrm{s}^{\mathsf{T}};e;\mathbb{T}}\right] \cup e;\mathbb{T}$
$= \left[\overline{e;\mathbb{T}} \cap \overline{\mathcal{D}_\mathrm{s}^{\mathsf{T}};\overline{e;\mathbb{T}}}\right] \cup e;\mathbb{T}$ Prop. 7.4.2.iii
$= (\overline{e;\mathbb{T}} \cup e;\mathbb{T}) \cap (\overline{\mathcal{D}_\mathrm{s}^{\mathsf{T}};\overline{e;\mathbb{T}}} \cup e;\mathbb{T})$
$= \overline{\mathcal{D}_\mathrm{s}^{\mathsf{T}};\overline{e;\mathbb{T}}}$ since $\mathcal{D}_\mathrm{s};e;\mathbb{T} = \perp\!\!\!\perp \subseteq e;\mathbb{T} \implies \mathcal{D}_\mathrm{s}^{\mathsf{T}};\overline{e;\mathbb{T}} \subseteq \overline{e;\mathbb{T}} \iff e;\mathbb{T} \subseteq \overline{\mathcal{D}_\mathrm{s}^{\mathsf{T}};\overline{e;\mathbb{T}}}$
$= \overline{\mathcal{D}_\mathrm{s}^{\mathsf{T}};\overline{\mathbb{I}};e;\mathbb{T}}$
$= \mathtt{DIV};e;\mathbb{T}$ □

In view of Fig. 9.3.2, we aim at proving the following:

9.3.4 Proposition. The constituents $\wp_1$, $z;\mathbb{T};z^{\mathsf{T}}$, $\wp_2;\mathbb{T};e^{\mathsf{T}}$ are disjoint in the following:

i) $\mathtt{DIA} = \wp_1$

ii) $\mathtt{MIQ} = \wp_1 \cup z;\mathbb{T};z^{\mathsf{T}} \cup \wp_2;\mathbb{T};e^{\mathsf{T}}$

iii) $\mathtt{MTI} = \wp_1 \cup z;\mathbb{T};z^{\mathsf{T}} \cup \wp_2;\mathbb{T};e^{\mathsf{T}} \cup e;\mathbb{T}$

iv) $\mathtt{ZPE} = z;\mathbb{T};z^{\mathsf{T}} \cup \wp_2;\mathbb{T};e^{\mathsf{T}} \cup e;\mathbb{T}$

v) $\mathtt{DIV} = \wp_2;\mathbb{T};e^{\mathsf{T}} \cup e;\mathbb{T}$

vi) $\mathtt{SYD} = \wp_2;\mathbb{T};e^{\mathsf{T}}$

Proof: i) reflects the Def. 9.3.3 directly.

We are going to prove "$\supseteq$" for the items $\wp_1,\ e\,;\mathbb{T},\ z\,;\mathbb{T}\,;z^\intercal,\ \wp_2\,;\mathbb{T}\,;e^\intercal$ first and "$\subseteq$" afterwards.

Together with Prop. 9.3.2, we thus have $\wp_1 \subseteq \mathtt{MIQ} \subseteq \mathtt{MTI}$.

Next, we recall $e\,;\mathbb{T} \subseteq \mathtt{DIV}$ from Prop. 9.3.1.iv, and thus, with Prop. 9.3.1.v and Prop. 9.3.2.i, $\ldots \subseteq \mathtt{ZPE} \subseteq \mathtt{MTI}$.

Now $z\,;\mathbb{T}\,;z^\intercal \subseteq \mathtt{MIQ}$, and thus $\ldots \subseteq \mathtt{MTI}$: We start from

$\mathbb{M}\,;z\,;\mathbb{T} \subseteq \overline{(z\,;\mathbb{T} \oslash \overline{z\,;\mathbb{T}})} = \overline{(\overline{\mathbb{I}} \oslash \overline{\mathbb{I}})}\,;z\,;\mathbb{T}$ which follows from the equality in Prop. 7.1.1.ii.c

$\iff$ $\mathbb{M}\,;z\,;\mathbb{T}\,;z^\intercal \subseteq \overline{(\overline{\mathbb{I}} \oslash \overline{\mathbb{I}})}$ shunting the point $z\,;\mathbb{T}$

$\iff$ $\overline{\mathtt{MIQ}} = \mathbb{M}^\intercal\,;\overline{(\overline{\mathbb{I}} \oslash \overline{\mathbb{I}})} \subseteq \overline{z\,;\mathbb{T}\,;z^\intercal}$

The inclusion $z\,;\mathbb{T}\,;z^\intercal \subseteq \mathtt{ZPE}$ requires a separate proof. It means that for all $n \geq 2$ we have $z\,;\mathbb{T}\,;z^\intercal \subseteq \overline{k_n^\intercal\,;\overline{\mathbb{I}}}$, which is trivial recalling $k_n\,;z\,;\mathbb{T} = z\,;\mathbb{T}$ from Lemma 6.4.2.i.c.

The next point concerns on the one hand $\wp_2\,;\mathbb{T}\,;e^\intercal \subseteq \mathtt{MIQ}$

$\wp_2\,;\mathbb{T}\,;e^\intercal \subseteq \wp_1\,;\mathbb{T}\,;e^\intercal \subseteq \mathtt{MIQ} \iff \wp_1 \subseteq \mathtt{MIQ}\,;e\,;\mathbb{T}$ shunting the point $e\,;\mathbb{T}$

and on the other $\wp_2\,;\mathbb{T}\,;e^\intercal \subseteq \mathtt{SYD}$, and thus $\ldots \subseteq \mathtt{DIV} \subseteq \mathtt{ZPE} \subseteq \mathtt{MTI}$:

$\wp_2\,;\mathbb{T}\,;e^\intercal \subseteq \mathtt{SYD} \iff \wp_2\,;\mathbb{T} \subseteq \mathtt{SYD}\,;e\,;\mathbb{T}$ simply shunting

---

It remains the task to show the reverse direction "$\subseteq$". For (i), this is already established. The last one, (vi), is shown recalling that SYD is a univalent relation. Then namely, since $\wp_2\,;\mathbb{T}\,;e^\intercal \subseteq \mathtt{SYD}$ has already been shown, it suffices to prove using Def. 9.3.3 that their domains coincide

$\wp_2\,;\mathbb{T}\,;e^\intercal\,;\mathbb{T} = \wp_2\,;\mathbb{T} = \mathtt{SYD}\,;\mathbb{T}$.

For (iv,v), we use Prop. 9.3.2.v. □

Now we study the mutual relationships between the possible definitions. The following is a result that a sane mathematician would hardly deny: Odd primes are indeed odd. So far the oddness of primes is, however, just denotation based on the fact that the definition excludes $\mathbb{S}^{\intercal 2}\,;z\,;\mathbb{T}$. For all the other even numbers it has to be proved, which still turns out to be ridiculously complicated — far from our attempted purely relational way.

9.3.5 Proposition. i) $\wp\,;\mathbb{T} \subseteq \mathtt{odd}\,;\mathbb{T}$ $\quad \wp \subseteq \mathtt{odd}$ $\quad \mathbb{S}\,;\wp\,;\mathbb{T} \subseteq \mathtt{eve}\,;\mathbb{T}$ $\quad \mathbb{S}^\intercal\,;\wp\,;\mathbb{T} \subseteq \mathtt{eve}\,;\mathbb{T}$

ii) m odd $\implies$ $\wp\,;\mathbb{S}^m\,;\wp = \mathbb{\perp\!\!\!\perp}$

iii) m odd $\implies$ $\mathbb{S}^{\intercal m}\,;\wp\,;\mathbb{S}^m \cap \mathbb{T}\,;\wp = \mathbb{\perp\!\!\!\perp}$

Proof: i) We start with Def. 9.3.3

$\wp\,;\mathbb{T} = \mathtt{SYD}\,;\mathbb{T} \cap \overline{\mathbb{S}^\intercal\,;e\,;\mathbb{T}}$

$= \left[\mathcal{D}_\mathrm{s}{}^\intercal \cap \overline{\mathcal{D}_\mathrm{s}{}^\intercal\,;\overline{\mathbb{I}}}\right]\,;\mathbb{T} \cap \overline{\mathbb{S}^\intercal\,;e\,;\mathbb{T}}$

$= \left[\mathtt{sup}_{n\geq 2,\ n\ \text{prime}}\ \mathbb{S}^{\intercal n}\,;z\,;\mathbb{T}\,;e^\intercal\right]\,;\mathbb{T} \cap \overline{\mathbb{S}^\intercal\,;e\,;\mathbb{T}}$ Prop. 7.4.2.ix

$= \left[\mathtt{sup}_{n\geq 2,\ n\ \text{prime}}\ \mathbb{S}^{\intercal n}\,;z\,;\mathbb{T}\,;e^\intercal\,;\mathbb{T}\right] \cap \overline{\mathbb{S}^\intercal\,;e\,;\mathbb{T}}$

$= \left[\mathtt{sup}_{n\geq 2,\ n\ \text{prime}}\ \mathbb{S}^{\intercal n}\,;z\,;\mathbb{T}\right] \cap \overline{\mathbb{S}^\intercal\,;e\,;\mathbb{T}}$ since e is a point

$= \big[\, \mathbb{S}^{\intercal 2} ; z ; \mathbb{T} \cup \mathtt{sup}_{n\geq 3,\ n \text{ prime}}\, \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \big] \cap \overline{\mathbb{S}^{\intercal 2} ; z ; \mathbb{T}}$ observe e vs. z
$\subseteq \mathtt{sup}_{n\geq 3,\ n \text{ prime}}\, \mathbb{S}^{\intercal n} ; z ; \mathbb{T}$
$\subseteq \mathtt{sup}_{n\geq 3,\ n \text{ odd}}\, \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \subseteq \mathtt{odd} ; \mathbb{T}$

This proves also the second claim.

Now

$\mathbb{S} ; \wp ; \mathbb{T} \subseteq \mathbb{S} ; \mathtt{odd} ; \mathbb{T} \subseteq \mathtt{eve} ; \mathbb{T}$ using Prop. 5.2.1.ii.g
$\mathbb{S}^{\intercal} ; \wp ; \mathbb{T} \subseteq \mathbb{S}^{\intercal} ; \mathtt{odd} ; \mathbb{T} \subseteq \mathtt{eve} ; \mathbb{T}$ again using Prop. 5.2.1.ii.g

---

ii) Using (i), we get

$\mathbb{S} ; \wp ; \mathbb{T} \subseteq \mathbb{S} ; \mathtt{odd} ; \mathbb{T} \subseteq \mathtt{eve} ; \mathbb{T} \quad \Longrightarrow \quad \mathbb{S}^{2*n} ; \mathbb{S} ; \wp ; \mathbb{T} \subseteq \mathbb{S}^{2*n} ; \mathtt{eve} ; \mathbb{T} \subseteq \mathtt{eve} ; \mathbb{T}$
$\wp ; \mathbb{S}^{2*n+1} ; \wp \subseteq \mathtt{odd} ; \mathbb{S}^{2*n+1} ; \wp \subseteq (\mathbb{I} \cap \mathtt{odd} ; \mathbb{T}) ; \mathtt{eve} ; \mathbb{T} = \mathbb{I} ; \mathtt{eve} ; \mathbb{T} \cap \mathtt{odd} ; \mathbb{T} = \mathbb{\bot}$

---

iii) We recall that $\wp = \wp^{\intercal} \subseteq \mathbb{I}$ and use (ii):

$\mathbb{T} ; \wp \cap \mathbb{S}^{\intercal m} ; \wp ; \mathbb{S}^{m} \subseteq (\mathbb{T} \cap \mathbb{S}^{\intercal m} ; \wp ; \mathbb{S}^{m} ; \wp) ; (\ldots)$ Dedekind rule
$\subseteq \mathbb{S}^{\intercal m} ; \wp ; \mathbb{S}^{m} ; \wp ; \mathbb{T} = \mathbb{S}^{\intercal m} ; \mathbb{\bot} ; \mathbb{T} = \mathbb{\bot}$ □

With the following proposition, we relate the colloquial version "p is prime" with the thus established relation-algebraic one.

9.3.6 Proposition. i) Assume $\mathbb{S}^{\intercal p} ; z ; \mathbb{T} \neq \mathbb{\bot}$; then

$\mathbb{S}^{\intercal p} ; z ; \mathbb{T} \subseteq \wp_1 ; \mathbb{T} \quad \Longleftrightarrow \quad p$ prime — possibly 1 or 2

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
6	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
7	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
8	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
9	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
10	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
11	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
12	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
13	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
14	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
15	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
16	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
17	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
17	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Fig. 9.3.3 Illustrating $\mathbb{I} \cup \mathbb{T} ; e^{\intercal}$ as well as $(\mathbb{I} \cup \mathbb{T} ; e^{\intercal}) \cap \wp_1 ; \mathbb{T}$ of Prop. 9.3.6.iii

ii) For $n \geq 2$:

$k_n ; \wp_1 ; \mathbb{T} = e ; \mathbb{T} \quad \Longleftrightarrow \quad \mathbb{\bot} \neq \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \subseteq \wp_1 ; \mathbb{T},$ otherwise $k_n ; \wp_1 ; \mathbb{T} = \mathbb{\bot}$

$k_n ; \wp_2 ; \mathbb{T} = e ; \mathbb{T} \quad \Longleftrightarrow \quad \mathbb{\bot} \neq \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \subseteq \wp_2 ; \mathbb{T},$ otherwise $k_n ; \wp_2 ; \mathbb{T} = \mathbb{\bot}$

$k_n ; \wp ; \mathbb{T} = e ; \mathbb{T} \iff \bot\!\!\!\bot \neq \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \subseteq \wp ; \mathbb{T}$, otherwise $k_n ; \wp ; \mathbb{T} = \bot\!\!\!\bot$

iii) $\mathcal{D}^{\intercal} \cap \wp_1 ; \mathbb{T} = (\mathbb{I} \cup \mathbb{T} ; e^{\intercal}) \cap \wp_1 ; \mathbb{T}$

iv) $\mathbb{M} ; \wp_1 ; \mathbb{T} = (e ; \mathbb{T} \circledcirc \wp_1 ; \mathbb{T}) \cup (\wp_1 ; \mathbb{T} \circledcirc e ; \mathbb{T})$
$\mathbb{M} ; \wp_2 ; \mathbb{T} = (e ; \mathbb{T} \circledcirc \wp_2 ; \mathbb{T}) \cup (\wp_2 ; \mathbb{T} \circledcirc e ; \mathbb{T})$
$\mathbb{M} ; \wp ; \mathbb{T} = (e ; \mathbb{T} \circledcirc \wp ; \mathbb{T}) \cup (\wp ; \mathbb{T} \circledcirc e ; \mathbb{T})$

For all three, this may be continued thus:

$$\ldots \subseteq (e ; \mathbb{T} \circledcirc \mathbb{T}) \cup (\mathbb{T} \circledcirc e ; \mathbb{T}) = (\pi \cup \rho) ; e ; \mathbb{T}$$

Furthermore

$\wp_1 ; \mathbb{T} = \overline{\mathbb{M}^{\intercal} ; \overline{\mathbb{M} ; \wp_1 ; \mathbb{T}}}$ $\quad \wp_2 ; \mathbb{T} = \overline{\mathbb{M}^{\intercal} ; \overline{\mathbb{M} ; \wp_2 ; \mathbb{T}}}$ $\quad \wp ; \mathbb{T} = \overline{\mathbb{M}^{\intercal} ; \overline{\mathbb{M} ; \wp ; \mathbb{T}}}$

Proof: i) Among the variants of Def. 9.3.3, we choose $\wp_1 ; \mathbb{T} = \mathtt{ZPE} ; e ; \mathbb{T}$, which leads to

$\mathbb{S}^{\intercal p} ; z ; \mathbb{T} \subseteq \mathtt{inf}_{n \geq 2} \overline{k_n^{\intercal} ; \overline{e ; \mathbb{T}}}$
$\forall n \geq 2 : \mathbb{S}^{\intercal p} ; z ; \mathbb{T} \subseteq \overline{k_n^{\intercal} ; \overline{e ; \mathbb{T}}}$
$\forall n \geq 2 : k_n^{\intercal} ; \overline{e ; \mathbb{T}} \subseteq \overline{\mathbb{S}^{\intercal p} ; z ; \mathbb{T}}$ negated
$\forall n \geq 2 : k_n ; \mathbb{S}^{\intercal p} ; z ; \mathbb{T} \subseteq e ; \mathbb{T}$ Schröder rule
$\forall n \geq 2 : \begin{cases} \bot\!\!\!\bot \subseteq e ; \mathbb{T} & \text{if } n \nmid p \\ \mathbb{S}^{\intercal \frac{p}{n}} ; z ; \mathbb{T} \subseteq e ; \mathbb{T} & \text{if } n \mid p \end{cases}$ see Lemma 6.4.2.i.c

If one assumes $p = 0$, only the second variant remains, so that as a consequence $\mathbb{S}^0 ; z ; \mathbb{T} \not\subseteq e ; \mathbb{T}$, conformant with the fact that the number $p = 0$ is not prime.

For all $p > 0$ the condition $\bot\!\!\!\bot \subseteq e ; \mathbb{T}$ is always satisfied. Should according to the second line, n happen to divide p, the containment

$\mathbb{S}^{\intercal \frac{p}{n}} ; z ; \mathbb{T} \subseteq e ; \mathbb{T}$

will hold only if $\frac{p}{n} = 1$, meaning that p is prime.

---

ii) We recall that $n \geq 2$ and show for the largest of the prime vectors that certainly holds

$k_n ; \wp_1 ; \mathbb{T} = k_n ; \mathtt{ZPE} ; e ; \mathbb{T} = \mathtt{inf}_{i \geq 2} k_n ; \overline{k_i^{\intercal} ; \overline{e ; \mathbb{T}}} \subseteq k_n ; \overline{k_n^{\intercal} ; \overline{e ; \mathbb{T}}} \subseteq e ; \mathbb{T}$,

where the last containment holds due to the Schröder rule. Should even equality hold, i.e.

$e ; \mathbb{T} \subseteq k_n ; \wp_1 ; \mathbb{T}$,

shunting the point $e ; \mathbb{T}$ gives $\mathbb{T} \subseteq \mathbb{T} ; e^{\intercal} ; k_n ; \wp_1 ; \mathbb{T} = \mathbb{T} ; z^{\intercal} ; \mathbb{S}^n ; \wp_1 ; \mathbb{T}$ due to Lemma 6.4.2.i.b. This means indeed that $\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \subseteq \wp_1 ; \mathbb{T}$ — otherwise the right side would be $\bot\!\!\!\bot$.

The two other ones need some tiny additional discussion wrt. their definition as

$\wp_2 ; \mathbb{T} := \wp_1 ; \mathbb{T} \cap \overline{e ; \mathbb{T}}$ $\quad \wp ; \mathbb{T} := \wp_1 ; \mathbb{T} \cap \overline{e ; \mathbb{T}} \cap \overline{\mathbb{S}^{\intercal 2} ; z ; \mathbb{T}}$

---

iii) $\mathcal{D}^{\intercal} \cap \wp_1 ; \mathbb{T} = \left[\mathtt{sup}_{n \geq 0} k_n^{\intercal} \right] \cap \wp_1 ; \mathbb{T}$
$= \mathtt{sup}_{n \geq 0} \left[k_n^{\intercal} \cap \wp_1 ; \mathbb{T} \right]$ choosing variant ZPE, now $\wp_1 ; \mathbb{T} = \mathtt{inf}_{i \geq 2} \overline{k_i^{\intercal} ; \overline{e ; \mathbb{T}}} = \overline{k_n^{\intercal} ; \overline{e ; \mathbb{T}}} \cap \wp_1 ; \mathbb{T}$
$= \left[z ; \mathbb{T} \cap \wp_1 ; \mathbb{T} \right] \cup \left[\mathbb{I} \cap \wp_1 ; \mathbb{T} \right] \cup \mathtt{sup}_{n \geq 2} \left[k_n^{\intercal} \cap \overline{k_n^{\intercal} ; \overline{e ; \mathbb{T}}} \cap \wp_1 ; \mathbb{T} \right]$
$= \bot\!\!\!\bot \cup \left[\mathbb{I} \cap \wp_1 ; \mathbb{T} \right] \cup \mathtt{sup}_{n \geq 2} \left[\mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; e^{\intercal} \cap \wp_1 ; \mathbb{T} \right]$ Lemma 6.4.2.ii.k
$= \wp_1 ; \mathbb{T} \cap \left[\mathbb{I} \cup (\mathtt{sup}_{n \geq 2} \mathbb{S}^{\intercal n} ; z ; \mathbb{T}) ; e^{\intercal} \right]$
$= \wp_1 ; \mathbb{T} \cap \left[\mathbb{I} \cup (\mathbb{T} \cap \overline{z ; \mathbb{T}} \cap \overline{e ; \mathbb{T}}) ; e^{\intercal} \right]$ because $\mathbb{T} = \mathtt{sup}_{n \geq 0} \mathbb{S}^{\intercal n} ; z ; \mathbb{T}$

$= \big[\wp_1 ; \mathbb{T} \cap \mathbb{I}\big] \cup \big[\wp_1 ; \mathbb{T} \cap \{\overline{z ; \mathbb{T}} \cap \overline{e ; \mathbb{T}} ; e^\intercal\}\big]$ distributive composition and masking
$= \big[\wp_1 ; \mathbb{T} \cap \mathbb{I}\big] \cup \big[\wp_1 ; \mathbb{T} \cap \overline{e ; \mathbb{T}} ; e^\intercal\big]$ since $\overline{z ; \mathbb{T}} \supseteq \wp_1 ; \mathbb{T}$
$= \wp_1 ; \mathbb{T} \cap \big[\mathbb{I} \cup \overline{e ; \mathbb{T}} ; e^\intercal\big]$
$= \wp_1 ; \mathbb{T} \cap (\mathbb{I} \cup e ; \mathbb{T} ; e^\intercal \cup \overline{e ; \mathbb{T}} ; e^\intercal)$ because $e ; \mathbb{T} ; e^\intercal \subseteq \mathbb{I}$
$= \wp_1 ; \mathbb{T} \cap (\mathbb{I} \cup \mathbb{T} ; e^\intercal)$

---

iv) The first steps of the proof are executed simultaneously for $\wp_1 ; \mathbb{T}, \wp_2 ; \mathbb{T}, \wp ; \mathbb{T}$, but written down just for $\wp ; \mathbb{T}$. For all three obviously $k_0 ; \wp ; \mathbb{T} = \bot\!\!\!\bot$.

$\mathbb{M} ; \wp ; \mathbb{T} = \mathtt{sup}_{n \geq 0} \, (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \leftthreetimes k_n) ; \wp ; \mathbb{T} = \mathtt{sup}_{n \geq 0} \, (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \leftthreetimes k_n ; \wp ; \mathbb{T})$ masking a join
$= (\mathbb{S}^{\intercal 0} ; z ; \mathbb{T} \leftthreetimes k_0 ; \wp ; \mathbb{T}) \cup (\mathbb{S}^{\intercal 1} ; z ; \mathbb{T} \leftthreetimes k_1 ; \wp ; \mathbb{T}) \cup \mathtt{sup}_{n \geq 2} \, (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \leftthreetimes k_n ; \wp ; \mathbb{T})$
$= \bot\!\!\!\bot \cup (e ; \mathbb{T} \leftthreetimes \wp ; \mathbb{T}) \cup \mathtt{sup}_{n \geq 2} \, (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \leftthreetimes k_n ; \wp ; \mathbb{T})$

Case $\wp_1 ; \mathbb{T}$:

$\ldots = (e ; \mathbb{T} \leftthreetimes \wp_1 ; \mathbb{T}) \cup \mathtt{sup}_{n \geq 2} \begin{cases} (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \leftthreetimes e ; \mathbb{T}) & \text{provided } \bot\!\!\!\bot \neq \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \subseteq \wp_1 ; \mathbb{T} \\ \bot\!\!\!\bot & \text{otherwise} \end{cases}$ see (ii)
$= (e ; \mathbb{T} \leftthreetimes \wp_1 ; \mathbb{T}) \cup \mathtt{sup}_{n \geq 2,\ \bot\!\!\!\bot \neq \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \subseteq \wp_1 ; \mathbb{T}} \, (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \leftthreetimes e ; \mathbb{T})$
$= (e ; \mathbb{T} \leftthreetimes \wp_1 ; \mathbb{T}) \cup (\wp_2 ; \overline{\mathbb{T}} \leftthreetimes e ; \mathbb{T})$
$= (e ; \mathbb{T} \leftthreetimes \wp_1 ; \mathbb{T}) \cup (\wp_1 ; \mathbb{T} \leftthreetimes e ; \mathbb{T})$ since already $(e ; \mathbb{T} \leftthreetimes e ; \mathbb{T}) \subseteq (e ; \mathbb{T} \leftthreetimes \wp_1 ; \mathbb{T})$

Case $\wp_2 ; \mathbb{T}$:

$\ldots = (e ; \mathbb{T} \leftthreetimes \wp_2 ; \mathbb{T}) \cup \mathtt{sup}_{n \geq 2} \begin{cases} (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \leftthreetimes e ; \mathbb{T}) & \text{provided } \bot\!\!\!\bot \neq \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \subseteq \wp_2 ; \mathbb{T} \\ \bot\!\!\!\bot & \text{otherwise} \end{cases}$ see (ii)
$= (e ; \mathbb{T} \leftthreetimes \wp_2 ; \mathbb{T}) \cup \mathtt{sup}_{n \geq 2,\ \bot\!\!\!\bot \neq \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \subseteq \wp_2 ; \mathbb{T}} \, (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \leftthreetimes e ; \mathbb{T})$
$= (e ; \mathbb{T} \leftthreetimes \wp_2 ; \mathbb{T}) \cup (\wp_2 ; \overline{\mathbb{T}} \leftthreetimes e ; \mathbb{T})$

Case $\wp ; \mathbb{T}$:

$\ldots = (e ; \mathbb{T} \leftthreetimes \wp ; \mathbb{T}) \cup \mathtt{sup}_{n \geq 2} \begin{cases} (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \leftthreetimes e ; \mathbb{T}) & \text{provided } \bot\!\!\!\bot \neq \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \subseteq \wp ; \mathbb{T} \\ \bot\!\!\!\bot & \text{otherwise} \end{cases}$ see (ii)
$= (e ; \mathbb{T} \leftthreetimes \wp ; \mathbb{T}) \cup \mathtt{sup}_{n \geq 2,\ \bot\!\!\!\bot \neq \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \subseteq \wp ; \mathbb{T}} \, (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \leftthreetimes e ; \mathbb{T})$
$= (e ; \mathbb{T} \leftthreetimes \wp ; \mathbb{T}) \cup (\wp ; \overline{\mathbb{T}} \leftthreetimes e ; \mathbb{T})$

$\overline{\mathbb{M}^\intercal ; \overline{\mathbb{M} ; \wp_1 ; \mathbb{T}}} = \overline{\mathbb{M}^\intercal ; \overline{\mathbb{M} ; \mathtt{MIQ} ; e ; \mathbb{T}}}$ choosing a definition variant
$= \overline{\mathbb{M}^\intercal ; \mathbb{M} ; \overline{\mathbb{M}^\intercal ; (\overline{\mathbb{I}} \leftthreetimes \overline{\mathbb{I}}) ; e ; \mathbb{T}}}$
$= \overline{\mathbb{M}^\intercal ; (\overline{\mathbb{I}} \leftthreetimes \overline{\mathbb{I}}) ; e ; \mathbb{T}}$ since $R^\intercal ; R ; \overline{R^\intercal ; X} = \overline{R^\intercal ; X}$, i.e. $\mathtt{ubd}_R(\mathtt{lbd}_R(\mathtt{ubd}_R(X))) = \mathtt{ubd}_R(X)$
$= \mathtt{MIQ} ; e ; \mathbb{T} = \wp_1 ; \mathbb{T}$

With tiny variations due to $\wp_2 ; \mathbb{T} = \wp_1 ; \mathbb{T} \cap \overline{e ; \mathbb{T}}$ and $\wp := \wp_1 ; \mathbb{T} \cap \overline{e ; \mathbb{T}} \cap \overline{\mathbb{S}^{\intercal 2} ; z ; \mathbb{T}}$, corresponding results follow for the others. Here it is shown just for $\wp_2 ; \mathbb{T}$:

$\overline{\mathbb{M}^\intercal ; \overline{\mathbb{M} ; \wp_2 ; \mathbb{T}}} = \overline{\mathbb{M}^\intercal ; \overline{\mathbb{M} ; [\wp_1 ; \mathbb{T} \cap \overline{e ; \mathbb{T}}]}} = \overline{\mathbb{M}^\intercal ; \overline{\mathbb{M} ; \wp_1 ; \mathbb{T}} \cap \mathbb{M} ; \overline{e ; \mathbb{T}}}$ with $\mathbb{M}$ univalent
$= \overline{\mathbb{M}^\intercal ; \Big[\overline{\mathbb{M} ; \wp_1 ; \mathbb{T}} \cup \overline{\mathbb{M} ; \overline{e ; \mathbb{T}}}\Big]}$
$= \overline{\mathbb{M}^\intercal ; \overline{\mathbb{M} ; \wp_1 ; \mathbb{T}}} \cap \overline{\mathbb{M}^\intercal ; \mathbb{M} ; \overline{e ; \mathbb{T}}}$
$= \wp_1 ; \mathbb{T} \cap \overline{\mathbb{M}^\intercal ; \overline{\mathbb{M} ; e ; \mathbb{T}} \cap \mathbb{M} ; \mathbb{T}}$ see above; $\mathbb{M}$ univalent
$= \wp_1 ; \mathbb{T} \cap \overline{\mathbb{M}^\intercal ; \big[\mathbb{M} ; e ; \mathbb{T} \cup \overline{\mathbb{M} ; \mathbb{T}}\big]}$
$= \wp_1 ; \mathbb{T} \cap \overline{\mathbb{M}^\intercal ; \mathbb{M} ; e ; \mathbb{T}} \cap \overline{\mathbb{M}^\intercal ; \overline{\mathbb{M} ; \mathbb{T}}}$

$= \wp_1 ; \mathbb{T} \cap \overline{\mathbb{M}^\intercal ; \mathbb{M} ; e ; \mathbb{T}}$ since $\wp_1 ; \mathbb{T} = \overline{\mathbb{M}^\intercal ; \overline{\mathbb{M} ; \wp_1 ; \mathbb{T}}} \subseteq \overline{\mathbb{M}^\intercal ; \overline{\mathbb{M} ; \mathbb{T}}}$
$= \wp_1 ; \mathbb{T} \cap \overline{e ; \mathbb{T}} = \wp_2 ; \mathbb{T}$

The last step uses $\overline{\mathbb{M}^\intercal ; \mathbb{M} ; e ; \mathbb{T}} = \overline{\mathbb{M}^\intercal ; (e ; \mathbb{T} \oslash e ; \mathbb{T})} = \overline{e ; \mathbb{T}}$ due to Prop. 7.1.1.i.e and ii.d. □

Interpretation of (iv) is as follows: If the result of some multiplication is prime, one of the factors will be 1. This result lets us evade from overly pointilistic proofs and elevates us to an algebraic level.

The following proposition may be seen as a description of the construct $\texttt{MIQ} = \overline{\mathbb{M}^\intercal ; (\mathbb{I} \oslash \mathbb{I})}$ depicted in Fig. 9.1.4, showing how columns look like:

9.3.7 Proposition. Given $\mathbb{S}^{\intercal^n} ; z ; \mathbb{T} \neq \bot\!\!\!\bot$, implying that $\mathbb{S}^{\intercal^n} ; z ; \mathbb{T}$ is a point, always

$$\texttt{MIQ} ; \mathbb{S}^{\intercal^n} ; z ; \mathbb{T} = \begin{cases} z ; \mathbb{T} & n = 0 \\ \wp_1 ; \mathbb{T} & n = 1 \\ \mathbb{S}^{\intercal^n} ; z ; \mathbb{T} & n \text{ prime} \\ \bot\!\!\!\bot & \text{otherwise} \end{cases}$$

Proof: The case $n = 1$ directly reflects the definition Def. 9.3.3 of $\wp_1 ; \mathbb{T}$.

To cope with $n \neq 1$, we evaluate the term. We have to split with respect to $k_0^\intercal$ which is not univalent.

$\texttt{MIQ} ; \mathbb{S}^{\intercal^n} ; z ; \mathbb{T} = \overline{\mathbb{M}^\intercal ; (\overline{\mathbb{I}} \oslash \overline{\mathbb{I}})} ; \mathbb{S}^{\intercal^n} ; z ; \mathbb{T}$
$= \overline{\left[\texttt{sup}_{i \geq 0} \ (\mathbb{T} ; z^\intercal ; \mathbb{S}^i \oslash k_i^\intercal) \ \right] ; (\overline{\mathbb{S}^{\intercal^n} ; z ; \mathbb{T}} \oslash \overline{\mathbb{S}^{\intercal^n} ; z ; \mathbb{T}})}$ Def. 7.1.1.i.a
$= \overline{(\mathbb{T} ; z^\intercal \oslash z ; \mathbb{T}) ; (\overline{\mathbb{S}^{\intercal^n} ; z ; \mathbb{T}} \oslash \overline{\mathbb{S}^{\intercal^n} ; z ; \mathbb{T}})} \cap \texttt{inf}_{i \geq 1} \overline{\mathbb{T} ; z^\intercal ; \mathbb{S}^i ; \overline{\mathbb{S}^{\intercal^n} ; z ; \mathbb{T}} \cap k_i^\intercal ; \overline{\mathbb{S}^{\intercal^n} ; z ; \mathbb{T}}}$
$= \overline{(\mathbb{T} ; z^\intercal ; \pi^\intercal \cap z ; \mathbb{T} ; \rho^\intercal) ; (\overline{\mathbb{S}^{\intercal^n} ; z ; \mathbb{T}} \oslash \overline{\mathbb{S}^{\intercal^n} ; z ; \mathbb{T}})} \cap \texttt{inf}_{i \geq 1} \overline{\overline{\mathbb{T} ; z^\intercal ; \mathbb{S}^i ; \mathbb{S}^{\intercal^n} ; z ; \mathbb{T}} \cap k_i^\intercal ; \overline{\mathbb{S}^{\intercal^n} ; z ; \mathbb{T}}}$
$= \overline{(\mathbb{T} ; z^\intercal ; \pi^\intercal \cap z ; \mathbb{T}) ; (\overline{\mathbb{S}^{\intercal^n} ; z ; \mathbb{T}} \oslash \overline{\mathbb{S}^{\intercal^n} ; z ; \mathbb{T}})} \cap \texttt{inf}_{i \geq 1} \left[\mathbb{T} ; z^\intercal ; \mathbb{S}^i ; \mathbb{S}^{\intercal^n} ; z ; \mathbb{T} \cup \overline{k_i^\intercal ; \overline{\mathbb{S}^{\intercal^n} ; z ; \mathbb{T}}} \right]$
$= \overline{\mathbb{T} ; z^\intercal ; \pi^\intercal ; (\overline{\mathbb{S}^{\intercal^n} ; z ; \mathbb{T}} \oslash \overline{\mathbb{S}^{\intercal^n} ; z ; \mathbb{T}}) \cap z ; \mathbb{T}} \cap X_n$ abbreviated; see below
$= \overline{\mathbb{T} ; z^\intercal ; (\overline{\mathbb{S}^{\intercal^n} ; z ; \mathbb{T}} \cap \mathbb{T} ; \overline{\mathbb{S}^{\intercal^n} ; z ; \mathbb{T}}) \cap z ; \mathbb{T}} \cap X_n$ destroy and append
$= \overline{\mathbb{T} ; z^\intercal ; \overline{\mathbb{S}^{\intercal^n} ; z ; \mathbb{T}} \cap \mathbb{T} ; \overline{\mathbb{S}^{\intercal^n} ; z ; \mathbb{T}} \cap z ; \mathbb{T}} \cap X_n$ masking
$= \overline{\overline{\mathbb{T} ; z^\intercal ; \mathbb{S}^{\intercal^n} ; z ; \mathbb{T}} \cap \mathbb{T} \cap z ; \mathbb{T}} \cap X_n$ where $X_n := \texttt{inf}_{i \geq 1} \left[\mathbb{T} ; z^\intercal ; \mathbb{S}^i ; \mathbb{S}^{\intercal^n} ; z ; \mathbb{T} \cup \overline{k_i^\intercal ; \overline{\mathbb{S}^{\intercal^n} ; z ; \mathbb{T}}} \right]$

Now we proceed according to the different values of $n \neq 1$.

$n = 0$:
$= \overline{\overline{\mathbb{T} ; z^\intercal ; z ; \mathbb{T}} \cap z ; \mathbb{T}} \cap \texttt{inf}_{i \geq 1} \overline{k_i^\intercal ; \overline{z ; \mathbb{T}}}$ since $\mathbb{S}^i ; \mathbb{S}^{\intercal^0} ; z ; \mathbb{T} = \bot\!\!\!\bot$ for $i \geq 1$
$= \mathbb{T} \cap \texttt{inf}_{i \geq 1} \overline{k_i^\intercal ; \overline{z ; \mathbb{T}}}$ because z is a point
$= z ; \mathbb{T}$ because $\overline{k_1^\intercal ; \overline{z ; \mathbb{T}}} = z ; \mathbb{T}$ and $z ; \mathbb{T} \subseteq \overline{k_i^\intercal ; \overline{z ; \mathbb{T}}}$ for $i > 1$

$n > 1$:
$= \overline{\overline{\mathbb{I}} \cap z ; \mathbb{T}} \cap \texttt{inf}_{1 \leq i \neq n} \overline{k_i^\intercal ; \overline{\mathbb{S}^{\intercal^n} ; z ; \mathbb{T}}}$ since $\mathbb{S}^{\intercal^n} ; z ; \mathbb{T} \neq \bot\!\!\!\bot \Longrightarrow \mathbb{T} ; z^\intercal ; \mathbb{S}^n ; \mathbb{S}^{\intercal^n} ; z ; \mathbb{T} = \mathbb{T}$
$= \overline{z ; \mathbb{T}} \cap \texttt{inf}_{1 \leq i \neq n} \left[k_i^\intercal ; \mathbb{S}^{\intercal^n} ; z ; \mathbb{T} \cup \overline{k_i^\intercal ; \mathbb{T}} \right]$ because k_i is injective
$= \overline{z ; \mathbb{T}} \cap \texttt{inf}_{1 \leq i \neq n} \left[\mathbb{S}^{\intercal^{n*i}} ; z ; \mathbb{T} \cup \overline{k_i^\intercal ; \mathbb{T}} \right]$ Prop. 6.4.2.i.b

$= \mathtt{inf}_{1\leq i\neq n}\left[\,\mathbb{S}^{\intercal n*i};z;\mathbb{T}\cup\overline{k_i^\intercal;\mathbb{T}}\,\right]$ since always $\mathbb{S}^{\intercal i*n};z;\mathbb{T}\cup\overline{k_i^\intercal;\mathbb{T}}\subseteq\overline{z;\mathbb{T}}$
$= \mathbb{S}^{\intercal n};z;\mathbb{T}\cap\mathtt{inf}_{2\leq i\neq n}\left[\mathbb{S}^{\intercal i*n};z;\mathbb{T}\cup\overline{k_i^\intercal;\mathbb{T}}\,\right]$ because $k_1=\mathbb{I}$

Now we use that $\mathbb{S}^{\intercal n};z;\mathbb{T}$ is a point and thus either totally contained in every other $\mathtt{inf}$-intersected term or not. Therefore, the result will be $\mathbb{S}^{\intercal n};z;\mathbb{T}$ or $\mathbb{\bot}$.

It is obviously impossible that the point $\mathbb{S}^{\intercal n};z;\mathbb{T}$ is contained in one of the $\mathbb{S}^{\intercal i*n};z;\mathbb{T}$, since the latter are either *other* points or $=\mathbb{\bot}$. So we study when

$\mathbb{S}^{\intercal n};z;\mathbb{T}\subseteq\overline{k_i^\intercal;\mathbb{T}}\iff k_i^\intercal;\mathbb{T}\subseteq\overline{\mathbb{S}^{\intercal n};z;\mathbb{T}}\iff k_i;\mathbb{S}^{\intercal n};z;\mathbb{T}\subseteq\mathbb{\bot}$,

for which the conditions may be found in Lemma 6.4.2.i.c; namely $i\nmid n$. Because the cases $i=1$ and $i=n$ are already expressly excluded, the fact $i\nmid n$ for $i=2,\dots n-1$ — and thus the result $\mathbb{S}^{\intercal n};z;\mathbb{T}$ — will be obtained precisely when n is a prime. □

9.3.8 Proposition. We assume regressive finiteness and consider the extrusions of primes.

i) For $\iota\in\{\iota_{\wp_1;\mathbb{T}},\iota_{\wp_2;\mathbb{T}},\iota_{\wp;\mathbb{T}}\}$ always $\iota;z;\mathbb{T}=\mathbb{\bot}$

$\iota_{\wp_1;\mathbb{T}};e;\mathbb{T}\neq\mathbb{\bot}$ $\iota_{\wp_1;\mathbb{T}};\mathbb{S}^{\intercal 2};z;\mathbb{T}\neq\mathbb{\bot}$ and for $n\geq 3$: $\iota_{\wp_1;\mathbb{T}};\mathbb{S}^{\intercal n};z;\mathbb{T}\neq\mathbb{\bot}\iff n$ is prime

$\iota_{\wp_2;\mathbb{T}};e;\mathbb{T}=\mathbb{\bot}$ $\iota_{\wp_2;\mathbb{T}};\mathbb{S}^{\intercal 2};z;\mathbb{T}\neq\mathbb{\bot}$ and for $n\geq 3$: $\iota_{\wp_2;\mathbb{T}};\mathbb{S}^{\intercal n};z;\mathbb{T}\neq\mathbb{\bot}\iff n$ is prime

$\iota_{\wp;\mathbb{T}};e;\mathbb{T}=\mathbb{\bot}$ $\iota_{\wp;\mathbb{T}};\mathbb{S}^{\intercal 2};z;\mathbb{T}=\mathbb{\bot}$ and for $n\geq 3$: $\iota_{\wp;\mathbb{T}};\mathbb{S}^{\intercal n};z;\mathbb{T}\neq\mathbb{\bot}\iff n$ is prime

ii) For $\iota\in\{\iota_{\wp_1;\mathbb{T}},\iota_{\wp_2;\mathbb{T}},\iota_{\wp;\mathbb{T}}\}$ always $\iota;k_n^\intercal=\iota;\mathbb{S}^{\intercal n};z;\mathbb{T};e^\intercal$ when $n\geq 2$.

iii) For $\iota\in\{\iota_{\wp_1;\mathbb{T}},\iota_{\wp_2;\mathbb{T}},\iota_{\wp;\mathbb{T}}\}$ always $\iota;\mathbb{M}^\intercal=(\iota\circledcirc\mathbb{T};e^\intercal)\cup(\mathbb{T};e^\intercal\circledcirc\iota)$

iv) For $\iota\in\{\iota_{\wp_1;\mathbb{T}},\iota_{\wp_2;\mathbb{T}},\iota_{\wp;\mathbb{T}}\}$ always $\iota;\mathcal{D}^\intercal=\iota;\mathbb{M}^\intercal;\pi=\iota;\mathbb{M}^\intercal;\rho=\mathbb{T};e^\intercal\cup\iota$

v) $\iota_{\wp_1;\mathbb{T}};\mathcal{D}^\intercal;\iota_{\wp_1;\mathbb{T}}^\intercal=\mathbb{I}\cup\mathbb{T};e^\intercal;\iota_{\wp_1;\mathbb{T}}^\intercal$ $\iota_{\wp_2;\mathbb{T}};\mathcal{D}^\intercal;\iota_{\wp_2;\mathbb{T}}^\intercal=\mathbb{I}$ $\iota_{\wp;\mathbb{T}};\mathcal{D}^\intercal;\iota_{\wp;\mathbb{T}}^\intercal=\mathbb{I}$

$$\begin{array}{c} \\ 1\to \\ 2\to \\ 3\to \\ 5\to \\ 7\to \\ 11\to \\ 13\to \\ 17\to \end{array}\begin{array}{c} \begin{array}{cccccccc} 1\uparrow & 2\uparrow & 3\uparrow & 5\uparrow & 7\uparrow & 11\uparrow & 13\uparrow & 17\uparrow \end{array} \\ \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \end{pmatrix} \end{array} \quad \begin{array}{c} \\ 2\to \\ 3\to \\ 5\to \\ 7\to \\ 11\to \\ 13\to \\ 17\to \end{array}\begin{array}{c} \begin{array}{ccccccc} 2\uparrow & 3\uparrow & 5\uparrow & 7\uparrow & 11\uparrow & 13\uparrow & 17\uparrow \end{array} \\ \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \end{pmatrix} \end{array} \quad \begin{array}{c} \\ 3\to \\ 5\to \\ 7\to \\ 11\to \\ 13\to \\ 17\to \end{array}\begin{array}{c} \begin{array}{cccccc} 3\uparrow & 5\uparrow & 7\uparrow & 11\uparrow & 13\uparrow & 17\uparrow \end{array} \\ \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} \end{pmatrix} \end{array}$$

Fig. 9.3.4 Illustrating (v): $\iota_{\wp_1;\mathbb{T}};\mathcal{D}^\intercal;\iota_{\wp_1;\mathbb{T}}^\intercal$ $\iota_{\wp_2;\mathbb{T}};\mathcal{D}^\intercal;\iota_{\wp_2;\mathbb{T}}^\intercal$ $\iota_{\wp;\mathbb{T}};\mathcal{D}^\intercal;\iota_{\wp;\mathbb{T}}^\intercal$

Proof: i) Since $\iota_{\wp;\mathbb{T}}^\intercal;\mathbb{T}=\wp;\mathbb{T}\subseteq\iota_{\wp_2;\mathbb{T}}^\intercal;\mathbb{T}=\wp_2;\mathbb{T}\subseteq\iota_{\wp_1;\mathbb{T}}^\intercal;\mathbb{T}=\wp_1;\mathbb{T}$, the proof of just $\iota_{\wp_1;\mathbb{T}};z;\mathbb{T}\subseteq\mathbb{\bot}$ modified to $\iota_{\wp_1;\mathbb{T}}^\intercal;\mathbb{T}\subseteq\overline{z;\mathbb{T}}$ will suffice:

$\iota_{\wp_1;\mathbb{T}}^\intercal;\mathbb{T}=\wp_1;\mathbb{T}=\mathtt{MIQ};e;\mathbb{T}\subseteq\overline{z;\mathbb{T}}$ according to Def. 9.3.3
$\iff$ $e;\mathbb{T}\subseteq\overline{\mathtt{MIQ}}^\intercal;z;\mathbb{T}$ negated, transposed and points z,e shunted
$=(\overline{\mathbb{I}}\circledcirc\overline{\mathbb{I}});\mathbb{M};z;\mathbb{T}$ expanded

$= (\overline{\mathbb{I}} \circledcirc \overline{\mathbb{I}}) ; (\pi \cup \rho) ; z ; \mathbb{T}$ Prop. 7.1.1.ii.c
$= (\overline{\mathbb{I}} \circledcirc \overline{\mathbb{I}}) ; \pi ; z ; \mathbb{T} \cup (\overline{\mathbb{I}} \circledcirc \overline{\mathbb{I}}) ; \rho ; z ; \mathbb{T}$
$= \left[\overline{\mathbb{I}} \cap \overline{\mathbb{I}} ; \mathbb{T}\right] ; z ; \mathbb{T} \cup \left[\overline{\mathbb{I}} ; \mathbb{T} \cap \overline{\mathbb{I}}\right] ; z ; \mathbb{T} = \overline{\mathbb{I}} ; z ; \mathbb{T} \cup \overline{\mathbb{I}} ; z ; \mathbb{T}$ destroy and append
$= \overline{z ; \mathbb{T}} \cup \overline{z ; \mathbb{T}} = \overline{z ; \mathbb{T}}$ which is satisfied

---

The more general versions are shown via Prop. 9.3.6.i:

$$\iota_{\wp_1;\mathbb{T}} ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \subseteq \mathbb{\perp} \iff \iota^{\intercal}_{\wp_1;\mathbb{T}} ; \mathbb{T} = \wp_1 ; \mathbb{T} \subseteq \overline{\mathbb{S}^{\intercal n} ; z ; \mathbb{T}} \iff \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \subseteq \overline{\wp_1 ; \mathbb{T}}$$

---

ii) We split $k_n^{\intercal}$ and use Lemma 6.4.2.ii.k

$$\iota ; k_n^{\intercal} = \iota ; \left[k_n^{\intercal} \cap \overline{k_n^{\intercal} ; \overline{e ; \mathbb{T}}}\right] \cup \iota ; \left[k_n^{\intercal} \cap k_n^{\intercal} ; \overline{e ; \mathbb{T}}\right] = \iota ; \left[k_n^{\intercal} \cap \overline{k_n^{\intercal} ; \overline{e ; \mathbb{T}}}\right] \cup \mathbb{\perp} = \iota ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; e^{\intercal},$$

which to complete $\iota ; k_n^{\intercal} ; \overline{e ; \mathbb{T}} \subseteq \mathbb{\perp}$ has yet to be shown. Using Prop. 9.3.3 for the biggest, indeed

$$\iota^{\intercal}_{\wp_1;\mathbb{T}} ; \mathbb{T} = \wp_1 ; \mathbb{T} = \mathtt{ZPE} ; e ; \mathbb{T} \subseteq \overline{k_n^{\intercal} ; \overline{\mathbb{I}}} ; e ; \mathbb{T} = \overline{k_n^{\intercal} ; \overline{e ; \mathbb{T}}}.$$

---

iii) $\iota ; \mathbb{M}^{\intercal} = \iota ; \left[\, \mathtt{sup}_{n \geq 0} \, (\mathbb{S}^{\intercal n} ; z ; \mathbb{T} \circledcirc k_n) \, \right]^{\intercal}$
$= \mathtt{sup}_{n \geq 0} \, \iota ; (\mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n} \circledcirc k_n^{\intercal})$
$= \mathtt{sup}_{n \geq 0} \, (\mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n} \circledcirc \iota ; k_n^{\intercal})$ map property of ι
$= (\mathbb{T} ; z^{\intercal} \circledcirc \iota ; k_0^{\intercal}) \cup (\mathbb{T} ; z^{\intercal} ; \mathbb{S} \circledcirc \iota ; k_1^{\intercal})$
$\cup \, \mathtt{sup}_{n \geq 2} \left[\mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n} ; \pi^{\intercal} \cap \iota ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; e^{\intercal} ; \rho^{\intercal}\right]$ Prop. 9.3.8.ii
$= (\mathbb{T} ; z^{\intercal} \circledcirc \iota ; z ; \mathbb{T}) \cup (\mathbb{T} ; e^{\intercal} \circledcirc \iota)$
$\cup \, \mathtt{sup}_{n \geq 2} \, \iota ; \left[\mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n} ; \pi^{\intercal} \cap \mathbb{S}^{\intercal n} ; z ; \mathbb{T} \cap \mathbb{T} ; e^{\intercal} ; \rho^{\intercal}\right]$
$= (\mathbb{T} ; z^{\intercal} \circledcirc \mathbb{\perp}) \cup (\mathbb{T} ; e^{\intercal} \circledcirc \iota)$ due to (i)
$\cup \, \iota ; \mathtt{sup}_{n \geq 2} \left[\, \mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n} ; \pi^{\intercal} \cap \mathbb{T} ; e^{\intercal} ; \rho^{\intercal}\right]$ masking
$= (\mathbb{T} ; e^{\intercal} \circledcirc \iota) \cup \iota ; (\mathtt{sup}_{n \geq 2} \, \mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{n} \circledcirc \mathbb{T} ; e^{\intercal})$
$= (\mathbb{T} ; e^{\intercal} \circledcirc \iota) \cup \iota ; (\mathbb{I} \cap \overline{z ; \mathbb{T} ; z^{\intercal}} \cap \overline{e ; \mathbb{T} ; e^{\intercal}} \circledcirc \mathbb{T} ; e^{\intercal})$ Prop. 5.2.2.ii; regressive finiteness
$= (\mathbb{T} ; e^{\intercal} \circledcirc \iota) \cup (\iota \cap \overline{\iota ; z ; \mathbb{T} ; z^{\intercal}} \cap \overline{\iota ; e ; \mathbb{T} ; e^{\intercal}} \circledcirc \mathbb{T} ; e^{\intercal})$
$= (\mathbb{T} ; e^{\intercal} \circledcirc \iota) \cup (\iota \cap \overline{\mathbb{\perp}} \cap \overline{\iota ; e ; \mathbb{T} ; e^{\intercal}} \circledcirc \mathbb{T} ; e^{\intercal})$ because $\iota ; z ; \mathbb{T} = \mathbb{\perp}$ due to (i)
$= (\mathbb{T} ; e^{\intercal} \circledcirc \iota) \cup (\iota ; e ; \mathbb{T} ; e^{\intercal} \circledcirc \mathbb{T} ; e^{\intercal}) \cup (\iota \cap \overline{\iota ; e ; \mathbb{T} ; e^{\intercal}} \circledcirc \mathbb{T} ; e^{\intercal})$ see below
$= (\mathbb{T} ; e^{\intercal} \circledcirc \iota) \cup (\iota \circledcirc \mathbb{T} ; e^{\intercal})$

The introduction of the tiny missing part was justified since it is contained in the first

$$(\iota ; e ; \mathbb{T} ; e^{\intercal} \circledcirc \mathbb{T} ; e^{\intercal}) \subseteq (\mathbb{T} ; e^{\intercal} \circledcirc \iota),$$

which holds because

$\iota ; e ; \mathbb{T} ; e^{\intercal} ; \pi^{\intercal} \cap \mathbb{T} ; e^{\intercal} ; \rho^{\intercal} = \iota ; e ; \mathbb{T} \cap \mathbb{T} ; e^{\intercal} ; \pi^{\intercal} \cap \mathbb{T} ; e^{\intercal} ; \rho^{\intercal}$
$= \mathbb{T} ; e^{\intercal} ; \pi^{\intercal} \cap \iota ; e ; \mathbb{T} ; e^{\intercal} ; \rho^{\intercal} \subseteq \mathbb{T} ; e^{\intercal} ; \pi^{\intercal} \cap \iota ; \rho^{\intercal}$

---

iv) $\iota ; \mathcal{D}^{\intercal} = \iota ; \mathbb{M}^{\intercal} ; \pi = \left[(\iota \circledcirc \mathbb{T} ; e^{\intercal}) \cup (\mathbb{T} ; e^{\intercal} \circledcirc \iota)\right] ; \pi$ using (iii)
$= (\iota \circledcirc \mathbb{T} ; e^{\intercal}) ; \pi \cup (\mathbb{T} ; e^{\intercal} \circledcirc \iota) ; \pi$
$= (\iota \cap \mathbb{T} ; e^{\intercal} ; \mathbb{T}) \cup (\mathbb{T} ; e^{\intercal} \cap \iota ; \mathbb{T})$
$= \iota \cup \mathbb{T} ; e^{\intercal}$

---

v) $\iota_{\wp_1;\mathbb{T}} ; \mathcal{D}^{\intercal} ; \iota^{\intercal}_{\wp_1;\mathbb{T}} = \iota_{\wp_1;\mathbb{T}} ; (\mathtt{sup}_{n \geq 0} \, k_n^{\intercal}) ; \iota^{\intercal}_{\wp_1;\mathbb{T}}$ Prop. 7.4.2.i
$= \mathtt{sup}_{n \geq 0} \, \iota_{\wp_1;\mathbb{T}} ; k_n^{\intercal} ; \iota^{\intercal}_{\wp_1;\mathbb{T}}$
$= \iota_{\wp_1;\mathbb{T}} ; k_0^{\intercal} ; \iota^{\intercal}_{\wp_1;\mathbb{T}} \cup \iota_{\wp_1;\mathbb{T}} ; k_1^{\intercal} ; \iota^{\intercal}_{\wp_1;\mathbb{T}} \cup \mathtt{sup}_{n \geq 2} \, \iota_{\wp_1;\mathbb{T}} ; k_n^{\intercal} ; \iota^{\intercal}_{\wp_1;\mathbb{T}}$
$= \iota_{\wp_1;\mathbb{T}} ; z ; \mathbb{T} ; \iota^{\intercal}_{\wp_1;\mathbb{T}} \cup \iota_{\wp_1;\mathbb{T}} ; \mathbb{I} ; \iota^{\intercal}_{\wp_1;\mathbb{T}} \cup \mathtt{sup}_{n \geq 2} \, \iota_{\wp_1;\mathbb{T}} ; \mathbb{S}^{\intercal n} ; z ; \mathbb{T} ; e^{\intercal} ; \iota^{\intercal}_{\wp_1;\mathbb{T}}$ due to (ii)

$= \mathbb{\perp} \cup \iota_{\wp_1;\mathbb{T}};\iota^{\mathsf{T}}_{\wp_1;\mathbb{T}} \cup \iota_{\wp_1;\mathbb{T}};\big[\,\mathtt{sup}_{i\geq 0}\,\mathbb{S}^{\,\mathsf{T}i};z;\mathbb{T}\big];e^{\mathsf{T}};\iota^{\mathsf{T}}_{\wp_1;\mathbb{T}}$
since $\iota_{\wp_1;\mathbb{T}};\mathbb{S}^{\,\mathsf{T}0};z;\mathbb{T};e^{\mathsf{T}};\iota^{\mathsf{T}}_{\wp_1;\mathbb{T}} = \mathbb{\perp}$ and $\iota_{\wp_1;\mathbb{T}};\mathbb{S}^{\,\mathsf{T}1};z;\mathbb{T};e^{\mathsf{T}};\iota^{\mathsf{T}}_{\wp_1;\mathbb{T}} \subseteq \iota_{\wp_1;\mathbb{T}};\mathbb{I};\iota^{\mathsf{T}}_{\wp_1;\mathbb{T}}$
$= \mathbb{I} \cup \iota_{\wp_1;\mathbb{T}};\mathbb{T};e^{\mathsf{T}};\iota^{\mathsf{T}}_{\wp_1;\mathbb{T}}$ when regressively finite; z is not a prime
$= \mathbb{I} \cup \mathbb{T};e^{\mathsf{T}};\iota^{\mathsf{T}}_{\wp_1;\mathbb{T}}$ natural injections are always total

---

For odd primes, and similarly for possibly even ones, abbreviating to ι:

$\iota;\mathcal{D}^{\mathsf{T}};\iota^{\mathsf{T}} = \iota;\mathbb{M}^{\,\mathsf{T}};\pi;\iota^{\mathsf{T}} = \iota;\mathtt{sup}_{n\geq 0}\,(\mathbb{T};z^{\mathsf{T}};\mathbb{S}^{\,n} \circledcirc k^{\mathsf{T}}_n);\pi;\iota^{\mathsf{T}} = \iota;\big[\,\mathtt{sup}_{n\geq 0}\,\mathbb{T};z^{\mathsf{T}};\mathbb{S}^{\,n} \cap k^{\mathsf{T}}_n;\mathbb{T}\big];\iota^{\mathsf{T}}$
$= \mathtt{sup}_{n\geq 0}\,\mathbb{T};z^{\mathsf{T}};\mathbb{S}^{\,n};\iota^{\mathsf{T}} \cap \iota;k^{\mathsf{T}}_n;\mathbb{T}$
$= \mathtt{sup}_{n\geq 0}\,\iota;k^{\mathsf{T}}_n;\mathbb{T};z^{\mathsf{T}};\mathbb{S}^{\,n};\iota^{\mathsf{T}}$ masking
$= \iota;k^{\mathsf{T}}_0;\mathbb{T};z^{\mathsf{T}};\mathbb{S}^{\,0};\iota^{\mathsf{T}} \cup \iota;k^{\mathsf{T}}_1;\mathbb{T};z^{\mathsf{T}};\mathbb{S};\iota^{\mathsf{T}} \cup \mathtt{sup}_{n\geq 2}\,\iota;k^{\mathsf{T}}_n;\mathbb{T};z^{\mathsf{T}};\mathbb{S}^{\,n};\iota^{\mathsf{T}}$
$= \iota;z;\mathbb{T};z^{\mathsf{T}};\iota^{\mathsf{T}} \cup \iota;\mathbb{T};e^{\mathsf{T}};\iota^{\mathsf{T}} \cup \mathtt{sup}_{n\geq 2}\,\iota;\mathbb{S}^{\,\mathsf{T}n};z;\mathbb{T};e^{\mathsf{T}};\mathbb{T};z^{\mathsf{T}};\mathbb{S}^{\,n};\iota^{\mathsf{T}}$ due to (ii)
$= \mathbb{\perp} \cup \mathbb{\perp} \cup \mathtt{sup}_{n\geq 2}\,\iota;\mathbb{S}^{\,\mathsf{T}n};z;\mathbb{T};e^{\mathsf{T}};\mathbb{T};z^{\mathsf{T}};\mathbb{S}^{\,n};\iota^{\mathsf{T}}$ due to (i)
$= \iota;\big[\,\mathtt{sup}_{n\geq 2}\,\mathbb{S}^{\,\mathsf{T}n};z;\mathbb{T};z^{\mathsf{T}};\mathbb{S}^{\,n}\big];\iota^{\mathsf{T}}$ since e is a point
$= \iota;\big[\,\mathtt{sup}_{i\geq 0}\,\mathbb{S}^{\,\mathsf{T}i};z;\mathbb{T};z^{\mathsf{T}};\mathbb{S}^{\,i}\big];\iota^{\mathsf{T}}$ because $\iota;z;\mathbb{T} = \mathbb{\perp}$ and $\iota;e;\mathbb{T} = \mathbb{\perp}$ for $\iota \in \{\iota_{\wp_2;\mathbb{T}}, \iota_{\wp;\mathbb{T}}\}$
$= \iota;\mathbb{I};\iota^{\mathsf{T}}$ Prop. 5.2.2.ii.a
$= \iota;\iota^{\mathsf{T}} = \mathbb{I}$ since ι is an injective map □

9.3.9 Lemma. $\overline{(\overline{\mathbb{I}} \circledcirc \overline{\mathbb{I}});e;\mathbb{T}} = \pi;e;\mathbb{T} \cup \rho;e;\mathbb{T} = \big[\,(\mathbb{I} \circledcirc e;\mathbb{T}) \cup (e;\mathbb{T} \circledcirc \mathbb{I})\,\big];\mathbb{T}$

Proof: In what follows, we use not least totality of projections:

$\overline{(\overline{\mathbb{I}} \circledcirc \overline{\mathbb{I}});e;\mathbb{T}} = \overline{(\overline{e;\mathbb{T}} \circledcirc \overline{e;\mathbb{T}})} = \overline{\pi;\overline{e;\mathbb{T}} \cap \rho;\overline{e;\mathbb{T}}} = \overline{\overline{\pi;e;\mathbb{T}} \cap \overline{\rho;e;\mathbb{T}}} = \pi;e;\mathbb{T} \cup \rho;e;\mathbb{T}$

$\big[\,(\mathbb{I} \circledcirc e;\mathbb{T}) \cup (e;\mathbb{T} \circledcirc \mathbb{I})\,\big];\mathbb{T} = (\mathbb{I} \circledcirc e;\mathbb{T});\mathbb{T} \cup (e;\mathbb{T} \circledcirc \mathbb{I});\mathbb{T}$
$= (\pi \cap \rho;e;\mathbb{T});\mathbb{T} \cup (\pi;e;\mathbb{T} \cap \rho);\mathbb{T} = (\pi;\mathbb{T} \cap \rho;e;\mathbb{T}) \cup (\pi;e;\mathbb{T} \cap \rho;\mathbb{T}) = \pi;e;\mathbb{T} \cup \rho;e;\mathbb{T}$ □

In total: So far the following tabulated representations for primes have been developed. Not all conceivable versions are listed.

$\wp_1 = \mathtt{DIA}$	$= \overline{\mathbb{M}^{\,\mathsf{T}};\overline{(\mathbb{I} \circledcirc e;\mathbb{T}) \cup (e;\mathbb{T} \circledcirc \mathbb{I})}}$
$= \mathtt{MIQ} \cap \mathbb{I} \cap \overline{z;\mathbb{T};z^{\mathsf{T}}}$	$= \overline{\mathbb{M}^{\,\mathsf{T}};(\overline{\mathbb{I}} \circledcirc \overline{\mathbb{I}})} \cap \mathbb{I} \cap \overline{z;\mathbb{T};z^{\mathsf{T}}}$
$= \mathtt{MTI} \cap \mathbb{I} \cap \overline{z;\mathbb{T};z^{\mathsf{T}}}$	$= \mathtt{inf}_{n\geq 2}\,\big[\mathbb{T};z^{\mathsf{T}};\mathbb{S}^{\,n} \cup \overline{k^{\mathsf{T}}_n;\overline{\mathbb{I}}}\,\big] \cap \mathbb{I} \cap \overline{z;\mathbb{T};z^{\mathsf{T}}}$
$\wp_2 = \mathtt{DIA} \cap \overline{e;\mathbb{T};e^{\mathsf{T}}}$	$= \overline{\mathbb{M}^{\,\mathsf{T}};\overline{(\mathbb{I} \circledcirc e;\mathbb{T}) \cup (e;\mathbb{T} \circledcirc \mathbb{I})}} \cap \overline{e;\mathbb{T};e^{\mathsf{T}}}$
$= \mathtt{MIQ} \cap \mathbb{I} \cap \overline{z;\mathbb{T};z^{\mathsf{T}}} \cap \overline{e;\mathbb{T};e^{\mathsf{T}}}$	$= \overline{\mathbb{M}^{\,\mathsf{T}};(\overline{\mathbb{I}} \circledcirc \overline{\mathbb{I}})} \cap \mathbb{I} \cap \overline{z;\mathbb{T};z^{\mathsf{T}}} \cap \overline{e;\mathbb{T};e^{\mathsf{T}}}$
$= \mathtt{MTI} \cap \mathbb{I} \cap \overline{z;\mathbb{T};z^{\mathsf{T}}} \cap \overline{e;\mathbb{T};e^{\mathsf{T}}}$	$= \mathtt{inf}_{n\geq 2}\,\big[\mathbb{T};z^{\mathsf{T}};\mathbb{S}^{\,n} \cup \overline{k^{\mathsf{T}}_n;\overline{\mathbb{I}}}\,\big] \cap \mathbb{I} \cap \overline{z;\mathbb{T};z^{\mathsf{T}}} \cap \overline{e;\mathbb{T};e^{\mathsf{T}}}$
$\wp = \wp_2 \cap \overline{\mathbb{S}^{\,\mathsf{T}};e;\mathbb{T};e^{\mathsf{T}};\mathbb{S}}$	

$\wp_1 ; \mathbb{T} = \mathtt{DIA} ; \mathbb{T}$	$= \overline{\mathbb{M}^{\mathsf{T}} ; \overline{(\mathbb{I} \circledcirc e ; \mathbb{T}) \cup (e ; \mathbb{T} \circledcirc \mathbb{I})}} ; \mathbb{T}$
$= \mathtt{DIV} ; e ; \mathbb{T}$	$= \overline{\mathcal{D}_{\mathrm{s}}^{\mathsf{T}} ; \overline{e ; \mathbb{T}}}$
$= \mathtt{DIV} ; \mathbb{T}$	$= \overline{\mathcal{D}_{\mathrm{s}}^{\mathsf{T}} ; \overline{\mathbb{I}}} ; \mathbb{T}$
$= \mathtt{ZPE} ; e ; \mathbb{T}$	$= \mathtt{inf}_{n \geq 2} \overline{k_n^{\mathsf{T}} ; \overline{\mathbb{I}}} ; e ; \mathbb{T}$
$= \mathtt{ZPE} ; \overline{z ; \mathbb{T}}$	$= \mathtt{inf}_{n \geq 2} \overline{k_n^{\mathsf{T}} ; \overline{\mathbb{I}}} ; \overline{z ; \mathbb{T}}$
$= \mathtt{MTI} ; e ; \mathbb{T}$	$= \mathtt{inf}_{n \geq 2} \left[\mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^n \cup \overline{k_n^{\mathsf{T}} ; \overline{\mathbb{I}}} \right] ; e ; \mathbb{T}$
$= \mathtt{MIQ} ; e ; \mathbb{T}$	$= \overline{\mathbb{M}^{\mathsf{T}} ; (\overline{\mathbb{I}} \circledcirc \overline{\mathbb{I}})} ; e ; \mathbb{T}$
$\wp_2 ; \mathbb{T} = \mathtt{SYD} ; e ; \mathbb{T}$	$= \mathtt{syq}(\mathcal{D}_{\mathrm{s}}, e ; \mathbb{T})$
$= \mathtt{HSD}$	$= \left[\mathcal{D}_{\mathrm{s}} \cap \overline{\mathcal{D}_{\mathrm{s}} ; \mathcal{D}_{\mathrm{s}}} \right]^{\mathsf{T}} ; e ; \mathbb{T}$
$= \mathtt{ZPE} ; e ; \mathbb{T} \cap \overline{e ; \mathbb{T}}$	$= \mathtt{inf}_{n \geq 2} \overline{k_n^{\mathsf{T}} ; \overline{\mathbb{I}}} ; e ; \mathbb{T} \cap \overline{e ; \mathbb{T}}$
$= \mathtt{DIV} ; e ; \mathbb{T} \cap \overline{e ; \mathbb{T}}$	$= \overline{\mathcal{D}_{\mathrm{s}}^{\mathsf{T}} ; \overline{e ; \mathbb{T}}} \cap \overline{e ; \mathbb{T}}$
$\wp ; \mathbb{T} = \mathtt{SYD} ; e ; \mathbb{T} \cap \overline{\mathbb{S}^{\mathsf{T}2} ; z ; \mathbb{T}}$	$= \mathtt{syq}(\mathcal{D}_{\mathrm{s}}, e ; \mathbb{T}) \cap \overline{\mathbb{S}^{\mathsf{T}2} ; z ; \mathbb{T}}$
$= \mathtt{HSD} \cap \overline{\mathbb{S}^{\mathsf{T}2} ; z ; \mathbb{T}}$	$= \left[\mathcal{D}_{\mathrm{s}} \cap \overline{\mathcal{D}_{\mathrm{s}} ; \mathcal{D}_{\mathrm{s}}} \right]^{\mathsf{T}} ; e ; \mathbb{T} \cap \overline{\mathbb{S}^{\mathsf{T}2} ; z ; \mathbb{T}}$
$= \mathtt{ZPE} ; e ; \mathbb{T} \cap \overline{e ; \mathbb{T}} \cap \overline{\mathbb{S}^{\mathsf{T}2} ; z ; \mathbb{T}}$	$= \mathtt{inf}_{n \geq 2} \overline{k_n^{\mathsf{T}} ; \overline{\mathbb{I}}} ; e ; \mathbb{T} \cap \overline{e ; \mathbb{T}} \cap \overline{\mathbb{S}^{\mathsf{T}2} ; z ; \mathbb{T}}$
$= \mathtt{DIV} ; e ; \mathbb{T} \cap \overline{e ; \mathbb{T}} \cap \overline{\mathbb{S}^{\mathsf{T}2} ; z ; \mathbb{T}}$	$= \overline{\mathcal{D}_{\mathrm{s}}^{\mathsf{T}} ; \overline{e ; \mathbb{T}}} \cap \overline{e ; \mathbb{T}} \cap \overline{\mathbb{S}^{\mathsf{T}2} ; z ; \mathbb{T}}$

9.4 Relations between primes, exponents and distances

Odd primes, i.e. primes starting at 3, have been defined in Def. 9.3.3 as vector

$\wp ; \mathbb{T} : \mathbb{N} \longrightarrow \mathbb{1}$

and correspondingly as diagonal $\wp \subseteq \mathbb{I}$. This allows to prove the following:

9.4.1 Proposition. Starting from $S^{\oplus} := \wp ; \mathbb{S}^{+} ; \wp$, one obtains

i) $\overline{S^{\oplus \mathsf{T}}} = \wp ; \mathbb{S}^{*} ; \wp \cup \overline{\wp ; \mathbb{T} ; \wp}$ $\qquad \wp ; \overline{S^{\oplus \mathsf{T}}} ; \wp = \wp ; \mathbb{S}^{*} ; \wp$

ii) $S^{\oplus}$ is a so-called intervalorder — by definition a strictorder satisfying the Ferrers condition, that is, however, neither semi-transitive nor negatively transitive. It thus satisfies

$S^{\oplus} \subseteq \overline{\mathbb{I}}$ $\qquad S^{\oplus} ; S^{\oplus} \subseteq S^{\oplus}$ $\qquad S^{\oplus} ; \overline{S^{\oplus \mathsf{T}}} ; S^{\oplus} \subseteq S^{\oplus}$

and possibly violates

$S^{\oplus} ; S^{\oplus} ; \overline{S^{\oplus \mathsf{T}}} \subseteq S^{\oplus}$ $\qquad \overline{S^{\oplus}} ; \overline{S^{\oplus}} \subseteq \overline{S^{\oplus}}$.

iii) $S^{\oplus}$ is discrete.

iv) $S^{\oplus}$ admits a Hasse relation, which we are going to call the next prime relation thus

$\mathtt{NP} := \mathtt{Hasse}(S^{\oplus}) = S^{\oplus} \cap \overline{S^{\oplus} ; S^{\oplus}}$.

It satisfies $\mathtt{NP}^{+} = S^{\oplus}$.

v) $\mathtt{NP} = \wp ; \left[\mathbb{S}^{+} \cap \overline{\mathbb{S}^{+} ; \wp ; \mathbb{S}^{+}} \right] ; \wp \subseteq \mathbb{S}^{+}$

vi) $\mathtt{NP} = \mathtt{NP} ; \wp = \wp ; \mathtt{NP}$

vii) $\mathtt{NP}$ is univalent as well as injective

Proof: i) We use that $\wp \subseteq \mathbb{I}$ is univalent and injective and apply Prop. 2.1.1.vi:

$\overline{S^{\oplus\mathsf{T}}} = \overline{\wp ; \mathbb{S}^{\mathsf{T}+} ; \wp}$ by definition
$= \wp ; \overline{\mathbb{S}^{\mathsf{T}+}} ; \wp \cup \overline{\wp ; \mathbb{T} ; \wp} = \wp ; \mathbb{S}^* ; \wp \cup \overline{\wp ; \mathbb{T} ; \wp}$ since $\mathbb{S}^{\mathsf{T}+}$ is a linear strictorder

---

$\wp ; \overline{S^{\oplus\mathsf{T}}} ; \wp = \wp ; \left[\wp ; \mathbb{S}^* ; \wp \cup \overline{\wp ; \mathbb{T} ; \wp} \right] ; \wp$ see before
$= \wp ; \mathbb{S}^* ; \wp \cup \wp ; \overline{\wp ; \mathbb{T} ; \wp} ; \wp = \wp ; \mathbb{S}^* ; \wp \cup \mathbb{\perp\!\!\!\perp}$ again Prop. 2.1.1.vi

---

ii) Irreflexivity and transitivity are obvious.

$S^{\oplus} ; \overline{S^{\oplus\mathsf{T}}} ; S^{\oplus} = \wp ; \mathbb{S}^+ ; \wp ; \overline{S^{\oplus\mathsf{T}}} ; \wp ; \mathbb{S}^+ ; \wp = \wp ; \mathbb{S}^+ ; \wp ; \mathbb{S}^* ; \wp ; \mathbb{S}^+ ; \wp$ applying (i)
$\subseteq \wp ; \mathbb{S}^+ ; \mathbb{S}^* ; \mathbb{S}^+ ; \wp$ since $\wp \subseteq \mathbb{I}$
$\subseteq \wp ; \mathbb{S}^+ ; \wp = S^{\oplus}$

For the other two statements, one will easily find counter-examples considering Fig. 9.4.1.

---

iii) Being contained in the discrete relation $\mathbb{S}$, also $S^{\oplus}$ is discrete; [11, 12, Prop. 6.5.5].

---

iv) The, thus, discrete strictorder $S^{\oplus}$ admits a Hasse relation due to [11, 12, Prop. 6.5.10].

---

v) With (iv), the existence of a Hasse relation has been shown. Here another representation is given.

$\wp ; \left[\mathbb{S}^+ \cap \overline{\mathbb{S}^+ ; \wp ; \mathbb{S}^+} \right] ; \wp$
$= \wp ; \mathbb{S}^+ ; \wp \cap \wp ; \overline{\mathbb{S}^+ ; \wp ; \mathbb{S}^+} ; \wp$ since $\wp$ is univalent and injective
$= \wp ; \mathbb{S}^+ ; \wp \cap \left[\wp ; \overline{\overline{\mathbb{S}^+ ; \wp ; \mathbb{S}^+}} ; \wp \cap \wp ; \mathbb{T} ; \wp \right]$ Prop. 2.1.1.vi
$= \wp ; \mathbb{S}^+ ; \wp \cap \overline{\wp ; \mathbb{S}^+ ; \wp ; \wp ; \mathbb{S}^+ ; \wp} = \mathtt{Hasse}(\wp ; \mathbb{S}^+ ; \wp) = \mathtt{Hasse}(\mathbb{S}^{\oplus}) = \mathtt{NP}$

---

vi) This follows immediately from (v).

---

vii) We will use Prop. 2.1.1.vi several times when reasoning that $A \cap B ; \wp = (A \cap B) ; \wp$.

$\mathtt{NP}^{\mathsf{T}} ; \mathtt{NP} \subseteq \mathbb{I} \quad \Longleftrightarrow \quad \mathtt{NP} ; \overline{\mathbb{I}} = \mathtt{NP} ; \left[\mathbb{S}^+ \cup \mathbb{S}^{\mathsf{T}+} \right] \subseteq \overline{S^{\oplus}} \cup S^{\oplus} ; S^{\oplus} = \overline{\mathtt{NP}}$
$\Longleftrightarrow \quad \left[\mathtt{NP} ; \mathbb{S}^+ \cup \mathtt{NP} ; \mathbb{S}^{\mathsf{T}+} \right] \cap S^{\oplus} \subseteq S^{\oplus} ; S^{\oplus}$

Now we check using the Dedekind rule for the first part:

$\mathtt{NP} ; \mathbb{S}^+ \cap S^{\oplus} \subseteq \left(\mathtt{NP} \cap S^{\oplus} ; \mathbb{S}^{\mathsf{T}+} \right) ; \left(\mathbb{S}^+ \cap \mathtt{NP}^{\mathsf{T}} ; S^{\oplus} \right) \subseteq S^{\oplus} ; \left(\mathbb{S}^+ \cap \mathtt{NP}^{\mathsf{T}} ; S^{\oplus} \right)$
$= S^{\oplus} ; \wp ; \left(\mathbb{S}^+ \cap \mathtt{NP}^{\mathsf{T}} ; S^{\oplus} ; \wp \right) = S^{\oplus} ; \wp ; \left(\mathbb{S}^+ \cap \mathtt{NP}^{\mathsf{T}} ; S^{\oplus} \right) ; \wp \subseteq S^{\oplus} ; \wp ; \mathbb{S}^+ ; \wp = S^{\oplus} ; S^{\oplus}$

and for the second

$\mathtt{NP} ; \mathbb{S}^{\mathsf{T}+} \cap S^{\oplus} = \mathtt{NP} ; \wp ; \mathbb{S}^{\mathsf{T}+} \cap S^{\oplus} ; \wp = \mathtt{NP} ; \wp ; \mathbb{S}^{\mathsf{T}+} ; \wp \cap S^{\oplus} ; \wp = \mathtt{NP} ; S^{\oplus\mathsf{T}} \cap S^{\oplus}$
$\subseteq \left(\mathtt{NP} \cap S^{\oplus} ; S^{\oplus} \right) ; \left(S^{\oplus\mathsf{T}} \cap \mathtt{NP}^{\mathsf{T}} ; S^{\oplus} \right) \subseteq \left(\overline{S^{\oplus} ; S^{\oplus}} \cap S^{\oplus} ; S^{\oplus} \right) ; \left(\dots \right) = \mathbb{\perp\!\!\!\perp}$

Injectivity:

$\mathtt{NP} ; \mathtt{NP}^{\mathsf{T}} \subseteq \mathbb{I} \quad \Longleftrightarrow \quad \overline{\mathbb{I}} ; \mathtt{NP} = \left[\mathbb{S}^+ \cup \mathbb{S}^{\mathsf{T}+} \right] ; \mathtt{NP} \subseteq \overline{S^{\oplus}} \cup S^{\oplus} ; S^{\oplus} = \overline{\mathtt{NP}}$
$\Longleftrightarrow \quad \left[\mathbb{S}^+ ; \mathtt{NP} \cup \mathbb{S}^{\mathsf{T}+} ; \mathtt{NP} \right] \cap S^{\oplus} \subseteq S^{\oplus} ; S^{\oplus}$

Now again using the Dedekind rule:

$\mathbb{S}^{+};\mathtt{NP} \cap S^{\oplus} \subseteq (\mathbb{S}^{+} \cap S^{\oplus};\mathtt{NP}^{\mathsf{T}});(\mathtt{NP} \cap \mathbb{S}^{\mathsf{T}+};S^{\oplus}) = (\mathbb{S}^{+} \cap \wp;S^{\oplus};\mathtt{NP}^{\mathsf{T}};\wp);(\mathtt{NP} \cap \mathbb{S}^{\mathsf{T}+};S^{\oplus})$ see (vi)
$= (\wp;\mathbb{S}^{+};\wp \cap \wp;S^{\oplus};\mathtt{NP}^{\mathsf{T}};\wp);(\mathtt{NP} \cap \mathbb{S}^{\mathsf{T}+};S^{\oplus})$
$\subseteq S^{\oplus};S^{\oplus}$

$\mathbb{S}^{\mathsf{T}+};\mathtt{NP} \cap S^{\oplus} \subseteq (\mathbb{S}^{\mathsf{T}+} \cap S^{\oplus};\mathtt{NP}^{\mathsf{T}});(\mathtt{NP} \cap \mathbb{S}^{+};S^{\oplus})$
$= (\ldots);(S^{\oplus} \cap \overline{S^{\oplus};S^{\oplus}} \cap S^{\oplus};S^{\oplus}) = \mathbb{\bot}$ following (iv) □

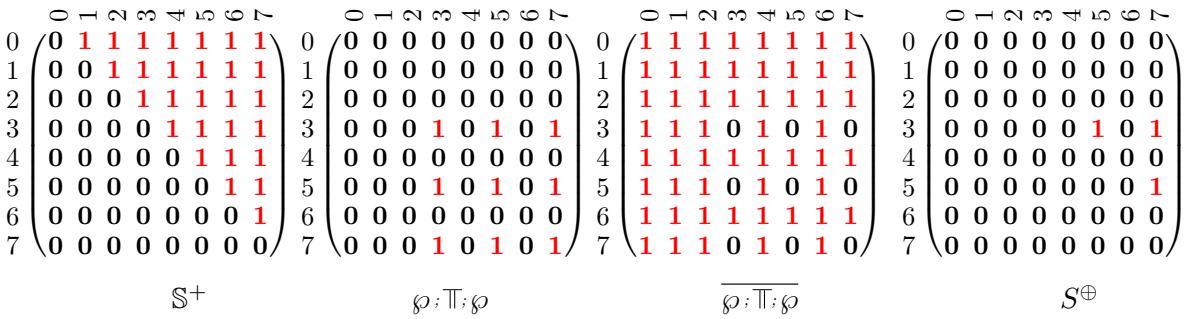


Fig. 9.4.1 Illustrating Prop. 9.4.1.i

A remark seems necessary to the not so well-known concepts of a non-transitive part of a homogeneous relation and its Hasse relation. The non-transitive part expressed by the term $\mathtt{Nontrans}(R) = R \cap \overline{R;R}$ can — of course — always be formed. To qualify it as the Hasse relation of R, the equality $\left[\mathtt{Hasse}(R)\right]^{+} = R$ has to be satisfied in addition; [11, 12, Chapter 6].

So far, all this has happened inside $\mathbb{N}$. Now we take also $\mathbb{P}$ into consideration, its connection with $\mathbb{N}$ as defined in Def. 9.3.3

$\iota_{\wp;\mathbb{T}} = \mathtt{Inject}(\wp;\mathbb{T}) : \mathbb{P} \longrightarrow \mathbb{N}$,

and study the ordering on the thus extruded set $\mathbb{P}$ of odd primes. Looking at Fig. 9.4.3, it is pictorially obvious that there will exist "a successor function $\mathbb{S}_{\mathbb{P}}$" on $\mathbb{P}$ as well. We are going to formally define it and to check how it behaves. Our expectation is that it subsumes under the definition of $\mathbb{S}$ in Def. 5.2.1. In addition, we will look for the relation-algebraic connections of $\mathbb{S}$ with $\mathbb{S}_{\mathbb{P}}$.

9.4.2 Definition. We define the successor relation on the extruded primes together with its zero, as well as the notation for any thus extruded odd prime p out of the naturals:

i) $\mathbb{S}_{\mathbb{P}} := \iota_{\wp;\mathbb{T}};\mathtt{NP};\iota^{\mathsf{T}}_{\wp;\mathbb{T}} : \mathbb{P} \longrightarrow \mathbb{P}$

ii) $z_{\mathbb{P}} := \overline{\mathbb{S}^{\mathsf{T}}_{\mathbb{P}};\mathbb{T}}$

iii) $[p^{\rightarrow}] := \iota_{\wp;\mathbb{T}};\mathbb{S}^{\mathsf{T}p};z;\mathbb{T} : \mathbb{P} \longrightarrow \mathbb{1}$ □

Next we study the properties of these constructs.

9.4.3 Proposition. i) $\mathbb{S}^{+}_{\mathbb{P}}$ is a linear strictorder. $z_{\mathbb{P}}$ and $[p^{\rightarrow}]$ are points.

$\mathbb{S}_{\mathbb{P}}^{+} = \iota_{\wp;\mathbb{T}}; \mathtt{NP}^{+}; \iota_{\wp;\mathbb{T}}^{\mathsf{T}} = \iota_{\wp;\mathbb{T}}; \mathbb{S}^{+}; \iota_{\wp;\mathbb{T}}^{\mathsf{T}}$ $\qquad$ $\mathbb{S}_{\mathbb{P}}^{*} = \iota_{\wp;\mathbb{T}}; \mathtt{NP}^{*}; \iota_{\wp;\mathbb{T}}^{\mathsf{T}} = \iota_{\wp;\mathbb{T}}; \mathbb{S}^{*}; \iota_{\wp;\mathbb{T}}^{\mathsf{T}}$

ii) $\mathbb{S}_{\mathbb{P}}$ is univalent and injective

iii) $\mathtt{NP} = \iota_{\wp;\mathbb{T}}^{\mathsf{T}}; \mathbb{S}_{\mathbb{P}}; \iota_{\wp;\mathbb{T}}$

iv) $\mathbb{S}_{\mathbb{P}}; \iota_{\wp;\mathbb{T}} = \iota_{\wp;\mathbb{T}}; \mathtt{NP} \subseteq \iota_{\wp;\mathbb{T}}; S^{\oplus} \subseteq \iota_{\wp;\mathbb{T}}; \mathbb{S}^{+}$

$\mathbb{S}_{\mathbb{P}}^{n}; \iota_{\wp;\mathbb{T}} = \iota_{\wp;\mathbb{T}}; \mathtt{NP}^{n}$ for $n \geq 0$

v) $\mathbb{S}_{\mathbb{P}}$ is regressively finite and discrete.

vi) $\mathbb{S}_{\mathbb{P}}$ enjoys all the properties demanded in Def. 5.1.1 for $\mathbb{S}$, namely, that it is injective, univalent, non-empty, regressively finite.

vii) $z_{\mathbb{P}} ; \mathbb{T} \subseteq \mathbb{S}_{\mathbb{P}}^{*}$ $\qquad$ $\mathbb{T}_{\mathbb{P}} = \mathtt{sup}_{i \geq 0}\, \mathbb{S}_{\mathbb{P}}^{\mathsf{T}^i}; z_{\mathbb{P}}; \mathbb{T}$

Proof: i) $\mathbb{S}_{\mathbb{P}}^{+}$ is by definition transitive. To show that it is in addition irreflexive and semi-connex, we begin as follows:

$\mathbb{S}_{\mathbb{P}}^{+} = \left[\iota_{\wp;\mathbb{T}}; \mathtt{NP}; \iota_{\wp;\mathbb{T}}^{\mathsf{T}}\right]^{+} = \iota_{\wp;\mathbb{T}}; \mathtt{NP}^{+}; \iota_{\wp;\mathbb{T}}^{\mathsf{T}}$ due to $\mathtt{NP}; \iota_{\wp;\mathbb{T}}^{\mathsf{T}}; \iota_{\wp;\mathbb{T}} = \mathtt{NP}; \wp = \mathtt{NP}$ and Prop. 9.4.1.vi
$= \iota_{\wp;\mathbb{T}}; S^{\oplus}; \iota_{\wp;\mathbb{T}}^{\mathsf{T}}$ due to Prop. 9.4.1.iv
$= \iota_{\wp;\mathbb{T}}; \wp ; \mathbb{S}^{+}; \wp; \iota_{\wp;\mathbb{T}}^{\mathsf{T}}$ as defined in Prop. 9.4.1
$= \iota_{\wp;\mathbb{T}}; \mathbb{S}^{+}; \iota_{\wp;\mathbb{T}}^{\mathsf{T}}$ since $\iota_{\wp;\mathbb{T}}$ extrudes $\wp ; \mathbb{T}$

Now irreflexivity is shown together with semi-connexity:

$\mathbb{S}_{\mathbb{P}}^{+} \cup \left[\mathbb{S}_{\mathbb{P}}^{+}\right]^{\mathsf{T}} = \iota_{\wp;\mathbb{T}}; \mathbb{S}^{+}; \iota_{\wp;\mathbb{T}}^{\mathsf{T}} \cup \left[\iota_{\wp;\mathbb{T}}; \mathbb{S}^{+}; \iota_{\wp;\mathbb{T}}^{\mathsf{T}}\right]^{\mathsf{T}} = \iota_{\wp;\mathbb{T}}; \mathbb{S}^{+}; \iota_{\wp;\mathbb{T}}^{\mathsf{T}} \cup \iota_{\wp;\mathbb{T}}; \left[\mathbb{S}^{+}\right]^{\mathsf{T}}; \iota_{\wp;\mathbb{T}}^{\mathsf{T}}$
$= \iota_{\wp;\mathbb{T}}; \left\{\mathbb{S}^{+} \cup \left[\mathbb{S}^{+}\right]^{\mathsf{T}}\right\}; \iota_{\wp;\mathbb{T}}^{\mathsf{T}} = \iota_{\wp;\mathbb{T}}; \overline{\mathbb{I}}; \iota_{\wp;\mathbb{T}}^{\mathsf{T}}$ because $\mathbb{S}^{+}$ is a linear strictorder
$= \overline{\iota_{\wp;\mathbb{T}}; \iota_{\wp;\mathbb{T}}^{\mathsf{T}}}$ since $\iota_{\wp;\mathbb{T}}$ is a map
$= \overline{\mathbb{I}_{\mathbb{P}}}$

The relation $z_{\mathbb{P}}$ is certainly row-constant and $\neq \mathbb{L}$. Moreover,

$z_{\mathbb{P}}; z_{\mathbb{P}}^{\mathsf{T}} = \overline{\mathbb{S}_{\mathbb{P}}^{\mathsf{T}}; \mathbb{T}} ; \overline{\mathbb{S}_{\mathbb{P}}^{\mathsf{T}}; \mathbb{T}}^{\mathsf{T}} \subseteq \mathbb{I}_{\mathbb{P}}$

because via the Schröder rule and $\mathbb{S}_{\mathbb{P}}^{+}$ being a linear strictorder

$\iff \quad \overline{\mathbb{I}_{\mathbb{P}}} ; \overline{\mathbb{S}_{\mathbb{P}}^{\mathsf{T}}; \mathbb{T}} = \left[\mathbb{S}_{\mathbb{P}}^{+\mathsf{T}} \cup \mathbb{S}_{\mathbb{P}}^{+}\right]; \overline{\mathbb{S}_{\mathbb{P}}^{\mathsf{T}}; \mathbb{T}} \subseteq \mathbb{S}_{\mathbb{P}}^{\mathsf{T}}; \mathbb{T}$

Now we consider the two cases separately. The first part is immediate. The second results from $\mathbb{S}_{\mathbb{P}}; \overline{\mathbb{S}_{\mathbb{P}}^{\mathsf{T}}; \mathbb{T}} \subseteq \mathbb{L}_{\mathbb{P}}$.

By definition $\mathbb{S}^{\mathsf{T}^p}; z ; \mathbb{T}$ is a point and so is $[p^{\rightarrow}]$, since injectivity, e.g., follows from

$[p^{\rightarrow}]; [p^{\rightarrow}]^{\mathsf{T}} = \iota_{\wp;\mathbb{T}}; \mathbb{S}^{\mathsf{T}^p}; z ; \mathbb{T} ; z^{\mathsf{T}}; \mathbb{S}^{p}; \iota_{\wp;\mathbb{T}}^{\mathsf{T}}$ by definition
$\subseteq \iota_{\wp;\mathbb{T}}; \mathbb{S}^{\mathsf{T}^p}; \mathbb{S}^{p}; \iota_{\wp;\mathbb{T}}^{\mathsf{T}}$ because $z ; \mathbb{T}$ as a point is injective
$\subseteq \iota_{\wp;\mathbb{T}}; \iota_{\wp;\mathbb{T}}^{\mathsf{T}}$ because $\mathbb{S}$ is univalent
$= \mathbb{I}_{\mathbb{P}}$ because the extrusion $\iota_{\wp;\mathbb{T}}$ is an injective map

---

ii) This follows immediately from the corresponding properties of $\mathtt{NP}$ in Prop. 9.4.1.vii.

---

iii) $\iota_{\wp;\mathbb{T}}^{\mathsf{T}}; \mathbb{S}_{\mathbb{P}}; \iota_{\wp;\mathbb{T}} = \iota_{\wp;\mathbb{T}}^{\mathsf{T}} \iota_{\wp;\mathbb{T}}; \mathtt{NP}; \iota_{\wp;\mathbb{T}}^{\mathsf{T}}; \iota_{\wp;\mathbb{T}} = \wp; \mathtt{NP}; \wp = \mathtt{NP}$

---

iv) $\mathbb{S}_{\mathbb{P}} ; \iota_{\wp,\mathbb{T}} = \iota_{\wp,\mathbb{T}} ; \mathtt{NP} ; \iota^{\mathsf{T}}_{\wp,\mathbb{T}} ; \iota_{\wp,\mathbb{T}} = \iota_{\wp,\mathbb{T}} ; \mathtt{NP} ; \wp = \iota_{\wp,\mathbb{T}} ; \mathtt{NP} \subseteq \iota_{\wp,\mathbb{T}} ; \mathbb{S}^{\oplus} = \iota_{\wp,\mathbb{T}} ; \wp ; \mathbb{S}^{+} ; \wp \subseteq \iota_{\wp,\mathbb{T}} ; \mathbb{S}^{+}$

The next result follows simply by induction.

---

v) According to (iv) the relation $\mathbb{S}_{\mathbb{P}}$ is mapped homomorphically into the regressively finite $\mathbb{S}^{+}$. Following Prop. 4.1.4, it will be regressively finite itself.

According to [11, 12, Def. 6.5.4], the relation $\mathbb{S}_{\mathbb{P}}$ is discrete by definition, if

$$\mathtt{sup}\left\{X \mid X \subseteq \tau_{\mathbb{S}_{\mathbb{P}}}(X)\right\} = \mathbb{1}, \quad \text{where} \quad \tau_{\mathbb{S}_{\mathbb{P}}}(X) := \mathbb{S}_{\mathbb{P}} \cap \left[\mathbb{S}_{\mathbb{P}} ; X \cup X ; \mathbb{S}_{\mathbb{P}}\right].$$

To check this, we consider

$$\begin{aligned}
&\iota^{\mathsf{T}}_{\wp,\mathbb{T}} ; X ; \iota_{\wp,\mathbb{T}} \subseteq \iota^{\mathsf{T}}_{\wp,\mathbb{T}} ; \tau_{\mathbb{S}_{\mathbb{P}}}(X) ; \iota_{\wp,\mathbb{T}} \\
&= \iota^{\mathsf{T}}_{\wp,\mathbb{T}} ; \left\{\mathbb{S}_{\mathbb{P}} \cap \left[\mathbb{S}_{\mathbb{P}} ; X \cup X ; \mathbb{S}_{\mathbb{P}}\right]\right\} ; \iota_{\wp,\mathbb{T}} = \iota^{\mathsf{T}}_{\wp,\mathbb{T}} ; \mathbb{S}_{\mathbb{P}} ; \iota_{\wp,\mathbb{T}} \cap \iota^{\mathsf{T}}_{\wp,\mathbb{T}} ; \left[\mathbb{S}_{\mathbb{P}} ; X \cup X ; \mathbb{S}_{\mathbb{P}}\right] ; \iota_{\wp,\mathbb{T}} \\
&\subseteq S^{\oplus} \cap \left[\iota^{\mathsf{T}}_{\wp,\mathbb{T}} ; \mathbb{S}_{\mathbb{P}} ; X ; \iota_{\wp,\mathbb{T}} \cup \iota^{\mathsf{T}}_{\wp,\mathbb{T}} ; X ; \mathbb{S}_{\mathbb{P}} ; \iota_{\wp,\mathbb{T}}\right] \quad \text{due to (iv) since } \iota_{\wp,\mathbb{T}} \text{ is univalent} \\
&\subseteq S^{\oplus} \cap \left[\iota^{\mathsf{T}}_{\wp,\mathbb{T}} ; \mathbb{S}_{\mathbb{P}} ; \iota_{\wp,\mathbb{T}} ; \iota^{\mathsf{T}}_{\wp,\mathbb{T}} ; X ; \iota_{\wp,\mathbb{T}} \cup \iota^{\mathsf{T}}_{\wp,\mathbb{T}} ; X ; \iota_{\wp,\mathbb{T}} ; \iota^{\mathsf{T}}_{\wp,\mathbb{T}} ; \mathbb{S}_{\mathbb{P}} ; \iota_{\wp,\mathbb{T}}\right] \quad \text{since } \iota_{\wp,\mathbb{T}} \text{ is an injective map} \\
&\subseteq S^{\oplus} \cap \left[S^{\oplus} ; \iota^{\mathsf{T}}_{\wp,\mathbb{T}} ; X ; \iota_{\wp,\mathbb{T}} \cup \iota^{\mathsf{T}}_{\wp,\mathbb{T}} ; X ; \iota_{\wp,\mathbb{T}} ; S^{\oplus}\right] \quad \text{as before}
\end{aligned}$$

and find out that $Y := \iota^{\mathsf{T}}_{\wp,\mathbb{T}} ; X ; \iota_{\wp,\mathbb{T}}$ satisfies

$$Y \subseteq S^{\oplus} \cap \left[S^{\oplus} ; Y \cup Y ; S^{\oplus}\right],$$

i.e. the corresponding condition for $S^{\oplus}$ which is already known to be discrete, so that $Y \subseteq \mathbb{1}$.

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vi) is now more or less obvious.

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vii) This is written down simply as a consequence of (vi). □

$$\mathbb{S}^{0}_{\mathbb{P}};\iota_{\wp;\mathbb{T}} = \begin{array}{c} \\ 3\to \\ 5\to \\ 7\to \\ 11\to \\ 13\to \\ 17\to \end{array}\!\!\begin{array}{c} \begin{array}{cccccccccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \end{array} \\ \begin{pmatrix} 0&0&0&\mathbf{1}&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&\mathbf{1}&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&\mathbf{1}&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&\mathbf{1}&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&\mathbf{1}&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\mathbf{1} \end{pmatrix} \end{array} = \iota_{\wp;\mathbb{T}};\mathtt{NP}^0$$

$$\mathbb{S}^{1}_{\mathbb{P}};\iota_{\wp;\mathbb{T}} = \begin{array}{c} \\ 3\to \\ 5\to \\ 7\to \\ 11\to \\ 13\to \\ 17\to \end{array}\!\!\begin{array}{c} \begin{array}{cccccccccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \end{array} \\ \begin{pmatrix} 0&0&0&0&0&\mathbf{1}&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&\mathbf{1}&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&\mathbf{1}&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&\mathbf{1}&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\mathbf{1} \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \end{array} = \iota_{\wp;\mathbb{T}};\mathtt{NP}^1$$

$$\mathbb{S}^{2}_{\mathbb{P}};\iota_{\wp;\mathbb{T}} = \begin{array}{c} \\ 3\to \\ 5\to \\ 7\to \\ 11\to \\ 13\to \\ 17\to \end{array}\!\!\begin{array}{c} \begin{array}{cccccccccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \end{array} \\ \begin{pmatrix} 0&0&0&0&0&0&0&\mathbf{1}&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&\mathbf{1}&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&\mathbf{1}&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\mathbf{1} \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \end{array} = \iota_{\wp;\mathbb{T}};\mathtt{NP}^2$$

$$\mathbb{S}^{3}_{\mathbb{P}};\iota_{\wp;\mathbb{T}} = \begin{array}{c} \\ 3\to \\ 5\to \\ 7\to \\ 11\to \\ 13\to \\ 17\to \end{array}\!\!\begin{array}{c} \begin{array}{cccccccccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \end{array} \\ \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&\mathbf{1}&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&\mathbf{1}&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\mathbf{1} \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \end{array} = \iota_{\wp;\mathbb{T}};\mathtt{NP}^3$$

Fig. 9.4.2 Illustrating successors of primes

Several other properties of $\mathbb{S}_{\mathbb{P}}$ in similarity to $\mathbb{S}$ follow immediately:

9.4.4 Proposition. Let be given an odd prime p inside the present limitation, corresponding to a point $\mathbb{S}^{\intercal p};z;\mathbb{T}$, together with its extruded version $[p\!\!\rightarrow]$. Then

i) $\iota^{\intercal}_{\wp;\mathbb{T}};z_{\mathbb{P}};\mathbb{T} = \mathbb{S}^{\intercal 3};z;\mathbb{T} \qquad z_{\mathbb{P}};\mathbb{T} = [3\!\!\rightarrow] \qquad \iota^{\intercal}_{\wp;\mathbb{T}};[p\!\!\rightarrow] = \mathbb{S}^{\intercal p};z;\mathbb{T}$

ii) $\mathbb{S}^{\intercal}_{\mathbb{P}};\mathbb{S}_{\mathbb{P}} = \mathbb{I}_{\mathbb{P}} \cap \mathbb{S}^{\intercal}_{\mathbb{P}};\mathbb{T} = \mathbb{I}_{\mathbb{P}} \cap \overline{[3\!\!\rightarrow]}$

iii) For $i \geq 0$ always $\mathbb{S}^{\intercal *}_{\mathbb{P}};\mathbb{S}^{\intercal i}_{\mathbb{P}};z_{\mathbb{P}};\mathbb{T} = \mathbb{S}^{\intercal i}_{\mathbb{P}};\mathbb{T}$.

iv) For $k \geq 0$ always $\mathtt{NP}^{\intercal k};\mathbb{S}^{\intercal 3};z;\mathbb{T} = \iota^{\intercal}_{\wp;\mathbb{T}};\mathbb{S}^{\intercal k}_{\mathbb{P}};z_{\mathbb{P}};\mathbb{T}$ and $\iota_{\wp;\mathbb{T}};\mathtt{NP}^{\intercal k};\mathbb{S}^{\intercal 3};z;\mathbb{T} = \mathbb{S}^{\intercal k}_{\mathbb{P}};z_{\mathbb{P}};\mathbb{T}$

v) $\mathbb{S}^{\intercal n};z;\mathbb{T} = \mathbb{\perp} \implies \mathbb{S}^{\intercal n}_{\mathbb{P}};z_{\mathbb{P}};\mathbb{T} = \mathbb{\perp}$ and $\mathbb{S}^{\intercal n}_{\mathbb{P}} = \mathbb{\perp}$

Proof: i) $\iota^{\intercal}_{\wp;\mathbb{T}};z_{\mathbb{P}};\mathbb{T} = \iota^{\intercal}_{\wp;\mathbb{T}};\overline{\mathbb{S}^{\intercal}_{\mathbb{P}};\mathbb{T}} = \iota^{\intercal}_{\wp;\mathbb{T}};\overline{\iota_{\wp;\mathbb{T}};\mathtt{NP}^{\intercal};\iota^{\intercal}_{\wp;\mathbb{T}};\mathbb{T}}$ Def. 9.4.2
$= \iota^{\intercal}_{\wp;\mathbb{T}};\iota_{\wp;\mathbb{T}};\overline{\mathtt{NP}^{\intercal};\iota^{\intercal}_{\wp;\mathbb{T}};\mathbb{T}}$ because $\iota_{\wp;\mathbb{T}}$ is a map
$= \wp;\overline{\mathtt{NP}^{\intercal};\wp;\mathbb{T}}$ since $\iota_{\wp;\mathbb{T}}$ extrudes $\wp;\mathbb{T}$
$\supseteq \wp;\overline{\mathbb{S}^{+\intercal};\wp;\mathbb{T}}$ due to Prop. 9.4.1.v
$= \wp;\overline{\mathbb{S}^{+\intercal};\mathbb{S}^{\intercal 3};z;\mathbb{T}}$ since for $n \geq 3$ always $\mathbb{S}^{+\intercal};\mathbb{S}^{\intercal n};z;\mathbb{T} \subseteq \mathbb{S}^{+\intercal};\mathbb{S}^{\intercal 3};z;\mathbb{T}$
$= \wp;\big[z;\mathbb{T} \cup e;\mathbb{T} \cup \mathbb{S}^{\intercal 2};z;\mathbb{T} \cup \mathbb{S}^{\intercal 3};z;\mathbb{T}\big] = \mathbb{S}^{\intercal 3};z;\mathbb{T}$

Equality follows from Prop. 2.2.2.i because $\iota^{\intercal}_{\wp;\mathbb{T}};z_{\mathbb{P}}$ is a point.

$[3^{\rightarrow}] = \iota_{\wp;\mathbb{T}}; \mathbb{S}^{\intercal 3}; z_{;}\mathbb{T}$ by definition
$= \iota_{\wp;\mathbb{T}}; \iota^{\intercal}_{\wp;\mathbb{T}}; z_{\mathbb{P}\,;}\mathbb{T}$ previous result
$= z_{\mathbb{P}\,;}\mathbb{T}$ since $\iota_{\wp;\mathbb{T}}$ is an injective map

---

$\iota^{\intercal}_{\wp;\mathbb{T}}[p^{\rightarrow}] = \iota^{\intercal}_{\wp;\mathbb{T}}; \iota_{\wp;\mathbb{T}}; \mathbb{S}^{\intercal p}; z_{;}\mathbb{T}$ by definition
$\subseteq \mathbb{S}^{\intercal p}; z_{;}\mathbb{T}$ because $\iota_{\wp;\mathbb{T}}$ is univalent

Recalling Prop. 2.2.2.i, we use that $\iota^{\intercal}_{\wp;\mathbb{T}}[p^{\rightarrow}] \neq \perp\!\!\!\perp$ and, thus, equal to the point $\mathbb{S}^{\intercal p}; z_{;}\mathbb{T}$.

---

ii) According to Prop. 9.4.3.vi, the relation $\mathbb{S}_{\mathbb{P}}$ behaves like $\mathbb{S}$ as explained in Def. 5.1.1. Therefore, not least following Def. 5.1.1.i.i

$\mathbb{S}^{\intercal}_{\mathbb{P}}; \mathbb{S}_{\mathbb{P}} = \mathbb{I}_{\mathbb{P}} \cap \mathbb{S}^{\intercal}_{\mathbb{P}}; \mathbb{T} = \mathbb{I}_{\mathbb{P}} \cap \overline{z_{\mathbb{P}\,;}\mathbb{T}} = \mathbb{I}_{\mathbb{P}} \cap \overline{[3^{\rightarrow}]}$

---

iii) The statement is simply what Prop. 5.2.1.ii.j says, however, now transferred into the present setting.

---

iv) We prove this by induction, starting with $k = 0$:

$\mathtt{NP}^{\intercal 0}; \mathbb{S}^{\intercal 3}; z_{;}\mathbb{T} = \iota^{\intercal}_{\wp;\mathbb{T}}; \mathbb{S}^{\intercal 0}_{\mathbb{P}}; z_{\mathbb{P}\,;}\mathbb{T}$

holds due to (i). Induction step $k \mapsto k+1$:

$\mathtt{NP}^{\intercal k+1}; \mathbb{S}^{\intercal 3}; z_{;}\mathbb{T} = \mathtt{NP}^{\intercal}; \mathtt{NP}^{\intercal k}; \mathbb{S}^{\intercal 3}; z_{;}\mathbb{T} = \mathtt{NP}^{\intercal}; \iota^{\intercal}_{\wp;\mathbb{T}}; \mathbb{S}^{\intercal k}_{\mathbb{P}}; z_{\mathbb{P};}\mathbb{T}$ hypothesis
$= \iota^{\intercal}_{\wp;\mathbb{T}}; \mathbb{S}^{\intercal}_{\mathbb{P}}; \iota_{\wp;\mathbb{T}}; \iota^{\intercal}_{\wp;\mathbb{T}}; \mathbb{S}^{\intercal k}_{\mathbb{P}}; z_{\mathbb{P};}\mathbb{T}$ following Prop. 9.4.3.iii
$= \iota^{\intercal}_{\wp;\mathbb{T}}; \mathbb{S}^{\intercal}_{\mathbb{P}}; \mathbb{S}^{\intercal k}_{\mathbb{P}}; z_{\mathbb{P};}\mathbb{T}$ since $\iota_{\wp;\mathbb{T}}$ is sn injective map
$= \iota^{\intercal}_{\wp;\mathbb{T}}; \mathbb{S}^{\intercal k+1}_{\mathbb{P}}; z_{\mathbb{P};}\mathbb{T}$

The second version results from $\iota_{\wp;\mathbb{T}}$ being an injective map.

---

v) We start with the observation that in the given situation with $\mathbb{S}$ regressively finite and have $\mathbb{S}^{\,n} = \perp\!\!\!\perp$ according to Prop. 5.2.1.ii.d. Then

$\mathbb{S}^{\intercal n}_{\mathbb{P}}; z_{\mathbb{P}\,;}\mathbb{T} = \left[\iota_{\wp;\mathbb{T}}; \mathtt{NP}^{\intercal}; \iota^{\intercal}_{\wp;\mathbb{T}}\right]^{n}; z_{\mathbb{P}\,;}\mathbb{T}$ Def. 9.4.2
$= \left[\iota_{\wp;\mathbb{T}}; \mathtt{NP}^{\intercal}; \iota^{\intercal}_{\wp;\mathbb{T}}\right]^{n}; \iota_{\wp;\mathbb{T}}; \mathbb{S}^{\intercal 3}; z_{;}\mathbb{T}$ due to (i) and Def. 9.4.2.iii
$\subseteq \iota_{\wp;\mathbb{T}}; \mathtt{NP}^{\intercal n}; \iota^{\intercal}_{\wp;\mathbb{T}}; \iota_{\wp;\mathbb{T}}; \mathbb{S}^{\intercal 3}; z_{;}\mathbb{T}$
$\subseteq \iota_{\wp;\mathbb{T}}; \mathbb{S}^{+\intercal n}; \mathbb{S}^{\intercal 3}; z_{;}\mathbb{T} = \iota_{\wp;\mathbb{T}}; \mathtt{sup}_{i \geq n+3}\, \mathbb{S}^{\intercal i}; z_{;}\mathbb{T}$ Prop. 9.4.1.v
$= \left(\iota_{\wp;\mathbb{T}}; \mathtt{sup}_{i \geq 3}\, \mathbb{S}^{\intercal i}\right); \mathbb{S}^{\intercal n}; z_{;}\mathbb{T} = \perp\!\!\!\perp$

The second result now follows also via Prop. 5.2.1.ii.d. □

To define it in a relationally acceptable way seems to require consideration of the strictorder on $\mathbb{P}$ first.

The relation $\mathbb{S}^+$ on $\mathbb{N}$ is a regressively finite linear strictorder, and thus Ferrers, semi-transitive, negatively transitive, and semi-connex. Of course, we expect this correspondingly to happen for the strictorder on $\mathbb{P}$. Afterwards the successor function on $\mathbb{P}$ may be defined as the Hasse relation thereof.

We are going to follow the layers of orderings as presented in [9, Fig. 12.5] in formally establishing these properties step by step. In doing so, there will frequently occur such concepts as regressive finiteness, regressive boundedness, admitting a Hasse relation, being path-length bounded or discrete, which are far from being known as relational concepts. Most often people handle them via intuition in a non-formal way. Decades ago, however, these concepts have already been developed and presented in full precision with [11, 12]. Earlier in this text, we have referred to these concepts via the footnote in Def. 5.1.1, and will now use more of it, not least [11, 12, Prop. 6.3.4, Prop. 6.5.3 and 6.5.5].

9.4.5 Proposition. We define $W := \iota_{\wp;\mathbb{T}};\mathbb{S}^+;\iota^\mathsf{T}_{\wp;\mathbb{T}}$ and prove several properties beyond the obvious asymmetry and transitivity:

i) W is a Ferrers relation, i.e. satisfies $W;\overline{W^\mathsf{T}};W \subseteq W$.

ii) W is semi-transitive and negatively transitive, i.e. $W;W;\overline{W^\mathsf{T}} = W$ and $\overline{W};\overline{W} = \overline{W}$.

iii) W is semi-connex, i.e. satisfies $\overline{\mathbb{I}_\mathbb{P}} \subseteq W \cup W^\mathsf{T}$.

iv) Altogether, W is a linear strictorder.

v) W is regressively finite, regressively bounded, path-length bounded and discrete.[1]

Proof: Obviously $W = \iota_{\wp;\mathbb{T}};\mathbb{S}^+;\iota^\mathsf{T}_{\wp;\mathbb{T}} = \iota_{\wp;\mathbb{T}};\wp;\mathbb{S}^+;\wp;\iota^\mathsf{T}_{\wp;\mathbb{T}} = \iota_{\wp;\mathbb{T}};S^\oplus;\iota^\mathsf{T}_{\wp;\mathbb{T}}$.

i) $W;\overline{W^\mathsf{T}};W = \iota_{\wp;\mathbb{T}};S^\oplus;\iota^\mathsf{T}_{\wp;\mathbb{T}};\overline{\iota_{\wp;\mathbb{T}};S^{\oplus\mathsf{T}};\iota^\mathsf{T}_{\wp;\mathbb{T}}};\iota_{\wp;\mathbb{T}};S^\oplus;\iota^\mathsf{T}_{\wp;\mathbb{T}}$
$= \iota_{\wp;\mathbb{T}};S^\oplus;\iota^\mathsf{T}_{\wp;\mathbb{T}};\iota_{\wp;\mathbb{T}};\overline{S^{\oplus\mathsf{T}}};\iota^\mathsf{T}_{\wp;\mathbb{T}};\iota_{\wp;\mathbb{T}};S^\oplus;\iota^\mathsf{T}_{\wp;\mathbb{T}} = \iota_{\wp;\mathbb{T}};S^\oplus;\wp;\overline{S^{\oplus\mathsf{T}}};\wp;S^\oplus;\iota^\mathsf{T}_{\wp;\mathbb{T}}$ because $\iota_{\wp;\mathbb{T}}$ is a map
$\subseteq \iota_{\wp;\mathbb{T}};S^\oplus;\overline{S^{\oplus\mathsf{T}}};S^\oplus;\iota^\mathsf{T}_{\wp;\mathbb{T}}$
$\subseteq \iota_{\wp;\mathbb{T}};S^\oplus;\iota^\mathsf{T}_{\wp;\mathbb{T}} = W$ applying that $S^\oplus$ is Ferrers; Prop. 9.4.1.ii

---

ii) $W;W;\overline{W^\mathsf{T}} = \iota_{\wp;\mathbb{T}};S^\oplus;\iota^\mathsf{T}_{\wp;\mathbb{T}};\iota_{\wp;\mathbb{T}};S^\oplus;\iota^\mathsf{T}_{\wp;\mathbb{T}};\overline{\iota_{\wp;\mathbb{T}};S^{\oplus\mathsf{T}};\iota^\mathsf{T}_{\wp;\mathbb{T}}}$
$= \iota_{\wp;\mathbb{T}};S^\oplus;\wp;S^\oplus;\iota^\mathsf{T}_{\wp;\mathbb{T}};\iota_{\wp;\mathbb{T}};\overline{S^{\oplus\mathsf{T}}};\iota^\mathsf{T}_{\wp;\mathbb{T}}$ because $\iota_{\wp;\mathbb{T}}$ is a map
$= \iota_{\wp;\mathbb{T}};[\wp;\mathbb{S}^+;\wp];\wp;[\wp;\mathbb{S}^+;\wp];\wp;[\wp;\overline{S^{\oplus\mathsf{T}}};\wp];\iota^\mathsf{T}_{\wp;\mathbb{T}}$
$\subseteq \iota_{\wp;\mathbb{T}};\wp;\mathbb{S}^+;\mathbb{S}^+;\mathbb{S}^*;\wp;\iota^\mathsf{T}_{\wp;\mathbb{T}}$ using Prop. 9.4.1.i
$\subseteq \iota_{\wp;\mathbb{T}};\wp;\mathbb{S}^+;\wp;\iota^\mathsf{T}_{\wp;\mathbb{T}}$ since $\mathbb{S}^+$ is a linear strictorder
$= \iota_{\wp;\mathbb{T}};S^\oplus;\iota^\mathsf{T}_{\wp;\mathbb{T}} = W$

$\overline{W};\overline{W} = \overline{\iota_{\wp;\mathbb{T}};\mathbb{S}^+;\iota^\mathsf{T}_{\wp;\mathbb{T}}};\overline{\iota_{\wp;\mathbb{T}};\mathbb{S}^+;\iota^\mathsf{T}_{\wp;\mathbb{T}}}$
$= \iota_{\wp;\mathbb{T}};\overline{\mathbb{S}^+};\iota^\mathsf{T}_{\wp;\mathbb{T}};\iota_{\wp;\mathbb{T}};\overline{\mathbb{S}^+};\iota^\mathsf{T}_{\wp;\mathbb{T}} = \iota_{\wp;\mathbb{T}};\mathbb{S}^{*\mathsf{T}};\wp;\mathbb{S}^{*\mathsf{T}};\iota^\mathsf{T}_{\wp;\mathbb{T}}$ since $\mathbb{S}^+$ is a linear strictorder
$\subseteq \iota_{\wp;\mathbb{T}};\mathbb{S}^{*\mathsf{T}};\iota^\mathsf{T}_{\wp;\mathbb{T}} = \iota_{\wp;\mathbb{T}};\overline{\mathbb{S}^+};\iota^\mathsf{T}_{\wp;\mathbb{T}}$ since $\mathbb{S}^*$ is a linear order
$= \overline{\iota_{\wp;\mathbb{T}};\mathbb{S}^+;\iota^\mathsf{T}_{\wp;\mathbb{T}}} = \overline{W}$

---

[1] A relation B is discrete provided $\mathtt{sup}\{\, X \mid X \subseteq B \cap (B;X \cup X;B)\} = \mathbb{1}$.

iii) $W \cup W^{\intercal} = \iota_{\wp;\mathbb{T}};\mathbb{S}^{+};\iota^{\intercal}_{\wp;\mathbb{T}} \cup \iota_{\wp;\mathbb{T}};\mathbb{S}^{+\intercal};\iota^{\intercal}_{\wp;\mathbb{T}}$
$= \iota_{\wp;\mathbb{T}};\big[\mathbb{S}^{+} \cup \mathbb{S}^{\intercal+}\big];\iota^{\intercal}_{\wp;\mathbb{T}} \supseteq \iota_{\wp;\mathbb{T}};\overline{\mathbb{I}};\iota^{\intercal}_{\wp;\mathbb{T}}$ since $\mathbb{S}^{+}$ is semi-connex
$= \overline{\iota_{\wp;\mathbb{T}};\mathbb{I};\iota^{\intercal}_{\wp;\mathbb{T}}} = \overline{\mathbb{I}_{\mathbb{P}}}$ because $\iota_{\wp;\mathbb{T}}$ is an injective map

---

iv) This follows finally from [9, Def. 12.4].

---

v) First we recall Prop. 4.1.3, namely that $\mathbb{S}^{+}$ is regressively finite precisely if $\mathbb{S}$ is. Then we observe that $\iota_{\wp;\mathbb{T}}$ is a homomorphism sending W into $\mathbb{S}^{+}$, i.e. $W;\iota_{\wp;\mathbb{T}} \subseteq \iota_{\wp;\mathbb{T}}\mathbb{S}^{+}$. Due to Prop. 4.1.4, then also W is regressively finite.

Referring to [11, 12, Def. 6.3.1], we get that W is regressively bounded because

$\mathtt{sup}_{h\geq 0}\,\overline{W^{\intercal h};\mathbb{T}} = \mathtt{sup}_{h\geq 0}\,\overline{\big[\iota_{\wp;\mathbb{T}};\mathbb{S}^{\intercal+};\iota^{\intercal}_{\wp;\mathbb{T}}\big]^{h};\mathbb{T}}$
$\supseteq \mathtt{sup}_{h\geq 0}\,\iota_{\wp;\mathbb{T}};\overline{\big[\mathbb{S}^{\intercal+}\big]^{h};\iota^{\intercal}_{\wp;\mathbb{T}};\mathbb{T}}$ because $\iota^{\intercal}_{\wp;\mathbb{T}};\iota_{\wp;\mathbb{T}} \subseteq \mathbb{I}$
$= \iota_{\wp;\mathbb{T}};\mathtt{sup}_{h\geq 0}\,\overline{\big[\mathbb{S}^{\intercal+}\big]^{h};\iota^{\intercal}_{\wp;\mathbb{T}};\mathbb{T}}$ since $\iota_{\wp;\mathbb{T}}$ is a map
$\supseteq \iota_{\wp;\mathbb{T}};\mathtt{sup}_{h\geq 0}\,\overline{\big[\mathbb{S}^{\intercal+}\big]^{h};\mathbb{T}}$
$= \iota_{\wp;\mathbb{T}};\mathtt{sup}_{h\geq 0}\,\overline{\mathbb{S}^{\intercal h};\mathbb{T}}$ because $\mathbb{S}^{h};\mathbb{T} = \mathbb{S}^{+h};\mathbb{T}$
$= \iota_{\wp;\mathbb{T}};\big[\mathbb{\perp} \cup \mathtt{sup}_{m\geq 1}\,\overline{\mathbb{S}^{\intercal m};\mathbb{T}}\,\big]$
$= \iota_{\wp;\mathbb{T}};\big[\mathbb{\perp} \cup \mathtt{sup}_{m\geq 1}\{\mathtt{sup}_{i<m}\,\mathbb{S}^{\intercal i};z;\mathbb{T}\}\big]$ due to Prop. 5.2.1.ii.c
$= \iota_{\wp;\mathbb{T}};\mathtt{sup}_{i\geq 0}\,\mathbb{S}^{\intercal i};z;\mathbb{T} = \iota_{\wp;\mathbb{T}};\mathbb{T} = \mathbb{T}_{\mathbb{P}}$ Exhaustion Theorem of [11, 12, Prop. 6.3.4] □

Our goal to identify a linear strictorder on $\mathbb{P}$ has thus been achieved. As a discrete strictordering, W has, due to [11, 12, Prop. 6.5.10], indeed a Hasse relation, i.e. a relation $\mathtt{Hasse}(W)$ that satisfies

$\mathtt{Hasse}(W) = W \cap \overline{W;W}$ as well as $W = \big[\mathtt{Hasse}(W)\big]^{+}$.

This Hasse relation will now be identified to be equal to be the $\mathbb{S}_{\mathbb{P}}$ of Def. 9.4.2:

9.4.6 Proposition. $\mathtt{Hasse}(\iota_{\wp;\mathbb{T}};\mathbb{S}^{+};\iota^{\intercal}_{\wp;\mathbb{T}}) = \iota_{\wp;\mathbb{T}};\mathtt{NP};\iota^{\intercal}_{\wp;\mathbb{T}}$

Proof: $\iota_{\wp;\mathbb{T}};\mathtt{NP};\iota^{\intercal}_{\wp;\mathbb{T}}$ Def. 9.4.2.i
$= \iota_{\wp;\mathbb{T}};\big[S^{\oplus} \cap \overline{S^{\oplus};S^{\oplus}}\big];\iota^{\intercal}_{\wp;\mathbb{T}}$ Prop. 9.4.1.iv
$= \iota_{\wp;\mathbb{T}};S^{\oplus};\iota^{\intercal}_{\wp;\mathbb{T}} \cap \iota_{\wp;\mathbb{T}};\overline{S^{\oplus};S^{\oplus}};\iota^{\intercal}_{\wp;\mathbb{T}}$
$= \iota_{\wp;\mathbb{T}};\wp;\mathbb{S}^{+};\wp;\iota^{\intercal}_{\wp;\mathbb{T}} \cap \iota_{\wp;\mathbb{T}};\overline{\wp;\mathbb{S}^{+};\wp;\wp;\mathbb{S}^{+};\wp};\iota^{\intercal}_{\wp;\mathbb{T}}$ see definition in Prop. 9.4.1
$= \iota_{\wp;\mathbb{T}};\mathbb{S}^{+};\iota^{\intercal}_{\wp;\mathbb{T}} \cap \overline{\iota_{\wp;\mathbb{T}};\wp;\mathbb{S}^{+};\wp;\mathbb{S}^{+};\wp;\iota^{\intercal}_{\wp;\mathbb{T}}}$ since $\iota_{\wp;\mathbb{T}}$ is the extrusion of $\wp;\mathbb{T}$ and a map
$= \iota_{\wp;\mathbb{T}};\mathbb{S}^{+};\iota^{\intercal}_{\wp;\mathbb{T}} \cap \overline{\iota_{\wp;\mathbb{T}};\mathbb{S}^{+};\iota^{\intercal}_{\wp;\mathbb{T}};\iota_{\wp;\mathbb{T}};\mathbb{S}^{+};\iota^{\intercal}_{\wp;\mathbb{T}}}$
$= W \cap \overline{W;W} = \mathtt{Hasse}(W)$ □

So finally $\mathbb{S}_{\mathbb{P}} = \mathtt{Hasse}(W)$ with $W = \mathbb{S}^{+}_{\mathbb{P}}$ and $\mathbb{S}_{\mathbb{P}} := \mathtt{Hasse}(\mathbb{S}^{+}_{\mathbb{P}})$. We are now sufficiently prepared to present the essential properties of $\mathbb{S}_{\mathbb{P}}$:

In $\mathbb{P}$, a prime has also a denotation that corresponds to $\mathbb{S}^{\intercal p}{;}\, z \,;\mathbb{T}$ in $\mathbb{N}$, namely as $\mathbb{S}_{\mathbb{P}}^{\intercal c}{;}\, z_{\mathbb{P}}{;}\,\mathbb{T}$. So we have also to relate these exponents somehow. We are going to provide a standard notation for these transitions and will refer to the following as "exponent conversion from odd primes $\mathbb{P}$ to naturals $\mathbb{N}$" or correspondingly as (partial) "exponent conversion from naturals $\mathbb{N}$ to odd primes $\mathbb{P}$". This brings a twofold notation, namely — when inside limitation — as a point

$$\mathbb{S}^{\intercal p}{;}\, z \,;\mathbb{T} : \mathbb{N} \longrightarrow \mathbb{1}, \quad \text{satisfying} \quad \mathbb{S}^{\intercal p}{;}\, z \,;\mathbb{T} \subseteq \wp \,;\mathbb{T},$$

but also its extruded version according to Def. 9.4.2, namely the point

$$[p\!\rightarrow] = \iota_{\wp;\mathbb{T}};\mathbb{S}^{\intercal p}{;}\, z \,;\mathbb{T} : \mathbb{P} \longrightarrow \mathbb{1}.$$

Of course, the two are heavily interrelated as indicated with Fig. 9.4.3.

$$(\wp \,;\mathbb{T})^{\intercal} = \begin{array}{c} \begin{array}{cccccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \end{array} \\ \{\}\,(\mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{1}\ \mathbf{0}\ \mathbf{1}\ \mathbf{0}\ \mathbf{1}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{1}\ \mathbf{0}\ \mathbf{1}) \end{array}$$

$$\iota_{\wp;\mathbb{T}} = \begin{array}{c} \begin{array}{cccccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \end{array} \\ \begin{array}{r} 3\!\rightarrow \\ 5\!\rightarrow \\ 7\!\rightarrow \\ 11\!\rightarrow \\ 13\!\rightarrow \end{array} \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix} \end{array}$$

$$[5\!\rightarrow] = \begin{array}{r} 3\!\rightarrow \\ 5\!\rightarrow \\ 7\!\rightarrow \\ 11\!\rightarrow \\ 13\!\rightarrow \end{array} \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

$$\mathbb{S}_{\mathbb{P}} = \begin{array}{c} \begin{array}{ccccc} 3\!\rightarrow & 5\!\rightarrow & 7\!\rightarrow & 11\!\rightarrow & 13\!\rightarrow \end{array} \\ \begin{array}{r} 3\!\rightarrow \\ 5\!\rightarrow \\ 7\!\rightarrow \\ 11\!\rightarrow \\ 13\!\rightarrow \end{array} \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{array}$$

Fig. 9.4.3 $\wp \,;\mathbb{T}$ extruded by $\iota_{\wp;\mathbb{T}}$ together with $\mathbb{S}_{\mathbb{P}}$ — limited by 13

The following are several useful definitions:

9.4.7 Definition (*Relations on odd primes* inside $\mathbb{N}$)**.**

$\mathtt{ND} := (\mathtt{NP} \circledcirc \mathbb{I})\,;\mathfrak{S} : \mathbb{N} \longrightarrow \mathbb{N}$ the **distance of a prime to the next prime** function

$f := \iota_{\wp;\mathbb{T}};\mathtt{ND} : \mathbb{P} \longrightarrow \mathbb{N}$ next prime distance when starting in $\mathbb{P}$

$f' := f\,;\iota_{\mathbb{E}_2}^{\intercal} : \mathbb{P} \longrightarrow \mathbb{E}_2$ next prime distance when additionally required to end in $\mathbb{E}_2$ □

These prime distances are visualized with Fig. 9.4.4. There, the number 13 has no next prime inside the present naturals — here limited to 16. On the right, therefore, no distance is assigned to 13.

$$\mathtt{NP} = \iota^{\mathsf{T}}_{\wp;\mathbb{T}};\mathbb{S}_{\mathbb{P}};\iota_{\wp;\mathbb{T}}$$

$$\mathtt{ND} = (\mathtt{NP} \circledcirc \mathbb{I});\mathfrak{S}$$

Fig. 9.4.4 $f = \iota_{\wp;\mathbb{T}};\mathtt{ND}$ $f' = \iota_{\wp;\mathbb{T}};\mathtt{ND};\iota^{\mathsf{T}}_{\mathbb{E}_2}$

9.5 Illustrating prime distances

In view of the gaps between primes, it seems a good idea to study what happens with tiny HASKELL programs. The first is intended to specify those primes for which another — not necessarily consecutive — prime shows up in a prespecified distance. From this idea, we are lead to define

$\wp_{2-D} := \{3, 5, 11, 17, 29, \ldots\}$ because, e.g. $29 + 2 = 31$ is prime
$\wp_{4-D} := \{3, 7, 13, 19, \ldots\}$ because, e.g. $19 + 4 = 23$ is prime
$\wp_{6-D} := \{5, 7, 11, 13, 17, 23, \ldots\}$ because, e.g. $23 + 6 = 29$ is prime
$\wp_{8-D} := \{3, 5, 11, 23, 29, \ldots\}$ because, e.g. $29 + 8 = 37$ is prime
...

A formal definition would be for n with $n \geq 0$

$$\wp_{2n-D} := \wp\,;\mathbb{T} \cap \mathbb{S}^{\,2n};\wp\,;\mathbb{T} = (\wp\,;\mathbb{T} \cap \mathbb{S}^{\,2n});\wp\,;\mathbb{T}$$

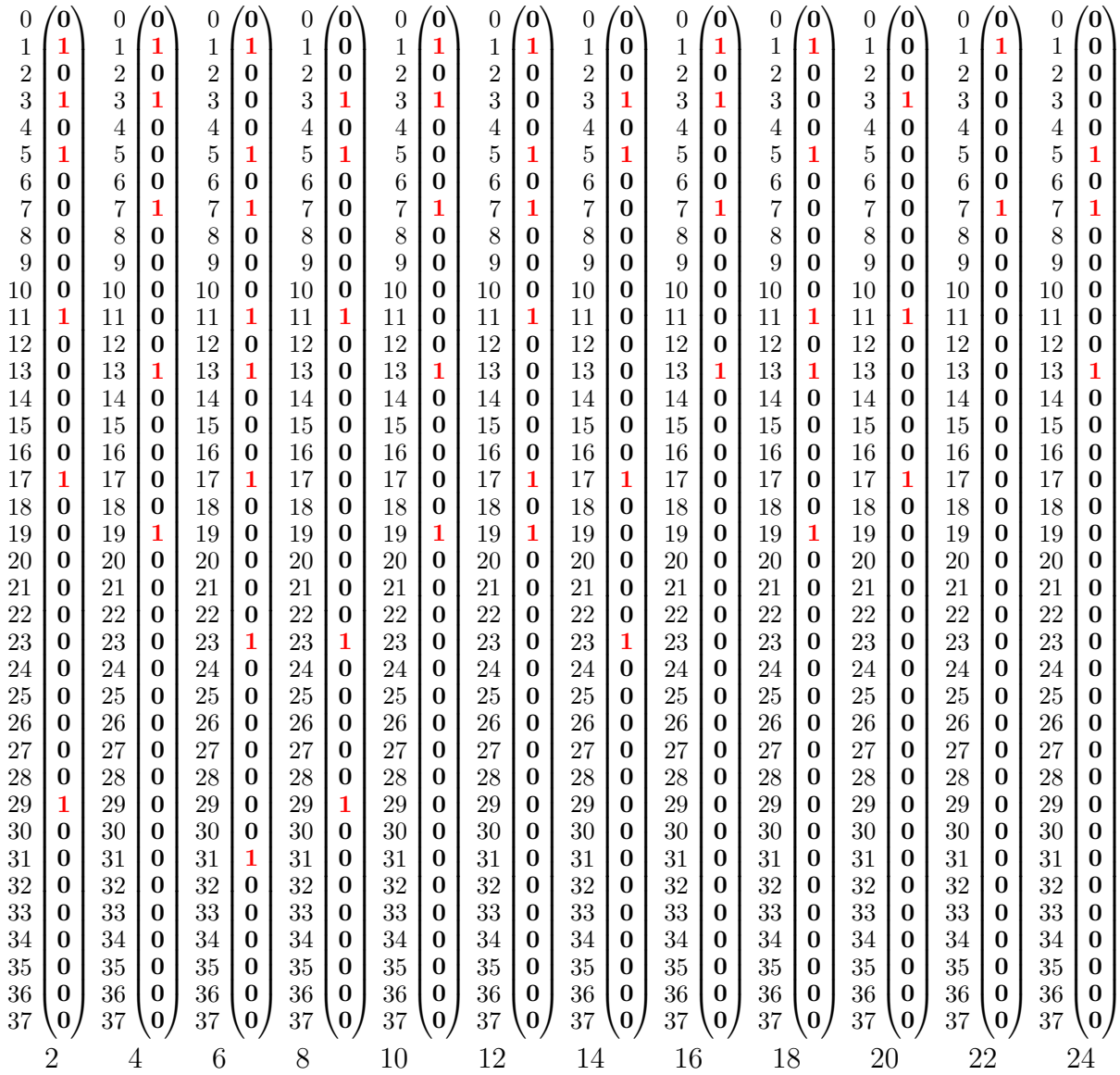


Fig. 9.5.1 The vectors $\wp_{2-D} \ldots \wp_{24-D}$ for numbers limited by 37

An important point: In $\wp_{4-D}$, the value 37 is not marked, although we know that $37 + 4 = 41$ is prime. However, when restricting the discussion to 37 (as in this case), reasoning about 41 cannot yet take place. In the same way, $\wp_{10-D}$ does neither mark 31 nor 37, although $31 + 10 = 41$ as well as $37 + 10 = 47$ are both prime.

When considering the first primes with distances 2 and 4, we are slightly mislead: Distances between consecutive primes will be much longer for bigger primes. An example which to evaluate took quite some time: The largest distance among the first 800000 primes[2] is 154, between 11113933 and 11114087. The tiny HASKELL program

```
map (\u -> (u,isPrime u)) [11113933,11113935..11114087]
```

shows indeed

[2]The last one out of these is 12195263.

```
[(11113933,True),
(11113935,False),(11113937,False),(11113939,False),(11113941,False),
(11113943,False),(11113945,False),(11113947,False),(11113949,False),
(11113951,False),(11113953,False),(11113955,False),(11113957,False),
(11113959,False),(11113961,False),(11113963,False),(11113965,False),
(11113967,False),(11113969,False),(11113971,False),(11113973,False),
(11113975,False),(11113977,False),(11113979,False),(11113981,False),
...
(11114039,False),(11114041,False),(11114043,False),(11114045,False),
(11114047,False),(11114049,False),(11114051,False),(11114053,False),
(11114055,False),(11114057,False),(11114059,False),(11114061,False),
(11114063,False),(11114065,False),(11114067,False),(11114069,False),
(11114071,False),(11114073,False),(11114075,False),(11114077,False),
(11114079,False),(11114081,False),(11114083,False),(11114085,False),
(11114087,True)]
```

The program `nextPrimeDiff` shows the list of primes up to `n` together with the distance to the respective next higher one.

```
nextPrimeDiff n = map (\(a,b) -> (a,b-a)) ppairs
   where primesBelow = oddPrimesBELOW n
         ppairs = zip primesBelow (tail primesBelow)
```

An example result is this, with, e.g., $3+2=5$, $5+2=7$, $7+4=11$:

```
nextPrimeDiff 400 = [(3,2),(5,2),(7,4),(11,2),(13,4),(17,2),(19,4),(23,6),
    (29,2),(31,6),(37,4),(41,2),(43,4),(47,6),(53,6),(59,2),(61,6),(67,4),
    (71,2),(73,6),(79,4),(83,6),(89,8),(97,4),(101,2),(103,4),(107,2),(109,4),
    (113,14),(127,4),(131,6),(137,2),(139,10),(149,2),(151,6),(157,6),(163,4),
    (167,6),(173,6),(179,2),(181,10),(191,2),(193,4),(197,2),(199,12),(211,12),
    (223,4),(227,2),(229,4),(233,6),(239,2),(241,10),(251,6),(257,6),(263,6),
    (269,2),(271,6),(277,4),(281,2),(283,10),(293,14),(307,4),(311,2),(313,4),
    (317,14),(331,6),(337,10),(347,2),(349,4),(353,6),(359,8),(367,6),(373,6),
    (379,4),(383,6),(389,8)]
```

The distances occurring as second component in the result of `nextPrimeDiff` may be sorted to obtain an impression of how this is distributed. The following program `maxPrimeDist` shows the maximum prime distance until `n`:

```
maxPrimeDist :: Int -> Int
maxPrimeDist n = case n > 1 of True  -> maximum primeDists
                               False -> error "Argument should be greater than 1"
   where op = oddPrimesBELOW n
         primePairs = zip op (tail op)
         primeDists = map (\(p,q) -> q-p) primePairs
```

It delivers for instance `maxPrimeDist 400 = 14` resembling $113 + 14 = 127$ or $293 + 14 = 307$ or $317 + 14 = 331$; see above.

The program `primeDistances` shows all prime distances of the first n primes in decreasing order.

```
primeDistances n = ssort $ map (\(p,q) -> (q-p,p)) primePairs
   where primeList = take n oddPrimes
         primePairs = zip primeList (tail primeList)
         ssort ((p,q):xs) = lesser ++ (p,q) : greater
           where lesser  = ssort [ (a,b) | (a,b) <- xs, a >= p]
                 greater = ssort [ (a,b) | (a,b) <- xs, a <  p]
         ssort _ = []

primeDistances 100 =
   [(18,523),(14,317),(14,293),(14,113),(12,509),(12,467),(12,211),(12,199),
   (10,421),(10,409),(10,337),(10,283),(10,241),(10,181),(10,139),(8,491),
   (8,479),(8,449),(8,401),(8,389),(8,359),(8,89),(6,541),(6,503),(6,443),
   (6,433),(6,383),(6,373),(6,367),(6,353),(6,331),(6,271),(6,263),(6,257),
   (6,251),(6,233),(6,173),(6,167),(6,157),(6,151),(6,131),(6,83),(6,73),(6,61),
   (6,53),(6,47),(6,31),(6,23),(4,499),(4,487),(4,463),(4,457),(4,439),(4,397),
   (4,379),(4,349),(4,313),(4,307),(4,277),(4,229),(4,223),(4,193),(4,163),
   (4,127),(4,109),(4,103),(4,97),(4,79),(4,67),(4,43),(4,37),(4,19),(4,13),
   (4,7),(2,521),(2,461),(2,431),(2,419),(2,347),(2,311),(2,281),(2,269),
   (2,239),(2,227),(2,197),(2,191),(2,179),(2,149),(2,137),(2,107),(2,101),
   (2,71),(2,59),(2,41),(2,29),(2,17),(2,11),(2,5),(2,3)]
```

```
actuallyOccuringPrimeDistances n = nub $ map fst pdList
   where pdList = primeDistances n
```

The distance of 72 occurs for the first time between the primes 31397 and 31469. It stays fairly long as the exceptional distance: among the first 4000 as well as among the first 10000 primes. The gaps are gradually filled with, e.g. 64.

```
actuallyOccuringPrimeDistances 4000 = [72,62,54,52,50,48,
    44,42,40,38,36,34,32,30,28,26,24,22,20,18,16,14,12,10,8,6,4,2]
actuallyOccuringPrimeDistances 10000 = [72,64,62,60,58,56,54,52,50,48,46,
    44,42,40,38,36,34,32,30,28,26,24,22,20,18,16,14,12,10,8,6,4,2]
actuallyOccuringPrimeDistances 100000 = [114,112,106,100,98,96,94,92,90,
    88,86,84,82,80,78,76,74,72,70,68,66,64,62,60,58,56,54,52,50,48,46,
    44,42,40,38,36,34,32,30,28,26,24,22,20,18,16,14,12,10,8,6,4,2]
actuallyOccuringPrimeDistances 500000 = [154,148,146,138,136,134,132,130,128,
    126,124,122,120,118,116,114,112,110,108,106,104,102,100,98,96,94,92,90,
    88,86,84,82,80,78,76,74,72,70,68,66,64,62,60,58,56,54,52,50,48,46,
    44,42,40,38,36,34,32,30,28,26,24,22,20,18,16,14,12,10,8,6,4,2]
```

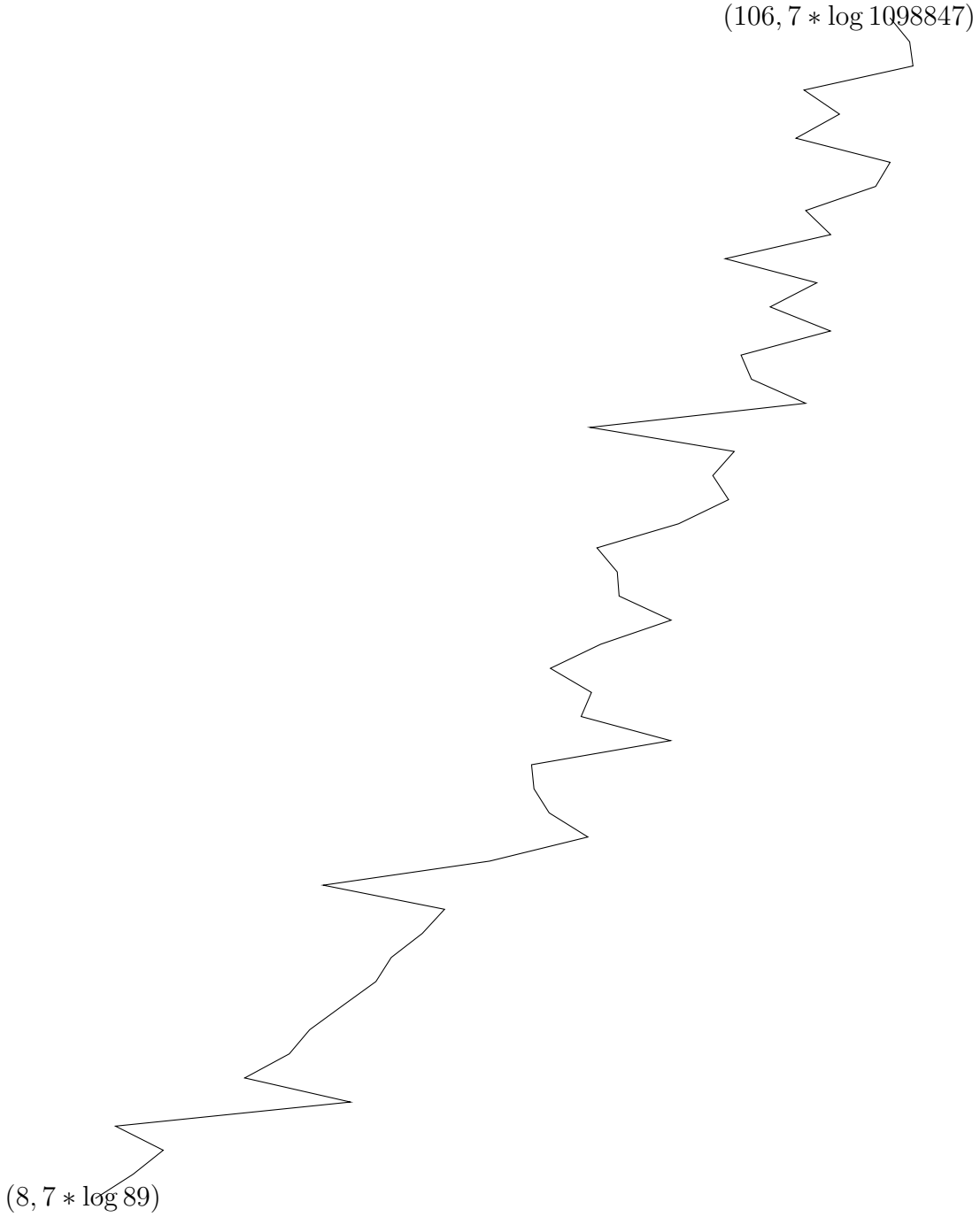
An observation is immediate: The sequence $2, 4, 6, \ldots$ with just evens to follow up seems to be finally be achieved, but only hesitantly will its upper even numbers become integrated.

Yet another idea of this sort is it to ask when some given prime distance occurs for the first time.

```
firstOccurrenceOfPrimeDist n = fst $ head rest
   where pairs = zip oddPrimes (tail oddPrimes)
         pairDiffs = map (\(a,b) -> (a,b-a)) pairs
         rest = dropWhile (\(a,d) -> d /= n) pairDiffs

listOfFirstPrimeDistOccurrence max = zip l distancesFirst
   where l = take max [8,10..]
         distancesFirst = map firstOccurrenceOfPrimeDist l
```

This is far from isotonic. The following picture shows the function
$n \mapsto 7 * \log(\texttt{firstOccurrenceOfPrimeDist}\ n) : \mathbb{N} \longrightarrow \mathbb{R}$.


(106, 7 ∗ log 1098847)
(8, 7 ∗ log 89)

4	7
6	23
8	89
10	139
12	199
14	113
16	1831
18	523
20	887
22	1129
24	1669
26	2477
28	2971
30	4297
32	5591
34	1327
36	9551
38	30593

40	19333
42	16141
44	15683
46	81463
48	28229
50	31907
52	19609
54	35617
56	82073
58	44293
60	43331
62	34061
64	89689
66	162143
68	134513
70	173359
72	31397
74	404597

76	212701
78	188029
80	542603
82	265621
84	461717
86	155921
88	544279
90	404851
92	927869
94	1100977
96	360653
98	604073
100	396733
102	1444309
104	1388483
106	1098847
108	2238823
110	1468277

112	370261
114	492113
116	5845193
118	1349533
120	1895359
122	3117299
124	6752623
126	1671781
128	3851459
130	5518687
132	1357201
134	6958667
136	6371401
138	3826019
140	7621259
142	10343761
144	11981443
146	6034247
148	2010733
150	6034247

```
listOfFirstPrimeDistOccurrence 150
   [(8,89),(10,139),(12,199),(14,113),
       (16,1831),(18,523),(20,887),(22,1129),(24,1669),(26,2477),(28,2971),
       (30,4297),(32,5591),(34,1327),(36,9551),(38,30593),(40,19333),
       (42,16141),(44,15683),(46,81463),(48,28229),(50,31907),(52,19609),
       (54,35617),(56,82073),(58,44293),(60,43331),(62,34061),
       (64,89689),(66,162143),(68,134513),(70,173359),(72,31397),
       (74,404597),(76,212701),(78,188029),(80,542603),(82,265621),
       (84,461717),(86,155921),(88,544279),(90,404851),(92,927869),
       (94,1100977),(96,360653),(98,604073),(100,396733),(102,1444309),
       (104,1388483),(106,1098847),(108,2238823),(110,1468277),(112,370261),
       (114,492113),(116,5845193),(118,1349533),(120,1895359),(122,3117299),
       (124,6752623),(126,1671781),(128,3851459),(130,5518687),(132,1357201),
       (134,6958667),(136,6371401),(138,3826019),(140,7621259),(142,10343761),
       (144,11981443),(146,6034247),(148,2010733),(150,6034247)]
```

9.6 Further on primes and their distances

A problem offer the varying prime distances. To get hold of these, we define as follows:
The map psn will assign to an odd prime out of $\mathbb{P}$ its exponent starting with $3\rightarrow$ mapped to 0:

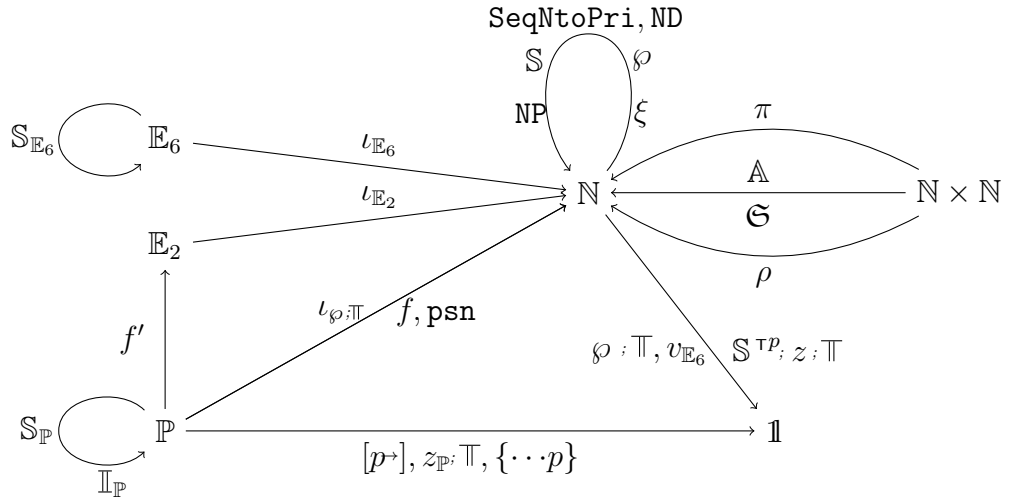


Fig. 9.6.1 Typing for primes and even numbers

9.6.1 Definition and Proposition. Given the successor relation $\mathbb{S}$ assumed to be regressively finite, together with $\mathbb{S}_{\mathbb{P}}$, we consider the functional

$$\mu(X) := \mathtt{psn}_0 \cup \mu_H(X) \quad \text{with} \quad \mathtt{psn}_0 := z_{\mathbb{P}};\mathbb{T};z^{\mathsf{T}} \quad \text{and} \quad \mu_H(X) := \mathbb{S}^{\mathsf{T}}_{\mathbb{P}};X;\mathbb{S}$$

and define a relation $\mathtt{psn} : \mathbb{P} \longrightarrow \mathbb{N}$ — to be called **prime sequence exponent** — as

$$\mathtt{psn} := \mathtt{psn}_{\mathbb{S}} := \mathtt{inf}\,\{\, X \mid \mu(X) \subseteq X\}.$$

Then:

i) $\mathtt{psn} = \mu(\mathtt{psn}) = z_{\mathbb{P}};\mathbb{T};z^{\mathsf{T}} \cup \mathbb{S}^{\mathsf{T}}_{\mathbb{P}};\mathtt{psn};\mathbb{S} = \mathtt{sup}_{n\geq 0}\,\mu^n_H(z_{\mathbb{P}};\mathbb{T};z^{\mathsf{T}}) = \mathtt{sup}_{n\geq 0}\,\mathbb{S}^{\mathsf{T}n}_{\mathbb{P}};z_{\mathbb{P}};\mathbb{T};z^{\mathsf{T}};\mathbb{S}^{\,n}$

ii) psn is an injective map

iii) Given an odd prime p inside the present limitation, there will always exist the point

$$\mathtt{psn}^{\mathsf{T}};[p\!\rightarrow] =: \mathbb{S}^{\,\tau\{\cdots p\}};\,z\,;\mathbb{T}, \qquad \text{satisfying also} \qquad [p\!\rightarrow] = \mathtt{psn}\,;\mathbb{S}^{\,\tau\{\cdots p\}};\,z\,;\mathbb{T}.$$

Proof: We observe in advance that the functional μ_H is

$$\begin{aligned}
&\text{isotonic,} && \text{i.e.}\ \ A \subseteq B \implies \mathbb{S}^{\mathsf{T}}_{\mathbb{P}};\,A\,;\mathbb{S} \subseteq \mathbb{S}^{\mathsf{T}}_{\mathbb{P}};\,B\,;\mathbb{S}\ \ \text{i.e.}\ \ \mu_H(A) \subseteq \mu_H(B),\\
&\text{additive,} && \text{i.e.}\ \ \mu_H(X \cup Y) = \mathbb{S}^{\mathsf{T}}_{\mathbb{P}};\,(X \cup Y)\,;\mathbb{S}\\
&&&\qquad\qquad = \mathbb{S}^{\mathsf{T}}_{\mathbb{P}};\,X;\mathbb{S} \cup \mathbb{S}^{\mathsf{T}}_{\mathbb{P}};\,Y;\mathbb{S} = \mu_H(X) \cup \mu_H(Y),\\
&\cap\text{-hereditary, i.e.} && \mu_H(X \cap Y) = \mathbb{S}^{\mathsf{T}}_{\mathbb{P}};\,(X \cap Y)\,;\mathbb{S} \quad \text{with } \mathbb{S} \text{ injective}\\
&&&\qquad\qquad = \mathbb{S}^{\mathsf{T}}_{\mathbb{P}};\,X;\mathbb{S} \cap \mathbb{S}^{\mathsf{T}}_{\mathbb{P}};\,Y;\mathbb{S} = \mu_H(X) \cap \mu_H(Y),
\end{aligned}$$

so that Lemma 4.2.1 may indeed be applied.

---

i) This follows from Lemma 4.2.1.i,iv.

---

ii) $\mathtt{psn}\fatsemi\mathbb{T} = \big[\mathtt{sup}_{n\geq 0}\,\mathbb{S}_{\mathbb{P}}^{\intercal n}\fatsemi z_{\mathbb{P}}\fatsemi\mathbb{T}\fatsemi z^{\intercal}\fatsemi\mathbb{S}^{n}\big]\fatsemi\mathbb{T} = \mathtt{sup}_{n\geq 0}\,\mathbb{S}_{\mathbb{P}}^{\intercal n}\fatsemi z_{\mathbb{P}}\fatsemi\mathbb{T}\fatsemi z^{\intercal}\fatsemi\mathbb{S}^{n}\fatsemi\mathbb{T}$
$= \mathtt{sup}_{n\geq 0}\,\mathbb{S}_{\mathbb{P}}^{\intercal n}\fatsemi z_{\mathbb{P}}\fatsemi\mathbb{T}\cap\mathbb{T}\fatsemi z^{\intercal}\fatsemi\mathbb{S}^{n}\fatsemi\mathbb{T}$ masking
$= \mathtt{sup}_{n\geq 0}\,\mathbb{S}_{\mathbb{P}}^{\intercal n}\fatsemi z_{\mathbb{P}}\fatsemi\mathbb{T}$ because due to Prop. 9.4.4.v $\mathbb{S}^{\intercal n}\fatsemi z\fatsemi\mathbb{T} = \mathbb{\perp} \implies \mathbb{S}_{\mathbb{P}}^{\intercal n}\fatsemi z_{\mathbb{P}}\fatsemi\mathbb{T} = \mathbb{\perp}$
$= \mathbb{T}_{\mathbb{P}}$ Prop. 5.2.1.ii.a; so psn is total.

$\mathtt{psn}\fatsemi\mathtt{psn}^{\intercal} = \mathtt{sup}_{n\geq 0}\,\mathbb{S}_{\mathbb{P}}^{\intercal n}\fatsemi z_{\mathbb{P}}\fatsemi\mathbb{T}\fatsemi z^{\intercal}\fatsemi\mathbb{S}^{n}\fatsemi\big[\mathtt{sup}_{i\geq 0}\,\mathbb{S}_{\mathbb{P}}^{\intercal i}\fatsemi z_{\mathbb{P}}\fatsemi\mathbb{T}\fatsemi z^{\intercal}\fatsemi\mathbb{S}^{i}\big]^{\intercal}$
$= \mathtt{sup}_{n\geq 0}\,\mathbb{S}_{\mathbb{P}}^{\intercal n}\fatsemi z_{\mathbb{P}}\fatsemi\mathbb{T}\fatsemi z^{\intercal}\fatsemi\mathbb{S}^{n}\fatsemi\mathbb{S}^{\intercal n}\fatsemi z\fatsemi\mathbb{T}\fatsemi z_{\mathbb{P}}^{\intercal}\fatsemi\mathbb{S}_{\mathbb{P}}^{n}$ with $i := n$ since otherwise $\mathbb{S}\fatsemi z = \mathbb{\perp}$
$\subseteq \mathtt{sup}_{n\geq 0}\,\mathbb{S}_{\mathbb{P}}^{\intercal n}\fatsemi z_{\mathbb{P}}\fatsemi\mathbb{T}\fatsemi z_{\mathbb{P}}^{\intercal}\fatsemi\mathbb{S}_{\mathbb{P}}^{n}$
$= \mathbb{I}_{\mathbb{P}}$ Prop. 5.2.2.ii.a; so psn is injective.

$\mathtt{psn}^{\intercal}\fatsemi\mathtt{psn} = \big[\mathtt{sup}_{i\geq 0}\,\mathbb{S}_{\mathbb{P}}^{\intercal i}\fatsemi z_{\mathbb{P}}\fatsemi\mathbb{T}\fatsemi z^{\intercal}\fatsemi\mathbb{S}^{i}\big]^{\intercal}\fatsemi\mathtt{sup}_{n\geq 0}\,\mathbb{S}_{\mathbb{P}}^{\intercal n}\fatsemi z_{\mathbb{P}}\fatsemi\mathbb{T}\fatsemi z^{\intercal}\fatsemi\mathbb{S}^{n}$
$= \mathtt{sup}_{n\geq 0}\,\mathbb{S}^{\intercal n}\fatsemi z\fatsemi\mathbb{T}\fatsemi z_{\mathbb{P}}^{\intercal}\fatsemi\mathbb{S}_{\mathbb{P}}^{n}\fatsemi\mathbb{S}_{\mathbb{P}}^{\intercal n}\fatsemi z_{\mathbb{P}}\fatsemi\mathbb{T}\fatsemi z^{\intercal}\fatsemi\mathbb{S}^{n}$ with $i := n$ since otherwise $\mathbb{S}_{\mathbb{P}}\fatsemi z_{\mathbb{P}} = \mathbb{\perp}$
$\subseteq \mathtt{sup}_{n\geq 0}\,\mathbb{S}^{\intercal n}\fatsemi z\fatsemi\mathbb{T}\fatsemi z^{\intercal}\fatsemi\mathbb{S}^{n}$
$= \mathbb{I}$ Prop. 5.2.2.ii.a; so psn is univalent.

---

iii) The construct $\mathtt{psn}^{\intercal}\fatsemi[p\!\rightarrow]$ is row-constant since $[p\!\rightarrow]$ is. It is injective because
$\mathtt{psn}^{\intercal}\fatsemi[p\!\rightarrow]\fatsemi\big(\mathtt{psn}^{\intercal}\fatsemi[p\!\rightarrow]\big)^{\intercal} = \mathtt{psn}^{\intercal}\fatsemi[p\!\rightarrow]\fatsemi[p\!\rightarrow]^{\intercal}\fatsemi\mathtt{psn} \subseteq \mathtt{psn}^{\intercal}\fatsemi\mathtt{psn}$ since $[p\!\rightarrow]$ is a point
$\subseteq \mathbb{I}$ due to (ii)

That the construct is $\neq \mathbb{\perp}$ follows since p has been chosen inside the present limits.

We evaluate

$[p\!\rightarrow] = \mathtt{psn}\fatsemi\mathtt{psn}^{\intercal}\fatsemi[p\!\rightarrow]$ since psn is following (ii) an injective map
$= \mathtt{psn}\fatsemi\mathbb{S}^{\intercal\{\cdots p\}}\fatsemi z\fatsemi\mathbb{T}$ by definition □

$$\texttt{psn} = \begin{array}{c} \\ 3\rightarrow \\ 5\rightarrow \\ 7\rightarrow \\ 11\rightarrow \\ 13\rightarrow \\ 17\rightarrow \end{array} \begin{array}{c} \begin{array}{cccccccccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$\iota_{\wp;\mathbb{T}} = \begin{array}{c} \\ 3\rightarrow \\ 5\rightarrow \\ 7\rightarrow \\ 11\rightarrow \\ 13\rightarrow \\ 17\rightarrow \end{array} \begin{array}{c} \begin{array}{cccccccccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$\texttt{SeqNtoPri} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \\ 16 \\ 17 \end{array} \begin{array}{c} \begin{array}{cccccccccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \quad \begin{array}{c|l} 0 & \{\} \\ 1 & \{\} \\ 2 & \{\} \\ 3 & \{0\} \\ 4 & \{\} \\ 5 & \{1\} \\ 6 & \{\} \\ 7 & \{2\} \\ 8 & \{\} \\ 9 & \{\} \\ 10 & \{\} \\ 11 & \{3\} \\ 12 & \{\} \\ 13 & \{4\} \\ 14 & \{\} \\ 15 & \{\} \\ 16 & \{\} \\ 17 & \{5\} \end{array} = \xi$$

Fig. 9.6.2 Typing for prime exponents `psn` together with injection $\iota_{\wp;\mathbb{T}}$, `SeqNtoPri`, and ξ

9.6.2 Definition. There is a relation assigning to a prime sequence exponent the corresponding prime

$\texttt{SeqNtoPri} := \texttt{psn}^{\mathsf{T}};\iota_{\wp;\mathbb{T}} : \mathbb{N} \longrightarrow \mathbb{N}$

which in transposed form will sometimes be used as a partial function with an argument in the normal mathematical sense and then be denoted as

$\xi : \mathbb{N} \longrightarrow \mathbb{N}$. □

Nevertheless certain relational rules have to be derived as follows.

9.6.3 Proposition. `SeqNtoPri` is an injective function. It is a map in case $\mathbb{S}$ is total.

Proof: $\texttt{SeqNtoPri}^{\mathsf{T}};\texttt{SeqNtoPri} = \iota^{\mathsf{T}}_{\wp;\mathbb{T}};\texttt{psn};\texttt{psn}^{\mathsf{T}};\iota_{\wp;\mathbb{T}}$

$= \iota^{\mathsf{T}}_{\wp;\mathbb{T}};\iota_{\wp;\mathbb{T}}$ since `psn` is an injective map

$\subseteq \mathbb{I}_{\mathbb{P}}$ since $\iota_{\wp;\mathbb{T}}$ is univalent

$\mathtt{SeqNtoPri}\fatsemi\mathtt{SeqNtoPri}^\intercal = \mathtt{psn}^\intercal\fatsemi\iota_{\wp;\mathbb{T}}\fatsemi\iota^\intercal_{\wp;\mathbb{T}}\fatsemi\mathtt{psn}$
$= \mathtt{psn}^\intercal\fatsemi\mathtt{psn}$ since $\iota_{\wp;\mathbb{T}}$ is an injective map
$\subseteq \mathbb{I}$ since $\mathtt{psn}$ is univalent □

Slightly more difficult to handle are two *consecutive* odd primes $p < q$, such as $3, 5$ or $31, 37$, e.g., and their distances. Having both of them in — a possibly finite initial section of — the naturals, means two points $\mathbb{S}^{\intercal p}\fatsemi z\fatsemi\mathbb{T}$ and $\mathbb{S}^{\intercal q}\fatsemi z\fatsemi\mathbb{T}$ in $\mathbb{N}$, or else two points $[p^{\rightarrow}], [q^{\rightarrow}]$ in $\mathbb{P}$. Their interrelation is presented in the following proposition.

9.6.4 Proposition. Let $p < q$ be two *consecutive* odd primes, both inside the actual limitation with $d := q - p$ or — more formally as in (vi). Then they will satisfy

i) $\mathbb{S}^{d}\fatsemi\mathbb{S}^{\intercal q}\fatsemi z\fatsemi\mathbb{T} = \mathbb{S}^{\intercal p}\fatsemi z\fatsemi\mathbb{T}$ $\qquad$ $\mathbb{S}^{\intercal d}\fatsemi\mathbb{S}^{\intercal p}\fatsemi z\fatsemi\mathbb{T} = \mathbb{S}^{\intercal q}\fatsemi z\fatsemi\mathbb{T}$

ii) $\mathtt{NP}\fatsemi\mathbb{S}^{\intercal q}\fatsemi z\fatsemi\mathbb{T} = \mathbb{S}^{\intercal p}\fatsemi z\fatsemi\mathbb{T}$ $\qquad$ $\mathtt{NP}^\intercal\fatsemi\mathbb{S}^{\intercal p}\fatsemi z\fatsemi\mathbb{T} = \mathbb{S}^{\intercal q}\fatsemi z\fatsemi\mathbb{T}$

iii) $\mathbb{S}_{\mathbb{P}}\fatsemi[q^{\rightarrow}] = [p^{\rightarrow}]$ $\qquad$ $\mathbb{S}^\intercal_{\mathbb{P}}\fatsemi[p^{\rightarrow}] = [q^{\rightarrow}]$

iv) $\iota_{\wp;\mathbb{T}}\fatsemi\mathtt{NP}^{\intercal\xi(p)}\fatsemi\mathbb{S}^{\intercal 3}\fatsemi z\fatsemi\mathbb{T} = [p^{\rightarrow}]$

$[p^{\rightarrow}] = \mathbb{S}^{\intercal\xi(p)}_{\mathbb{P}}\fatsemi z_{\mathbb{P}}\fatsemi\mathbb{T}$ $\qquad$ $[q^{\rightarrow}] = \mathbb{S}^{\intercal 1+\xi(p)}_{\mathbb{P}}\fatsemi z_{\mathbb{P}}\fatsemi\mathbb{T}$

v) $\mathbb{S}^{\intercal *}_{\mathbb{P}}\fatsemi[p^{\rightarrow}] = \mathbb{S}^{\intercal\xi(p)}_{\mathbb{P}}\fatsemi\mathbb{T}$.

vi) $f^\intercal\fatsemi[p^{\rightarrow}] = \mathbb{S}^{\intercal d}\fatsemi z\fatsemi\mathbb{T}$

Proof: i) Obviously $\mathbb{S}^{\intercal q}\fatsemi z\fatsemi\mathbb{T} = \mathbb{S}^{\intercal d+p}\fatsemi z\fatsemi\mathbb{T} = \mathbb{S}^{\intercal d}\fatsemi\mathbb{S}^{\intercal p}\fatsemi z\fatsemi\mathbb{T}$, which implies
$\mathbb{S}^{d}\fatsemi\mathbb{S}^{\intercal q}\fatsemi z\fatsemi\mathbb{T} = \mathbb{S}^{d}\fatsemi\mathbb{S}^{\intercal d}\fatsemi\mathbb{S}^{\intercal p}\fatsemi z\fatsemi\mathbb{T} \subseteq \mathbb{S}^{\intercal p}\fatsemi z\fatsemi\mathbb{T}$ because $\mathbb{S}$ is injective.
Again, we recall Prop. 2.2.2.i, and find that these are two points one contained in the other and, thus, equal. The construct $\mathbb{S}^{d}\fatsemi\mathbb{S}^{\intercal q}\fatsemi z\fatsemi\mathbb{T}$ is certainly injective and also $\neq \perp\!\!\!\perp$ as a consequence of $d < q$.

---

ii) We start considering what will occur several times
$S^{\oplus}\fatsemi\mathbb{S}^{\intercal q}\fatsemi z\fatsemi\mathbb{T} = \wp\fatsemi\mathbb{S}^{+}\fatsemi\wp\fatsemi\mathbb{S}^{\intercal q}\fatsemi z\fatsemi\mathbb{T}$
$= \wp\fatsemi\mathbb{S}^{+}\fatsemi\mathbb{S}^{\intercal q}\fatsemi z\fatsemi\mathbb{T}$ because q is prime by assumption
$= \wp\fatsemi\mathtt{sup}_{0\leq x<q}\,\mathbb{S}^{\intercal x}\fatsemi z\fatsemi\mathbb{T}$
$= \mathtt{sup}_{0\leq x<q,\ x\text{ prime}}\,\mathbb{S}^{\intercal x}\fatsemi z\fatsemi\mathbb{T}$
and proceed thus
$\mathtt{NP}\fatsemi\mathbb{S}^{\intercal q}\fatsemi z\fatsemi\mathbb{T} = \mathtt{Hasse}(S^{\oplus})\fatsemi\mathbb{S}^{\intercal q}\fatsemi z\fatsemi\mathbb{T}$ see Prop. 9.4.1.iv
$= \left[S^{\oplus}\cap\overline{S^{\oplus}\fatsemi S^{\oplus}}\right]\fatsemi\mathbb{S}^{\intercal q}\fatsemi z\fatsemi\mathbb{T}$
$= S^{\oplus}\fatsemi\mathbb{S}^{\intercal q}\fatsemi z\fatsemi\mathbb{T}\cap\overline{S^{\oplus}\fatsemi S^{\oplus}}\fatsemi\mathbb{S}^{\intercal q}\fatsemi z\fatsemi\mathbb{T} = S^{\oplus}\fatsemi\mathbb{S}^{\intercal q}\fatsemi z\fatsemi\mathbb{T}\cap\overline{S^{\oplus}\fatsemi S^{\oplus}\fatsemi\mathbb{S}^{\intercal q}\fatsemi z\fatsemi\mathbb{T}}$
$= \mathtt{sup}_{0\leq x<q,\ x\text{ prime}}\,\mathbb{S}^{\intercal x}\fatsemi z\fatsemi\mathbb{T}\cap\overline{S^{\oplus}\fatsemi\mathtt{sup}_{0\leq y<q,\ y\text{ prime}}\,\mathbb{S}^{\intercal y}\fatsemi z\fatsemi\mathbb{T}}$
$= \mathtt{sup}_{0\leq x<q,\ x\text{ prime}}\,\mathbb{S}^{\intercal x}\fatsemi z\fatsemi\mathbb{T}\cap\overline{\mathtt{sup}_{0\leq y<q,\ y\text{ prime}}\,S^{\oplus}\fatsemi\mathbb{S}^{\intercal y}\fatsemi z\fatsemi\mathbb{T}}$
$= \mathtt{sup}_{0\leq x<q,\ x\text{ prime}}\,\mathbb{S}^{\intercal x}\fatsemi z\fatsemi\mathbb{T}\cap\overline{\mathtt{sup}_{0\leq y<q,\ y\text{ prime}}\,\mathtt{sup}_{0\leq w<y,\ w\text{ prime}}\,\mathbb{S}^{\intercal w}\fatsemi z\fatsemi\mathbb{T}}$
$= \mathtt{sup}_{0\leq x<q,\ x\text{ prime}}\,\mathbb{S}^{\intercal x}\fatsemi z\fatsemi\mathbb{T}\cap\overline{\mathtt{sup}_{0\leq w<p,\ w\text{ prime}}\,\mathbb{S}^{\intercal w}\fatsemi z\fatsemi\mathbb{T}}$

since $p < q$ consecutive and $\mathtt{sup}_{0\leq w<y,\ w\text{ prime}}\ \mathbb{S}^{\mathsf{T}w};z;\mathbb{T} \subseteq \mathtt{sup}_{0\leq y<q,\ y\text{ prime}}\ \mathbb{S}^{\mathsf{T}y};z;\mathbb{T}$

$= \mathbb{S}^{\mathsf{T}p};z;\mathbb{T}$

We start again considering what will occur several times

$S^{\oplus\mathsf{T}};\mathbb{S}^{\mathsf{T}p};z;\mathbb{T} = \wp;\mathbb{S}^{\mathsf{T}+};\wp;\mathbb{S}^{\mathsf{T}p};z;\mathbb{T}$

$= \wp;\mathbb{S}^{\mathsf{T}+};\mathbb{S}^{\mathsf{T}p};z;\mathbb{T}$ because p is prime by assumption

$= \wp;\mathtt{sup}_{x>p}\ \mathbb{S}^{\mathsf{T}x};z;\mathbb{T}$

$= \mathtt{sup}_{x>p,\ x\text{ prime}}\ \mathbb{S}^{\mathsf{T}x};z;\mathbb{T}$

and proceed thus

$\mathtt{NP}^{\mathsf{T}};\mathbb{S}^{\mathsf{T}p};z;\mathbb{T} = \left[\mathtt{Hasse}(S^{\oplus})\right]^{\mathsf{T}};\mathbb{S}^{\mathsf{T}p};z;\mathbb{T}$ see Prop. 9.4.1.iv

$= \left[S^{\oplus} \cap \overline{S^{\oplus};S^{\oplus}}\right]^{\mathsf{T}};\mathbb{S}^{\mathsf{T}p};z;\mathbb{T}$

$= S^{\oplus\mathsf{T}};\mathbb{S}^{\mathsf{T}p};z;\mathbb{T} \cap \overline{S^{\oplus\mathsf{T}};S^{\oplus\mathsf{T}};\mathbb{S}^{\mathsf{T}p};z;\mathbb{T}} = S^{\oplus\mathsf{T}};\mathbb{S}^{\mathsf{T}p};z;\mathbb{T} \cap \overline{S^{\oplus\mathsf{T}};S^{\oplus\mathsf{T}};\mathbb{S}^{\mathsf{T}p};z;\mathbb{T}}$

$= \mathtt{sup}_{x>p,\ x\text{ prime}}\ \mathbb{S}^{\mathsf{T}x};z;\mathbb{T} \cap \overline{S^{\oplus\mathsf{T}};\mathtt{sup}_{y>p,\ y\text{ prime}}\ \mathbb{S}^{\mathsf{T}y};z;\mathbb{T}}$

$= \mathtt{sup}_{x>p,\ x\text{ prime}}\ \mathbb{S}^{\mathsf{T}x};z;\mathbb{T} \cap \overline{\mathtt{sup}_{y>p,\ y\text{ prime}}\ S^{\oplus\mathsf{T}};\mathbb{S}^{\mathsf{T}y};z;\mathbb{T}}$

$= \mathtt{sup}_{x>p,\ x\text{ prime}}\ \mathbb{S}^{\mathsf{T}x};z;\mathbb{T} \cap \overline{\mathtt{sup}_{y>p,\ y\text{ prime}}\ \mathtt{sup}_{w>y,\ w\text{ prime}}\ \mathbb{S}^{\mathsf{T}w};z;\mathbb{T}}$

$= \mathtt{sup}_{x>p,\ x\text{ prime}}\ \mathbb{S}^{\mathsf{T}x};z;\mathbb{T} \cap \overline{\mathtt{sup}_{w>q,\ w\text{ prime}}\ \mathbb{S}^{\mathsf{T}w};z;\mathbb{T}}$

since $p < q$ consecutive and $\mathtt{sup}_{w>y,\ w\text{ prime}}\ \mathbb{S}^{\mathsf{T}w};z;\mathbb{T} \subseteq \mathtt{sup}_{y>p,\ y\text{ prime}}\ \mathbb{S}^{\mathsf{T}y};z;\mathbb{T}$

$= \mathbb{S}^{\mathsf{T}q};z;\mathbb{T}$

---

iii) $\mathbb{S}_{\mathbb{P}};[q\rightarrowtail] = \iota_{\wp;\mathbb{T}};\mathtt{NP};\iota^{\mathsf{T}}_{\wp;\mathbb{T}};\iota_{\wp;\mathbb{T}};\mathbb{S}^{\mathsf{T}q};z;\mathbb{T}$ Def. 9.4.2

$\subseteq \iota_{\wp;\mathbb{T}};\mathtt{NP};\mathbb{S}^{\mathsf{T}q};z;\mathbb{T}$ since $\iota_{\wp;\mathbb{T}}$ is univalent

$= \iota_{\wp;\mathbb{T}};\mathbb{S}^{\mathsf{T}p};z;\mathbb{T}$ due to (ii)

$= [p\rightarrowtail]$

Since both are points, they are nevertheless equal due to Prop. 2.2.2.i

---

$\mathbb{S}^{\mathsf{T}}_{\mathbb{P}};[p\rightarrowtail] = \iota_{\wp;\mathbb{T}};\mathtt{NP}^{\mathsf{T}};\iota^{\mathsf{T}}_{\wp;\mathbb{T}};\iota_{\wp;\mathbb{T}};\mathbb{S}^{\mathsf{T}p};z;\mathbb{T}$ Def. 9.4.2

$\subseteq \iota_{\wp;\mathbb{T}};\mathtt{NP}^{\mathsf{T}};\mathbb{S}^{\mathsf{T}p};z;\mathbb{T}$ since $\iota_{\wp;\mathbb{T}}$ is univalent

$= \iota_{\wp;\mathbb{T}};\mathbb{S}^{\mathsf{T}q};z;\mathbb{T}$ due to (ii)

$= [q\rightarrowtail]$

Since both are points, they are nevertheless equal due to Prop. 2.2.2.i

---

iv) Proof by induction over the sequence $[3\rightarrowtail], [5\rightarrowtail], [7\rightarrowtail], [11\rightarrowtail], \ldots$ of extruded odd primes:

Case $[3\rightarrowtail]$:

$\iota_{\wp;\mathbb{T}};\mathtt{NP}^{\mathsf{T}\xi(3)};\mathbb{S}^{\mathsf{T}3};z;\mathbb{T} = \iota_{\wp;\mathbb{T}};\mathtt{NP}^{\mathsf{T}0};\mathbb{S}^{\mathsf{T}3};z;\mathbb{T} = \iota_{\wp;\mathbb{T}};\mathbb{S}^{\mathsf{T}3};z;\mathbb{T} = [3\rightarrowtail]$

Step $[p\rightarrowtail] \longrightarrow [q\rightarrowtail]$ for $p < q$ consecutive primes:

$\iota_{\wp;\mathbb{T}};\mathtt{NP}^{\mathsf{T}\xi(q)};\mathbb{S}^{\mathsf{T}3};z;\mathbb{T} = \iota_{\wp;\mathbb{T}};\mathtt{NP}^{\mathsf{T}1+\xi(p)};\mathbb{S}^{\mathsf{T}3};z;\mathbb{T} = \iota_{\wp;\mathbb{T}};\mathtt{NP}^{\mathsf{T}};\mathtt{NP}^{\mathsf{T}\xi(p)};\mathbb{S}^{\mathsf{T}3};z;\mathbb{T}$

$= \iota_{\wp;\mathbb{T}};\iota^{\mathsf{T}}_{\wp;\mathbb{T}};\mathbb{S}^{\mathsf{T}}_{\mathbb{P}};\iota_{\wp;\mathbb{T}};\mathtt{NP}^{\mathsf{T}\xi(p)};\mathbb{S}^{\mathsf{T}3};z;\mathbb{T}$ Prop. 9.4.1.iv

$= \iota_{\wp;\mathbb{T}};\iota^{\mathsf{T}}_{\wp;\mathbb{T}};\mathbb{S}^{\mathsf{T}}_{\mathbb{P}};[p\rightarrowtail]$ hypothesis

$= \mathbb{S}^{\mathsf{T}}_{\mathbb{P}};[p\rightarrowtail]$ since $\iota_{\wp;\mathbb{T}}$ is an injective map

$= [q\rightarrowtail]$ because $p < q$ are assumed to be consecutive primes

---

v) This is mainly Prop. 9.4.3.i, but now with exponents denoted explicitly:

$\mathbb{S}^{\mathsf{T}*}_{\mathbb{P}};[p\rightarrowtail] = \mathbb{S}^{\mathsf{T}*}_{\mathbb{P}};\mathbb{S}^{\mathsf{T}\xi(p)}_{\mathbb{P}};z_{\mathbb{P}};\mathbb{T}$ due to (iv)

$= \mathbb{S}^{\mathsf{T}\xi(p)}_{\mathbb{P}};\mathbb{T}$ Prop. 5.2.1.ii.k

vi) $f^{\mathsf{T}_i}[p\!\rightarrow] = \mathtt{ND}^{\mathsf{T}_i}\iota^{\mathsf{T}}_{\wp;\mathbb{T}};\iota_{\wp;\mathbb{T}};\mathbb{S}^{\mathsf{T}p};z;\mathbb{T}$ both expanded
$= \mathfrak{S}^{\mathsf{T}_i}(\mathtt{NP}^{\mathsf{T}} \bigotimes \mathbb{I});\wp;\mathbb{S}^{\mathsf{T}p};z;\mathbb{T}$ Def. 9.4.7
$= \mathfrak{S}^{\mathsf{T}_i}(\mathtt{NP}^{\mathsf{T}} \bigotimes \mathbb{I});\mathbb{S}^{\mathsf{T}p};z;\mathbb{T}$ since p was assumed to be an odd prime
$= \mathfrak{S}^{\mathsf{T}_i}(\mathtt{NP}^{\mathsf{T}};\mathbb{S}^{\mathsf{T}p};z;\mathbb{T} \bigotimes \mathbb{S}^{\mathsf{T}p};z;\mathbb{T})$
$= \mathfrak{S}^{\mathsf{T}_i}(\mathbb{S}^{\mathsf{T}q};z;\mathbb{T} \bigotimes \mathbb{S}^{\mathsf{T}p};z;\mathbb{T})$ due to (ii)
$= \mathbb{S}^{\mathsf{T}q-p};z;\mathbb{T}$ Prop. 6.3.1.ii.i □

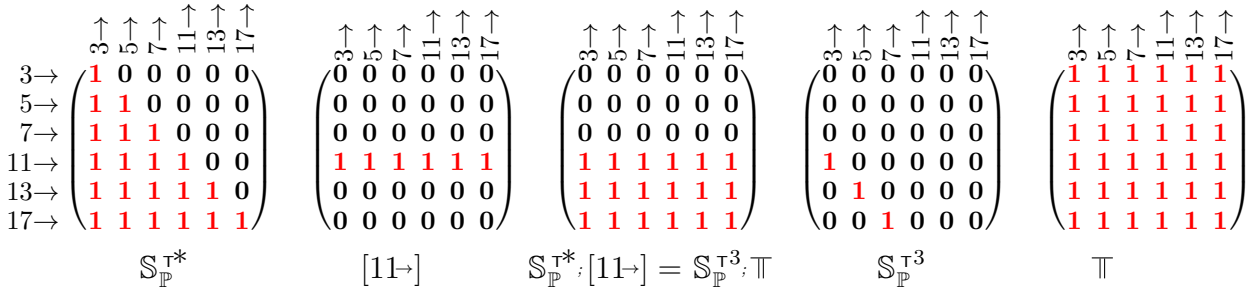


Fig. 9.6.3 Illustrating Prop. 9.6.4.v: Compare with Fig. 5.2.2!

In the proof of (ii), the construct $\iota_{\wp;\mathbb{T}};\left[\mathbb{S}^+ \cap \overline{\mathbb{S}^+;\wp;\mathbb{S}^+}\right]$ is interesting. By $\iota_{\wp;\mathbb{T}}$ it is expressed that the start into the bracketed expression is restricted to an odd prime. Then it will be proceeded by $\mathbb{S}^+$, but never a prime must be met in between. Thus, one proceeds from the initial prime to the next prime. This will in the following be investigated in more detail.

Problems in number theory — when focussing on primes — have their origin in the non-algebraic handling of the prime coefficients just mentioned. They appear as exponents, as normal numbers, and as elements of $\mathbb{P}$. They are mostly handled free-style.

In oder to acquire intuition, we will now work with powers $\mathbb{S}^3$, $\mathbb{S}^5$ of $\mathbb{S}$ and also with the power of their difference $\mathbb{S}^{5-3} = \mathbb{S}^2$. Standard mathematics doesn't provide algebraic or relational means to cope with this situation. We will study powers of $\mathbb{S}$ and the vectorizations thereof and compare this with the relation add.

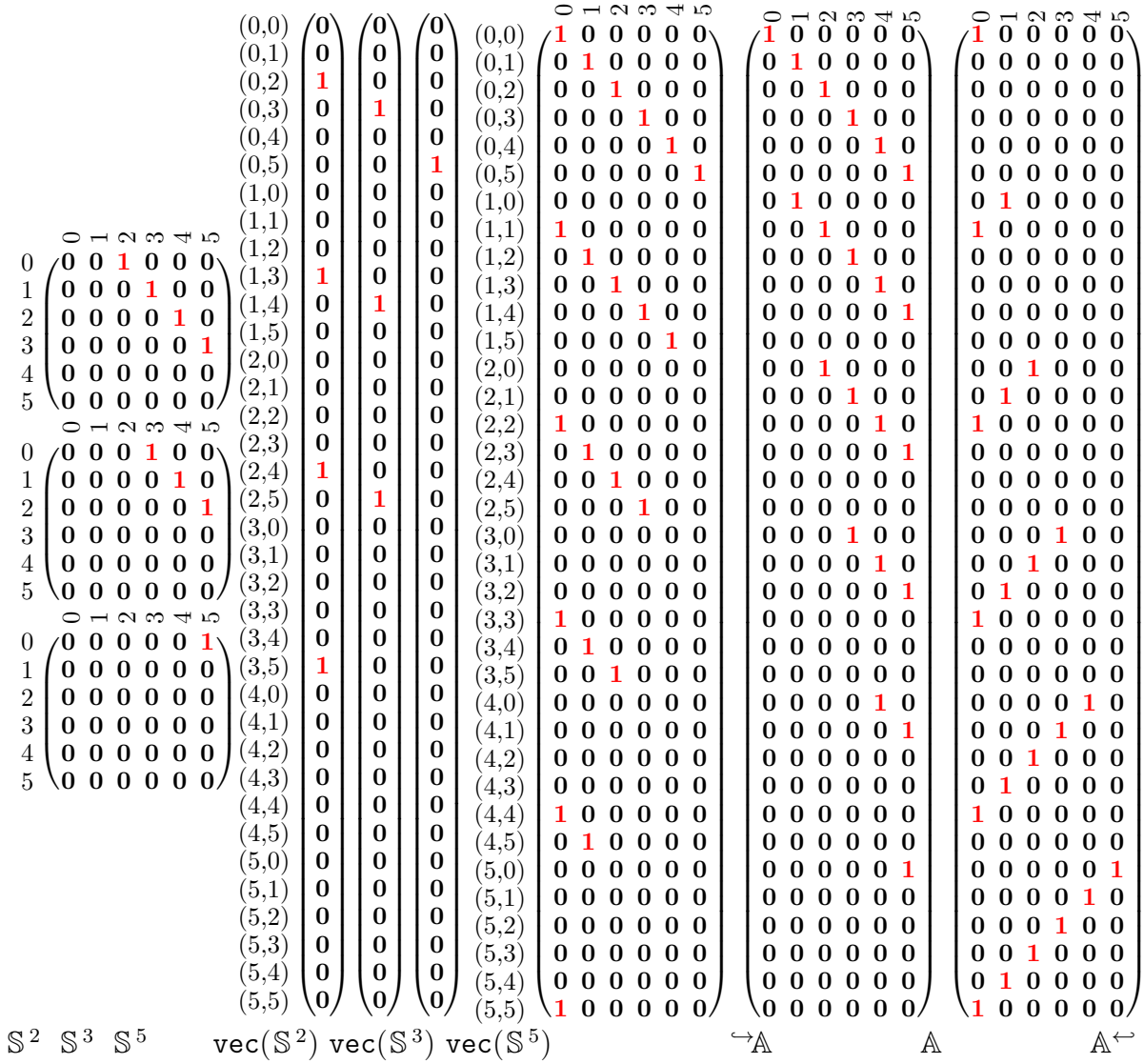


Fig. 9.6.4 Vectorizations of powers of $\mathbb{S}$ compared with addition $\mathbb{A}$ and its inverses

Looking at this Fig. 9.6.4, we feel triggered to look at the left inverse of $\mathbb{A}$ — according to Sect. 3.2.

At last, we find out that the relation we need is just the middle one representing addition $\mathbb{A}$ on naturals. So we will try to proceed in this way:

9.6.5 Definition and Proposition. The relation $\hookrightarrow\mathbb{A} = \mathtt{S}$ satisfies the following:
i) $\hookrightarrow\mathbb{A} = \left[\hookrightarrow\mathbb{A}\right]^{\hookleftarrow}$
ii) $\mathbb{S}^{\,d} = \mathtt{rel}(\hookrightarrow\mathbb{A} \,;\, f^{\mathsf{T}} \,;\, [p\!\rightarrow])$ with d the distance of p to the next higher prime

Proof: i) $\left[\hookrightarrow\mathbb{A}\right]^{\hookleftarrow} = \mathtt{S}^{\hookleftarrow} = \mathtt{S} = \hookrightarrow\mathbb{A}$ applying Prop. 6.3.1.ii.e

ii) $\mathtt{rel}(\hookrightarrow\mathbb{A} \,;\, f^{\mathsf{T}} \,;\, [p\!\rightarrow]) = \mathtt{rel}(\hookrightarrow\mathbb{A} \,;\, \mathtt{ND}^{\mathsf{T}} \,;\, \iota^{\mathsf{T}}_{\wp\mathbb{T}} \,;\, [p\!\rightarrow])$ following Def. 9.4.7
$= \mathtt{rel}(\hookrightarrow\mathbb{A} \,;\, \mathtt{ND}^{\mathsf{T}} \,;\, \mathbb{S}^{\,\mathsf{T}p} \,;\, z \,;\, \mathbb{T})$ Prop. 9.6.4.ii
$= \mathtt{rel}(\mathtt{S} \,;\, \mathfrak{S}^{\mathsf{T}} \,;\, (\mathtt{NP}^{\mathsf{T}} \circledcirc \mathbb{I}) \,;\, \mathbb{S}^{\,\mathsf{T}p} \,;\, z \,;\, \mathbb{T})$ by definition above; and again following Def. 9.4.7
$= \mathtt{rel}(\mathtt{S} \,;\, \mathfrak{S}^{\mathsf{T}} \,;\, (\mathtt{NP}^{\mathsf{T}} \,;\, \mathbb{S}^{\,\mathsf{T}p} \,;\, z \,;\, \mathbb{T} \circledcirc \mathbb{S}^{\,\mathsf{T}p} \,;\, z \,;\, \mathbb{T}))$

$= \mathtt{rel}(\mathtt{S}\,;\mathfrak{S}^{\mathsf{T}}\,;\,(\mathbb{S}^{\mathsf{T}p}\,;z\,;\mathbb{T} \circledcirc \mathbb{S}^{\mathsf{T}p}\,;z\,;\mathbb{T}))$ using Prop. 9.6.4.ii with q the next prime after p
$= \mathtt{rel}(\mathtt{S}\,;\mathbb{S}^{\mathsf{T}q-p}\,;z\,;\mathbb{T}) = \mathtt{rel}(\mathtt{S}\,;\mathbb{S}^{\mathsf{T}d}\,;z\,;\mathbb{T})$ with Prop. 6.3.1.ii.i
$= \mathtt{rel}(\mathtt{vec}(\mathbb{S}^{d})) = \mathbb{S}^{d}$ Prop. 6.3.1.ii.e □

We will now try to transfer the bulk of these definitions into a coherent system, enabling us to go from any version of defining a prime to every other. The start will be some given number $n \in \mathbb{N}$ in the classical way. From this n we immediately arrive at the vector

$\mathbb{S}^{\mathsf{T}n}\,;z\,;\mathbb{T} = \underline{|n|}$.

The first notation has so far been used everywhere in relational context. The variant doesn't use that much space and is here introduced due to this purpose.

If we assume the given number to be a prime p inside the present limits, this will work in the same way. We may in addition obtain the point

$[p^{\rightarrow}] := \iota_{\wp;\mathbb{T}}\,;\mathbb{S}^{\mathsf{T}p}\,;z\,;\mathbb{T} = \iota_{\wp;\mathbb{T}}\,;\underline{|p|}$,

which is — of course — also representable with some exponent $c \in \mathbb{N}$ as

$[p^{\rightarrow}] = \mathbb{S}_{\mathbb{P}}^{\mathsf{T}c}\,;z_{\mathbb{P}}\,;\mathbb{T}$,

thus „defining“ this necessary exponent as "prime sequence exponent". Be aware that the prime sequence exponent starts with 0 for the prime 3. Of course, as just a number this c has also a representation as $\mathbb{S}^{\mathsf{T}c}\,;z\,;\mathbb{T} = \underline{|c|}$, and we ought to be able to switch back and forth between these notations.

If the p is originally given, we will be able to obtain

$\mathbb{S}^{\mathsf{T}p}\,;z\,;\mathbb{T} = \underline{|p|} \qquad [p^{\rightarrow}] = \iota_{\wp;\mathbb{T}}\,;\underline{|p|}$,

together with its prime sequence exponent

$\{\cdots p\} = \mathtt{psn}^{\mathsf{T}}\,;[p^{\rightarrow}]$.

Of course, there exist other interrelationships expressed via

$\underline{|p|} = \iota_{\wp;\mathbb{T}}^{\mathsf{T}}\,;[p^{\rightarrow}] \qquad [p^{\rightarrow}] = \mathtt{psn}\,;\{\cdots p\} \qquad \mathtt{SeqNtoPri} = \mathtt{psn}^{\mathsf{T}}\,;\iota_{\wp;\mathbb{T}}$

We may, however start with the prime sequence exponent c and obtain all the others as in Fig. 9.6.1: $[p^{\rightarrow}] = \mathtt{psn}\,;\underline{|c|} \qquad \{\cdots p\}$

We will from now on use only these three versions in order to stay completely in the realm of relations.

$$[p^{\rightarrow}] = \iota_{\wp;\mathbb{T}}\,;\mathbb{S}^{\mathsf{T}p}\,;z\,;\mathbb{T}$$

Because $\iota_{\wp;\mathbb{T}}$ is an injective map, we may also obtain $\mathbb{S}^{\mathsf{T}p}\,;z\,;\mathbb{T}$ back using Prop. 9.6.4.ii

$\mathbb{S}^{\mathsf{T}p}\,;z\,;\mathbb{T} = \iota_{\wp;\mathbb{T}}^{\mathsf{T}}\,;[p^{\rightarrow}]$:

$$\begin{array}{c} \\ 0\\1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15\\16\\17\\18\\19\\20\\21\\22\\23\\24\\25\\26\\27\\28\\29\\30 \end{array}\begin{pmatrix}\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\mathbf{1}\\\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\mathbf{0}\end{pmatrix} = \begin{array}{c} \\ 0\\1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15\\16\\17\\18\\19\\20\\21\\22\\23\\24\\25\\26\\27\\28\\29\\30 \end{array}\begin{array}{c} \begin{matrix}3{\rightarrow}&5{\rightarrow}&7{\rightarrow}&11{\rightarrow}&13{\rightarrow}&17{\rightarrow}&19{\rightarrow}&23{\rightarrow}&29{\rightarrow}\end{matrix}\\ \begin{pmatrix} \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0} \end{pmatrix} \end{array}\; ; \quad \begin{array}{c} \\ 3{\rightarrow}\\5{\rightarrow}\\7{\rightarrow}\\11{\rightarrow}\\13{\rightarrow}\\17{\rightarrow}\\19{\rightarrow}\\23{\rightarrow}\\29{\rightarrow} \end{array}\begin{pmatrix}\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\mathbf{1}\\\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\mathbf{0}\end{pmatrix}$$

$$\mathbb{S}^{\intercal p}{}_{;}\, z \,{}_{;}\mathbb{T} = \iota^{\intercal}_{\wp;\mathbb{T}}[p{\mapsto}]$$

$$\begin{array}{c} \\ 3{\rightarrow}\\5{\rightarrow}\\7{\rightarrow}\\11{\rightarrow}\\13{\rightarrow}\\17{\rightarrow}\\19{\rightarrow}\\23{\rightarrow}\\29{\rightarrow} \end{array}\begin{pmatrix}\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\mathbf{1}\\\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\mathbf{0}\end{pmatrix} = \begin{array}{c} \begin{matrix}0&1&2&3&4&5&6&7&8&9&10&11&12&13&14&15&16&17&18&19&20&21&22&23&24&25&26&27&28&29&30\end{matrix}\\ \begin{pmatrix} \mathbf{1}&\mathbf{0}\\ \mathbf{0}&\mathbf{1}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0} \end{pmatrix} \end{array}\; ; \quad \begin{array}{c} \\ 0\\1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15\\16\\17\\18\\19\\20\\21\\22\\23\\24\\25\\26\\27\\28\\29\\30 \end{array}\begin{pmatrix}\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\mathbf{1}\\\mathbf{0}\end{pmatrix}$$

$$[p{\mapsto}] = \mathtt{psn}{}_{;}\, \mathbb{S}^{\intercal b}{}_{;}\, z \,{}_{;}\mathbb{T}$$

There is also the possibility to number the sequence of primes. This gives rise to define as follows:

Of course also $\mathbb{S}^{\mathsf{T}^b}{}_{;} z_{;} \mathbb{T} = \mathtt{psn}^{\mathsf{T}}{}_{;}[p\!\!\rightarrow]$, because $\mathtt{psn}$ is an injective mapping.

$$
\begin{array}{c|c}
 & \begin{array}{ccccccccccccccccccccccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 & 27 & 28 & 29 & 30 \end{array} \\
\hline
0 & \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{1}\ \mathbf{0} \\
1 & \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{1}\ \mathbf{0} \\
2 & \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{1}\ \mathbf{0} \\
3 & \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{1}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0} \\
4 & \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{1}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0} \\
5 & \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{1}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0} \\
6 & \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{1}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0} \\
7 & \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{1}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0} \\
8 & \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{1}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0} \\
9 & \mathbf{0}\ \mathbf{0} \\
10 & \mathbf{0}\ \mathbf{0} \\
11 & \mathbf{0}\ \mathbf{0} \\
12 & \mathbf{0}\ \mathbf{0} \\
13 & \mathbf{0}\ \mathbf{0} \\
14 & \mathbf{0}\ \mathbf{0} \\
15 & \mathbf{0}\ \mathbf{0} \\
16 & \mathbf{0}\ \mathbf{0} \\
17 & \mathbf{0}\ \mathbf{0} \\
18 & \mathbf{0}\ \mathbf{0} \\
19 & \mathbf{0}\ \mathbf{0} \\
20 & \mathbf{0}\ \mathbf{0} \\
21 & \mathbf{0}\ \mathbf{0} \\
22 & \mathbf{0}\ \mathbf{0} \\
23 & \mathbf{0}\ \mathbf{0} \\
24 & \mathbf{0}\ \mathbf{0} \\
25 & \mathbf{0}\ \mathbf{0} \\
26 & \mathbf{0}\ \mathbf{0} \\
27 & \mathbf{0}\ \mathbf{0} \\
28 & \mathbf{0}\ \mathbf{0} \\
29 & \mathbf{0}\ \mathbf{0} \\
30 & \mathbf{0}\ \mathbf{0}
\end{array}
\quad = \mathtt{psn}^{\mathsf{T}}{}_{;} \iota_{\wp;\mathbb{T}}
$$

gives the relation from numbers n to the nth odd prime. (Again numbering ise from 0 on.)

Now we recall prime distances in a relational way according to Defs. 9.4.1 and 9.4.7

$\mathtt{NP} = \iota^{\mathsf{T}}_{\wp;\mathbb{T}}{}_{;} \mathbb{S}_{\mathbb{P}}{}_{;} \iota_{\wp;\mathbb{T}} : \mathbb{N} \longrightarrow \mathbb{N}$ the successor relation inside $\mathbb{N}$ concerning odd primes ,

$\mathtt{ND} = (\mathtt{NP} \circledcirc \mathbb{I})_{;} \mathfrak{S} : \mathbb{N} \longrightarrow \mathbb{N}$ the next prime distance function

Also for these two relations $\mathtt{NP}$, $\mathtt{ND}$, we now collect their interrelationships:

$\mathbb{S}_{\mathbb{P}}{}_{;} \iota_{\wp;\mathbb{T}} = \iota_{\wp;\mathbb{T}}{}_{;} \mathtt{NP}$ $\qquad\qquad$ $\iota_{\wp;\mathbb{T}}{}_{;} \mathtt{NP}_{;} \iota^{\mathsf{T}}_{\wp;\mathbb{T}} = \mathbb{S}_{\mathbb{P}}$

making $\iota_{\wp;\mathbb{T}}$ a homomorphism from $\mathbb{S}_{\mathbb{P}}$ to $\mathtt{NP}$, i.e. from $\mathbb{P}$ to $\mathbb{N}$.

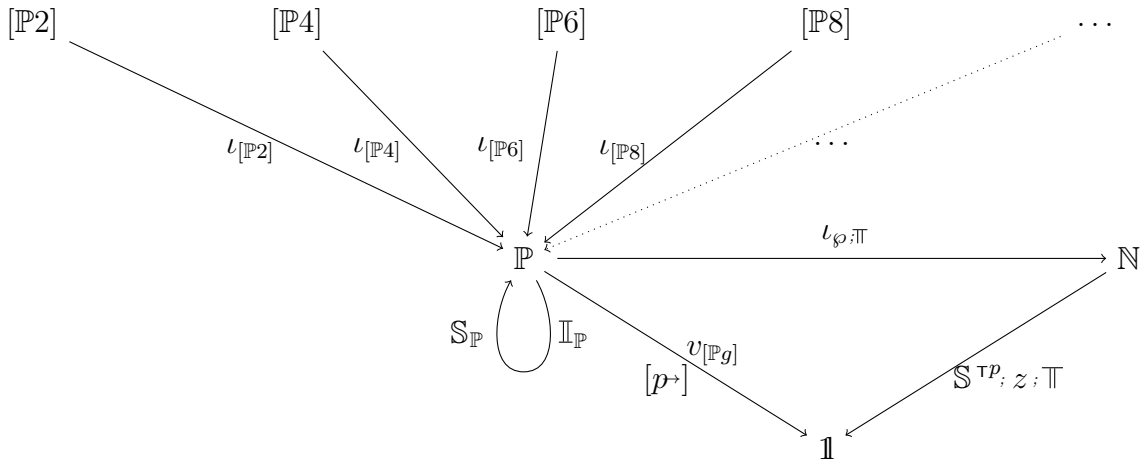


Fig. 9.6.5 Decomposing $\mathbb{P}$ according to prime distances

Obviously for primes below 400

$$\begin{aligned}
[\mathbb{P}2] &= \{3{,}5,\ 11,\ 17,\ 29,\ 41,\ 59,\ 71,\ 101,\ \dots\}\\
[\mathbb{P}4] &= \{\ 7,\ 13,\ 19,\ 37,\ 43,\ 67,\ 79,\ 97,\ 103{,}\dots\}\\
[\mathbb{P}6] &= \{\ 23,\ 31,\ 47{,}53,\ 61,\ 73,\ 83,\ \dots\}\\
[\mathbb{P}8] &= \{\ 89,\ \dots\}
\end{aligned}$$

and later

$$\begin{aligned}
[\mathbb{P}2] &= \{\dots 107, 137, 149, 179, 191, 197, 227, 239, 269, 281, 311, 347, \dots\}\\
[\mathbb{P}4] &= \{\dots 109, 127, 163, 193, 223, 229, 277, 307, 313, 349, 379, \dots\}\\
[\mathbb{P}6] &= \{\dots 131, 151, 157, 167, 173, 233, 251, 257, 263, 271, 331, 353, 367, 373, 383, \dots\}\\
[\mathbb{P}8] &= \{\dots 359, 389, \dots\}\\
[\mathbb{P}10] &= \{139, 181, 241, 283, 337, \dots\}\\
[\mathbb{P}12] &= \{199, 211, \dots\}\\
[\mathbb{P}14] &= \{113, 293, 317, \dots\}\\
&\dots
\end{aligned}$$

One will hardly detect any regularity. Of course, 397 is also prime, the biggest among those below 400 and, thus, without successor in this initial section. Not even the initial occurences grow monotonically as shown here:

[ℙ2]	[ℙ4]	[ℙ6]	[ℙ8]	[ℙ10]	[ℙ12]	[ℙ14]	...
3	7	23	89	139	199	113	...

9.6.6 Definition (*Properties of prime distances*)**.** For odd primes of $\mathbb{P}$ the corresponding distances $f = \iota_{\wp;\mathbb{T}};\mathtt{PD} = \iota_{\wp;\mathbb{T}};(\mathtt{NP} \oslash \mathbb{I});\mathfrak{S} : \mathbb{P} \longrightarrow \mathbb{N}$ have been defined in Def. 9.4.7. For $g = 2, 4, 6, 8, \dots$, the vectors

$\upsilon_{[\mathbb{P}g]} := \iota_{\wp;\mathbb{T}};(\mathtt{NP} \oslash \mathbb{I});\mathfrak{S};\mathbb{S}^{\mathsf{T}g};z;\mathbb{T}$ are considered and — if nonempty — extruded as

$\iota_{[\mathbb{P}g]} := \mathtt{Inject}(\upsilon_{[\mathbb{P}g]}) : [\mathbb{P}g] \longrightarrow \mathbb{P}$, thus *defining* $[\mathbb{P}g]$ □

In the finite case, the greatest prime will have no successor, and, thus, no prime distance attached.

9.6.7 Proposition (*Relations concerning prime distances*)**.**

i) $\mathtt{sup}\,\big\{\, \iota^{\mathsf{T}}_{[\mathbb{P}g]\,;}\iota_{[\mathbb{P}g]} \;\big|\; v_{[\mathbb{P}g]} \neq \perp\!\!\!\perp \;\big|\; g = 2,4,6,8\,\ldots\,\big\} \subseteq \mathbb{I}_{\mathbb{P}}$ in general

$\mathtt{sup}_{g=2,4,6,8\,\ldots}\, v_{[\mathbb{P}g]\,;}\mathbb{T} \subseteq \mathbb{T}_{\mathbb{P}}$

ii) $\mathtt{sup}\,\big\{\, \iota^{\mathsf{T}}_{[\mathbb{P}g]\,;}\iota_{[\mathbb{P}g]} \;\big|\; v_{[\mathbb{P}g]} \neq \perp\!\!\!\perp \;\big|\; g = 2,4,6,8\,\ldots\,\big\} = \mathbb{I}_{\mathbb{P}}$ if $\mathbb{S}$ is total

$\mathtt{sup}_{g=2,4,6,8\,\ldots}\, v_{[\mathbb{P}g]\,;}\mathbb{T} = \mathbb{T}_{\mathbb{P}}$

iii) If $\mathbb{S}$ is not total, there exists a maximal prime q or $\mathbb{S}^{\mathsf{T}q\,;}\, z\,;\mathbb{T}$ or $[q^{\rightarrow}] := \overline{\mathbb{S}_{\mathbb{P}\,;}\mathbb{T}}$. Then $[q^{\rightarrow}]_{;}[q^{\rightarrow}]^{\mathsf{T}} \cup \mathtt{sup}\,\big\{\, \iota^{\mathsf{T}}_{[\mathbb{P}g]\,;}\iota_{[\mathbb{P}g]} \;\big|\; v_{[\mathbb{P}g]} \neq \perp\!\!\!\perp \;\big|\; g = 2,4,6,8\,\ldots\,\big\} = \mathbb{I}_{\mathbb{P}}$.

$[q^{\rightarrow}]\,;\mathbb{T} \cup \mathtt{sup}_{g=2,4,6,8\,\ldots}\, v_{[\mathbb{P}g]\,;}\mathbb{T} = \mathbb{T}_{\mathbb{P}}$ $\qquad$ $\mathtt{sup}_{g=2,4,6,8\,\ldots}\, v_{[\mathbb{P}g]\,;}\mathbb{T} = \overline{[q^{\rightarrow}]\,;\mathbb{T}} = \mathbb{S}_{\mathbb{P}\,;}\mathbb{T}$

iv) As a generalization of Stäckel's conjecture[3] in case $\mathbb{S}$ should be total:
$[\mathbb{P}2], [\mathbb{P}4], [\mathbb{P}6], \ldots$ are all non-empty and infinite sets

v) $v_{[\mathbb{P}g]} = \big[\mathbb{I}_{\mathbb{P}} \cap \mathbb{S}_{\mathbb{P}\,;}\iota_{\wp;\mathbb{T}\,;}\mathbb{S}^{\mathsf{T}g\,;}\iota^{\mathsf{T}}_{\wp;\mathbb{T}}\big]\,;\mathbb{T}$ $\qquad$ $\iota^{\mathsf{T}}_{[\mathbb{P}g]\,;}\iota_{[\mathbb{P}g]} = \mathbb{I}_{\mathbb{P}} \cap \mathbb{S}_{\mathbb{P}\,;}\iota_{\wp;\mathbb{T}\,;}\mathbb{S}^{\mathsf{T}g\,;}\iota^{\mathsf{T}}_{\wp;\mathbb{T}}$

vi) $\iota^{\mathsf{T}}_{[\mathbb{P}g]\,;}\iota_{[\mathbb{P}g]\,;}\iota_{\wp;\mathbb{T}\,;}\mathbb{S}^{\,g} = \iota^{\mathsf{T}}_{[\mathbb{P}g]\,;}\iota_{[\mathbb{P}g]\,;}\mathbb{S}_{\mathbb{P}\,;}\iota_{\wp;\mathbb{T}}$ $\qquad$ $\iota_{\wp;\mathbb{T}\,;}\mathbb{S}^{\,g} \cap v_{[\mathbb{P}g]\,;}\mathbb{T} = \mathbb{S}_{\mathbb{P}\,;}\iota_{\wp;\mathbb{T}} \cap v_{[\mathbb{P}g]\,;}\mathbb{T}$

Proof: i) In general, we may have just a finite initial subsection of the naturals, so that the greatest prime therein cannot have a prime distance attached; see the prime 29 in the Fig. 9.6.6. The $\iota_{[\mathbb{P}g]}$ are all injections, so that "$\subseteq \mathbb{I}_{\mathbb{P}}$" is obtained.

ii) is obvious, since with regressive finiteness we get the usual naturals. For these, it is evident that every prime has indeed a prime distance attached.

iii) Here, it is managed to have also an entry for the greatest prime $[q^{\rightarrow}]$ by simply adding this point explicitly.

iv) This is a research problem! Its solution will in particular establish that there exist infinitely many prime twins such as $3, 5, \ldots 17, 19, \ldots 41, 43 \ldots$, e.g.

---

v) $v_{[\mathbb{P}g]} = \iota_{\wp;\mathbb{T}\,;}\,(\mathtt{NP} \circledcirc \mathbb{I})\,;\mathfrak{S}\,;\mathbb{S}^{\mathsf{T}g\,;}\, z\,;\mathbb{T} = \iota_{\wp;\mathbb{T}\,;}\,(\iota^{\mathsf{T}}_{\wp;\mathbb{T}\,;}\mathbb{S}_{\mathbb{P}\,;}\iota_{\wp;\mathbb{T}} \circledcirc \mathbb{I})\,;\mathfrak{S}\,;\mathbb{S}^{\mathsf{T}g\,;}\, z\,;\mathbb{T}$ expanded
$= (\iota_{\wp;\mathbb{T}\,;}\iota^{\mathsf{T}}_{\wp;\mathbb{T}\,;}\mathbb{S}_{\mathbb{P}\,;}\iota_{\wp;\mathbb{T}} \circledcirc \iota_{\wp;\mathbb{T}})\,;\mathfrak{S}\,;\mathbb{S}^{\mathsf{T}g\,;}\, z\,;\mathbb{T}$ since $\iota_{\wp;\mathbb{T}}$ is univalent
$= (\mathbb{S}_{\mathbb{P}\,;}\iota_{\wp;\mathbb{T}} \circledcirc \iota_{\wp;\mathbb{T}})\,;\mathfrak{S}\,;\mathbb{S}^{\mathsf{T}g\,;}\, z\,;\mathbb{T}$ since $\iota_{\wp;\mathbb{T}}$ is an injective map
$= (\mathbb{S}_{\mathbb{P}\,;}\iota_{\wp;\mathbb{T}} \circledcirc \iota_{\wp;\mathbb{T}})\,;\big[\,\mathtt{sup}_{n\geq 0}\,(\mathbb{S}^{\mathsf{T}n} \circledcirc \mathbb{I})\,;\mathbb{T}\,;z^{\mathsf{T}}\,;\mathbb{S}^{\,n}\big]\,;\mathbb{S}^{\mathsf{T}g\,;}\, z\,;\mathbb{T}$ due to Prop. 6.3.1.ii.e together with i.e
$= (\mathbb{S}_{\mathbb{P}\,;}\iota_{\wp;\mathbb{T}} \circledcirc \iota_{\wp;\mathbb{T}})\,;\big[\,\mathtt{sup}_{n\geq 0}\,(\mathbb{S}^{\mathsf{T}n} \circledcirc \mathbb{I})\,;\mathbb{T}\,;z^{\mathsf{T}}\,;\mathbb{S}^{\,n}\,;\mathbb{S}^{\mathsf{T}g\,;}\, z\,;\mathbb{T}\big]$ implies $n := g$ to be $\neq \perp\!\!\!\perp$
$= (\mathbb{S}_{\mathbb{P}\,;}\iota_{\wp;\mathbb{T}} \circledcirc \iota_{\wp;\mathbb{T}})\,;(\mathbb{S}^{\mathsf{T}g} \circledcirc \mathbb{I})\,;\mathbb{T}\,;z^{\mathsf{T}}\,;\mathbb{S}^{\,g}\,;\mathbb{S}^{\mathsf{T}g\,;}\, z\,;\mathbb{T}$
$= (\mathbb{S}_{\mathbb{P}\,;}\iota_{\wp;\mathbb{T}} \circledcirc \iota_{\wp;\mathbb{T}})\,;(\mathbb{S}^{\mathsf{T}g} \circledcirc \mathbb{I})\,;\mathbb{T}$
$= \big[\mathbb{S}_{\mathbb{P}\,;}\iota_{\wp;\mathbb{T}\,;}\mathbb{S}^{\mathsf{T}g} \cap \iota_{\wp;\mathbb{T}}\big]\,;\mathbb{T}$ see [13, Prop. 7.5]
$= \big[\mathbb{S}_{\mathbb{P}\,;}\iota_{\wp;\mathbb{T}\,;}\mathbb{S}^{\mathsf{T}g\,;}\iota^{\mathsf{T}}_{\wp;\mathbb{T}} \cap \mathbb{I}_{\mathbb{P}}\big]\,;\iota_{\wp;\mathbb{T}\,;}\,\mathbb{T}$ destroy and append in reverse direction
$= \big[\mathbb{S}_{\mathbb{P}\,;}\iota_{\wp;\mathbb{T}\,;}\mathbb{S}^{\mathsf{T}g\,;}\iota^{\mathsf{T}}_{\wp;\mathbb{T}} \cap \mathbb{I}_{\mathbb{P}}\big]\,;\mathbb{T}$ because $\iota_{\wp;\mathbb{T}}$ is total

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[3] See [17].

$\iota^{\mathsf{T}}_{[\mathbb{P}g]};\iota_{[\mathbb{P}g]} = \mathbb{I}_{\mathbb{P}} \cap \iota^{\mathsf{T}}_{[\mathbb{P}g]};\mathbb{T} = \mathbb{I}_{\mathbb{P}} \cap v_{[\mathbb{P}g]};\mathbb{T}$
$= \mathbb{I}_{\mathbb{P}} \cap \left[\mathbb{I}_{\mathbb{P}} \cap \mathbb{S}_{\mathbb{P}};\iota_{\wp;\mathbb{T}};\mathbb{S}^{\mathsf{T}g};\iota^{\mathsf{T}}_{\wp;\mathbb{T}}\right];\mathbb{T}$ see above
$= \mathbb{I}_{\mathbb{P}} \cap \mathbb{S}_{\mathbb{P}};\iota_{\wp;\mathbb{T}};\mathbb{S}^{\mathsf{T}g};\iota^{\mathsf{T}}_{\wp;\mathbb{T}}$ Prop. 2.1.1.v

---

vi) We abbreviate $D := \iota^{\mathsf{T}}_{[\mathbb{P}g]};\iota_{[\mathbb{P}g]}$. Then the claim reads $D;\iota_{\wp;\mathbb{T}};\mathbb{S}^{g} = D;\mathbb{S}_{\mathbb{P}};\iota_{\wp;\mathbb{T}}$. We start with
$D;\iota_{\wp;\mathbb{T}};\mathbb{S}^{g} = D;D;\iota_{\wp;\mathbb{T}};\mathbb{S}^{g} = D;\left[\mathbb{I}_{\mathbb{P}} \cap \mathbb{S}_{\mathbb{P}};\iota_{\wp;\mathbb{T}};\mathbb{S}^{\mathsf{T}g};\iota^{\mathsf{T}}_{\wp;\mathbb{T}}\right];\iota_{\wp;\mathbb{T}};\mathbb{S}^{g}$ see (v)
$= D;\left[\iota_{\wp;\mathbb{T}};\mathbb{S}^{g} \cap \mathbb{S}_{\mathbb{P}};\iota_{\wp;\mathbb{T}}\right]$ destroy and append twice
$\subseteq D;\mathbb{S}_{\mathbb{P}};\iota_{\wp;\mathbb{T}}$,
but also
$D;\mathbb{S}_{\mathbb{P}};\iota_{\wp;\mathbb{T}} = D;D^{\mathsf{T}};\mathbb{S}_{\mathbb{P}};\iota_{\wp;\mathbb{T}} = D;\left[\mathbb{I}_{\mathbb{P}} \cap \iota_{\wp;\mathbb{T}};\mathbb{S}^{g};\iota^{\mathsf{T}}_{\wp;\mathbb{T}};\mathbb{S}^{\mathsf{T}}_{\mathbb{P}}\right];\mathbb{S}_{\mathbb{P}};\iota_{\wp;\mathbb{T}}$ again (v)
$= D;\left[\mathbb{S}_{\mathbb{P}};\iota_{\wp;\mathbb{T}} \cap \iota_{\wp;\mathbb{T}};\mathbb{S}^{g}\right]$ destroy and append twice
$\subseteq D;\iota_{\wp;\mathbb{T}};\mathbb{S}^{g}$

Because $\iota_{[\mathbb{P}g]} = \mathtt{Inject}(v_{[\mathbb{P}g]})$ means $v_{[\mathbb{P}g]} = \iota^{\mathsf{T}}_{[\mathbb{P}g]};\mathbb{T}$ and $\iota^{\mathsf{T}}_{[\mathbb{P}g]};\iota_{[\mathbb{P}g]} \subseteq \mathbb{I}$, we have in general
$X \cap v_{[\mathbb{P}g]};\mathbb{T} = \iota^{\mathsf{T}}_{[\mathbb{P}g]};\iota_{[\mathbb{P}g]};X$. □

So far we mainly had prime distances from one prime to the next consecutive one. Now we begin to consider all upwards prime distances.

9.6.8 Definition. We define two types of **prime distance relations** for $d \geq 0$ and $g \geq 0$:
${}^{d}\mathtt{D} := \left[\mathtt{rel}(\,(\iota_{\wp;\mathbb{T}} \otimes \iota_{\wp;\mathbb{T}});\mathfrak{S};\mathbb{S}^{\mathsf{T}d};z\,;\mathbb{T})\right]^{\mathsf{T}} : \mathbb{P} \longrightarrow \mathbb{P}$
${}^{g}\mathtt{B} := \mathbb{S}_{\mathbb{P}} \cap \iota_{\wp;\mathbb{T}};\mathbb{S}^{g};\iota^{\mathsf{T}}_{\wp;\mathbb{T}} = \mathbb{S}_{\mathbb{P}} \cap v_{[\mathbb{P}g]} : \mathbb{P} \longrightarrow \mathbb{P}$. □

The difference between the two concepts is visualized in Fig. 9.6.6 below and then the following relations.

9.6.9 Proposition. i) ${}^{0}\mathtt{D} = \mathbb{I}$

ii) ${}^{2*n+1}\mathtt{D} = \perp\!\!\!\perp$ for $n \geq 0$

iii) $\mathtt{sup}_{d\geq 0}\,{}^{d}\mathtt{D} = \mathbb{S}_{\mathbb{P}}{}^{*}$ $\mathtt{sup}_{d>0}\,{}^{d}\mathtt{D} = \mathbb{S}_{\mathbb{P}}{}^{+}$

iv) $\mathbb{S}_{\mathbb{P}} = \mathtt{sup}_{g=2,4,6,8\ldots}\,{}^{g}\mathtt{B}$

Proof: i) $\left[{}^{0}\mathtt{D}\right]^{\mathsf{T}} = \mathtt{rel}(\,(\iota_{\wp;\mathbb{T}} \otimes \iota_{\wp;\mathbb{T}});\mathfrak{S};\mathbb{S}^{\mathsf{T}0};z\,;\mathbb{T}) = \mathtt{rel}(\,(\iota_{\wp;\mathbb{T}} \otimes \iota_{\wp;\mathbb{T}});\mathfrak{S};z\,;\mathbb{T})$
$= \mathtt{rel}(\,(\iota_{\wp;\mathbb{T}} \otimes \iota_{\wp;\mathbb{T}});(\mathbb{I} \circledcirc \mathbb{I});\mathbb{T})$ following Prop. 6.3.1.ii.a
$= \mathtt{rel}(\,(\iota_{\wp;\mathbb{T}} \circledcirc \iota_{\wp;\mathbb{T}});\mathbb{T}) = \mathtt{rel}(\,(\mathbb{I} \circledcirc \mathbb{I});\iota_{\wp;\mathbb{T}};\mathbb{T})$ because $\iota_{\wp;\mathbb{T}}$ is injective
$= \mathtt{rel}(\,(\mathbb{I} \circledcirc \mathbb{I});\mathbb{T})$ because $\iota_{\wp;\mathbb{T}}$ is a map
$= \mathbb{I}$ due to Prop. 2.2.6.i

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ii) Distances between odd primes are always even. This doesn't seem to need a formal proof.

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iii) We start the proof in transposed form, beginning with Def. 9.6.8:

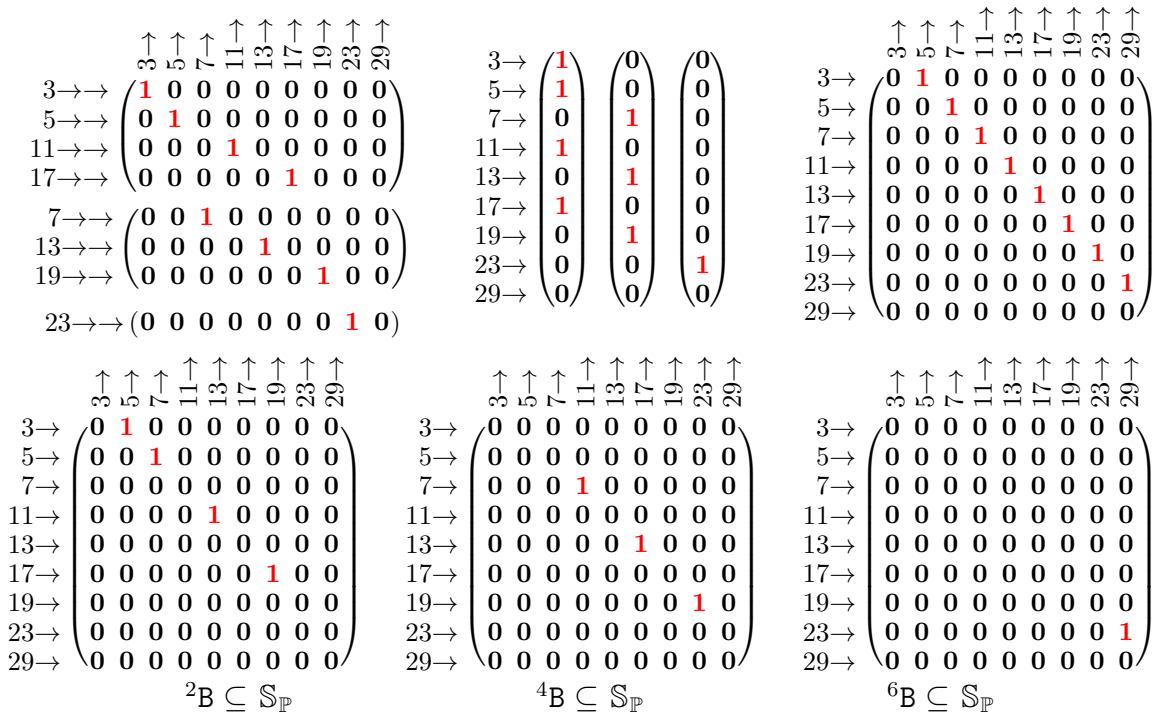


Fig. 9.6.6 Grouping primes into classes as to their prime distances: $\iota_{[\mathbb{P}2]}, \iota_{[\mathbb{P}4]}$ and $\iota_{[\mathbb{P}6]}$

$$
\begin{aligned}
&\big[\mathtt{sup}_{d>0}\,{}^{d}\mathtt{D}\big]^{\mathsf{T}}\\
&= \mathtt{sup}_{d>0}\big\{\mathtt{rel}\big(\,(\iota_{\wp;\mathbb{T}} \otimes \iota_{\wp;\mathbb{T}})\,;\mathfrak{S};\mathbb{S}^{\mathsf{T}d};z\,;\mathbb{T}\big)\big\}\\
&= \mathtt{rel}\big(\mathtt{sup}_{d>0}\,(\iota_{\wp;\mathbb{T}} \otimes \iota_{\wp;\mathbb{T}})\,;\mathfrak{S};\mathbb{S}^{\mathsf{T}d};z\,;\mathbb{T}\big) \quad \text{additivity according to Sect. 2.2}\\
&= \mathtt{rel}\big(\,(\iota_{\wp;\mathbb{T}} \otimes \iota_{\wp;\mathbb{T}})\,;\mathfrak{S};\big\{\,\mathtt{sup}_{d>0}\,\mathbb{S}^{\mathsf{T}d};z\,;\mathbb{T}\big\}\big)\\
&= \mathtt{rel}\big(\,(\iota_{\wp;\mathbb{T}} \otimes \iota_{\wp;\mathbb{T}})\,;\mathfrak{S};\overline{z\,;\mathbb{T}}\big)\\
&= \pi^{\mathsf{T}};\big\{\rho \cap \,(\iota_{\wp;\mathbb{T}} \otimes \iota_{\wp;\mathbb{T}})\,;\mathfrak{S};\overline{z\,;\mathbb{T}}\big\} \quad \text{according to Prop. 2.2.5}\\
&= \pi^{\mathsf{T}};\big\{\rho \cap \,(\iota_{\wp;\mathbb{T}} \otimes \iota_{\wp;\mathbb{T}})\,;\mathtt{sup}_{n\geq 0}\,(\mathbb{S}^{\mathsf{T}n} \circledcirc \mathbb{S}^{\mathsf{T}n};z\,;\mathbb{T})\,;\overline{z\,;\mathbb{T}}\big\} \quad \text{with } \mathfrak{S} \text{ expanded, Def. 6.3.1.i.a}\\
&= \pi^{\mathsf{T}};\big\{\rho \cap \mathtt{sup}_{n\geq 0}\,(\iota_{\wp;\mathbb{T}};\,\mathbb{S}^{\mathsf{T}n} \circledcirc \iota_{\wp;\mathbb{T}};\,\mathbb{S}^{\mathsf{T}n};z\,;\mathbb{T})\,;\overline{z\,;\mathbb{T}}\big\}\\
&= \pi^{\mathsf{T}};\big\{\rho \cap \mathtt{sup}_{n\geq 0}\,(\iota_{\wp;\mathbb{T}};\,\mathbb{S}^{\mathsf{T}n};\overline{z\,;\mathbb{T}} \circledcirc \iota_{\wp;\mathbb{T}};\,\mathbb{S}^{\mathsf{T}n};z\,;\mathbb{T})\,\big\} \quad \text{masking a join}\\
&= \mathtt{sup}_{n\geq 0}\,\pi^{\mathsf{T}};\big\{\rho \cap \pi;\iota_{\wp;\mathbb{T}};\,\mathbb{S}^{\mathsf{T}n};\overline{z\,;\mathbb{T}} \cap \rho\,;\iota_{\wp;\mathbb{T}};\,\mathbb{S}^{\mathsf{T}n};z\,;\mathbb{T}\big\}\\
&= \mathtt{sup}_{n\geq 0}\,\pi^{\mathsf{T}};\big\{\pi;\iota_{\wp;\mathbb{T}};\,\mathbb{S}^{\mathsf{T}n};\overline{z\,;\mathbb{T}} \cap \rho\,;(\mathbb{I} \cap \iota_{\wp;\mathbb{T}};\,\mathbb{S}^{\mathsf{T}n};z\,;\mathbb{T})\big\} \quad \text{rearranged}\\
&= \mathtt{sup}_{n\geq 0}\,\iota_{\wp;\mathbb{T}};\,\mathbb{S}^{\mathsf{T}n};\overline{z\,;\mathbb{T}} \cap \mathbb{T};(\mathbb{I} \cap \iota_{\wp;\mathbb{T}};\,\mathbb{S}^{\mathsf{T}n};z\,;\mathbb{T}) \quad \text{destroy and append}\\
&= \mathtt{sup}_{n\geq 0}\,\iota_{\wp;\mathbb{T}};\,\mathbb{S}^{\mathsf{T}n};\overline{z\,;\mathbb{T}} \cap \mathbb{T};(\mathbb{I} \cap \mathbb{T}\,;z^{\mathsf{T}};\mathbb{S}^{n};\iota^{\mathsf{T}}_{\wp;\mathbb{T}}) \quad \text{transposing a partial identity}\\
&= \mathtt{sup}_{n\geq 0}\,\iota_{\wp;\mathbb{T}};\,\mathbb{S}^{\mathsf{T}n};\overline{z\,;\mathbb{T}} \cap \mathbb{T};\mathbb{I} \cap \mathbb{T}\,;z^{\mathsf{T}};\mathbb{S}^{n};\iota^{\mathsf{T}}_{\wp;\mathbb{T}} \quad \text{masking}\\
&= \mathtt{sup}_{n\geq 0}\,\iota_{\wp;\mathbb{T}};\,\mathbb{S}^{\mathsf{T}n};\overline{z\,;\mathbb{T}};\mathbb{T}\,;z^{\mathsf{T}};\mathbb{S}^{n};\iota^{\mathsf{T}}_{\wp;\mathbb{T}} \quad \text{masking again}\\
&= \iota_{\wp;\mathbb{T}};\,\big[\mathtt{sup}_{n\geq 0}\,\mathbb{S}^{\mathsf{T}n};\overline{z\,;\mathbb{T}};\mathbb{T}\,;z^{\mathsf{T}};\mathbb{S}^{n}\big];\iota^{\mathsf{T}}_{\wp;\mathbb{T}} = \iota_{\wp;\mathbb{T}};\,\mathbb{S}^{\mathsf{T}+};\iota^{\mathsf{T}}_{\wp;\mathbb{T}} \quad \text{following Prop. 5.3.1.ii.b}\\
&= \mathbb{S}^{\mathsf{T}+}_{\mathbb{P}} \quad \text{due to Prop. 9.4.3.i}
\end{aligned}
$$

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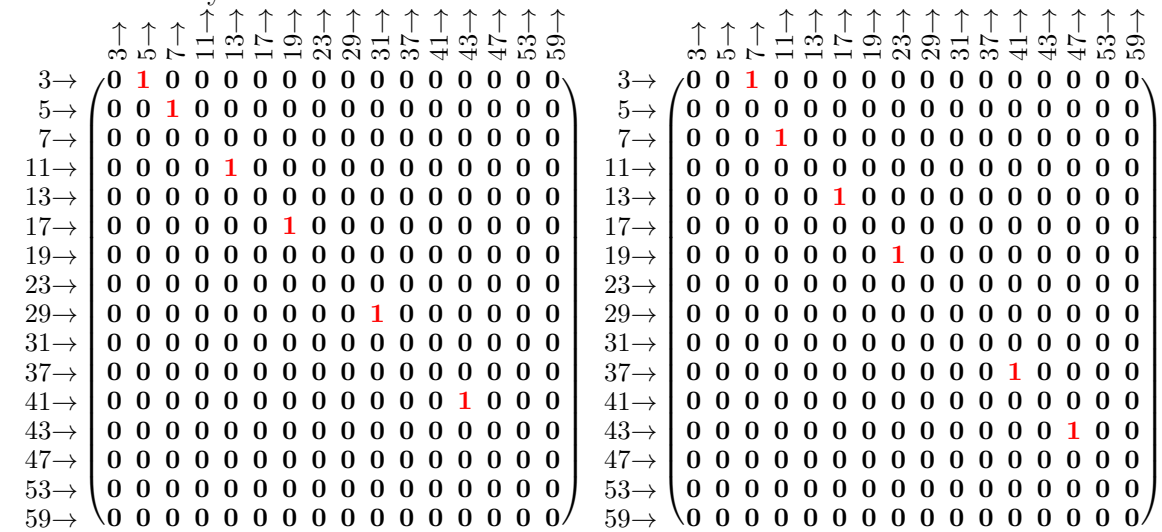
iv) Direction "$\supseteq$" is obvious by definition.

$$
\begin{aligned}
&\mathtt{sup}_{g=2,4,6,8\,\ldots}\,{}^{g}\mathtt{B} = \mathtt{sup}_{g=2,4,6,8\,\ldots}\,\big[\mathbb{S}_{\mathbb{P}} \cap v_{[\mathbb{P}g]}\big] \quad \text{by definition}\\
&= \mathbb{S}_{\mathbb{P}} \cap \mathtt{sup}_{g=2,4,6,8\,\ldots}\,v_{[\mathbb{P}g]} = \mathbb{S}_{\mathbb{P}} \cap \mathbb{S}_{\mathbb{P}};\mathbb{T} = \mathbb{S}_{\mathbb{P}} \quad \text{due to (iii)}
\end{aligned}
$$

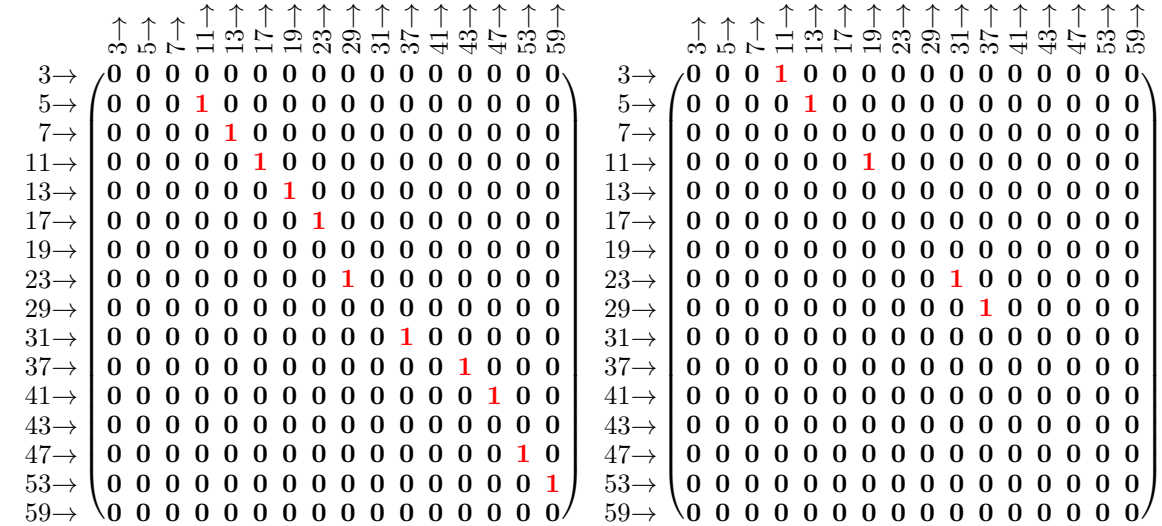
□

The following is a visualization of the prime distances ${}^{d}\mathtt{D}$. One will figure out that Prop. 9.6.9.iii is indeed satisfied.

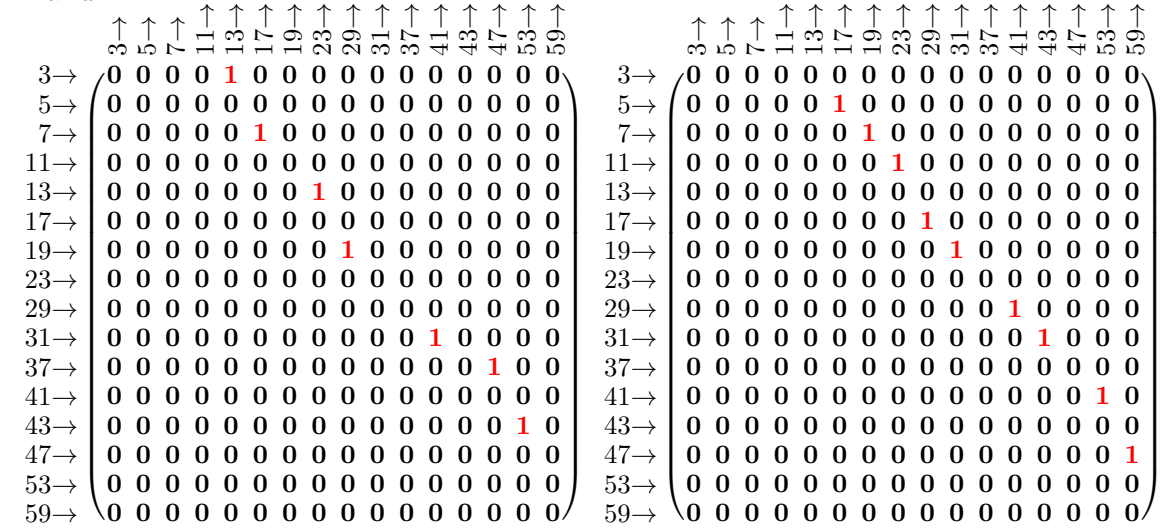
${}^{0}\mathtt{D}$ is the identity relation on $\mathbb{P}$. So we start with ${}^{2}\mathtt{D}$ and ${}^{4}\mathtt{D}$



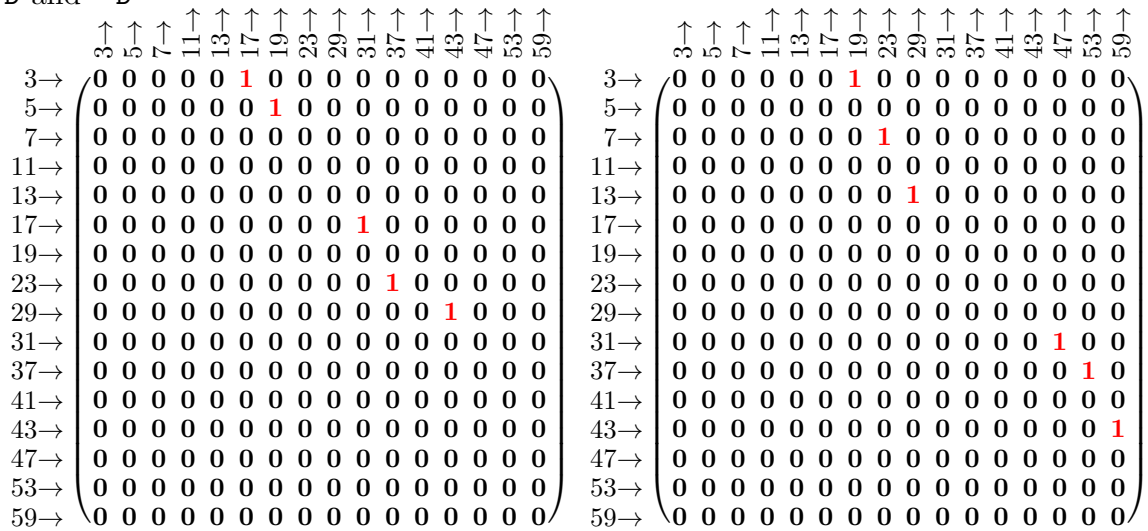
${}^{6}\mathtt{D}$ and ${}^{8}\mathtt{D}$



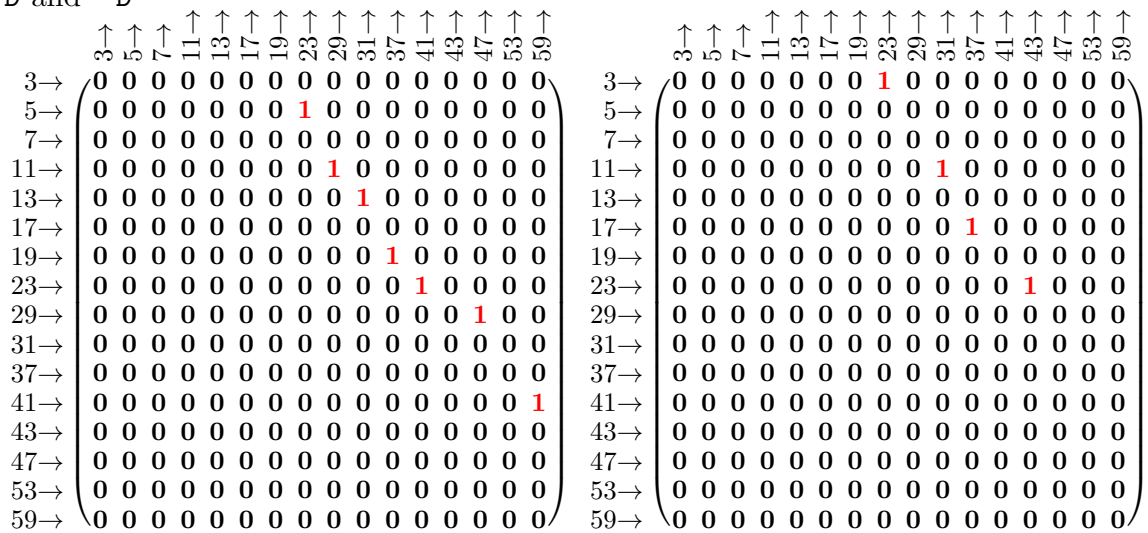
${}^{10}\mathtt{D}$ and ${}^{12}\mathtt{D}$



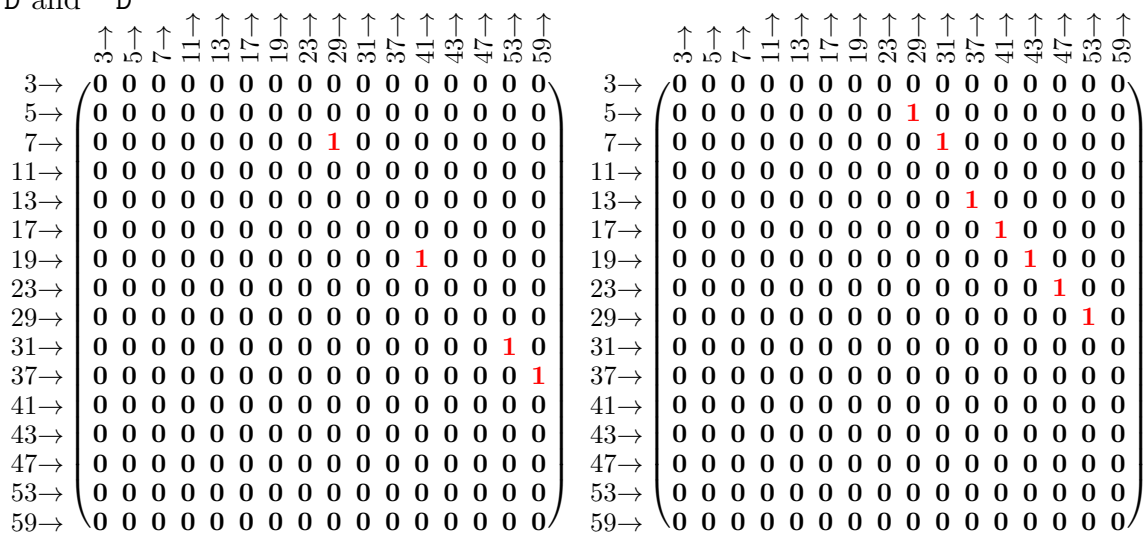
$^{14}\mathtt{D}$ and $^{16}\mathtt{D}$



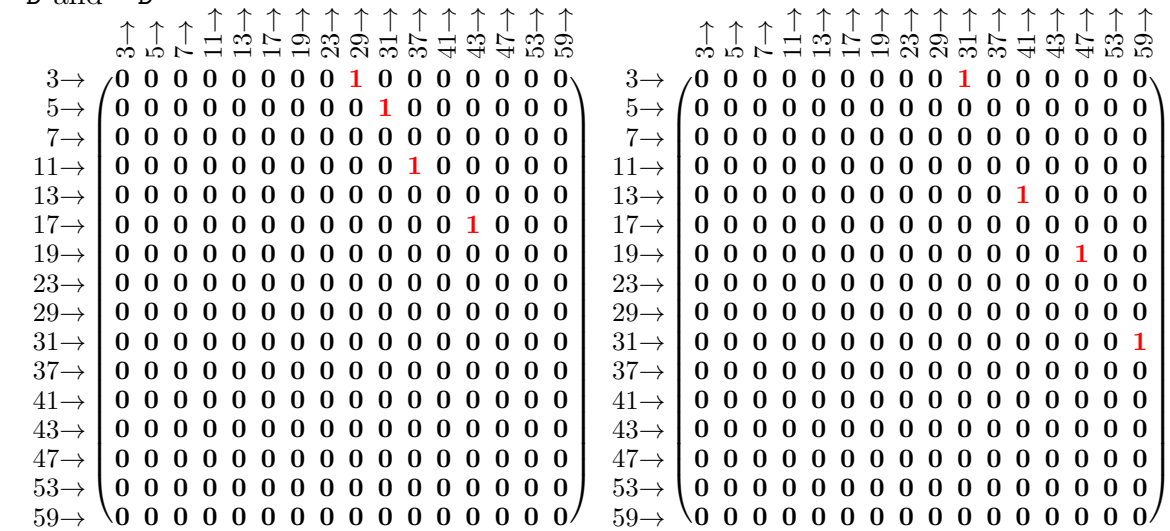
$^{18}\mathtt{D}$ and $^{20}\mathtt{D}$



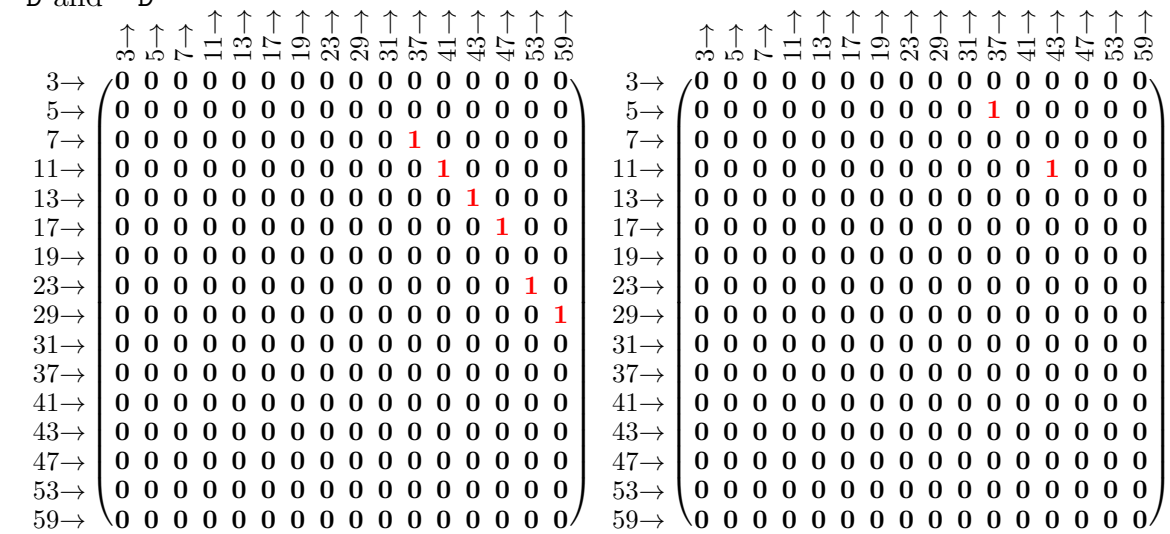
$^{22}\mathtt{D}$ and $^{24}\mathtt{D}$



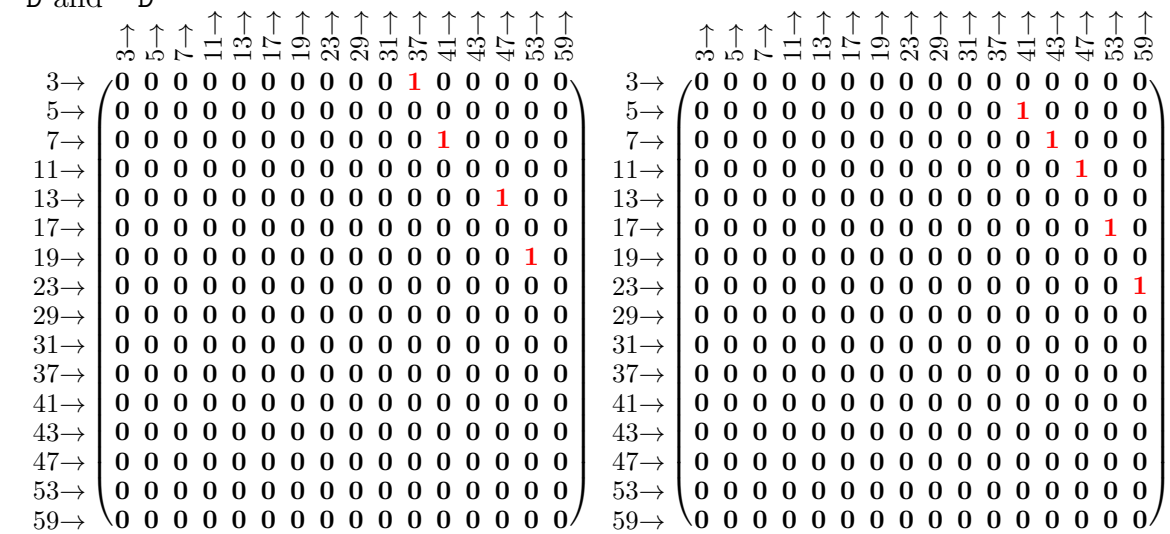
$^{26}\mathtt{D}$ and $^{28}\mathtt{D}$



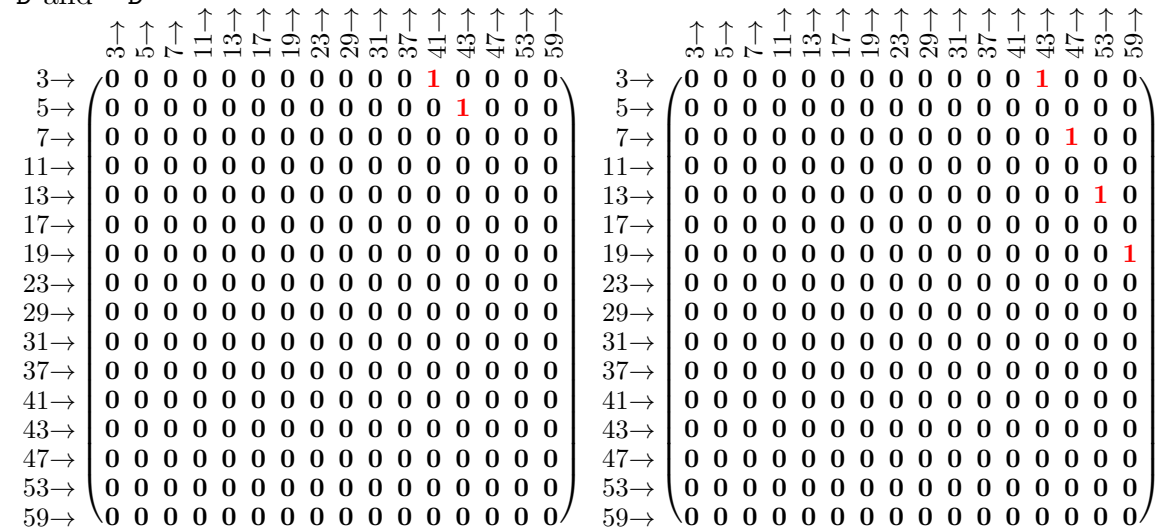
$^{30}\mathtt{D}$ and $^{32}\mathtt{D}$



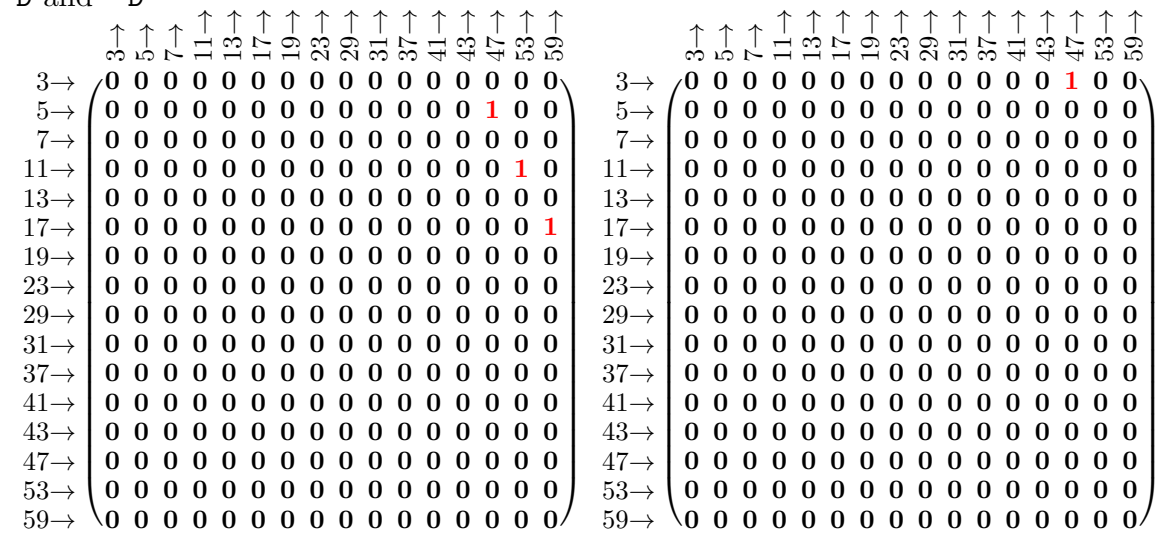
$^{34}\mathtt{D}$ and $^{36}\mathtt{D}$



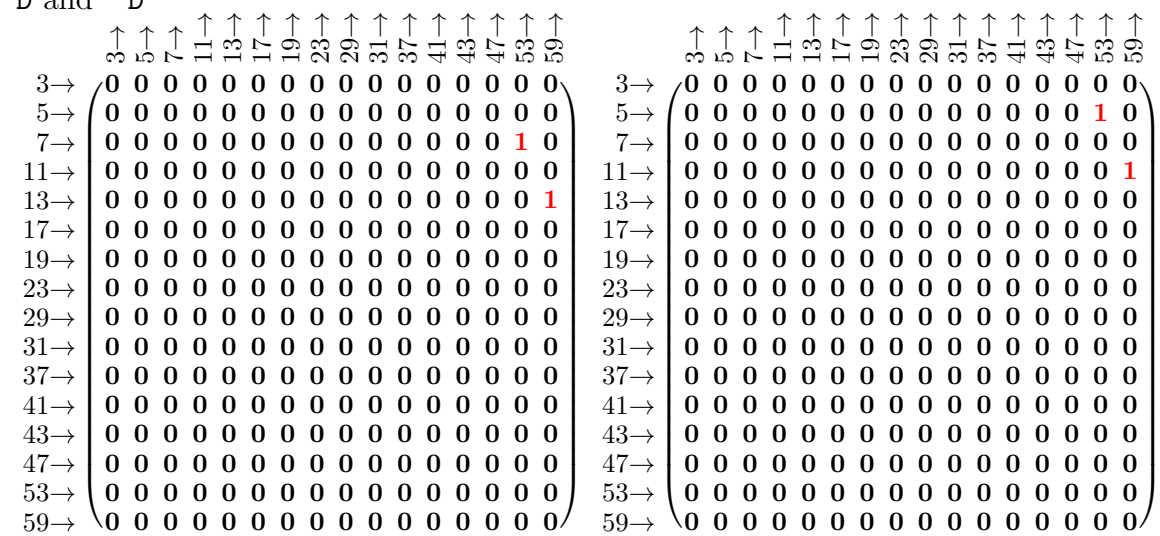
$^{38}\mathtt{D}$ and $^{40}\mathtt{D}$



$^{42}\mathtt{D}$ and $^{44}\mathtt{D}$



$^{46}\mathtt{D}$ and $^{48}\mathtt{D}$



$^{50}\mathtt{D}$ and $^{52}\mathtt{D}$

	3→	5→	7→	11→	13→	17→	19→	23→	29→	31→	37→	41→	43→	47→	53→	59→
3→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**1**	0
5→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
17→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
29→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
31→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
41→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
43→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
47→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
53→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
59→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

	3→	5→	7→	11→	13→	17→	19→	23→	29→	31→	37→	41→	43→	47→	53→	59→
3→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**1**
11→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
17→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
29→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
31→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
41→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
43→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
47→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
53→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
59→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$^{54}\mathtt{D}$ and $^{56}\mathtt{D}$

	3→	5→	7→	11→	13→	17→	19→	23→	29→	31→	37→	41→	43→	47→	53→	59→
3→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**1**
7→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
17→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
29→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
31→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
41→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
43→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
47→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
53→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
59→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

	3→	5→	7→	11→	13→	17→	19→	23→	29→	31→	37→	41→	43→	47→	53→	59→
3→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**1**
5→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
17→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
29→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
31→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
41→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
43→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
47→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
53→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
59→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Now, obviously, $\mathtt{sup}_{e=2\ldots\mathtt{ND}} = \mathbb{S}_{\mathbb{P}}{}^{+}$.

10 Selfsimilarity $\longleftrightarrow$ Goldbach

When inspector Lewis of the well-known TV-series returned from Australia, and for the first time cooperated with Hathaway in Oxford,[1] he had to hunt for a multiple murderer. It turned out, that it was a mathematician, who had worked on the Goldbach conjecture — even a Fields medallist. Somebody had reported flaws in his arguments, but he did not like to be torn on the grounds. This brought him to extinguish this person — and others in consequence; finally himself and also his wife (heavily disabled; to protect her from being without help should all this be detected).

For those spotting any problem in this text: I definitely do not intend to act that way!

10.1 Goldbach's conjecture

Goldbach conjectured that every even number is the sum of two primes. Following [17], his conjecture dates back to letters between Euler and Goldbach of June 7 and 30, 1742 that appeared in *Correspondance mathématique et physique de quelque célèbres géomètres du XVIII ième siècle* (p. 127 and p. 135). Also Edward Waring[2] is said to have uttered such a conjecture in his *Meditationes algebraicae* of 1770, [18].

In Wikipedia of 2017 it is reported, that the conjecture has been shown to hold for all even integers less than or equal to 4×10^{18}. For any conceivable practical purpose this will suffice. We are, however, mathematicians and so cultivate a gnawing doubt as to the numbers

$$4 \times 10^{18} + 2, \qquad 4 \times 10^{18} + 4, \qquad 4 \times 10^{18} + 6, \qquad \ldots$$

The question may obviously be studied already for any initial section of the naturals. In our relational context, it requires a presentation in some finer granulation.

In a first approach, we accept *all* primes as a vector $\wp_1 : \mathbb{T}$ or as a diagonal $\wp_1$ and find us confronted with Figs. 10.1.1 and 10.1.2 on the left.

[1] Hints to be found via Google searching for "goldbach-conjecture hathaway".
[2] 1736–1798

$$
\begin{aligned}
&2 = 1 + 1\\
&4 = 1 + 3 = 2 + 2\\
&6 = 1 + 5 = 3 + 3\\
&8 = 1 + 7 = 3 + 5\\
&10 = 3 + 7 = 5 + 5\\
&12 = 1 + 11 = 5 + 7\\
&14 = 1 + 13 = 3 + 11 = 7 + 7\\
&16 = 3 + 13 = 5 + 11\\
&18 = 1 + 17 = 5 + 13 = 7 + 11\\
&20 = 1 + 19 = 3 + 17 = 7 + 13\\
&22 = 3 + 19 = 5 + 17 = 11 + 11\\
&24 = 1 + 23 = 5 + 19 = 7 + 17 = 11 + 13\\
&26 = 3 + 23 = 7 + 19 = 13 + 13\\
&28 = 5 + 23 = 11 + 17
\end{aligned}
\qquad
\mathbb{A}^{\mathsf{T}_;}\,(\wp_1 \,;\, \mathbb{T} \oslash \wp_1 \,;\, \mathbb{T}) \;=\;
\begin{array}{cc}
0 & \mathbf{0}\\
1 & \mathbf{0}\\
2 & \mathbf{1}\\
3 & \mathbf{0}\\
4 & \mathbf{1}\\
5 & \mathbf{0}\\
6 & \mathbf{1}\\
7 & \mathbf{0}\\
8 & \mathbf{1}\\
9 & \mathbf{0}\\
10 & \mathbf{1}\\
11 & \mathbf{0}\\
12 & \mathbf{1}\\
13 & \mathbf{0}\\
14 & \mathbf{1}\\
15 & \mathbf{0}\\
16 & \mathbf{1}\\
17 & \mathbf{0}\\
18 & \mathbf{1}\\
19 & \mathbf{0}
\end{array}
$$

Fig. 10.1.1 Goldbach decompositions for $\wp_1 \,;\, \mathbb{T}$ as well as a vector of decomposable numbers

The Goldbach conjecture applied also to primes conceived as $\wp_1 \,;\, \mathbb{T}$ and $\wp_2 \,;\, \mathbb{T}$ is slightly less expressive, since every successor of a prime p automatically admits $1 + p$, i.e. not just a proper decomposition into two primes greater than 2. The 2 as an even prime does not introduce any further problem since there are no other even primes. The number 4 has in cases of $\wp_1 \,;\, \mathbb{T}$ or $\wp_2 \,;\, \mathbb{T}$, however, two different decompositions.

These tiny observations concerning numbers 1 or 2 to be prime lead us to reduce our endeavour to the odd primes — conceived as a diagonal $\wp$ or as a vector $\wp \,;\, \mathbb{T}$.

10.1.1 Definition. i) We are going to call

$$\mathbb{A}^{\mathsf{T}_;}\,(\wp \,;\, \mathbb{T} \oslash \wp \,;\, \mathbb{T})$$

the vector of **Goldbach decomposable numbers** and

$$\mathtt{GD} \;:=\; \mathbb{A}^{\mathsf{T}_;}\,(\wp \otimes \wp)$$

the **Goldbach decompositions relation**.[3]

ii) Based thereon, we define as **Goldbach reduction** the (obviously equivalent) relations

$$\mathtt{GR} := \mathtt{GD} \,;\, \pi = \mathtt{GD} \,;\, \rho = \mathbb{A}^{\mathsf{T}_;}\,(\wp \oslash \wp \,;\, \mathbb{T}) \;=\; \mathbb{A}^{\mathsf{T}_;}\,(\wp \,;\, \mathbb{T} \oslash \wp)$$

that assign to every even number all the odd primes involved in establishing one of its Goldbach decompositions. □

[3] See e.g. Fig. 10.1.2, where for space reasons not the here omnipresent matrix representation has been chosen, but the automatically generated mapping into the powerset $\mathtt{GD} : \mathbb{N} \longrightarrow \mathbf{2}^{\mathbb{N}\times\mathbb{N}}$.

0	{}
1	{}
2	{(1,1)}
3	{(1,2),(2,1)}
4	{(1,3),(2,2),(3,1)}
5	{(2,3),(3,2)}
6	{(1,5),(3,3),(5,1)}
7	{(2,5),(5,2)}
8	{(1,7),(3,5),(5,3),(7,1)}
9	{(2,7),(7,2)}
10	{(3,7),(5,5),(7,3)}
11	{}
12	{(1,11),(5,7),(7,5),(11,1)}
13	{(2,11),(11,2)}
14	{(1,13),(3,11),(7,7),(11,3),(13,1)}
15	{(2,13),(13,2)}
16	{(3,13),(5,11),(11,5),(13,3)}
17	{}
18	{(1,17),(5,13),(7,11),(11,7),(13,5),(17,1)}
19	{(2,17),(17,2)}
20	{(1,19),(3,17),(7,13),(13,7),(17,3),(19,1)}
21	{(2,19),(19,2)}
22	{(3,19),(5,17),(11,11),(17,5),(19,3)}
23	{}
24	{(1,23),(5,19),(7,17),(11,13),(13,11),(17,7),(19,5),(23,1)}
25	{(2,23),(23,2)}
26	{(3,23),(7,19),(13,13),(19,7),(23,3)}
27	{}
28	{(5,23),(11,17),(17,11),(23,5)}
29	{}
30	{(1,29),(7,23),(11,19),(13,17),(17,13),(19,11),(23,7),(29,1)}
31	{(2,29),(29,2)}
32	{(1,31),(3,29),(13,19),(19,13),(29,3),(31,1)}
33	{(2,31),(31,2)}
34	{(3,31),(5,29),(11,23),(17,17),(23,11),(29,5),(31,3)}
35	{}
36	{(5,31),(7,29),(13,23),(17,19),(19,17),(23,13),(29,7),(31,5)}
37	{}
38	{(1,37),(7,31),(19,19),(31,7),(37,1)}
39	{(2,37),(37,2)}
40	{(3,37),(11,29),(17,23),(23,17),(29,11),(37,3)}

0	{}
1	{}
2	{}
3	{}
4	{}
5	{}
6	{(3,3)}
7	{}
8	{(3,5),(5,3)}
9	{}
10	{(3,7),(5,5),(7,3)}
11	{}
12	{(5,7),(7,5)}
13	{}
14	{(3,11),(7,7),(11,3)}
15	{}
16	{(3,13),(5,11),(11,5),(13,3)}
17	{}
18	{(5,13),(7,11),(11,7),(13,5)}
19	{}
20	{(3,17),(7,13),(13,7),(17,3)}
21	{}
22	{(3,19),(5,17),(11,11),(17,5),(19,3)}
23	{}
24	{(5,19),(7,17),(11,13),(13,11),(17,7),(19,5)}
25	{}
26	{(3,23),(7,19),(13,13),(19,7),(23,3)}
27	{}
28	{(5,23),(11,17),(17,11),(23,5)}
29	{}
30	{(7,23),(11,19),(13,17),(17,13),(19,11),(23,7)}
31	{}
32	{(3,29),(13,19),(19,13),(29,3)}
33	{}
34	{(3,31),(5,29),(11,23),(17,17),(23,11),(29,5),(31,3)}
35	{}
36	{(5,31),(7,29),(13,23),(17,19),(19,17),(23,13),(29,7),(31,5)}
37	{}
38	{(7,31),(19,19),(31,7)}
39	{}
40	{(3,37),(11,29),(17,23),(23,17),(29,11),(37,3)}

Fig. 10.1.2 Goldbach decompositions relations $\mathbb{A}^{\mathsf{T}_;}(\wp_1 \otimes \wp_1)$ and $\mathbb{A}^{\mathsf{T}_;}(\wp \otimes \wp) =: \mathtt{GD}$

Our visualizations occur mostly via Boolean matrices. It is — just for space reasons — highly advisable to eliminate all the rows or columns of just **0**'s in the matrix representation of such relations and in addition those for numbers $0 \ldots 5$. To this end we extrude them purely mechanically, which will be indicated by the "$\rightarrow$" symbols. Only then will we start a major rigorous effort.

Relations concerning even numbers shall in particular be these:

10.1.2 Definition (*Relations concerning evens* ≥ 6)**.**

$v_{\mathbb{E}_6} := \mathbb{S}^{\mathsf{T}4}{}_; [\mathbb{S}^{\mathsf{T}2}]^+{}_; z : \mathbb{N} \longrightarrow \mathbb{1}$ selecting all the evens from 6 on — visualized transposed

$\iota_{\mathbb{E}_6} := \mathtt{Inject}(v_{\mathbb{E}_6}) : \mathbb{E}_6 \longrightarrow \mathbb{N}$, thus *defining* the extruded set $\mathbb{E}_6$

$\mathbb{S}_{\mathbb{E}_6} := \iota_{\mathbb{E}_6;} \mathbb{S}^2{}_; \iota^{\mathsf{T}}_{\mathbb{E}_6} : \mathbb{E}_6 \longrightarrow \mathbb{E}_6$ successor relation on the thus selected evens □

$$v^{\intercal}_{\mathbb{E}_6} = \begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \end{matrix} \\ \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \end{pmatrix} \end{array}$$

$$\iota_{\mathbb{E}_6} = \begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \end{matrix} \\ \begin{matrix} 6\rightarrow \\ 8\rightarrow \\ 10\rightarrow \\ 12\rightarrow \end{matrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \end{pmatrix} \end{array}$$

$$\mathbb{S}_{\mathbb{E}_6} = \begin{array}{c} \begin{matrix} 6\rightarrow & 8\rightarrow & 10\rightarrow & 12\rightarrow & 14\rightarrow & 16\rightarrow \end{matrix} \\ \begin{matrix} 6\rightarrow \\ 8\rightarrow \\ 10\rightarrow \\ 12\rightarrow \\ 14\rightarrow \\ 16\rightarrow \end{matrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{array} \qquad z_{\mathbb{E}_6} = \begin{array}{c} \\ \begin{matrix} 6\rightarrow \\ 8\rightarrow \\ 10\rightarrow \\ 12\rightarrow \\ 14\rightarrow \\ 16\rightarrow \end{matrix} \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} \end{array}$$

Fig. 10.1.3 $v_{\mathbb{E}_6}, \iota_{\mathbb{E}_6}$ and $\mathbb{S}_{\mathbb{E}_6}$, taken for limitations at 13, resp. 17

Already now may we show the Goldbach reduction GR of Fig. 10.1.4.

$$\begin{array}{c} \begin{matrix} 3\rightarrow & 5\rightarrow & 7\rightarrow & 11\rightarrow & 13\rightarrow & 17\rightarrow & 19\rightarrow & 23\rightarrow & 29\rightarrow & 31\rightarrow & 37\rightarrow \end{matrix} \\ \begin{matrix} 6\rightarrow \\ 8\rightarrow \\ 10\rightarrow \\ 12\rightarrow \\ 14\rightarrow \\ 16\rightarrow \\ 18\rightarrow \\ 20\rightarrow \\ 22\rightarrow \\ 24\rightarrow \\ 26\rightarrow \\ 28\rightarrow \\ 30\rightarrow \\ 32\rightarrow \\ 34\rightarrow \\ 36\rightarrow \\ 38\rightarrow \\ 40\rightarrow \end{matrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} \end{pmatrix} \end{array}$$

Fig. 10.1.4 GR in a space-reduced presentation: $\iota_{\mathbb{E}_6} ; \mathtt{GR} ; \iota^{\intercal}_{\wp\mathbb{T}}$

We will need a tiny but helpful result. The following (i) is obvious: Arriving with $\iota_{\mathbb{E}_6}$ means an even number ≥ 6 that is further increased with $\mathbb{S}^{\,2}$ and, thus, stays even and ≥ 6.

10.1.3 Proposition. i) $\iota_{\mathbb{E}_6} ; \mathbb{S}^{\,2} = \iota_{\mathbb{E}_6} ; \mathbb{S}^{\,2} ; \iota^{\intercal}_{\mathbb{E}_6} ; \iota_{\mathbb{E}_6} = \mathbb{S}_{\mathbb{E}_6} ; \iota_{\mathbb{E}_6}$

ii) $(\iota_{\mathbb{E}_6} ; \mathbb{S}^{\,2} ; \iota^{\intercal}_{\mathbb{E}_6})^n ; \iota_{\mathbb{E}_6} = \iota_{\mathbb{E}_6} ; \mathbb{S}^{\,2*n}$ for $n \geq 0$ $\qquad (\iota_{\mathbb{E}_6} ; \mathbb{S}^{\,2} ; \iota^{\intercal}_{\mathbb{E}_6})^n = \iota_{\mathbb{E}_6} ; \mathbb{S}^{\,2*n} ; \iota^{\intercal}_{\mathbb{E}_6}$ for $n \geq 0$

iii) $\mathbb{S}^{\,n}_{\mathbb{E}_6} = \iota_{\mathbb{E}_6} ; \mathbb{S}^{\,2*n} ; \iota^{\intercal}_{\mathbb{E}_6}$ $\qquad \mathbb{S}^{*}_{\mathbb{E}_6} = \iota_{\mathbb{E}_6} ; [\mathbb{S}^{\,2}]^{*} ; \iota^{\intercal}_{\mathbb{E}_6}$

iv) $\mathbb{S}_{\mathbb{E}_6}$ enjoys all the properties demanded in Def. 5.1.1 for $\mathbb{S}$, namely, that $\mathbb{S}_{\mathbb{E}_6}$ is injective, univalent and non-empty, that $z_{\mathbb{E}_6} := \overline{\mathbb{S}^{\intercal}_{\mathbb{E}_6} ; \mathbb{T}}$ is a point, and that $\mathbb{S}_{\mathbb{E}_6}$ is regressively finite.

In particular: $z_{\mathbb{E}_6} ; \mathbb{T} \subseteq \mathbb{S}^{*}_{\mathbb{E}_6}$

v) $\iota^{\intercal}_{\mathbb{E}_6} ; z_{\mathbb{E}_6} ; \mathbb{T} = \mathbb{S}^{\,\intercal 6} ; z ; \mathbb{T}$

vi) $\iota_{\mathbb{E}_6} ; \mathbb{S}^{\,\intercal 2*n} \cap \overline{\mathbb{T} ; z^{\intercal} ; \mathbb{S}^{\,4} \cup \mathbb{T} ; z^{\intercal} ; \mathbb{S}^{\,2} \cup \mathbb{T} ; z^{\intercal}} = \mathbb{S}^{\intercal\,n}_{\mathbb{E}_6} ; \iota_{\mathbb{E}_6}$ for all $n \geq 0$

$\iota_{\mathbb{E}_2} ; \mathbb{S}^{\,\intercal 2*n} \cap \overline{\mathbb{T} ; z^{\intercal}} = \mathbb{S}^{\intercal\,n}_{\mathbb{E}_2} ; \iota_{\mathbb{E}_2}$ for all $n \geq 0$

vii) $\iota_{\mathbb{E}_6};\mathbb{S}^{\intercal k};z;\mathbb{T} = \begin{cases} \mathbb{S}_{\mathbb{E}_6}^{\intercal\ \frac{k-6}{2}};z_{\mathbb{E}_6};\mathbb{T} & \text{if } k \geq 6 \text{ and even} \\ \perp\!\!\!\perp & \text{otherwise} \end{cases}$

$\mathbb{S}_{\mathbb{E}_6}^{\intercal\ a};z_{\mathbb{E}_6};\mathbb{T} = \iota_{\mathbb{E}_6};\mathbb{S}^{\intercal 2*a+6};z;\mathbb{T}$

viii) $\iota_{\mathbb{E}_6};\mathbb{S}^{2*n} = \mathbb{S}_{\mathbb{E}_6}^{n};\iota_{\mathbb{E}_6}$ for all $n \geq 0$

ix) $\iota_{\mathbb{E}_6}^{\intercal};\mathbb{S}_{\mathbb{E}_6}^{n} = \mathbb{S}^{2*n};\iota_{\mathbb{E}_6}^{\intercal} \cap v_{\mathbb{E}_6};\mathbb{T}$ for all $n \geq 0$

Proof: i) We have $\iota_{\mathbb{E}_6};\mathbb{S}^{2};\iota_{\mathbb{E}_6}^{\intercal};\iota_{\mathbb{E}_6} = \mathbb{S}_{\mathbb{E}_6};\iota_{\mathbb{E}_6}$ by definition of $\mathbb{S}_{\mathbb{E}_6}$. Direction "$\supseteq$" is evident since $\iota_{\mathbb{E}_6}$ is univalent. So we look for

$\iota_{\mathbb{E}_6};\mathbb{S}^{2} \subseteq \iota_{\mathbb{E}_6};\mathbb{S}^{2};\iota_{\mathbb{E}_6}^{\intercal};\iota_{\mathbb{E}_6} \iff \iota_{\mathbb{E}_6}^{\intercal};\iota_{\mathbb{E}_6};\mathbb{S}^{2} \subseteq \mathbb{S}^{2};\iota_{\mathbb{E}_6}^{\intercal};\iota_{\mathbb{E}_6}$

and — after thus having shunted the map $\iota_{\mathbb{E}_6}$ — prove as follows:

$\iota_{\mathbb{E}_6}^{\intercal};\iota_{\mathbb{E}_6};\mathbb{S}^{2} = (\mathbb{I} \cap v_{\mathbb{E}_6};\mathbb{T});\mathbb{S}^{2}$ by definition since $\iota_{\mathbb{E}_6}$ extrudes $v_{\mathbb{E}_6};\mathbb{T}$
$= \mathbb{S}^{2} \cap v_{\mathbb{E}_6};\mathbb{T}$ masking
$\subseteq \mathbb{S}^{2} \cap \mathbb{T};v_{\mathbb{E}_6}^{\intercal}$ concerning "$\subseteq \mathbb{T};v_{\mathbb{E}_6}^{\intercal}$" see below
$= \mathbb{S}^{2};(\mathbb{I} \cap \mathbb{T};v_{\mathbb{E}_6}^{\intercal})$ masking
$= \mathbb{S}^{2};\iota_{\mathbb{E}_6}^{\intercal};\iota_{\mathbb{E}_6}$ by definition since $\iota_{\mathbb{E}_6}$ extrudes $v_{\mathbb{E}_6};\mathbb{T}$

Now the remaining part:

$\mathbb{S}^{2} \cap v_{\mathbb{E}_6};\mathbb{T} = \mathbb{S}^{2} \cap \mathbb{S}^{\intercal 4};[\mathbb{S}^{\intercal 2}]^{+};z;\mathbb{T}$ by definition of $v_{\mathbb{E}_6};\mathbb{T}$
$\subseteq (\mathbb{S}^{\intercal 4} \cap \mathbb{S}^{2};\mathbb{T};z^{\intercal};[\mathbb{S}^{2}]^{+});([\mathbb{S}^{\intercal 2}]^{+};z;\mathbb{T} \cap \mathbb{S}^{4};\mathbb{S}^{2})$ Dedekind rule
$\subseteq \mathbb{T};z^{\intercal};[\mathbb{S}^{2}]^{+};\mathbb{S}^{6} = \mathbb{T};z^{\intercal};[\mathbb{S}^{2}]^{+};\mathbb{S}^{2};\mathbb{S}^{4} \subseteq \mathbb{T};z^{\intercal};[\mathbb{S}^{2}]^{+};\mathbb{S}^{4} = \mathbb{T};v_{\mathbb{E}_6}^{\intercal}$

---

ii) This is certainly satisfied for $n = 0$. Induction $n \mapsto n+1$:

$(\iota_{\mathbb{E}_6};\mathbb{S}^{2};\iota_{\mathbb{E}_6}^{\intercal})^{n+1};\iota_{\mathbb{E}_6} = (\iota_{\mathbb{E}_6};\mathbb{S}^{2};\iota_{\mathbb{E}_6}^{\intercal});(\iota_{\mathbb{E}_6};\mathbb{S}^{2};\iota_{\mathbb{E}_6}^{\intercal})^{n};\iota_{\mathbb{E}_6} = \iota_{\mathbb{E}_6};\mathbb{S}^{2};\iota_{\mathbb{E}_6}^{\intercal};\iota_{\mathbb{E}_6};\mathbb{S}^{2*n}$ hypothesis
$= \iota_{\mathbb{E}_6};\mathbb{S}^{2};\mathbb{S}^{2*n}$ due to (i)
$= \iota_{\mathbb{E}_6};\mathbb{S}^{2*(n+1)}$

The other claim is proved by induction with start at $n = 0$, which is obvious because $\iota_{\mathbb{E}_6}$ is an injective map. Then

$(\iota_{\mathbb{E}_6};\mathbb{S}^{2};\iota_{\mathbb{E}_6}^{\intercal})^{n+1} = \iota_{\mathbb{E}_6};\mathbb{S}^{2};\iota_{\mathbb{E}_6}^{\intercal};(\iota_{\mathbb{E}_6};\mathbb{S}^{2};\iota_{\mathbb{E}_6}^{\intercal})^{n}$
$= \iota_{\mathbb{E}_6};\mathbb{S}^{2};\iota_{\mathbb{E}_6}^{\intercal};\iota_{\mathbb{E}_6};\mathbb{S}^{2*n};\iota_{\mathbb{E}_6}^{\intercal}$ hypothesis
$= \iota_{\mathbb{E}_6};\mathbb{S}^{2};\mathbb{S}^{2*n};\iota_{\mathbb{E}_6}^{\intercal}$ due to (i)
$= \iota_{\mathbb{E}_6};\mathbb{S}^{2+2*n};\iota_{\mathbb{E}_6}^{\intercal} = \iota_{\mathbb{E}_6};\mathbb{S}^{2*(n+1)};\iota_{\mathbb{E}_6}^{\intercal}$

---

iii) Proof by induction with case $n = 0$ trivial because $\iota_{\mathbb{E}_6}$ is an injective map.

$\mathbb{S}_{\mathbb{E}_6}^{n+1} = \mathbb{S}_{\mathbb{E}_6};\mathbb{S}_{\mathbb{E}_6}^{n} = \mathbb{S}_{\mathbb{E}_6};\iota_{\mathbb{E}_6};\mathbb{S}^{2*n};\iota_{\mathbb{E}_6}^{\intercal}$ hypothesis
$= \iota_{\mathbb{E}_6};\mathbb{S}^{2};\mathbb{S}^{2*n};\iota_{\mathbb{E}_6}^{\intercal}$ due to (i)
$= \iota_{\mathbb{E}_6};\mathbb{S}^{2*(n+1)};\iota_{\mathbb{E}_6}^{\intercal}$

$\mathbb{S}_{\mathbb{E}_6}^{*} = \mathbb{S}_{\mathbb{E}_6}^{*};\iota_{\mathbb{E}_6};\iota_{\mathbb{E}_6}^{\intercal}$ since $\iota_{\mathbb{E}_6}$ as injection of an extruded vector is total and injective
$= \mathtt{sup}_{n\geq 0}\ \mathbb{S}_{\mathbb{E}_6}^{n};\iota_{\mathbb{E}_6};\iota_{\mathbb{E}_6}^{\intercal} = \mathtt{sup}_{n\geq 0}(\iota_{\mathbb{E}_6};\mathbb{S}^{2};\iota_{\mathbb{E}_6}^{\intercal})^{n};\iota_{\mathbb{E}_6};\iota_{\mathbb{E}_6}^{\intercal}$ by Def. 10.1.2
$= \mathtt{sup}_{n\geq 0}\ \iota_{\mathbb{E}_6};\mathbb{S}^{2*n};\iota_{\mathbb{E}_6}^{\intercal}$ following (ii)
$= \iota_{\mathbb{E}_6};(\mathtt{sup}_{n\geq 0}\ \mathbb{S}^{2*n});\iota_{\mathbb{E}_6}^{\intercal} = \iota_{\mathbb{E}_6};(\mathtt{sup}_{n\geq 0}[\mathbb{S}^{2}]^{n});\iota_{\mathbb{E}_6}^{\intercal} = \iota_{\mathbb{E}_6};[\mathbb{S}^{2}]^{*};\iota_{\mathbb{E}_6}^{\intercal}$

---

iv) is fairly obviously deduced from the corresponding properties of $\mathbb{S}$. The particular result following therefrom stems from Prop. 5.2.1.ii.b.

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v) $\iota_{\mathbb{E}_6}^{\mathsf{T}};z_{\mathbb{E}_6};\mathbb{T} = \iota_{\mathbb{E}_6}^{\mathsf{T}};\overline{\mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}};\mathbb{T}} = \iota_{\mathbb{E}_6}^{\mathsf{T}};\overline{\iota_{\mathbb{E}_6};\mathbb{S}^{\mathsf{T}^2};\iota_{\mathbb{E}_6}^{\mathsf{T}};\mathbb{T}}$ Def. 10.1.2
$= \iota_{\mathbb{E}_6}^{\mathsf{T}};\iota_{\mathbb{E}_6};\overline{\mathbb{S}^{\mathsf{T}^2};v_{\mathbb{E}_6};\mathbb{T}} = (\mathbb{I}\cap v_{\mathbb{E}_6};\mathbb{T});\overline{\mathbb{S}^{\mathsf{T}^2};v_{\mathbb{E}_6};\mathbb{T}}$ since the map $\iota_{\mathbb{E}_6}$ extrudes $v_{\mathbb{E}_6};\mathbb{T}$
$= \overline{\mathbb{S}^{\mathsf{T}^2};v_{\mathbb{E}_6};\mathbb{T}}\cap v_{\mathbb{E}_6};\mathbb{T}$ masking
$= \overline{\mathbb{S}^{\mathsf{T}^2};\mathbb{S}^{\mathsf{T}^4};[\mathbb{S}^{\mathsf{T}^2}]^+;z;\mathbb{T}}\cap \mathbb{S}^{\mathsf{T}^4};[\mathbb{S}^{\mathsf{T}^2}]^+;z;\mathbb{T}$ expanded
$= \left\{\overline{\mathbb{S}^{\mathsf{T}^2};\mathbb{S}^{\mathsf{T}^4};[\mathbb{S}^{\mathsf{T}^2}]^+}\cap \mathbb{S}^{\mathsf{T}^4};[\mathbb{S}^{\mathsf{T}^2}]^+\right\};z;\mathbb{T}$ since $z;\mathbb{T}$ is a point
$= \mathbb{S}^{\mathsf{T}^6};z;\mathbb{T}$ even powers of $\mathbb{S}^{\mathsf{T}}$ of 6 or more, but not 8 or more

---

vi) We prove direction "$\supseteq$" first:
$\mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\ n};\iota_{\mathbb{E}_6} = (\iota_{\mathbb{E}_6};\mathbb{S}^{\mathsf{T}^2};\iota_{\mathbb{E}_6}^{\mathsf{T}})^n;\iota_{\mathbb{E}_6}$ by definition of $\mathbb{S}_{\mathbb{E}_6}$
$\subseteq \iota_{\mathbb{E}_6};(\mathbb{S}^{\mathsf{T}^2})^n = \iota_{\mathbb{E}_6};\mathbb{S}^{\mathsf{T}^{2*n}}$ because $\iota_{\mathbb{E}_6}$ is univalent
$\mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\ n};\iota_{\mathbb{E}_6}\subseteq\overline{\mathbb{T};z^{\mathsf{T}};\mathbb{S}^4}$ $\iff$ $\mathbb{T};z^{\mathsf{T}};\mathbb{S}^4;\iota_{\mathbb{E}_6}^{\mathsf{T}}\subseteq\overline{\mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\ n}}$ with $\iota_{\mathbb{E}_6};\mathbb{S}^{\mathsf{T}^4};z;\mathbb{T}=\perp\!\!\!\perp$, e.g.

Now direction "$\subseteq$" via induction after having modified it slightly:
$\iota_{\mathbb{E}_6};\mathbb{S}^{\mathsf{T}^{2*n}}\cap\overline{\mathbb{T};z^{\mathsf{T}};\mathbb{S}^4\cup\mathbb{T};z^{\mathsf{T}};\mathbb{S}^2\cup\mathbb{T};z^{\mathsf{T}}}\subseteq\mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\ n};\iota_{\mathbb{E}_6}$
$\iff$ $\iota_{\mathbb{E}_6};\mathbb{S}^{\mathsf{T}^{2*n}}\subseteq\mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\ n};\iota_{\mathbb{E}_6}\cup\mathbb{T};z^{\mathsf{T}};\mathbb{S}^4\cup\mathbb{T};z^{\mathsf{T}};\mathbb{S}^2\cup\mathbb{T};z^{\mathsf{T}}$
The case $n=0$ is obvious and we prove $n\mapsto n+1$:
$\iota_{\mathbb{E}_6};\mathbb{S}^{\mathsf{T}^{2*(n+1)}} = \iota_{\mathbb{E}_6};\mathbb{S}^{\mathsf{T}^{2*n}};\mathbb{S}^{\mathsf{T}^2}\subseteq\left[\mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\ n};\iota_{\mathbb{E}_6}\cup\mathbb{T};z^{\mathsf{T}};\mathbb{S}^4\cup\mathbb{T};z^{\mathsf{T}};\mathbb{S}^2\cup\mathbb{T};z^{\mathsf{T}}\right];\mathbb{S}^{\mathsf{T}^2}$ hypothesis
$= \mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\ n};\iota_{\mathbb{E}_6};\mathbb{S}^{\mathsf{T}^2}\cup\left(\mathbb{T};z^{\mathsf{T}};\mathbb{S}^4\cup\mathbb{T};z^{\mathsf{T}};\mathbb{S}^2\cup\mathbb{T};z^{\mathsf{T}}\right);\mathbb{S}^{\mathsf{T}^2}$
$\subseteq \mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\ n};\iota_{\mathbb{E}_6};\mathbb{S}^{\mathsf{T}^2};\left[\iota_{\mathbb{E}_6}^{\mathsf{T}};\iota_{\mathbb{E}_6}\cup(\mathbb{I}\cap\overline{\iota_{\mathbb{E}_6}^{\mathsf{T}};\mathbb{T}})\right]\cup\mathbb{T};z^{\mathsf{T}};\mathbb{S}^2\cup\mathbb{T};z^{\mathsf{T}};\mathbb{S}^0\cup\perp\!\!\!\perp$
since $\iota_{\mathbb{E}_6}^{\mathsf{T}};\iota_{\mathbb{E}_6}=\mathbb{I}\cap\iota_{\mathbb{E}_6}^{\mathsf{T}};\mathbb{T}$ and $\mathbb{S}$ injective
$= \mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\ n};\iota_{\mathbb{E}_6};\mathbb{S}^{\mathsf{T}^2};\iota_{\mathbb{E}_6}^{\mathsf{T}};\iota_{\mathbb{E}_6}\cup\mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\ n};\iota_{\mathbb{E}_6};\mathbb{S}^{\mathsf{T}^2};(\mathbb{I}\cap\overline{\iota_{\mathbb{E}_6}^{\mathsf{T}};\mathbb{T}})\cup\mathbb{T};z^{\mathsf{T}};\mathbb{S}^2\cup\mathbb{T};z^{\mathsf{T}}$
$= \mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\ n};\mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}};\iota_{\mathbb{E}_6}\cup\mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\ n};\iota_{\mathbb{E}_6};\mathbb{S}^{\mathsf{T}^2};(\mathbb{I}\cap\overline{\iota_{\mathbb{E}_6}^{\mathsf{T}};\mathbb{T}})\cup\mathbb{T};z^{\mathsf{T}};\mathbb{S}^2\cup\mathbb{T};z^{\mathsf{T}}$ by definition of $\mathbb{S}_{\mathbb{E}_6}$
$= \mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\ n+1};\iota_{\mathbb{E}_6}\cup\mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\ n};\iota_{\mathbb{E}_6};\mathbb{S}^{\mathsf{T}^2};(\mathbb{I}\cap\overline{\iota_{\mathbb{E}_6}^{\mathsf{T}};\mathbb{T}})\cup\mathbb{T};z^{\mathsf{T}};\mathbb{S}^2\cup\mathbb{T};z^{\mathsf{T}}$
Of this only the medium term needs further discussion:
$\mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\ n};\iota_{\mathbb{E}_6};\mathbb{S}^{\mathsf{T}^2};(\mathbb{I}\cap\overline{\iota_{\mathbb{E}_6}^{\mathsf{T}};\mathbb{T}})$
$= \mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\ n};\iota_{\mathbb{E}_6};\mathbb{S}^{\mathsf{T}^2};(\mathbb{I}\cap[z;\mathbb{T}\cup\mathbb{S}^{\mathsf{T}^2};z;\mathbb{T}\cup\mathbb{S}^{\mathsf{T}^4};z;\mathbb{T}\cup\mathtt{sup}_i\,\mathbb{S}^{\mathsf{T}^{2*i+1}};z;\mathbb{T}])$ i.e. odd or <6
$= \mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\ n};\iota_{\mathbb{E}_6};\mathbb{S}^{\mathsf{T}^2};\left[(\mathbb{I}\cap z;\mathbb{T})\cup(\mathbb{I}\cap\mathbb{S}^{\mathsf{T}^2};z;\mathbb{T})\cup(\mathbb{I}\cap\mathbb{S}^{\mathsf{T}^4};z;\mathbb{T})\cup\mathtt{sup}_i(\mathbb{I}\cap\mathbb{S}^{\mathsf{T}^{2*i+1}};z;\mathbb{T})\right]$
$= \mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\ n};\iota_{\mathbb{E}_6};\mathbb{S}^{\mathsf{T}^2};\left[z;\mathbb{T};z^{\mathsf{T}}\cup\mathbb{S}^{\mathsf{T}^2};z;\mathbb{T};z^{\mathsf{T}};\mathbb{S}^{\mathsf{T}^2}\cup\mathbb{S}^{\mathsf{T}^4};z;\mathbb{T};z^{\mathsf{T}};\mathbb{S}^{\mathsf{T}^4}\cup\mathtt{sup}_i\,\mathbb{S}^{\mathsf{T}^{2*i+1}};z;\mathbb{T};z^{\mathsf{T}};\mathbb{S}^{\mathsf{T}^{2*i+1}}\right]$
$= \mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\ n};\iota_{\mathbb{E}_6};\mathbb{S}^{\mathsf{T}^2};\left[z;\mathbb{T};z^{\mathsf{T}}\cup\mathbb{S}^{\mathsf{T}^2};z;\mathbb{T};z^{\mathsf{T}};\mathbb{S}^{\mathsf{T}^2}\cup\mathbb{S}^{\mathsf{T}^4};z;\mathbb{T};z^{\mathsf{T}};\mathbb{S}^{\mathsf{T}^4}\right]$ as $\iota_{\mathbb{E}_6};\mathbb{S}^{\mathsf{T}^2}$ ends even
$= \mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\ n};\iota_{\mathbb{E}_6};\mathbb{S}^{\mathsf{T}^2};\mathbb{S}^{\mathsf{T}^4};z;\mathbb{T};z^{\mathsf{T}};\mathbb{S}^{\mathsf{T}^4}$ since $\iota_{\mathbb{E}_6}$ ends with a number ≥ 6
$\subseteq \mathbb{T};z^{\mathsf{T}};\mathbb{S}^4$

---

vii) $\iota_{\mathbb{E}_6} = \iota_{\mathbb{E}_6};\iota_{\mathbb{E}_6}^{\mathsf{T}};\iota_{\mathbb{E}_6} = \iota_{\mathbb{E}_6};(\mathbb{I}\cap v_{\mathbb{E}_6};\mathbb{T})$
$= \iota_{\mathbb{E}_6};(\mathbb{I}\cap\mathbb{S}^{\mathsf{T}^4};[\mathbb{S}^{\mathsf{T}^2}]^+;z;\mathbb{T}) = \iota_{\mathbb{E}_6};(\mathbb{I}\cap\mathbb{T};z^{\mathsf{T}};[\mathbb{S}^2]^+;\mathbb{S}^4)$ partial identity transposed
$= \iota_{\mathbb{E}_6}\cap\mathbb{T};z^{\mathsf{T}};[\mathbb{S}^2]^+;\mathbb{S}^4$ explains what happens if not "$k\geq 6$ and even"
Therefore with mask shifting

$\iota_{\mathbb{E}_6};\mathbb{S}^{\intercal k};z;\mathbb{T} = \big[\iota_{\mathbb{E}_6}\cap\mathbb{T};z^{\intercal};[\mathbb{S}^2]^+;\mathbb{S}^4\big];\mathbb{S}^{\intercal k};z;\mathbb{T} = \iota_{\mathbb{E}_6};\mathbb{S}^{\intercal k};z;\mathbb{T}\cap\mathbb{T};z^{\intercal};[\mathbb{S}^2]^+;\mathbb{S}^4;\mathbb{S}^{\intercal k};z;\mathbb{T}$,
i.e. the last term is $=\mathbb{\perp}$ unless $2*a+6=k$ for some $a\geq 0$, because $\mathbb{S}$ is injective, $\mathbb{S};z=\mathbb{\perp}$, and using (v). The rest follows from $\mathbb{S}_{\mathbb{E}_6}=\iota_{\mathbb{E}_6};\mathbb{S}^2;\iota_{\mathbb{E}_6}^{\intercal}$ in Def. 10.1.2.

The question of being even cannot pop up. The function $k\mapsto a:=\frac{k-6}{2}$ may be inverted to $a\mapsto k:=2*a+6$.

---

viii) Proof by induction with case $n=0$ absolutely trivial.
$\mathbb{S}_{\mathbb{E}_6}^{n+1};\iota_{\mathbb{E}_6} = \mathbb{S}_{\mathbb{E}_6};\mathbb{S}_{\mathbb{E}_6}^{n};\iota_{\mathbb{E}_6} = \mathbb{S}_{\mathbb{E}_6};\iota_{\mathbb{E}_6};\mathbb{S}^{2*n}$ hypothesis
$= \iota_{\mathbb{E}_6};\mathbb{S}^2;\iota_{\mathbb{E}_6}^{\intercal};\iota_{\mathbb{E}_6};\mathbb{S}^{2*n}$ expanding $\mathbb{S}_{\mathbb{E}_6}$
$= \iota_{\mathbb{E}_6};\mathbb{S}^2;\mathbb{S}^{2*n}$ following (i)
$= \iota_{\mathbb{E}_6};\mathbb{S}^{2*(n+1)}$

---

ix) Proof for $n=0$:
$\iota_{\mathbb{E}_6}^{\intercal};\mathbb{S}_{\mathbb{E}_6}^0 = \iota_{\mathbb{E}_6}^{\intercal} = \mathbb{S}^{2*0};\iota_{\mathbb{E}_6}^{\intercal}\cap\iota_{\mathbb{E}_6}^{\intercal};\mathbb{T} = \mathbb{S}^{2*0};\iota_{\mathbb{E}_6}^{\intercal}\cap v_{\mathbb{E}_6};\mathbb{T}$ since $\iota_{\mathbb{E}_6}$ extrudes $v_{\mathbb{E}_6};\mathbb{T}$
Now the inductive step $n\mapsto n+1$:
$\iota_{\mathbb{E}_6}^{\intercal};\mathbb{S}_{\mathbb{E}_6}^{n+1} = \iota_{\mathbb{E}_6}^{\intercal};\mathbb{S}_{\mathbb{E}_6}^{n};\mathbb{S}_{\mathbb{E}_6} = \big[\mathbb{S}^{2*n};\iota_{\mathbb{E}_6}^{\intercal}\cap v_{\mathbb{E}_6};\mathbb{T}\big];\mathbb{S}_{\mathbb{E}_6}$ hypothesis
$= \mathbb{S}^{2*n};\iota_{\mathbb{E}_6}^{\intercal};\mathbb{S}_{\mathbb{E}_6}\cap v_{\mathbb{E}_6};\mathbb{T}$ masking
$= \mathbb{S}^{2*n};\iota_{\mathbb{E}_6}^{\intercal};\iota_{\mathbb{E}_6};\mathbb{S}^2;\iota_{\mathbb{E}_6}^{\intercal}\cap v_{\mathbb{E}_6};\mathbb{T}$ Def. 10.1.2
$= \mathbb{S}^{2*n};\big[\mathbb{I}\cap v_{\mathbb{E}_6};\mathbb{T}\big];\mathbb{S}^2;\iota_{\mathbb{E}_6}^{\intercal}\cap v_{\mathbb{E}_6};\mathbb{T} = \mathbb{S}^{2*n};\big[\mathbb{S}^2;\iota_{\mathbb{E}_6}^{\intercal}\cap v_{\mathbb{E}_6};\mathbb{T}\big]\cap v_{\mathbb{E}_6};\mathbb{T}$ masking again
$= \mathbb{S}^{2*n+2};\iota_{\mathbb{E}_6}^{\intercal}\cap\mathbb{S}^{2*n};v_{\mathbb{E}_6};\mathbb{T}\cap v_{\mathbb{E}_6};\mathbb{T}$,
together with the observation that the middle term contains the intersection of the outer two and may, thus, be skipped. This holds certainly in case $n=0$, but also in general because
$\mathbb{S}^{2*n+2};\iota_{\mathbb{E}_6}^{\intercal}\cap v_{\mathbb{E}_6};\mathbb{T} = \mathbb{S}^{2*n};\mathbb{S}^2;\iota_{\mathbb{E}_6}^{\intercal}\cap v_{\mathbb{E}_6};\mathbb{T}$
$\subseteq (\mathbb{S}^{2*n}\cap v_{\mathbb{E}_6};\mathbb{T};\iota_{\mathbb{E}_6};\mathbb{S}^{\intercal 2});(\mathbb{S}^2;\iota_{\mathbb{E}_6}^{\intercal}\cap\mathbb{S}^{\intercal 2*n};v_{\mathbb{E}_6};\mathbb{T})$ Dedekind rule
$\subseteq \mathbb{S}^{2*n};\mathbb{S}^{\intercal 2*n};v_{\mathbb{E}_6};\mathbb{T} = \mathbb{S}^{2*n};\mathbb{S}^{\intercal 2*n};\mathtt{sup}_{m\geq 3}\,\mathbb{S}^{\intercal 2*m};z;\mathbb{T}$
$= \mathbb{S}^{2*n};\mathtt{sup}_{m\geq 3}\,\mathbb{S}^{\intercal 2*(n+m)};z;\mathbb{T}\subseteq\mathbb{S}^{2*n};\mathtt{sup}_{i\geq 3}\,\mathbb{S}^{\intercal 2*i};z;\mathbb{T} = \mathbb{S}^{2*n};v_{\mathbb{E}_6};\mathbb{T}$ □

We provide a standard notation for these transitions.

10.1.4 Definition. Given $\mathbb{S}$, assume $\mathbb{S}_{\mathbb{E}_6}^{\intercal\,a};z_{\mathbb{E}_6};\mathbb{T}$ as well as $\mathbb{S}^{\intercal n};z;\mathbb{T}$ to be points and consider the sets of naturals $a\geq 0$ and $n\geq 0$ thus introduced. We define
$$\mathtt{Exp6N}(a):=2*a+6 \qquad \mathtt{ExpN6}(n):=\begin{cases}\frac{n-6}{2} & \text{if } n\geq 6 \text{ and even}\\ \text{undefined} & \text{otherwise}\end{cases}$$ □

Then $\mathtt{Exp6N}(a)$ is a mapping while $\mathtt{ExpN6}(n)$ is just a function; always
$$\mathtt{ExpN6}(\mathtt{Exp6N}(a))=a \qquad \mathtt{Exp6N}(\mathtt{ExpN6}(n))=\begin{cases}n & \text{if } n\geq 6 \text{ and even}\\ \text{undefined} & \text{otherwise}\end{cases}$$

Above, it has been said that $\mathbb{S}_{\mathbb{E}_6}$ enjoys all the properties we have had for $\mathbb{S}$. So we may also define an addition $\mathbb{A}_{\mathbb{E}_6}$. Would we follow this idea and execute it precisely as earlier in

Prop. 6.2.1,[4] we would arrive at the left relation of Fig. 10.1.5. This addition will, however, bring

$\mathbb{A}^{\mathsf{T}}_{\mathbb{E}_6}{:}\,(z_{\mathbb{E}_6} \oslash z_{\mathbb{E}_6}) = z_{\mathbb{E}_6}$, meaning more or less $6+6=6$

— and this is certainly not what we are interested in.

	6→	8→	10→	12→	14→	16→
(6→,6→)	1	0	0	0	0	0
(6→,8→)	0	1	0	0	0	0
(6→,10→)	0	0	1	0	0	0
(6→,12→)	0	0	0	1	0	0
(6→,14→)	0	0	0	0	1	0
(6→,16→)	0	0	0	0	0	1
(8→,6→)	0	1	0	0	0	0
(8→,8→)	0	0	1	0	0	0
(8→,10→)	0	0	0	1	0	0
(8→,12→)	0	0	0	0	1	0
(8→,14→)	0	0	0	0	0	1
(8→,16→)	0	0	0	0	0	0
(10→,6→)	0	0	1	0	0	0
(10→,8→)	0	0	0	1	0	0
(10→,10→)	0	0	0	0	1	0
(10→,12→)	0	0	0	0	0	1
(10→,14→)	0	0	0	0	0	0
(10→,16→)	0	0	0	0	0	0
(12→,6→)	0	0	0	1	0	0
(12→,8→)	0	0	0	0	1	0
(12→,10→)	0	0	0	0	0	1
(12→,12→)	0	0	0	0	0	0
(12→,14→)	0	0	0	0	0	0
(12→,16→)	0	0	0	0	0	0
(14→,6→)	0	0	0	0	1	0
(14→,8→)	0	0	0	0	0	1
(14→,10→)	0	0	0	0	0	0
(14→,12→)	0	0	0	0	0	0
(14→,14→)	0	0	0	0	0	0
(14→,16→)	0	0	0	0	0	0
(16→,6→)	0	0	0	0	0	1
(16→,8→)	0	0	0	0	0	0
(16→,10→)	0	0	0	0	0	0
(16→,12→)	0	0	0	0	0	0
(16→,14→)	0	0	0	0	0	0
(16→,16→)	0	0	0	0	0	0

	6→	8→	10→	12→	14→	16→
(6→,6→)	0	0	0	1	0	0
(6→,8→)	0	0	0	0	1	0
(6→,10→)	0	0	0	0	0	1
(6→,12→)	0	0	0	0	0	0
(6→,14→)	0	0	0	0	0	0
(6→,16→)	0	0	0	0	0	0
(8→,6→)	0	0	0	0	1	0
(8→,8→)	0	0	0	0	0	1
(8→,10→)	0	0	0	0	0	0
(8→,12→)	0	0	0	0	0	0
(8→,14→)	0	0	0	0	0	0
(8→,16→)	0	0	0	0	0	0
(10→,6→)	0	0	0	0	0	1
(10→,8→)	0	0	0	0	0	0
(10→,10→)	0	0	0	0	0	0
(10→,12→)	0	0	0	0	0	0
(10→,14→)	0	0	0	0	0	0
(10→,16→)	0	0	0	0	0	0
(12→,6→)	0	0	0	0	0	0
(12→,8→)	0	0	0	0	0	0
(12→,10→)	0	0	0	0	0	0
(12→,12→)	0	0	0	0	0	0
(12→,14→)	0	0	0	0	0	0
(12→,16→)	0	0	0	0	0	0
(14→,6→)	0	0	0	0	0	0
(14→,8→)	0	0	0	0	0	0
(14→,10→)	0	0	0	0	0	0
(14→,12→)	0	0	0	0	0	0
(14→,14→)	0	0	0	0	0	0
(14→,16→)	0	0	0	0	0	0
(16→,6→)	0	0	0	0	0	0
(16→,8→)	0	0	0	0	0	0
(16→,10→)	0	0	0	0	0	0
(16→,12→)	0	0	0	0	0	0
(16→,14→)	0	0	0	0	0	0
(16→,16→)	0	0	0	0	0	0

Fig. 10.1.5 $\mathbb{A}_{\mathbb{E}_6}$ in a naïve approach — compare with Fig. 6.2.2 — and the useful one

Far more useful is this:

10.1.5 Definition. We define as **evens-addition** belonging to $\mathbb{S}_{\mathbb{E}_6}$ the construct

$\mathbb{A}_{\mathbb{E}_6} := (\iota_{\mathbb{E}_6} \otimes \iota_{\mathbb{E}_6}){:}\mathbb{A}{:}\iota^{\mathsf{T}}_{\mathbb{E}_6}$.

There exists a corresponding **evens-subtraction** that more or less follows the concepts of Sect. 6.3.

[4] — and do not elaborate what happens when $\mathbb{S}_{\mathbb{E}_6}$ or $\mathbb{S}_{\mathbb{P}}$ should not be regressively finite

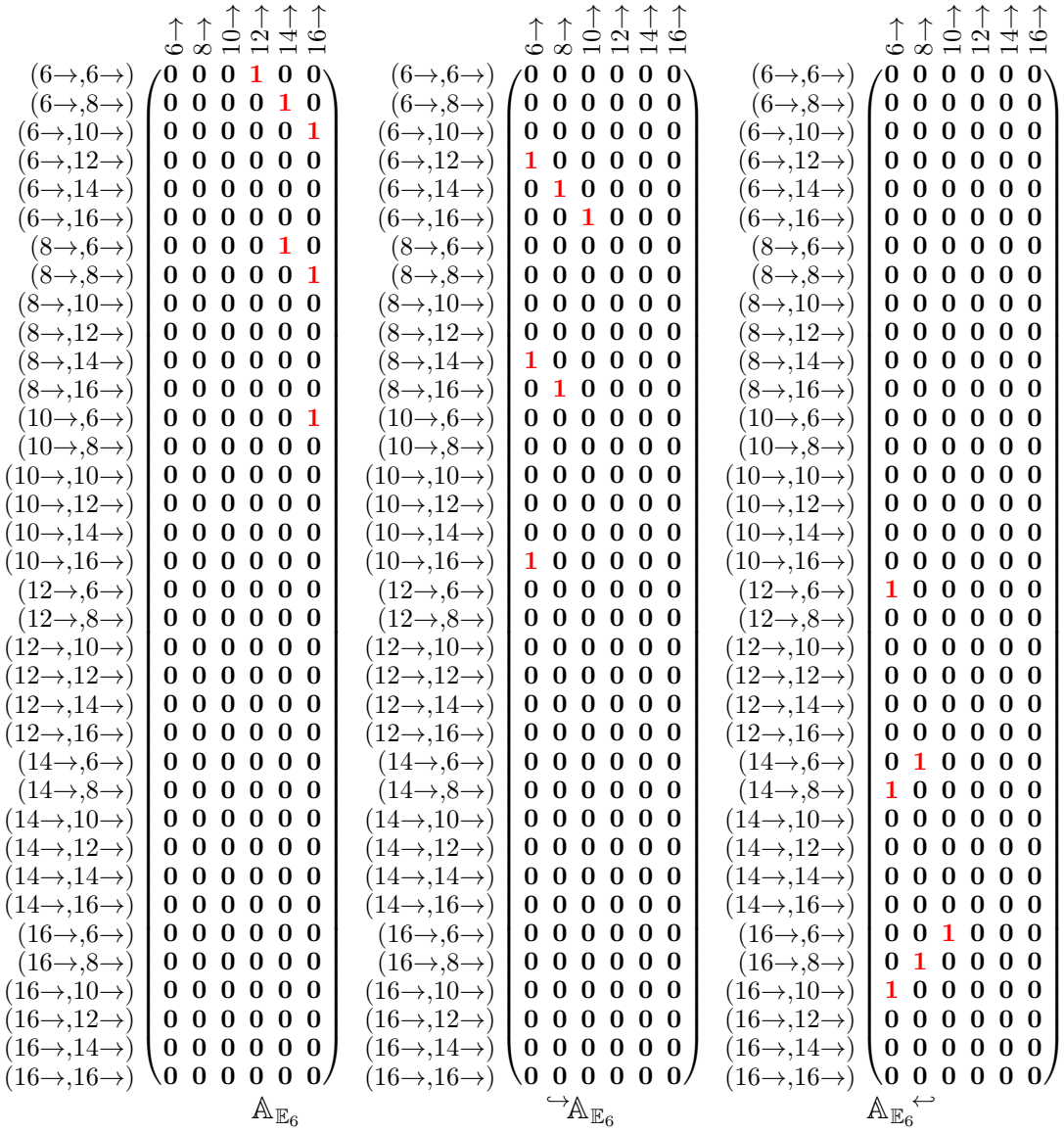


Fig. 10.1.6 $\mathrm{Evens}_{\geq 6}$-addition together with its left and right inverse

The following is hard to interpret in the above figure. It is again the difference between row inscription $8\rightarrow$, row number 2, and exponent 1 in the expression $8\rightarrow = \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{1}; 6\rightarrow$. Adding $6\rightarrow$ and $8\rightarrow$ shall result in $14\rightarrow$, which is achieved via the exponents $(0+1)+3=4$.

10.1.6 Proposition. $\mathbb{A}^{\mathsf{T}}_{\mathbb{E}_6}; (\mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{a}; z_{\mathbb{E}_6}; \mathbb{T} \between \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{b}; z_{\mathbb{E}_6}; \mathbb{T}) = \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{(a+b)+3}; z_{\mathbb{E}_6}; \mathbb{T}$

Proof: $\mathbb{A}^{\mathsf{T}}_{\mathbb{E}_6}; (\mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{a}; z_{\mathbb{E}_6}; \mathbb{T} \between \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{b}; z_{\mathbb{E}_6}; \mathbb{T}) = \iota_{\mathbb{E}_6}; \mathbb{A}^{\mathsf{T}}; (\iota^{\mathsf{T}}_{\mathbb{E}_6} \otimes \iota^{\mathsf{T}}_{\mathbb{E}_6}); (\mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{a}; z_{\mathbb{E}_6}; \mathbb{T} \between \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{b}; z_{\mathbb{E}_6}; \mathbb{T})$

$= \iota_{\mathbb{E}_6}; \mathbb{A}^{\mathsf{T}}; (\iota^{\mathsf{T}}_{\mathbb{E}_6}; \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{a}; z_{\mathbb{E}_6}; \mathbb{T} \between \iota^{\mathsf{T}}_{\mathbb{E}_6}; \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{b}; z_{\mathbb{E}_6}; \mathbb{T})$

$= \iota_{\mathbb{E}_6}; \mathbb{A}^{\mathsf{T}}; (\iota^{\mathsf{T}}_{\mathbb{E}_6}; \iota_{\mathbb{E}_6}; \mathbb{S}^{\mathsf{T}2*a+6}; z; \mathbb{T} \between \iota^{\mathsf{T}}_{\mathbb{E}_6}; \iota_{\mathbb{E}_6}; \mathbb{S}^{\mathsf{T}2*b+6}; z; \mathbb{T})$ Prop. 10.1.3.vii

$= \iota_{\mathbb{E}_6}; \mathbb{A}^{\mathsf{T}}; (\mathbb{S}^{\mathsf{T}2*a+6}; z; \mathbb{T} \between \mathbb{S}^{\mathsf{T}2*b+6}; z; \mathbb{T})$ since, e.g. $2*a+6$ is even and ≥ 6

$= \iota_{\mathbb{E}_6}; \mathbb{S}^{\mathsf{T}(2*a+6)+(2*b+6)}; z; \mathbb{T}$

$= \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{\frac{(2*a+6)+(2*b+6)-6}{2}}; z_{\mathbb{E}_6}; \mathbb{T}$ Prop. 10.1.3.vii, since $(2*a+6)+(2*b+6)$ is even and ≥ 6

$= \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{(a+b)+3}; z_{\mathbb{E}_6}; \mathbb{T}$ □

10.1.7 Proposition. i) In connection with $\mathbb{S}_{\mathbb{E}_6}$, we consider the functional

$a_{\mathbb{E}_6}(X) := \mathbb{A}_{\mathbb{E}_6 0} \cup a_{\mathbb{E}_6 H}(X)$,

built with

$\mathbb{A}_{\mathbb{E}_6 0} := (z_{\mathbb{E}_6}; z^{\mathsf{T}}_{\mathbb{E}_6} \otimes \mathbb{I}_{\mathbb{E}_6}); \mathbb{A}_{\mathbb{E}_6}$ and $a_{\mathbb{E}_6 H}(X) := (\mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6} \otimes \mathbb{I}_{\mathbb{E}_6}); X; \mathbb{S}_{\mathbb{E}_6}$.

Then the following is well-defined:

$\mathbb{A}'' := \mathtt{inf}\, \{\, X \mid a_{\mathbb{E}_6}(X) \subseteq X \}$

ii) The $\mathbb{A}_{\mathbb{E}_6}$ defined in Def. 10.1.5 satisfies $a_{\mathbb{E}_6 H}(\mathbb{A}_{\mathbb{E}_6}) \subseteq \mathbb{A}_{\mathbb{E}_6}$, and thus $\mathbb{A}'' \subseteq \mathbb{A}_{\mathbb{E}_6}$.

Proof: i) We easily convince ourselves that Prop. 4.2.1 may be applied, i.e. that the functional $X \mapsto a_{\mathbb{E}_6}(X)$ is monotonic, additive, and $\cap$-sub-hereditary.

The standard according to Prop. 4.2.1 thus establishes an iteration to obtain

$\mathbb{A}'' = \mathtt{sup}_{n\geq 0}\, a^n_{\mathbb{E}_6 H}(\mathbb{A}_{\mathbb{E}_6 0})$.

ii) $a_{\mathbb{E}_6}(\mathbb{A}_{\mathbb{E}_6}) = \mathbb{A}_{\mathbb{E}_6 0} \cup a_{\mathbb{E}_6 H}(\mathbb{A}_{\mathbb{E}_6}) = (z_{\mathbb{E}_6}; z^{\mathsf{T}}_{\mathbb{E}_6} \otimes \mathbb{I}_{\mathbb{E}_6}); \mathbb{A}_{\mathbb{E}_6} \cup (\mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6} \otimes \mathbb{I}_{\mathbb{E}_6}); \mathbb{A}_{\mathbb{E}_6}; \mathbb{S}_{\mathbb{E}_6}$
$\subseteq (\mathbb{I}_{\mathbb{E}_6} \otimes \mathbb{I}_{\mathbb{E}_6}); \mathbb{A}_{\mathbb{E}_6} \cup (\mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6} \otimes \mathbb{I}_{\mathbb{E}_6}); (\iota_{\mathbb{E}_6} \otimes \iota_{\mathbb{E}_6}); \mathbb{A}; \iota^{\mathsf{T}}_{\mathbb{E}_6}; \mathbb{S}_{\mathbb{E}_6}$ since $z_{\mathbb{E}_6}$ is injective, expanded
$= \mathbb{I}_{\mathbb{E}_6 \times \mathbb{E}_6}; \mathbb{A}_{\mathbb{E}_6} \cup (\iota_{\mathbb{E}_6}; \mathbb{S}^{\mathsf{T}2}; \iota^{\mathsf{T}}_{\mathbb{E}_6}; \iota_{\mathbb{E}_6} \otimes \iota_{\mathbb{E}_6}); \mathbb{A}; \iota^{\mathsf{T}}_{\mathbb{E}_6}; \iota_{\mathbb{E}_6}; \mathbb{S}^{2}; \iota^{\mathsf{T}}_{\mathbb{E}_6}$ by definition
$\subseteq \mathbb{A}_{\mathbb{E}_6} \cup (\iota_{\mathbb{E}_6}; \mathbb{S}^{\mathsf{T}2} \otimes \iota_{\mathbb{E}_6}); \mathbb{A}; \mathbb{S}^{2}; \iota^{\mathsf{T}}_{\mathbb{E}_6}$ since $\iota_{\mathbb{E}_6}$ is univalent
$= \mathbb{A}_{\mathbb{E}_6} \cup (\iota_{\mathbb{E}_6} \otimes \iota_{\mathbb{E}_6}); (\mathbb{S}^{\mathsf{T}2} \otimes \mathbb{I}); (\mathbb{S}^{2} \otimes \mathbb{I}); \mathbb{A}; \iota^{\mathsf{T}}_{\mathbb{E}_6}$ Prop. 6.2.1.ii.d
$\subseteq \mathbb{A}_{\mathbb{E}_6} \cup (\iota_{\mathbb{E}_6} \otimes \iota_{\mathbb{E}_6}); (\mathbb{I} \otimes \mathbb{I}); \mathbb{A}; \iota^{\mathsf{T}}_{\mathbb{E}_6}$ since $\mathbb{S}$ is univalent
$= \mathbb{A}_{\mathbb{E}_6} \cup (\iota_{\mathbb{E}_6} \otimes \iota_{\mathbb{E}_6}); \mathbb{A}; \iota^{\mathsf{T}}_{\mathbb{E}_6} = \mathbb{A}_{\mathbb{E}_6} \cup \mathbb{A}_{\mathbb{E}_6} = \mathbb{A}_{\mathbb{E}_6}$ □

In order to get rid of $\mathbb{A}_{\mathbb{E}_6}$ already in the definition of $\mathbb{A}_{\mathbb{E}_6 0}$, we may also introduce it differently: We might cobble together

$\mathbb{A}_{\mathbb{E}_6 0} := (\iota_{\mathbb{E}_6} \otimes \iota_{\mathbb{E}_6}); (\mathbb{S}^{\mathsf{T}6}; z; \mathbb{T}; z^{\mathsf{T}}; \mathbb{S}^{6} \otimes \mathbb{I}_{\mathbb{N}}); \mathbb{A}; \iota^{\mathsf{T}}_{\mathbb{E}_6}$ or $:= (z_{\mathbb{E}_6}; \mathbb{T} \circledcirc z_{\mathbb{E}_6}; \mathbb{T}); z^{\mathsf{T}}_{\mathbb{E}_6}; \mathbb{S}^{3}_{\mathbb{E}_6}$.

It is highly probable that in addition $\mathbb{A}'' = \mathbb{A}_{\mathbb{E}_6}$ will hold. A proof has not yet been executed.

10.2 Adding primes

In Def. 9.4.2, it has been said that $\mathbb{S}_{\mathbb{P}}$ enjoys all the properties we had in Sect. 5.1 postulated for $\mathbb{S}$. So we may also define an addition $\mathbb{A}_{\mathbb{P}}$. When we follow this idea and execute it as earlier in Prop. 6.2.1 and Fig. 6.2.2,[5] we would arrive at the left relation of Fig. 10.2.1. This is not useful. Recalling that the sum of two odd primes is even, we decide for a different typing on the right, namely $\mathbb{A}_{\mathbb{P}} : \mathbb{P} \times \mathbb{P} \longrightarrow \mathbb{E}_6$.

10.2.1 Definition. We define as **prime-addition** the construct

$\mathbb{A}_{\mathbb{P}} := (\iota_{\wp;\mathbb{T}} \otimes \iota_{\wp;\mathbb{T}}); \mathbb{A}; \iota^{\mathsf{T}}_{\mathbb{E}_6}$ addition for pairs of primes

In analogy

$\mathtt{S}_{\mathbb{P}} := (\iota_{\wp;\mathbb{T}} \otimes \iota_{\wp;\mathbb{T}}); \mathtt{S}; \iota^{\mathsf{T}}_{\mathbb{E}_6}$ subtraction for pairs of primes □

The Goldbach-question is then whether $\mathbb{A}_{\mathbb{P}}$ is surjective.

[5]— and do not elaborate what happens when $\mathbb{S}_{\mathbb{P}}$ should not be regressively finite

We avoid, thus, a detailed construction as in Prop. 6.2.1. First, we observe how it works:

$$\begin{array}{r} \\ (3\to,3\to)\\(3\to,5\to)\\(3\to,7\to)\\(3\to,11\to)\\(3\to,13\to)\\(5\to,3\to)\\(5\to,5\to)\\(5\to,7\to)\\(5\to,11\to)\\(5\to,13\to)\\(7\to,3\to)\\(7\to,5\to)\\(7\to,7\to)\\(7\to,11\to)\\(7\to,13\to)\\(11\to,3\to)\\(11\to,5\to)\\(11\to,7\to)\\(11\to,11\to)\\(11\to,13\to)\\(13\to,3\to)\\(13\to,5\to)\\(13\to,7\to)\\(13\to,11\to)\\(13\to,13\to)\end{array}\begin{array}{c}\begin{pmatrix}3\to & 5\to & 7\to & 11\to & 13\to\\ \mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}\\ \mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\end{pmatrix}\end{array}\qquad \begin{array}{r} \\ (3\to,3\to)\\(3\to,5\to)\\(3\to,7\to)\\(3\to,11\to)\\(3\to,13\to)\\(5\to,3\to)\\(5\to,5\to)\\(5\to,7\to)\\(5\to,11\to)\\(5\to,13\to)\\(7\to,3\to)\\(7\to,5\to)\\(7\to,7\to)\\(7\to,11\to)\\(7\to,13\to)\\(11\to,3\to)\\(11\to,5\to)\\(11\to,7\to)\\(11\to,11\to)\\(11\to,13\to)\\(13\to,3\to)\\(13\to,5\to)\\(13\to,7\to)\\(13\to,11\to)\\(13\to,13\to)\end{array}\begin{array}{c}\begin{pmatrix}6\to & 8\to & 10\to & 12\to & 14\to\\ \mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\end{pmatrix}\end{array}$$

Fig. 10.2.1 $\mathbb{A}_{\mathbb{P}}$ in a naïve useless approach (compare with Fig. 6.2.2); $\mathbb{A}_{\mathbb{P}}$ as intended

This addition on the right of Fig. 10.2.1 brings $\mathbb{A}^{\intercal}_{\mathbb{P}\,;}\,(z_{\mathbb{P}} \leqslant z_{\mathbb{P}}) = z_{\mathbb{E}_6}$, meaning more or less $3+3=6\in\mathbb{E}_6$ with $3\in\mathbb{P}$ — and this is closer to what we are interested in.

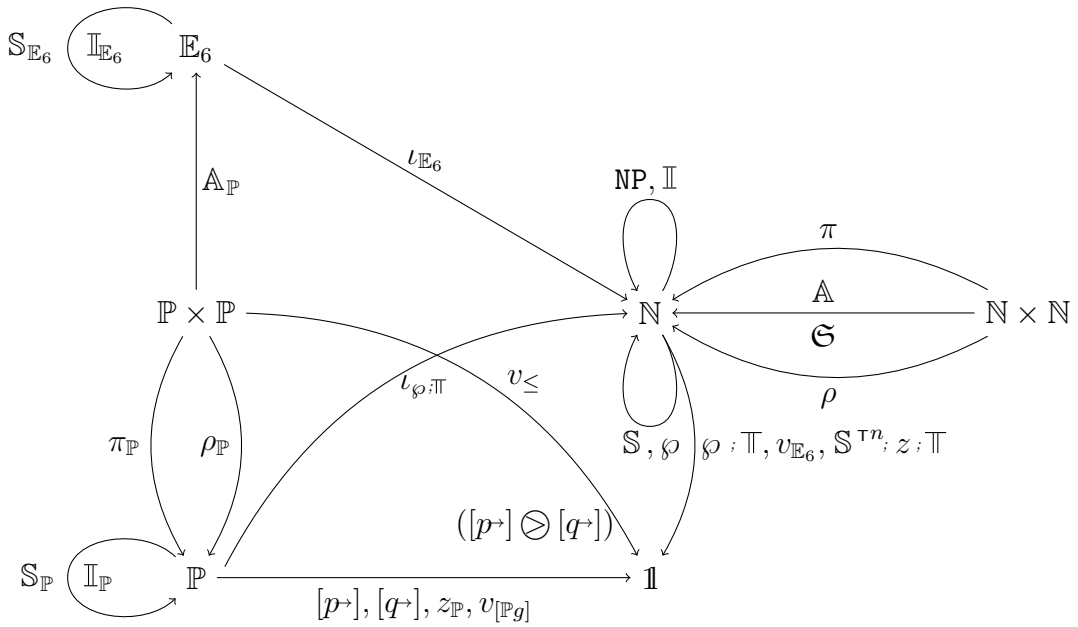


Fig. 10.2.2 Typing around the Goldbach setting

10.2.2 Proposition. We assume given arbitrarily two odd primes p, q — and thus $p+q \geq 6$ and even. This means in our present context points $\mathbb{S}^{\intercal p}; z; \mathbb{T}$ and $\mathbb{S}^{\intercal q}; z; \mathbb{T}$ in $\mathbb{N}$ or equivalently points $[p^{\rightarrow}], [q^{\rightarrow}]$ in $\mathbb{P}$, and we have always

i) $\mathbb{A}^{\intercal}_{\mathbb{P}}; ([p^{\rightarrow}] \oslash [q^{\rightarrow}]) = \mathbb{S}^{\intercal}_{\mathbb{E}_6}{}^{\frac{p+q-6}{2}}; z_{\mathbb{E}_6}; \mathbb{T}$.

ii) $\mathtt{S}^{\intercal}_{\mathbb{P}}; ([p^{\rightarrow}] \oslash [q^{\rightarrow}]) = \mathbb{S}^{\frac{p-6}{2}}_{\mathbb{E}_6}; \mathbb{S}^{\intercal}_{\mathbb{E}_6}{}^{\frac{q-6}{2}}; z_{\mathbb{E}_6}; \mathbb{T} = \begin{cases} \mathbb{S}^{\intercal}_{\mathbb{E}_6}{}^{\frac{q-p-6}{2}}; z_{\mathbb{E}_6}; \mathbb{T} & \text{provided } 6 \leq q-p \\ \perp\!\!\!\perp & \text{otherwise} \end{cases}$

iii) $\mathbb{A}^{\intercal}_{\mathbb{P}}; \pi_{\mathbb{P}} = \iota_{\mathbb{E}_6}; \mathtt{GR}; \iota^{\intercal}_{\wp;\mathbb{T}} = \mathbb{A}^{\intercal}_{\mathbb{P}}; \rho_{\mathbb{P}}$

iv) For $g = 2, 4, 6, \ldots$ always[6]

$(\mathbb{S}_{\mathbb{P}} \otimes \mathbb{I}_{\mathbb{P}}); \mathbb{A}_{\mathbb{P}} \cap \pi_{\mathbb{P}}; v_{[\mathbb{P}g]}; \mathbb{T} = \mathbb{A}_{\mathbb{P}}; \mathbb{S}^{\frac{g}{2}}_{\mathbb{E}_6} \cap \pi_{\mathbb{P}}; v_{[\mathbb{P}g]}; \mathbb{T}$

$(\mathbb{S}_{\mathbb{P}} \otimes \mathbb{I}_{\mathbb{P}}); \mathbb{A}_{\mathbb{P}} = \mathtt{sup}_{g=2,4,6,\ldots} \left[\mathbb{A}_{\mathbb{P}}; \mathbb{S}^{\frac{g}{2}}_{\mathbb{E}_6} \cap \pi_{\mathbb{P}}; v_{[\mathbb{P}g]}; \mathbb{T} \right]$

v) For every natural number i the construct $M := \mathtt{rel}(\overline{\mathbb{A}_{\mathbb{P}}; \overline{\mathbb{S}^{\intercal}_{\mathbb{E}_6}{}^{i}; z_{\mathbb{E}_6}; \mathbb{T}}})$ is a matching.

Proof: i) $\mathbb{A}^{\intercal}_{\mathbb{P}}; ([p^{\rightarrow}] \oslash [q^{\rightarrow}]) = \iota_{\mathbb{E}_6}; \mathbb{A}^{\intercal}; (\iota^{\intercal}_{\wp;\mathbb{T}} \otimes \iota^{\intercal}_{\wp;\mathbb{T}}); ([p^{\rightarrow}] \oslash [q^{\rightarrow}])$
$= \iota_{\mathbb{E}_6}; \mathbb{A}^{\intercal}; (\iota^{\intercal}_{\wp;\mathbb{T}}; [p^{\rightarrow}] \oslash \iota^{\intercal}_{\wp;\mathbb{T}}; [q^{\rightarrow}])$ following [13, Prop. 3.1.8.ii]
$= \iota_{\mathbb{E}_6}; \mathbb{A}^{\intercal}; (\iota^{\intercal}_{\wp;\mathbb{T}}; \iota_{\wp;\mathbb{T}}; \mathbb{S}^{\intercal p}; z; \mathbb{T} \oslash \iota^{\intercal}_{\wp;\mathbb{T}}; \iota_{\wp;\mathbb{T}}; \mathbb{S}^{\intercal q}; z; \mathbb{T})$ expanded
$= \iota_{\mathbb{E}_6}; \mathbb{A}^{\intercal}; (\wp; \mathbb{S}^{\intercal p}; z; \mathbb{T} \oslash \wp; \mathbb{S}^{\intercal q}; z; \mathbb{T})$ since $\iota_{\wp;\mathbb{T}}$ extrudes $\wp; \mathbb{T}$
$= \iota_{\mathbb{E}_6}; \mathbb{A}^{\intercal}; (\mathbb{S}^{\intercal p}; z; \mathbb{T} \oslash \mathbb{S}^{\intercal q}; z; \mathbb{T})$ because p, q are odd primes by assumption
$= \iota_{\mathbb{E}_6}; \mathbb{S}^{\intercal p+q}; z; \mathbb{T}$ Prop. 6.2.1.i.j
$= \mathbb{S}^{\intercal}_{\mathbb{E}_6}{}^{\frac{p+q-6}{2}}; z_{\mathbb{E}_6}; \mathbb{T}$ Prop. 10.1.3.vii

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ii) $\mathtt{S}^{\intercal}_{\mathbb{P}}; ([p^{\rightarrow}] \oslash [q^{\rightarrow}]) = \iota_{\mathbb{E}_6}; \mathtt{S}^{\intercal}; (\iota^{\intercal}_{\wp;\mathbb{T}} \otimes \iota^{\intercal}_{\wp;\mathbb{T}}); ([p^{\rightarrow}] \oslash [q^{\rightarrow}])$ by Def. 10.2.1
$= \iota_{\mathbb{E}_6}; \mathtt{S}^{\intercal}; (\iota^{\intercal}_{\wp;\mathbb{T}}; [p^{\rightarrow}] \oslash \iota^{\intercal}_{\wp;\mathbb{T}}; [q^{\rightarrow}])$ following [13, Prop. 3.1.8.ii]
$= \iota_{\mathbb{E}_6}; \mathtt{S}^{\intercal}; (\iota^{\intercal}_{\wp;\mathbb{T}}; \iota_{\wp;\mathbb{T}}; \mathbb{S}^{\intercal p}; z; \mathbb{T} \oslash \iota^{\intercal}_{\wp;\mathbb{T}}; \iota_{\wp;\mathbb{T}}; \mathbb{S}^{\intercal q}; z; \mathbb{T})$ expanded
$= \iota_{\mathbb{E}_6}; \mathtt{S}^{\intercal}; (\wp; \mathbb{S}^{\intercal p}; z; \mathbb{T} \oslash \wp; \mathbb{S}^{\intercal q}; z; \mathbb{T})$ since $\iota_{\wp;\mathbb{T}}$ extrudes $\wp; \mathbb{T}$
$= \iota_{\mathbb{E}_6}; \mathtt{S}^{\intercal}; (\mathbb{S}^{\intercal p}; z; \mathbb{T} \oslash \mathbb{S}^{\intercal q}; z; \mathbb{T})$ because p, q are odd primes by assumption
$= \iota_{\mathbb{E}_6}; \mathbb{S}^{p}; \mathbb{S}^{\intercal q}; z; \mathbb{T}$ Prop. 6.3.1.ii.i — in analogy to $\mathfrak{S}$
$= \begin{cases} \iota_{\mathbb{E}_6}; \mathbb{S}^{\intercal q-p}; z; \mathbb{T} & \text{provided } p \leq q \\ \perp\!\!\!\perp & \text{otherwise} \end{cases}$
$= \begin{cases} \mathbb{S}^{\intercal}_{\mathbb{E}_6}{}^{\frac{q-p-6}{2}}; z_{\mathbb{E}_6}; \mathbb{T} & \text{provided } 6 \leq q-p \\ \perp\!\!\!\perp & \text{otherwise} \end{cases}$ Prop. 10.1.3.vii

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iii) This reminds us of Fig. 10.1.4.

$\iota_{\mathbb{E}_6}; \mathtt{GR}; \iota^{\intercal}_{\wp;\mathbb{T}} = \iota_{\mathbb{E}_6}; \mathtt{GD}; \pi; \iota^{\intercal}_{\wp;\mathbb{T}} = \iota_{\mathbb{E}_6}; \mathbb{A}^{\intercal}; (\wp \otimes \wp); \pi; \iota^{\intercal}_{\wp;\mathbb{T}}$ by Def. 10.1.1
$= \iota_{\mathbb{E}_6}; \mathbb{A}^{\intercal}; (\wp \oslash \wp; \mathbb{T}); \iota^{\intercal}_{\wp;\mathbb{T}} = \iota_{\mathbb{E}_6}; \mathbb{A}^{\intercal}; (\wp; \iota^{\intercal}_{\wp;\mathbb{T}} \oslash \wp; \mathbb{T}; \iota^{\intercal}_{\wp;\mathbb{T}})$ since $\iota_{\wp;\mathbb{T}}$ is univalent
$= \iota_{\mathbb{E}_6}; \mathbb{A}^{\intercal}; (\iota^{\intercal}_{\wp;\mathbb{T}} \oslash \iota^{\intercal}_{\wp;\mathbb{T}}; \mathbb{T})$ since $\iota_{\wp;\mathbb{T}}$ extrudes $\wp; \mathbb{T}$ and $\iota_{\wp;\mathbb{T}}$ is total
$= \iota_{\mathbb{E}_6}; \mathbb{A}^{\intercal}; (\iota^{\intercal}_{\wp;\mathbb{T}} \otimes \iota^{\intercal}_{\wp;\mathbb{T}}); (\mathbb{I}_{\mathbb{P}} \oslash \mathbb{T}_{\mathbb{P}})$
$= \mathbb{A}^{\intercal}_{\mathbb{P}}; \pi_{\mathbb{P}}$

We use commutativity of $\mathbb{A}_{\mathbb{P}}$ for the second result.

---

[6] vgl. Prop. 6.2.1.ii.d

iv) $\mathbb{A}_{\mathbb{P}} ; \mathbb{S}_{\mathbb{E}_6}^{\frac{g}{2}} = (\iota_{\wp;\mathbb{T}} \bigotimes \iota_{\wp;\mathbb{T}}) ; \mathbb{A} ; \iota_{\mathbb{E}_6}^{\mathsf{T}} ; \iota_{\mathbb{E}_6} ; \mathbb{S}^{\,g} ; \iota_{\mathbb{E}_6}^{\mathsf{T}}$ by Def. 10.1.2, together with Prop. 10.1.3.ii
$= (\iota_{\wp;\mathbb{T}} \bigotimes \iota_{\wp;\mathbb{T}}) ; \mathbb{A} ; \mathbb{S}^{\,g} ; \iota_{\mathbb{E}_6}^{\mathsf{T}}$ since $(\iota_{\wp;\mathbb{T}} \bigotimes \iota_{\wp;\mathbb{T}}) ; \mathbb{A} \subseteq \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\,4} ; (\mathbb{S}^{\,2})^{+} = \mathbb{T} ; v_{\mathbb{E}_6}^{\mathsf{T}}$ (*)
$= (\iota_{\wp;\mathbb{T}} \bigotimes \iota_{\wp;\mathbb{T}}) ; (\mathbb{S}^{\,g} \bigotimes \mathbb{I}) ; \mathbb{A} ; \iota_{\mathbb{E}_6}^{\mathsf{T}}$ Prop. 6.2.1.ii.d
$= (\iota_{\wp;\mathbb{T}} \bigotimes \iota_{\wp;\mathbb{T}}) ; (\mathbb{S}^{\,g} \bigotimes \mathbb{I}) ; \mathbb{A} ; \iota_{\mathbb{E}_6}^{\mathsf{T}}$
$= (\iota_{\wp;\mathbb{T}} ; \mathbb{S}^{\,g} \bigotimes \iota_{\wp;\mathbb{T}}) ; \mathbb{A} ; \iota_{\mathbb{E}_6}^{\mathsf{T}}$

In the situation arrived at, we take care of the intersecting term:

$\mathbb{A}_{\mathbb{P}} ; \mathbb{S}_{\mathbb{E}_6}^{\frac{g}{2}} \cap \pi_{\mathbb{P}} ; v_{[\mathbb{P}g]} ; \mathbb{T} = (\iota_{\wp;\mathbb{T}} ; \mathbb{S}^{\,g} \bigotimes \iota_{\wp;\mathbb{T}}) ; \mathbb{A} ; \iota_{\mathbb{E}_6}^{\mathsf{T}} \cap \pi_{\mathbb{P}} ; v_{[\mathbb{P}g]} ; \mathbb{T}$
$= (\iota_{\wp;\mathbb{T}} ; \mathbb{S}^{\,g} \cap v_{[\mathbb{P}g]} ; \mathbb{T} \bigotimes \iota_{\wp;\mathbb{T}}) ; \mathbb{A} ; \iota_{\mathbb{E}_6}^{\mathsf{T}}$ masking
$= (\mathbb{S}_{\mathbb{P}} ; \iota_{\wp;\mathbb{T}} \cap v_{[\mathbb{P}g]} ; \mathbb{T} \bigotimes \iota_{\wp;\mathbb{T}}) ; \mathbb{A} ; \iota_{\mathbb{E}_6}^{\mathsf{T}}$ Prop. 9.6.7.vi
$= (\mathbb{S}_{\mathbb{P}} ; \iota_{\wp;\mathbb{T}} \bigotimes \iota_{\wp;\mathbb{T}}) ; \mathbb{A} ; \iota_{\mathbb{E}_6}^{\mathsf{T}} \cap \pi_{\mathbb{P}} ; v_{[\mathbb{P}g]} ; \mathbb{T}$ masking
$= (\mathbb{S}_{\mathbb{P}} \bigotimes \mathbb{I}_{\mathbb{P}}) ; (\iota_{\wp;\mathbb{T}} \bigotimes \iota_{\wp;\mathbb{T}}) ; \mathbb{A} ; \iota_{\mathbb{E}_6}^{\mathsf{T}} \cap \pi_{\mathbb{P}} ; v_{[\mathbb{P}g]} ; \mathbb{T}$
$= (\mathbb{S}_{\mathbb{P}} \bigotimes \mathbb{I}_{\mathbb{P}}) ; \mathbb{A}_{\,\mathbb{P}} \cap \pi_{\mathbb{P}} ; v_{[\mathbb{P}g]} ; \mathbb{T}$

The result marked $(*)$ seems obvious, but it shall here get a more formal proof:

$(\iota_{\wp;\mathbb{T}} \bigotimes \iota_{\wp;\mathbb{T}}) ; \mathbb{A} \subseteq (\mathbb{T} ; \iota_{\wp;\mathbb{T}} \bigotimes \mathbb{T} ; \iota_{\wp;\mathbb{T}}) ; \mathbb{A}$
$\subseteq (\mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\,3} ; (\mathbb{S}^{\,2})^{*} \bigotimes \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\,3} ; (\mathbb{S}^{\,2})^{*}) ; \mathbb{A}$
$= (\mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\,3} \bigotimes \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\,3}) ; ((\mathbb{S}^{\,2})^{*} \bigotimes \mathbb{I}) ; (\mathbb{I} \bigotimes (\mathbb{S}^{\,2})^{*}) ; \mathbb{A}$
$= (\mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\,3} \bigotimes \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\,3}) ; ((\mathbb{S}^{\,2})^{*} \bigotimes \mathbb{I}) ; \mathbb{A} ; (\mathbb{S}^{\,2})^{*}$ Prop. 6.2.1.ii.d
$= (\mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\,3} \bigotimes \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\,3}) ; \mathbb{A} ; (\mathbb{S}^{\,2})^{*} ; (\mathbb{S}^{\,2})^{*}$ again Prop. 6.2.1.ii.d
$= \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\,6} ; (\mathbb{S}^{\,2})^{*} = \mathbb{T} ; v_{\mathbb{E}_6}^{\mathsf{T}}$

The second statement follows directly herefrom. The only point to discuss is whether the greatest prime q — in the finite case — is handled correctly. We have then only

$[q^{\rightarrow}] ; \mathbb{T} \cup \mathtt{sup}_{\,g=2,4,6,8\,\ldots} \, v_{[\mathbb{P}g]} ; \mathbb{T} = \mathbb{T}_{\mathbb{P}}$,

but on the left side also $\mathbb{S}_{\mathbb{P}}$ leads to an empty row. See already Prop. 9.6.7.iii.

---

v) Since $\mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\,i} ; z_{\mathbb{E}_6} ; \mathbb{T}$ is a point representing an even number out of $\mathbb{E}_6$, the relation should match prime pairs with sum $2 * i + 6$. The proof consists of two estimations, namely

$M^{\mathsf{T}} ; M = \Big[\mathtt{rel}(\overline{\mathbb{A}_{\mathbb{P}} ; \overline{\mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\,i} ; z_{\mathbb{E}_6} ; \mathbb{T}}}) \Big]^{\mathsf{T}} ; \mathtt{rel}(\overline{\mathbb{A}_{\mathbb{P}} ; \overline{\mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\,i} ; z_{\mathbb{E}_6} ; \mathbb{T}}})$
$= \big[\pi^{\mathsf{T}} ; (\rho \cap \overline{\mathbb{A}_{\mathbb{P}} ; \overline{\mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\,i} ; z_{\mathbb{E}_6} ; \mathbb{T}}}) \big]^{\mathsf{T}} ; \big[\pi^{\mathsf{T}} ; (\rho \cap \overline{\mathbb{A}_{\mathbb{P}} ; \overline{\mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\,i} ; z_{\mathbb{E}_6} ; \mathbb{T}}}) \big]$
$= (\rho \cap \overline{\mathbb{A}_{\mathbb{P}} ; \overline{\mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\,i} ; z_{\mathbb{E}_6} ; \mathbb{T}}})^{\mathsf{T}} ; \pi ; \pi^{\mathsf{T}} ; (\rho \cap \overline{\mathbb{A}_{\mathbb{P}} ; \overline{\mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\,i} ; z_{\mathbb{E}_6} ; \mathbb{T}}})$
$= (\rho^{\mathsf{T}} \cap \overline{\mathbb{T} ; z_{\mathbb{E}_6}^{\mathsf{T}} ; \mathbb{S}_{\mathbb{E}_6}^{i} ; \mathbb{A}_{\mathbb{P}}^{\mathsf{T}}}) ; \pi ; \pi^{\mathsf{T}} ; (\rho \cap \overline{\mathbb{A}_{\mathbb{P}} ; \overline{\mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\,i} ; z_{\mathbb{E}_6} ; \mathbb{T}}})$
$= \rho^{\mathsf{T}} ; (\pi \cap \overline{\mathbb{A}_{\mathbb{P}} ; \overline{\mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\,i} ; z_{\mathbb{E}_6} ; \mathbb{T}}}) ; (\pi^{\mathsf{T}} \cap \overline{\mathbb{T} ; z_{\mathbb{E}_6}^{\mathsf{T}} ; \mathbb{S}_{\mathbb{E}_6}^{i} ; \mathbb{A}_{\mathbb{P}}^{\mathsf{T}}}) ; \rho$
$\ldots$ □

Although this does not directly follow the ideas for addition developed earlier in Prop. 6.2.1, it may be handled in a similar form. In the following proposition, we introduce an iteration in our standard form. Concerning $z_{\mathbb{P}}$, we refer back to Prop. 9.4.4.iv.

10.2.3 Proposition. i) Using the constructs

$\mathbb{A}_{\,\mathbb{P}0} := (\iota_{\wp;\mathbb{T}} \bigotimes \iota_{\wp;\mathbb{T}}) ; (\mathbb{S}^{\,\mathsf{T}3} ; z ; \mathbb{T} ; z^{\mathsf{T}} ; \mathbb{S}^{\,3} \bigotimes \mathbb{I}_{\mathbb{N}}) ; \mathbb{A} ; \iota_{\mathbb{E}_6}^{\mathsf{T}} = (z_{\mathbb{P}} ; z_{\mathbb{P}}^{\mathsf{T}} \bigotimes \mathbb{I}_{\mathbb{P}}) ; \mathbb{A}_{\mathbb{P}}$
$a_{\mathbb{P}H}(X) := ((\mathbb{S}_{\mathbb{P}}^{\mathsf{T}} \bigotimes \mathbb{I}_{\mathbb{P}}) ; X ; \iota_{\mathbb{E}_6} \leftthreetimes \pi_{\mathbb{P}} ; (\iota_{\wp;\mathbb{T}} \leftthreetimes \mathbb{S}_{\mathbb{P}}^{\mathsf{T}} ; \iota_{\wp;\mathbb{T}}) ; \mathfrak{S}) ; \mathbb{A} ; \iota_{\mathbb{E}_6}^{\mathsf{T}}$
$a_{\mathbb{P}}(X) := \mathbb{A}_{\,\mathbb{P}0} \cup a_{\mathbb{P}H}(X)$,

the following is well-defined

$\mathbb{A}'_{\mathbb{P}} := \mathtt{inf}\,\{\, X \mid a_{\mathbb{P}}(X) \subseteq X\}$

and

$\mathbb{A}'_{\mathbb{P}} = \mathtt{sup}_{n\geq 0}\, a^n_{\mathbb{P}H}(\mathbb{A}_{\mathbb{P}0})$.

ii) The $\mathbb{A}_{\mathbb{P}}$ defined in Def. 10.2.1 satisfies $a_{\mathbb{P}}(\mathbb{A}_{\mathbb{P}}) \subseteq \mathbb{A}_{\mathbb{P}}$, and thus $\mathbb{A}'_{\mathbb{P}} \subseteq \mathbb{A}_{\mathbb{P}}$, meaning $\mathtt{sup}_{n\geq 0}\, a^n_{\mathbb{P}H}(\mathbb{A}_{\mathbb{P}0}) \subseteq \mathbb{A}_{\mathbb{P}}$.

Proof: i) We convince ourselves that Prop. 4.2.1 may be applied. To this end, we simplify $a_{\mathbb{P}H}(X)$ to $a_{\mathbb{P}H}(X) = (A\,;X;B \oslash C)\,;D$ with B a mapping. Then it is more or less evident that the functional is monotone, additive, and $\cap$-sub-hereditary as required in Prop. 4.2.1:

$a_{\mathbb{P}H}(X) = (A\,;X;B \oslash C)\,;D = \big[A\,;X;B;\pi^{\intercal} \cap C;\rho^{\intercal}\big];\, D$

The standard according to Prop. 4.2.1 thus establishes an iteration to obtain

$\mathbb{A}'_{\mathbb{P}} = \mathtt{sup}_{n\geq 0}\, a^n_{\mathbb{P}H}(\mathbb{A}_{\mathbb{P}0})$.

---

ii) requires two estimations; first

$\mathbb{A}_{\mathbb{P}0} = (z_{\mathbb{P}};z^{\intercal}_{\mathbb{P}} \otimes \mathbb{I}_{\mathbb{P}})\,;\mathbb{A}_{\mathbb{P}} \subseteq (\mathbb{I}_{\mathbb{P}} \otimes \mathbb{I}_{\mathbb{P}})\,;\mathbb{A}_{\mathbb{P}} = \mathbb{I}_{\mathbb{P}\times\mathbb{P}};\, \mathbb{A}_{\mathbb{P}} = \mathbb{A}_{\mathbb{P}}$ since $z_{\mathbb{P}}$ is injective

The second one is so far executed only point-wise. It starts with a prime pair (p,q), i.e. the point

$([p^{\rightarrow}] \oslash [q^{\rightarrow}]) \in \mathbb{P}\times\mathbb{P}$ or $(\mathbb{S}^{\intercal p};z\,;\mathbb{T} \oslash \mathbb{S}^{\intercal q};z\,;\mathbb{T}) \in \mathbb{N}\times\mathbb{N}$

made up of any two points $[p^{\rightarrow}], [q^{\rightarrow}] \in \mathbb{P}$, or two points $\mathbb{S}^{\intercal p};z\,;\mathbb{T}, \mathbb{S}^{\intercal q};z\,;\mathbb{T} \in \mathbb{N}$, see Fig. 10.2.2.

$([p^{\rightarrow}] \oslash [q^{\rightarrow}])^{\intercal};\, a_{\mathbb{P}H}(\mathbb{A}_{\mathbb{P}})$
$= ([p^{\rightarrow}]^{\intercal} \oslash [q^{\rightarrow}]^{\intercal});\, ((\mathbb{S}^{\intercal}_{\mathbb{P}} \otimes \mathbb{I}_{\mathbb{P}})\,;\mathbb{A}_{\mathbb{P}};\iota_{\mathbb{E}_6} \oslash \pi_{\mathbb{P}};\, (\iota_{\wp;\mathbb{T}} \oslash \mathbb{S}^{\intercal}_{\mathbb{P}};\iota_{\wp;\mathbb{T}})\,;\mathfrak{S})\,;\mathbb{A}\,;\iota^{\intercal}_{\mathbb{E}_6}$
$= ((([p^{\rightarrow}]^{\intercal} \oslash [q^{\rightarrow}]^{\intercal});\, (\mathbb{S}^{\intercal}_{\mathbb{P}} \otimes \mathbb{I}_{\mathbb{P}})\,;\mathbb{A}_{\mathbb{P}};\iota_{\mathbb{E}_6} \oslash\ ([p^{\rightarrow}]^{\intercal} \oslash [q^{\rightarrow}]^{\intercal});\pi_{\mathbb{P}};\, (\iota_{\wp;\mathbb{T}} \oslash \mathbb{S}^{\intercal}_{\mathbb{P}};\iota_{\wp;\mathbb{T}})\,;\mathfrak{S})\,;\mathbb{A}\,;\iota^{\intercal}_{\mathbb{E}_6}$
since a transposed point is univalent
$= ((([p^{\rightarrow}]^{\intercal};\mathbb{S}^{\intercal}_{\mathbb{P}} \oslash [q^{\rightarrow}]^{\intercal})\,;\mathbb{A}_{\mathbb{P}};\iota_{\mathbb{E}_6} \oslash [p^{\rightarrow}]^{\intercal};\, (\iota_{\wp;\mathbb{T}} \oslash \mathbb{S}^{\intercal}_{\mathbb{P}};\iota_{\wp;\mathbb{T}})\,;\mathfrak{S})\,;\mathbb{A}\,;\iota^{\intercal}_{\mathbb{E}_6}$
again since the transposed point q is total
$= ((\big[\mathbb{S}_{\mathbb{P}};[p^{\rightarrow}]\big]^{\intercal} \oslash [q^{\rightarrow}]^{\intercal})\,;\mathbb{A}_{\mathbb{P}};\iota_{\mathbb{E}_6} \oslash\ ([p^{\rightarrow}]^{\intercal};\iota_{\wp;\mathbb{T}} \oslash [p^{\rightarrow}]^{\intercal};\mathbb{S}^{\intercal}_{\mathbb{P}};\iota_{\wp;\mathbb{T}})\,;\mathfrak{S})\,;\mathbb{A}\,;\iota^{\intercal}_{\mathbb{E}_6}$
$= ((\big[\mathbb{S}_{\mathbb{P}};[p^{\rightarrow}]\big]^{\intercal} \oslash [q^{\rightarrow}]^{\intercal});\, (\iota_{\wp;\mathbb{T}} \otimes \iota_{\wp;\mathbb{T}})\,;\mathbb{A}\,;\iota^{\intercal}_{\mathbb{E}_6};\iota_{\mathbb{E}_6} \oslash\ ([p^{\rightarrow}]^{\intercal};\iota_{\wp;\mathbb{T}} \oslash \big[\mathbb{S}_{\mathbb{P}};[p^{\rightarrow}]\big]^{\intercal};\iota_{\wp;\mathbb{T}})\,;\mathfrak{S})\,;\mathbb{A}\,;\iota^{\intercal}_{\mathbb{E}_6}$
$= ((\big[\mathbb{S}_{\mathbb{P}};[p^{\rightarrow}]\big]^{\intercal};\iota_{\wp;\mathbb{T}} \oslash [q^{\rightarrow}]^{\intercal};\iota_{\wp;\mathbb{T}})\,;\mathbb{A}\,;\iota^{\intercal}_{\mathbb{E}_6};\iota_{\mathbb{E}_6} \oslash\ ([p^{\rightarrow}]^{\intercal};\iota_{\wp;\mathbb{T}} \oslash \big[\mathbb{S}_{\mathbb{P}};[p^{\rightarrow}]\big]^{\intercal};\iota_{\wp;\mathbb{T}})\,;\mathfrak{S})\,;\mathbb{A}\,;\iota^{\intercal}_{\mathbb{E}_6}$
$= ((\mathbb{T}\,;z^{\intercal};\mathbb{S}^{\,p-d} \oslash \mathbb{T}\,;z^{\intercal};\mathbb{S}^{\,q})\,;\mathbb{A}\,;\iota^{\intercal}_{\mathbb{E}_6};\iota_{\mathbb{E}_6} \oslash\ (\mathbb{T}\,;z^{\intercal};\mathbb{S}^{\,p} \oslash \mathbb{T}\,;z^{\intercal};\mathbb{S}^{\,p-d})\,;\mathfrak{S})\,;\mathbb{A}\,;\iota^{\intercal}_{\mathbb{E}_6}$
recalling $[q^{\rightarrow}] = \iota_{\wp;\mathbb{T}};\mathbb{S}^{\intercal q};z\,;\mathbb{T}$, $[p^{\rightarrow}] = \iota_{\wp;\mathbb{T}};\mathbb{S}^{\intercal p};z\,;\mathbb{T}$ and $\mathbb{S}_{\mathbb{P}};[p^{\rightarrow}] =: \iota_{\wp;\mathbb{T}};\mathbb{S}^{\intercal p-d};z\,;\mathbb{T}$
thus p cannot be 3 — see below for this case
$\iota^{\intercal}_{\wp;\mathbb{T}};\iota_{\wp;\mathbb{T}}$ may be dropped since q,p are primes by assumption
$= (\mathbb{T}\,;z^{\intercal};\mathbb{S}^{\,(p-d)+q} \oslash \mathbb{T}\,;z^{\intercal};\mathbb{S}^{\,p-(p-d)})\,;\mathbb{A}\,;\iota^{\intercal}_{\mathbb{E}_6}$ since the even $(p-d)+q \geq 6$; Prop. 6.2.1.ii.e
$= \mathbb{T}\,;z^{\intercal};\mathbb{S}^{\,[(p-d)+q]+[p-(p-d)]};\iota^{\intercal}_{\mathbb{E}_6}$ again Prop. 6.2.1.ii.e and Prop. 6.3.1.ii.i
$= \mathbb{T}\,;z^{\intercal};\mathbb{S}^{\,[(p-d)+q]+d};\iota^{\intercal}_{\mathbb{E}_6} = \mathbb{T}\,;z^{\intercal};\mathbb{S}^{\,p+q};\iota^{\intercal}_{\mathbb{E}_6} = (\mathbb{T}\,;z^{\intercal};\mathbb{S}^{\,p} \oslash \mathbb{T}\,;z^{\intercal};\mathbb{S}^{\,q})\,;\mathbb{A}\,;\iota^{\intercal}_{\mathbb{E}_6}$
$= ([p^{\rightarrow}]^{\intercal};\iota_{\wp;\mathbb{T}} \oslash [q^{\rightarrow}]^{\intercal};\iota_{\wp;\mathbb{T}})\,;\mathbb{A}\,;\iota^{\intercal}_{\mathbb{E}_6}$
$= ([p^{\rightarrow}]^{\intercal} \oslash [q^{\rightarrow}]^{\intercal});\, (\iota_{\wp;\mathbb{T}} \otimes \iota_{\wp;\mathbb{T}})\,;\mathbb{A}\,;\iota^{\intercal}_{\mathbb{E}_6} = ([p^{\rightarrow}]^{\intercal} \oslash [q^{\rightarrow}]^{\intercal})\,;\mathbb{A}_{\mathbb{P}}$

Now the case $p = 3$ excluded above:

$([3^{\rightarrow}] \oslash [q^{\rightarrow}])^{\intercal};\, a_{\mathbb{P}H}(\mathbb{A}_{\mathbb{P}})$
$= ([3^{\rightarrow}]^{\intercal} \oslash [q^{\rightarrow}]^{\intercal});\, ((\mathbb{S}^{\intercal}_{\mathbb{P}} \otimes \mathbb{I}_{\mathbb{P}})\,;\mathbb{A}_{\mathbb{P}};\iota_{\mathbb{E}_6} \oslash \pi_{\mathbb{P}};\, (\iota_{\wp;\mathbb{T}} \oslash \mathbb{S}^{\intercal}_{\mathbb{P}};\iota_{\wp;\mathbb{T}})\,;\mathfrak{S})\,;\mathbb{A}\,;\iota^{\intercal}_{\mathbb{E}_6}$

$$
\begin{aligned}
&= \big(\big([3{\rightarrow}]^{\mathsf{T}} \between [q{\rightarrow}]^{\mathsf{T}}\big) ; \big(\mathbb{S}_{\mathbb{P}}^{\mathsf{T}} \otimes \mathbb{I}_{\mathbb{P}}\big) ; \mathbb{A}_{\mathbb{P}} ; \iota_{\mathbb{E}_6} \between \big([3{\rightarrow}]^{\mathsf{T}} \between [q{\rightarrow}]^{\mathsf{T}}\big) ; \pi_{\mathbb{P}} ; \big(\iota_{\wp;\mathbb{T}} \between \mathbb{S}_{\mathbb{P}}^{\mathsf{T}} ; \iota_{\wp;\mathbb{T}}\big) ; \mathfrak{S}\big) ; \mathbb{A} ; \iota_{\mathbb{E}_6}^{\mathsf{T}} \\
&= \big(\big([3{\rightarrow}]^{\mathsf{T}} ; \mathbb{S}_{\mathbb{P}}^{\mathsf{T}} \between [q{\rightarrow}]^{\mathsf{T}}\big) ; \mathbb{A}_{\mathbb{P}} ; \iota_{\mathbb{E}_6} \between \ldots\big) ; \mathbb{A} ; \iota_{\mathbb{E}_6}^{\mathsf{T}} \quad \text{since a transposed point is total} \\
&= \big(\big(\big[\mathbb{S}_{\mathbb{P}} ; [3{\rightarrow}]\big]^{\mathsf{T}} \between [q{\rightarrow}]^{\mathsf{T}}\big) ; \mathbb{A}_{\mathbb{P}} ; \iota_{\mathbb{E}_6} \between \ldots\big) ; \mathbb{A} ; \iota_{\mathbb{E}_6}^{\mathsf{T}} \\
&= \big(\big(\perp\!\!\!\perp \between [q{\rightarrow}]^{\mathsf{T}}\big) ; \mathbb{A}_{\mathbb{P}} ; \iota_{\mathbb{E}_6} \between \ldots\big) ; \mathbb{A} ; \iota_{\mathbb{E}_6}^{\mathsf{T}} = \perp\!\!\!\perp
\end{aligned}
$$

□

The present author has not gone so far as to prove in full detail that $\mathbb{A}'_{\mathbb{P}} = \mathbb{A}_{\mathbb{P}}$ — as of course should be assumed to hold.

	6→	8→	10→	12→	14→
(3→,3→)	**1**	0	0	0	0
(3→,5→)	0	**1**	0	0	0
(3→,7→)	0	0	**1**	0	0
(3→,11→)	0	0	0	0	**1**
(3→,13→)	0	0	0	0	0
(5→,3→)	0	0	0	0	0
(5→,5→)	0	0	0	0	0
(5→,7→)	0	0	0	0	0
(5→,11→)	0	0	0	0	0
(5→,13→)	0	0	0	0	0
(7→,3→)	0	0	0	0	0
(7→,5→)	0	0	0	0	0
(7→,7→)	0	0	0	0	0
(7→,11→)	0	0	0	0	0
(7→,13→)	0	0	0	0	0
(11→,3→)	0	0	0	0	0
(11→,5→)	0	0	0	0	0
(11→,7→)	0	0	0	0	0
(11→,11→)	0	0	0	0	0
(11→,13→)	0	0	0	0	0
(13→,3→)	0	0	0	0	0
(13→,5→)	0	0	0	0	0
(13→,7→)	0	0	0	0	0
(13→,11→)	0	0	0	0	0
(13→,13→)	0	0	0	0	0

	6→	8→	10→	12→	14→
(3→,3→)	0	0	0	0	0
(3→,5→)	0	0	0	0	0
(3→,7→)	0	0	0	0	0
(3→,11→)	0	0	0	0	0
(3→,13→)	0	0	0	0	0
(5→,3→)	**1**	0	0	0	0
(5→,5→)	0	**1**	0	0	0
(5→,7→)	0	0	**1**	0	0
(5→,11→)	0	0	0	**1**	0
(5→,13→)	0	0	0	0	0
(7→,3→)	0	0	0	0	0
(7→,5→)	0	0	0	0	0
(7→,7→)	0	0	0	0	0
(7→,11→)	0	0	0	0	0
(7→,13→)	0	0	0	0	0
(11→,3→)	0	0	0	0	0
(11→,5→)	0	0	0	0	0
(11→,7→)	0	0	0	0	0
(11→,11→)	0	0	0	0	0
(11→,13→)	0	0	0	0	0
(13→,3→)	0	0	0	0	0
(13→,5→)	0	0	0	0	0
(13→,7→)	0	0	0	0	0
(13→,11→)	0	0	0	0	0
(13→,13→)	0	0	0	0	0

	6→	8→	10→	12→	14→
(3→,3→)	0	0	0	0	0
(3→,5→)	0	0	0	0	0
(3→,7→)	0	0	0	0	0
(3→,11→)	0	0	0	0	0
(3→,13→)	0	0	0	0	0
(5→,3→)	0	0	0	0	0
(5→,5→)	0	0	0	0	0
(5→,7→)	0	0	0	0	0
(5→,11→)	0	0	0	0	0
(5→,13→)	0	0	0	0	0
(7→,3→)	0	0	**1**	0	0
(7→,5→)	0	0	0	**1**	0
(7→,7→)	0	0	0	0	**1**
(7→,11→)	0	0	0	0	0
(7→,13→)	0	0	0	0	0
(11→,3→)	0	0	0	0	0
(11→,5→)	0	0	0	0	0
(11→,7→)	0	0	0	0	0
(11→,11→)	0	0	0	0	0
(11→,13→)	0	0	0	0	0
(13→,3→)	0	0	0	0	0
(13→,5→)	0	0	0	0	0
(13→,7→)	0	0	0	0	0
(13→,11→)	0	0	0	0	0
(13→,13→)	0	0	0	0	0

	6→	8→	10→	12→	14→
(3→,3→)	0	0	0	0	0
(3→,5→)	0	0	0	0	0
(3→,7→)	0	0	0	0	0
(3→,11→)	0	0	0	0	0
(3→,13→)	0	0	0	0	0
(5→,3→)	0	0	0	0	0
(5→,5→)	0	0	0	0	0
(5→,7→)	0	0	0	0	0
(5→,11→)	0	0	0	0	0
(5→,13→)	0	0	0	0	0
(7→,3→)	0	0	0	0	0
(7→,5→)	0	0	0	0	0
(7→,7→)	0	0	0	0	0
(7→,11→)	0	0	0	0	0
(7→,13→)	0	0	0	0	0
(11→,3→)	0	0	0	0	**1**
(11→,5→)	0	0	0	0	0
(11→,7→)	0	0	0	0	0
(11→,11→)	0	0	0	0	0
(11→,13→)	0	0	0	0	0
(13→,3→)	0	0	0	0	0
(13→,5→)	0	0	0	0	0
(13→,7→)	0	0	0	0	0
(13→,11→)	0	0	0	0	0
(13→,13→)	0	0	0	0	0

	6→	8→	10→	12→	14→
(3→,3→)	0	0	0	0	0
(3→,5→)	0	0	0	0	0
(3→,7→)	0	0	0	0	0
(3→,11→)	0	0	0	0	0
(3→,13→)	0	0	0	0	0
(5→,3→)	0	0	0	0	0
(5→,5→)	0	0	0	0	0
(5→,7→)	0	0	0	0	0
(5→,11→)	0	0	0	0	0
(5→,13→)	0	0	0	0	0
(7→,3→)	0	0	0	0	0
(7→,5→)	0	0	0	0	0
(7→,7→)	0	0	0	0	0
(7→,11→)	0	0	0	0	0
(7→,13→)	0	0	0	0	0
(11→,3→)	0	0	0	0	0
(11→,5→)	0	0	0	0	0
(11→,7→)	0	0	0	0	0
(11→,11→)	0	0	0	0	0
(11→,13→)	0	0	0	0	0
(13→,3→)	0	0	0	0	0
(13→,5→)	0	0	0	0	0
(13→,7→)	0	0	0	0	0
(13→,11→)	0	0	0	0	0
(13→,13→)	0	0	0	0	0

Fig. 10.2.3 An approach to obtain $\mathbb{A}_{\mathbb{P}}$ by iteration — respecting prime distances

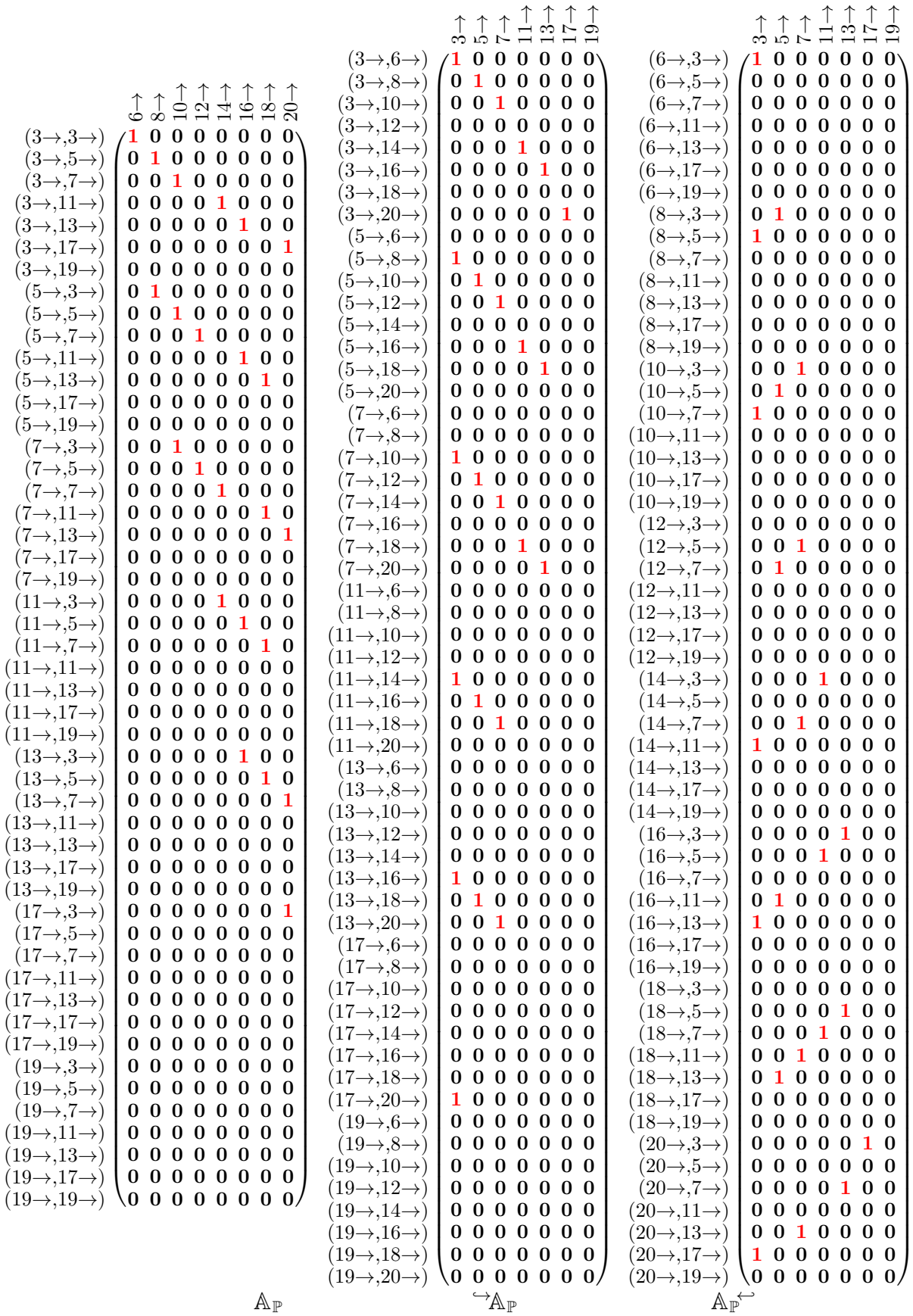


Fig. 10.2.4 Prime addition together with its left and right inverse

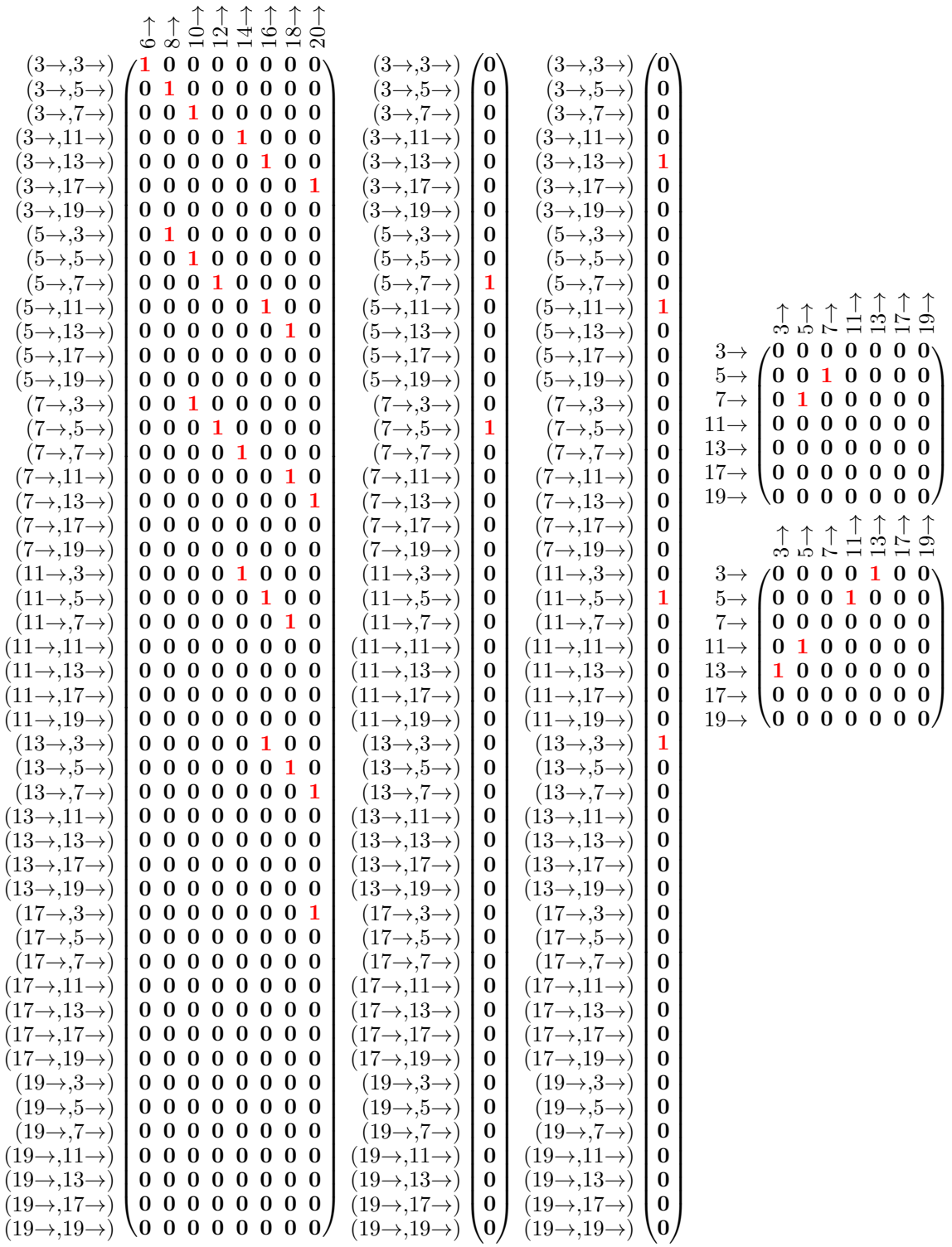


Fig. 10.2.5 Showing again $\mathbb{A}_{\mathbb{P}}$ with what will be extracted by 12 or 16 as vector or table

Still irritating is that the Goldbach decompositions all occur twice.

There is yet another way of adding, namely primes plus evens, resulting possibly in primes.

$$\mathbb{A}_{6\mathbb{P}} := (\iota_{\mathbb{E}_6} \otimes \iota_{\wp;\mathbb{T}}) ; \mathbb{A} ; \iota_{\wp;\mathbb{T}}^{\mathsf{T}}$$

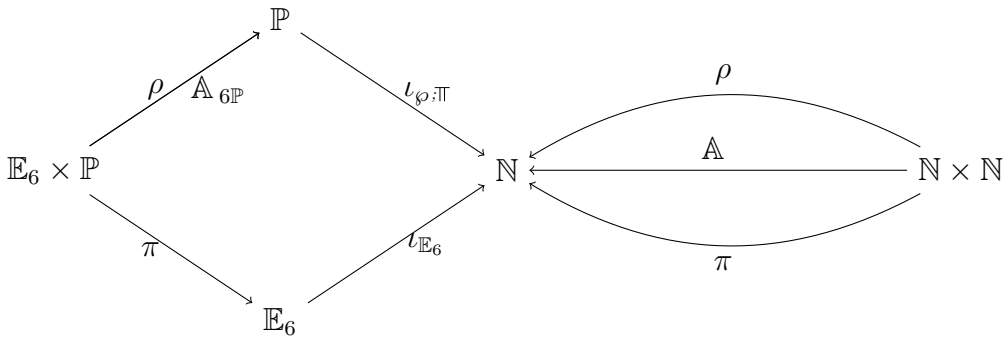


Fig. 10.2.6 Adding even numbers to primes

This is probably equal to the right-inverse $\mathbb{A}_{\mathbb{P}}^{\leftarrow}$ of $\mathbb{A}_{\mathbb{P}}$ — in view of Fig. 10.2.4.

10.3 Restricting to ordered pairs

Now follow relations concerning the lexicographic ordering of pairs of odd primes. We intend, thus, to get rid of all the fuzziness around $p \leq q$ and $p \geq q$.

10.3.1 Definition (*Lexicographic order on pairs of odd primes*)**.**

$$\begin{aligned} v_{\leq} &:= (\mathbb{S}_{\mathbb{P}}^{*} \oslash \mathbb{I}_{\mathbb{P}}) ; \mathbb{T} : \mathbb{P} \times \mathbb{P} \longrightarrow \mathbb{1} \quad \text{vector selecting pairs } (p,q) \text{ of primes with } p \leq q \\ &= (\mathbb{I}_{\mathbb{P}} \oslash \mathbb{S}_{\mathbb{P}}^{\intercal *}) ; \mathbb{T} \\ &= \mathtt{vec}(\mathbb{S}_{\mathbb{P}}^{*}) \end{aligned}$$

$\iota_{\times} := \mathtt{Inject}(v_{\leq}) : (\mathbb{P} \times \mathbb{P})_{\leq} \longrightarrow \mathbb{P} \times \mathbb{P}$, thus *defining* $(\mathbb{P} \times \mathbb{P})_{\leq}$ □

The equivalence of the first two variants follows from Prop. 2.1.1.iii. Using these, we redefine operations with respect to the lexicographic ordering accordingly:

10.3.2 Definition.

$$\begin{aligned} \mathfrak{F}_{\times} &:= \iota_{\times} ; (\mathbb{S}_{\mathbb{P}} \otimes \mathbb{I}_{\mathbb{P}}) ; \iota_{\times}^{\intercal} \quad \text{increases first component on ordered pairs of odd primes} \\ z_{\times} &:= \iota_{\times} ; (z_{\mathbb{P}} \oslash z_{\mathbb{P}}) \\ \mathbb{I}_{\times} &:= \iota_{\times} ; \iota_{\times}^{\intercal} \end{aligned}$$ □

Obviously, then $z_{\times} = \iota_{\times} ; ([3\rightarrow] \oslash [3\rightarrow])$.

10.3.3 Proposition. i) $\mathfrak{F}_{\times}$ is univalent and injective

ii) $\mathfrak{F}_{\times}^{\intercal} ; \mathfrak{F}_{\times} = \mathbb{I}_{\times} \cap \iota_{\times} ; \pi_{\mathbb{P}} ; \mathbb{S}_{\mathbb{P}}^{\intercal} ; \mathbb{T}$

iii) $v_{\leq} ; \mathbb{T} = (\iota_{\wp ; \mathbb{T}} \otimes \iota_{\wp ; \mathbb{T}}) ; (\mathbb{S}^{*} \oslash \mathbb{I}) ; \mathbb{T} = (\iota_{\wp ; \mathbb{T}} \otimes \iota_{\wp ; \mathbb{T}}) ; \mathtt{vec}(\mathbb{S}^{*})$

Proof: i) We show the proof of univalency; injectivity follows in a similar way.

$\mathfrak{F}_\times^\intercal ; \mathfrak{F}_\times = \iota_\times ; (\mathbb{S}_\mathbb{P}^\intercal \otimes \mathbb{I}_\mathbb{P}) ; \iota_\times^\intercal ; \iota_\times ; (\mathbb{S}_\mathbb{P} \otimes \mathbb{I}_\mathbb{P}) ; \iota_\times^\intercal$
$\subseteq \iota_\times ; (\mathbb{S}_\mathbb{P}^\intercal \otimes \mathbb{I}_\mathbb{P}) ; (\mathbb{S}_\mathbb{P} \otimes \mathbb{I}_\mathbb{P}) ; \iota_\times^\intercal$ as a natural injection $\iota_\times$ is univalent
$= \iota_\times ; (\mathbb{S}_\mathbb{P}^\intercal ; \mathbb{S}_\mathbb{P} \otimes \mathbb{I}_\mathbb{P}) ; \iota_\times^\intercal$ since $\mathbb{S}_\mathbb{P}$ is injective; see [14, Prop. 3.1.7.iv]
$\subseteq \iota_\times ; (\mathbb{I}_\mathbb{P} \otimes \mathbb{I}_\mathbb{P}) ; \iota_\times^\intercal = \iota_\times ; \iota_\times^\intercal$ since $\mathbb{S}_\mathbb{P}$ is univalent due to Prop. 9.4.3.ii
$\subseteq \mathbb{I}_\times$ because $\iota_\times$ is an injective map

---

ii) We recall that, following Prop. 9.4.3.vi, $\mathbb{S}_\mathbb{P}$ enjoys the same basic properties as $\mathbb{S}$, so that Prop. 5.2.1.ii.i may also be applied to $\mathbb{S}_\mathbb{P}$:

$(\mathbb{S}_\mathbb{P}^\intercal ; \mathbb{S}_\mathbb{P}^* \cap \mathbb{S}_\mathbb{P}^*) ; \mathbb{T} = \mathbb{S}_\mathbb{P}^\intercal ; \mathbb{T}$

Having this in mind, we proceed as follows.

$\mathfrak{F}_\times^\intercal ; \mathfrak{F}_\times = \iota_\times ; (\mathbb{S}_\mathbb{P}^\intercal \otimes \mathbb{I}_\mathbb{P}) ; \iota_\times^\intercal ; \iota_\times ; (\mathbb{S}_\mathbb{P} \otimes \mathbb{I}_\mathbb{P}) ; \iota_\times^\intercal$
$= \iota_\times ; (\mathbb{S}_\mathbb{P}^\intercal \otimes \mathbb{I}_\mathbb{P}) ; \big[\mathbb{I} \cap (\mathbb{S}_\mathbb{P}^* \oslash \mathbb{I}) ; \mathbb{T}\big] ; (\mathbb{S}_\mathbb{P} \otimes \mathbb{I}_\mathbb{P}) ; \iota_\times^\intercal$ because $\iota_\times$ extrudes $v_\leq ; \mathbb{T} = \mathtt{vec}(\mathbb{S}_\mathbb{P}^*)$
$= \iota_\times ; (\mathbb{S}_\mathbb{P}^\intercal \otimes \mathbb{I}_\mathbb{P}) ; \big[(\mathbb{S}_\mathbb{P} \otimes \mathbb{I}_\mathbb{P}) ; \iota_\times^\intercal \cap (\mathbb{S}_\mathbb{P}^* \oslash \mathbb{I}_\mathbb{P}) ; \mathbb{T}\big]$ masking
$= \iota_\times ; (\mathbb{S}_\mathbb{P}^\intercal \otimes \mathbb{I}_\mathbb{P}) ; (\mathbb{S}_\mathbb{P} \otimes \mathbb{I}_\mathbb{P}) ; \iota_\times^\intercal \cap \iota_\times ; (\mathbb{S}_\mathbb{P}^\intercal \otimes \mathbb{I}_\mathbb{P}) ; (\mathbb{S}_\mathbb{P}^* \oslash \mathbb{I}_\mathbb{P}) ; \mathbb{T}$ because $\mathbb{S}_\mathbb{P}$ is injective
$= \iota_\times ; (\mathbb{S}_\mathbb{P}^\intercal ; \mathbb{S}_\mathbb{P} \otimes \mathbb{I}_\mathbb{P}) ; \iota_\times^\intercal \cap \iota_\times ; (\mathbb{S}_\mathbb{P}^\intercal ; \mathbb{S}_\mathbb{P}^* \oslash \mathbb{I}_\mathbb{P}) ; \mathbb{T}$
$= \iota_\times ; (\mathbb{I}_\mathbb{P} \cap \mathbb{S}_\mathbb{P}^\intercal ; \mathbb{T} \otimes \mathbb{I}_\mathbb{P}) ; \iota_\times^\intercal \cap \iota_\times ; (\mathbb{S}_\mathbb{P}^\intercal ; \mathbb{S}_\mathbb{P}^* \oslash \mathbb{I}_\mathbb{P}) ; \mathbb{T}$ Prop. 9.4.4.ii
$= \iota_\times ; (\mathbb{I}_\mathbb{P} \otimes \mathbb{I}_\mathbb{P}) ; \iota_\times^\intercal \cap \iota_\times ; \pi_\mathbb{P} ; \mathbb{S}_\mathbb{P}^\intercal ; \mathbb{T} \cap \iota_\times ; (\mathbb{S}_\mathbb{P}^\intercal ; \mathbb{S}_\mathbb{P}^* \oslash \mathbb{I}_\mathbb{P}) ; \mathbb{T}$ masking
$= \iota_\times ; \mathbb{I}_{\mathbb{P}\times\mathbb{P}} ; \iota_\times^\intercal \cap \iota_\times ; (\mathbb{S}_\mathbb{P}^\intercal ; \mathbb{S}_\mathbb{P}^* \oslash \mathbb{I}_\mathbb{P}) ; \mathbb{T}$ third term contained in second
$= \iota_\times ; \iota_\times^\intercal \cap \iota_\times ; \mathtt{vec}(\mathbb{S}_\mathbb{P}^\intercal ; \mathbb{S}_\mathbb{P}^*)$ see Prop. 2.2.5
$= \iota_\times ; \iota_\times^\intercal \cap \iota_\times ; (\mathbb{I} \cap \mathtt{vec}(\mathbb{S}_\mathbb{P}^*)) ; \mathtt{vec}(\mathbb{S}_\mathbb{P}^\intercal ; \mathbb{S}_\mathbb{P}^*)$ since $\iota_\times$ extrudes $v_\leq ; \mathbb{T} = \mathtt{vec}(\mathbb{S}_\mathbb{P}^*)$
$= \mathbb{I}_\times \cap \iota_\times ; [\mathtt{vec}(\mathbb{S}_\mathbb{P}^*) \cap \mathtt{vec}(\mathbb{S}_\mathbb{P}^\intercal ; \mathbb{S}_\mathbb{P}^*)]$ masking
$= \mathbb{I}_\times \cap \iota_\times ; \mathtt{vec}(\mathbb{S}_\mathbb{P}^* \cap \mathbb{S}_\mathbb{P}^\intercal ; \mathbb{S}_\mathbb{P}^*)$ since $\mathtt{vec}(A \cap B) = \mathtt{vec}(A) \cap \mathtt{vec}(B)$; see page 15
$= \mathbb{I}_\times \cap \iota_\times ; \mathtt{vec}(\mathbb{S}_\mathbb{P}^\intercal ; \mathbb{T})$ see introductory remark above
$= \mathbb{I}_\times \cap \iota_\times ; (\pi_\mathbb{P} ; \mathbb{S}_\mathbb{P}^\intercal ; \mathbb{T} \cap \rho_\mathbb{P}) ; \mathbb{T} = \mathbb{I}_\times \cap \iota_\times ; (\pi_\mathbb{P} ; \mathbb{S}_\mathbb{P}^\intercal ; \mathbb{T} \cap \rho_\mathbb{P} ; \mathbb{T})$ masking
$= \mathbb{I}_\times \cap \iota_\times ; \pi_\mathbb{P} ; \mathbb{S}_\mathbb{P}^\intercal ; \mathbb{T}$ because $\rho_\mathbb{P}$ is total

---

iii) $v_\leq = (\mathbb{S}_\mathbb{P}^* \oslash \mathbb{I}_\mathbb{P}) ; \mathbb{T}$ by Def. 10.3.1
$= (\iota_{\wp;\mathbb{T}} ; \mathbb{S}^* ; \iota_{\wp;\mathbb{T}}^\intercal \oslash \mathbb{I}_\mathbb{P}) ; \mathbb{T}$ Prop. 9.4.3.i
$= (\iota_{\wp;\mathbb{T}} ; \mathbb{S}^* ; \iota_{\wp;\mathbb{T}}^\intercal \oslash \mathbb{I}_\mathbb{P}) ; \iota_{\wp;\mathbb{T}} ; \mathbb{T}$ since $\iota_{\wp;\mathbb{T}}$ is total
$= (\iota_{\wp;\mathbb{T}} ; \mathbb{S}^* \oslash \iota_{\wp;\mathbb{T}}) ; \mathbb{T}$ destroy and append
$= (\iota_{\wp;\mathbb{T}} \otimes \iota_{\wp;\mathbb{T}}) ; (\mathbb{S}^* \oslash \mathbb{I}) ; \mathbb{T} = (\iota_{\wp;\mathbb{T}} \otimes \iota_{\wp;\mathbb{T}}) ; \mathtt{vec}(\mathbb{S}^*)$ □

$$v_{\leq}^{\mathsf{T}} = \begin{array}{c} \begin{matrix} (3\to,3\to) & (3\to,5\to) & (3\to,7\to) & (3\to,11\to) & (3\to,13\to) & (5\to,3\to) & (5\to,5\to) & (5\to,7\to) & (5\to,11\to) & (5\to,13\to) & (7\to,3\to) & (7\to,5\to) & (7\to,7\to) & (7\to,11\to) & (7\to,13\to) & (11\to,3\to) & (11\to,5\to) & (11\to,7\to) & (11\to,11\to) & (11\to,13\to) & (13\to,3\to) & (13\to,5\to) & (13\to,7\to) & (13\to,11\to) & (13\to,13\to) \end{matrix} \\ (1\ 1\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 1) \end{array}$$

$$\iota_{\times} = \begin{array}{r} \\ (3\to,3\to)\to \\ (3\to,5\to)\to \\ (3\to,7\to)\to \\ (3\to,11\to)\to \\ (3\to,13\to)\to \\ (5\to,5\to)\to \\ (5\to,7\to)\to \\ (5\to,11\to)\to \\ (5\to,13\to)\to \\ (7\to,7\to)\to \\ (7\to,11\to)\to \\ (7\to,13\to)\to \\ (11\to,11\to)\to \\ (11\to,13\to)\to \\ (13\to,13\to)\to \end{array} \begin{array}{c} \begin{matrix} (3\to,3\to) & (3\to,5\to) & (3\to,7\to) & (3\to,11\to) & (3\to,13\to) & (5\to,3\to) & (5\to,5\to) & (5\to,7\to) & (5\to,11\to) & (5\to,13\to) & (7\to,3\to) & (7\to,5\to) & (7\to,7\to) & (7\to,11\to) & (7\to,13\to) & (11\to,3\to) & (11\to,5\to) & (11\to,7\to) & (11\to,11\to) & (11\to,13\to) & (13\to,3\to) & (13\to,5\to) & (13\to,7\to) & (13\to,11\to) & (13\to,13\to) \end{matrix} \\ \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \end{pmatrix} \end{array}$$

$$\mathfrak{F}_{\times} = \begin{array}{r} \\ (3\to,3\to)\to \\ (3\to,5\to)\to \\ (3\to,7\to)\to \\ (3\to,11\to)\to \\ (3\to,13\to)\to \\ (5\to,5\to)\to \\ (5\to,7\to)\to \\ (5\to,11\to)\to \\ (5\to,13\to)\to \\ (7\to,7\to)\to \\ (7\to,11\to)\to \\ (7\to,13\to)\to \\ (11\to,11\to)\to \\ (11\to,13\to)\to \\ (13\to,13\to)\to \end{array} \begin{array}{c} \begin{matrix} (3\to,3\to)\to & (3\to,5\to)\to & (3\to,7\to)\to & (3\to,11\to)\to & (3\to,13\to)\to & (5\to,5\to)\to & (5\to,7\to)\to & (5\to,11\to)\to & (5\to,13\to)\to & (7\to,7\to)\to & (7\to,11\to)\to & (7\to,13\to)\to & (11\to,11\to)\to & (11\to,13\to)\to & (13\to,13\to)\to \end{matrix} \\ \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \end{array}$$

Fig. 10.3.1 $v_{\leq}^{\mathsf{T}}$, $\iota_{\times}$ and $\mathfrak{F}_{\times}$

$$\mathfrak{F}_{\times};\mathfrak{F}_{\times}^{\intercal} = \begin{array}{r} \\ (3\to,3\to)\to \\ (3\to,5\to)\to \\ (3\to,7\to)\to \\ (3\to,11\to)\to \\ (3\to,13\to)\to \\ (5\to,5\to)\to \\ (5\to,7\to)\to \\ (5\to,11\to)\to \\ (5\to,13\to)\to \\ (7\to,7\to)\to \\ (7\to,11\to)\to \\ (7\to,13\to)\to \\ (11\to,11\to)\to \\ (11\to,13\to)\to \\ (13\to,13\to)\to \end{array} \begin{array}{c} \begin{array}{cccccccccccccccc} (3\to,3\to)\uparrow & (3\to,5\to)\uparrow & (3\to,7\to)\uparrow & (3\to,11\to)\uparrow & (3\to,13\to)\uparrow & (5\to,5\to)\uparrow & (5\to,7\to)\uparrow & (5\to,11\to)\uparrow & (5\to,13\to)\uparrow & (7\to,7\to)\uparrow & (7\to,11\to)\uparrow & (7\to,13\to)\uparrow & (11\to,11\to)\uparrow & (11\to,13\to)\uparrow & (13\to,13\to)\uparrow \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

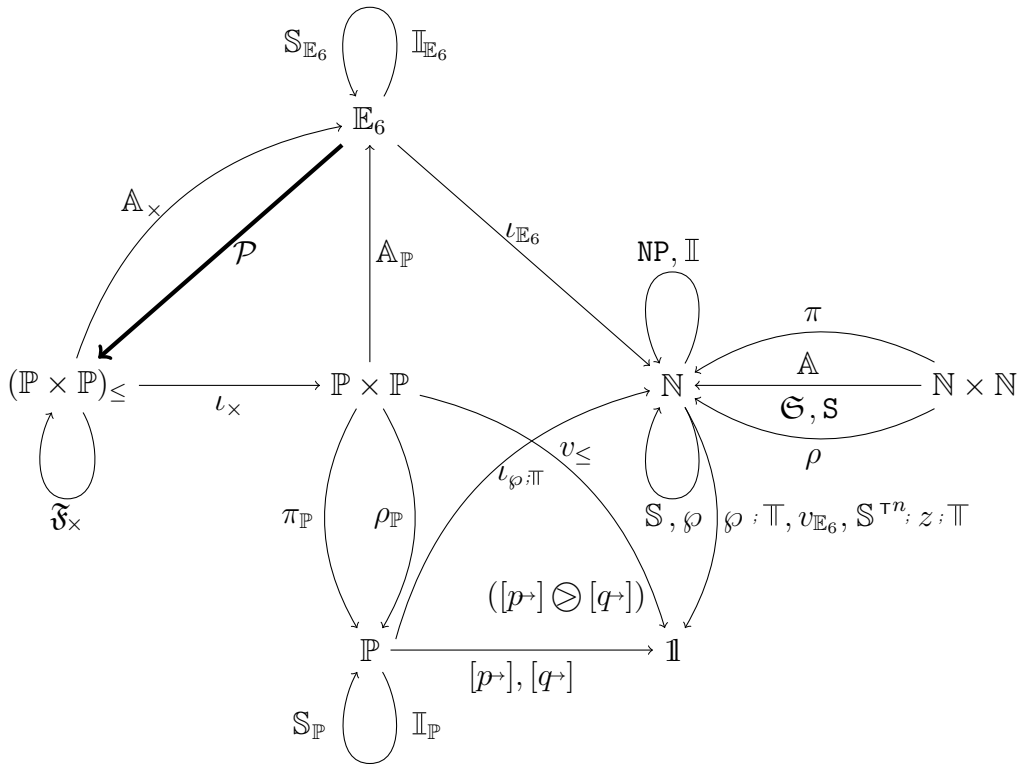


Fig. 10.3.2 Typing for primes and even numbers

10.3.4 Proposition. Let p, q be arbitrary odd primes, i.e. $\mathbb{S}^{\intercal p}{;}\, z \,;\mathbb{T}$ and $\mathbb{S}^{\intercal q}{;}\, z \,;\mathbb{T}$. Then

$$\iota_\times^\intercal{;}\,\iota_\times{;}\,([p^\rightarrow] \circledcirc [q^\rightarrow]) \;=\; \begin{cases} ([p^\rightarrow] \circledcirc [q^\rightarrow]) & \text{provided } p \le q \\ \perp\!\!\!\perp & \text{otherwise} \end{cases}$$

Proof: $\iota_\times^\intercal{;}\,\iota_\times{;}\,([p^\rightarrow] \circledcirc [q^\rightarrow]) = (\mathbb{I} \cap v_{\le}{;}\,\mathbb{T}){;}\,([p^\rightarrow] \circledcirc [q^\rightarrow])$ since $\iota_\times$ extrudes $v_{\le}{;}\,\mathbb{T}$
$= (\mathbb{I} \cap \mathtt{vec}(\mathbb{S}_{\mathbb{P}}^*){;}\,([p^\rightarrow] \circledcirc [q^\rightarrow])$ expanded; vectorization
$= ([p^\rightarrow] \circledcirc [q^\rightarrow]) \cap \mathtt{vec}(\mathbb{S}_{\mathbb{P}}^*)$ masking
$= \mathtt{vec}([p^\rightarrow]{;}\,\mathbb{T}{;}\,[q^\rightarrow]^\intercal) \cap \mathtt{vec}(\mathbb{S}_{\mathbb{P}}^*)$ Prop. 2.2.6.v
$= \mathtt{vec}([p^\rightarrow]{;}\,\mathbb{T}{;}\,[q^\rightarrow]^\intercal \cap \mathbb{S}_{\mathbb{P}}^*)$ since $\mathtt{vec}(A \cap B) = \mathtt{vec}(A) \cap \mathtt{vec}(B)$; see page 15

This leaves us with $[p^\rightarrow]{;}\,\mathbb{T}{;}\,[q^\rightarrow]^\intercal$ that as an entry according to Prop. 2.2.8.iii is either fully contained in $\mathbb{S}_{\mathbb{P}}^*$ or fully in the complement thereof. In the second case, we arrive indeed at $\perp\!\!\!\perp$. It remains to study the condition of the first case:

$[p^\rightarrow]{;}\,\mathbb{T}{;}\,[q^\rightarrow]^\intercal \subseteq \mathbb{S}_{\mathbb{P}}^*$ $\iff$ $\iota_{\wp;\mathbb{T}}{;}\,\mathbb{S}^{\intercal p}{;}\, z \,;\mathbb{T}{;}\, z^\intercal{;}\,\mathbb{S}^{q}{;}\,\iota_{\wp;\mathbb{T}}^\intercal \subseteq \iota_{\wp;\mathbb{T}}{;}\,\mathbb{S}^*{;}\,\iota_{\wp;\mathbb{T}}^\intercal$ Prop. 9.4.3.i
$\iff$ $\iota_{\wp;\mathbb{T}}^\intercal{;}\,\iota_{\wp;\mathbb{T}}{;}\,\mathbb{S}^{\intercal p}{;}\, z \,;\mathbb{T}{;}\, z^\intercal{;}\,\mathbb{S}^{q}{;}\,\iota_{\wp;\mathbb{T}}^\intercal{;}\,\iota_{\wp;\mathbb{T}} \subseteq \mathbb{S}^*$ shunting twice
$\iff$ $\wp{;}\,\mathbb{S}^{\intercal p}{;}\, z \,;\mathbb{T}{;}\, z^\intercal{;}\,\mathbb{S}^{q}{;}\,\wp \subseteq \mathbb{S}^*$ since $\iota_{\wp;\mathbb{T}}$ is an extrusion of $\wp \,;\mathbb{T}$
$\iff$ $\mathbb{S}^{\intercal p}{;}\, z \,;\mathbb{T}{;}\, z^\intercal{;}\,\mathbb{S}^{q} \subseteq \mathbb{S}^*$ since p, q are assumed to be odd primes
$\iff$ $\mathbb{S}^{\intercal p}{;}\, z \,;\mathbb{T} \subseteq \mathbb{S}^*{;}\,\mathbb{S}^{\intercal q}{;}\, z \,;\mathbb{T}$ shunted again

This delivers indeed $p \le q$ as the respective condition. □

10.3.5 Definition. We define as **lexi-prime-addition** the construct

$\mathbb{A}_\times := \iota_\times{;}\,\mathbb{A}_{\mathbb{P}}$ addition for lexicographically ordered pairs of primes

In analogy

$\mathtt{S}_\times := \iota_\times{;}\,\mathtt{S}_{\mathbb{P}}$ subtraction $(a, b) \mapsto b - a$ for lexicographically ordered pairs of primes □

The Goldbach-question is now whether $\mathbb{A}_\times$ is surjective.

	6→	8→	10→	12→	14→
(3→,3→)→	**1**	0	0	0	0
(3→,5→)→	0	**1**	0	0	0
(3→,7→)→	0	0	**1**	0	0
(3→,11→)→	0	0	0	0	**1**
(3→,13→)→	0	0	0	0	0
(5→,5→)→	0	0	0	0	0
(5→,7→)→	0	0	0	0	0
(5→,11→)→	0	0	0	0	0
(5→,13→)→	0	0	0	0	0
(7→,7→)→	0	0	0	0	0
(7→,11→)→	0	0	0	0	0
(7→,13→)→	0	0	0	0	0
(11→,11→)→	0	0	0	0	0
(11→,13→)→	0	0	0	0	0
(13→,13→)→	0	0	0	0	0

	6→	8→	10→	12→	14→
(3→,3→)→	**1**	0	0	0	0
(3→,5→)→	0	**1**	0	0	0
(3→,7→)→	0	0	**1**	0	0
(3→,11→)→	0	0	0	0	**1**
(3→,13→)→	0	0	0	0	0
(5→,5→)→	0	0	**1**	0	0
(5→,7→)→	0	0	0	**1**	0
(5→,11→)→	0	0	0	0	0
(5→,13→)→	0	0	0	0	0
(7→,7→)→	0	0	0	0	**1**
(7→,11→)→	0	0	0	0	0
(7→,13→)→	0	0	0	0	0
(11→,11→)→	0	0	0	0	0
(11→,13→)→	0	0	0	0	0
(13→,13→)→	0	0	0	0	0

Fig. 10.3.3 $\mathbb{A}_{\times 0}$ and $\mathbb{A}_\times$

10.3.6 Proposition. i) For two arbitrary odd primes p, q, which means in our present context points $\mathbb{S}^{\mathsf{T}p;}\, z\, ;\mathbb{T}$ and $\mathbb{S}^{\mathsf{T}q;}\, z\, ;\mathbb{T}$ in $\mathbb{N}$ or points $[p\rightarrow], [q\rightarrow]$ in $\mathbb{P}$, we have always

$$\mathbb{A}^{\mathsf{T}}_{\times;}\iota_{\times;}\,([p\rightarrow]\circledcirc[q\rightarrow]) \;=\; \begin{cases} \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{\frac{p+q-6}{2}}\, ; z_{\mathbb{E}_6;}\mathbb{T} & \text{provided } p\leq q \\ \perp\!\!\!\perp & \text{otherwise} \end{cases}$$

$$\text{ii)}\;\; \mathbb{S}^{\mathsf{T}}_{\times;}\iota_{\times;}\,([p\rightarrow]\circledcirc[q\rightarrow]) \;=\; \begin{cases} \iota_{\mathbb{E}_6;}\mathbb{S}^{\mathsf{T}q-p;}\, z\, ;\mathbb{T} & \text{provided } p\leq q \\ \perp\!\!\!\perp & \text{otherwise} \end{cases}$$

Proof: i) $\mathbb{A}^{\mathsf{T}}_{\times;}\iota_{\times;}\,([p\rightarrow]\circledcirc[q\rightarrow]) = \mathbb{A}^{\mathsf{T}}_{\mathbb{P};}\iota^{\mathsf{T}}_{\times;}\iota_{\times;}\,([p\rightarrow]\circledcirc[q\rightarrow])$ expanding $\mathbb{A}_\times$

$= \mathbb{A}^{\mathsf{T}}_{\mathbb{P};}\begin{cases} ([p\rightarrow]\circledcirc[q\rightarrow]) & \text{provided } p\leq q \\ \perp\!\!\!\perp & \text{otherwise} \end{cases}$ due to Prop. 10.3.4

$= \begin{cases} \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{\frac{p+q-6}{2}}\, ; z_{\mathbb{E}_6;}\mathbb{T} & \text{provided } p\leq q \\ \perp\!\!\!\perp & \text{otherwise} \end{cases}$ following Prop. 10.2.2.i

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ii) $\mathbb{S}^{\mathsf{T}}_{\times;}\iota_{\times;}\,([p\rightarrow]\circledcirc[q\rightarrow]) = \mathbb{S}^{\mathsf{T}}_{\mathbb{P};}\iota^{\mathsf{T}}_{\times;}\iota_{\times;}\,([p\rightarrow]\circledcirc[q\rightarrow])$ expanding $\mathbb{S}_\times$

$= \mathbb{S}^{\mathsf{T}}_{\mathbb{P};}\begin{cases} ([p\rightarrow]\circledcirc[q\rightarrow]) & \text{provided } p\leq q \\ \perp\!\!\!\perp & \text{otherwise} \end{cases}$ Prop. 10.3.4

$= \begin{cases} \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{\frac{q-p-6}{2}}\, ; z_{\mathbb{E}_6;}\mathbb{T} & \text{provided } p\leq q \\ \perp\!\!\!\perp & \text{otherwise} \end{cases}$ Prop. 10.2.2.ii

$= \begin{cases} \iota_{\mathbb{E}_6;}\mathbb{S}^{\mathsf{T}q-p;}\, z\, ;\mathbb{T} & \text{provided } p\leq q \\ \perp\!\!\!\perp & \text{otherwise} \end{cases}$ Prop. 10.1.3.vii □

Since it may happen in (ii) that $q-p<6$, the formulation according to (i) is not useful.

	6→	8→	10→	12→	14→	6→	8→	10→	12→	14→	6→	8→	10→	12→	14→	6→	8→	10→	12→	14→	6→	8→	10→	12→	14→
(3→,3→)→	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(3→,5→)→	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(3→,7→)→	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(3→,11→)→	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(3→,13→)→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(5→,5→)→	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(5→,7→)→	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(5→,11→)→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(5→,13→)→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(7→,7→)→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
(7→,11→)→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(7→,13→)→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(11→,11→)→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(11→,13→)→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(13→,13→)→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Fig. 10.3.4 An approach to obtain $\mathbb{A}_\times$ by iteration — respecting prime distances

In analogy to Prop. 10.2.3, we introduce an iteration for $\mathbb{A}_\times$:

10.3.7 Proposition. i) Using the constructs

$\mathbb{A}_{\times 0} := \iota_{\times;}\,(\iota_{\wp;\mathbb{T}} \otimes \iota_{\wp;\mathbb{T}})_{;}\,(\mathbb{S}^{\mathsf{T}3;}\, z\, ;\mathbb{T}\, ;z^{\mathsf{T}};\mathbb{S}^{\,3} \otimes \mathbb{I}_{\mathbb{N}})_{;}\mathbb{A}\, ;\iota^{\mathsf{T}}_{\mathbb{E}_6}$

$a_{\times H}(X) := (\mathfrak{F}^{\mathsf{T}}_{\times;}\, X\, ;\iota_{\mathbb{E}_6} \circledcirc \iota_{\times;}\pi_{\mathbb{P};}\,(\iota_{\wp;\mathbb{T}} \circledcirc \mathbb{S}^{\mathsf{T}}_{\mathbb{P};}\iota_{\wp;\mathbb{T}})_{;}\,\mathfrak{S})_{;}\mathbb{A}\, ;\iota^{\mathsf{T}}_{\mathbb{E}_6}$

$a_\times(X) := \mathbb{A}_{\times 0} \cup a_{\times H}(X),$

the following is well-defined

$$\mathbb{A}'_\times := \mathtt{inf}\,\{\, X \mid a_\times(X) \subseteq X\}.$$

ii) The $\mathbb{A}_\times$ defined in Def. 10.3.5 satisfies $a_\times(\mathbb{A}_\times) \subseteq \mathbb{A}_\times$, and thus $\mathbb{A}'_\times \subseteq \mathbb{A}_\times$.

Proof: i) Again, Prop. 4.2.1 may indeed be applied.

ii) The task to prove is mainly $a_{\times H}(\mathbb{A}_\times) \subseteq \mathbb{A}_\times$, which will again run via points. To this end, we assume a pair $p < q$ of odd primes, or else a point $\iota_\times ; ([p^\rightarrow] \oslash [q^\rightarrow])$ in $(\mathbb{P} \times \mathbb{P})_\leq$ and try to prove that then always

$$\Big[\iota_\times ; ([p^\rightarrow] \oslash [q^\rightarrow])\Big]^\mathsf{T} ; a_{\times H}(\mathbb{A}_\times) \subseteq \Big[\iota_\times ; ([p^\rightarrow] \oslash [q^\rightarrow])\Big]^\mathsf{T} ; \mathbb{A}_\times .$$

Since there occurs $\mathfrak{F}^\mathsf{T}_\times$, the case $p = 3$ requires a separate treatment:

$$\begin{aligned}
&\Big[\iota_\times ; ([3^\rightarrow] \oslash [q^\rightarrow])\Big]^\mathsf{T} ; \mathfrak{F}^\mathsf{T}_\times = ([3^\rightarrow]^\mathsf{T} \oslash [q^\rightarrow]^\mathsf{T}) ; \iota^\mathsf{T}_\times ; \iota_\times ; (\mathbb{S}^\mathsf{T}_\mathbb{P} \otimes \mathbb{I}_\mathbb{P}) ; \iota^\mathsf{T}_\times \quad \text{expanded}\\
&= ([3^\rightarrow]^\mathsf{T} \oslash [q^\rightarrow]^\mathsf{T}) ; (\mathbb{I} \cap v_\leq) ; (\mathbb{S}^\mathsf{T}_\mathbb{P} \otimes \mathbb{I}_\mathbb{P}) ; \iota^\mathsf{T}_\times\\
&= ([3^\rightarrow]^\mathsf{T} \oslash [q^\rightarrow]^\mathsf{T}) ; (\mathbb{S}^\mathsf{T}_\mathbb{P} \otimes \mathbb{I}_\mathbb{P}) ; \iota^\mathsf{T}_\times \quad \text{since obviously } 3 \leq q\\
&= ([3^\rightarrow]^\mathsf{T} ; \mathbb{S}^\mathsf{T}_\mathbb{P} \oslash [q^\rightarrow]^\mathsf{T}) ; \iota^\mathsf{T}_\times = (\perp\!\!\!\perp \oslash [q^\rightarrow]^\mathsf{T}) ; \iota^\mathsf{T}_\times = \perp\!\!\!\perp \quad \text{due to Prop. 9.4.4.iii and iv}
\end{aligned}$$

Now the general case $3 < p \leq q$:

$$\begin{aligned}
&\Big[\iota_\times ; ([p^\rightarrow] \oslash [q^\rightarrow])\Big]^\mathsf{T} ; a_{\times H}(\mathbb{A}_\times)\\
&= ([p^\rightarrow]^\mathsf{T} \oslash [q^\rightarrow]^\mathsf{T}) ; \iota^\mathsf{T}_\times ; (\mathfrak{F}^\mathsf{T}_\times ; \mathbb{A}_\times ; \iota_{\mathbb{E}_6} \oslash \iota_\times ; \pi_\mathbb{P} ; (\iota_{\wp;\mathbb{T}} \oslash \mathbb{S}^\mathsf{T}_\mathbb{P} ; \iota_{\wp;\mathbb{T}}) ; \mathfrak{S}) ; \mathbb{A} ; \iota^\mathsf{T}_{\mathbb{E}_6}\\
&= (([p^\rightarrow]^\mathsf{T} \oslash [q^\rightarrow]^\mathsf{T}) ; \iota^\mathsf{T}_\times ; \mathfrak{F}^\mathsf{T}_\times ; \mathbb{A}_\times ; \iota_{\mathbb{E}_6} \oslash ([p^\rightarrow]^\mathsf{T} \oslash [q^\rightarrow]^\mathsf{T}) ; \iota^\mathsf{T}_\times ; \iota_\times ; \pi_\mathbb{P} ; (\iota_{\wp;\mathbb{T}} \oslash \mathbb{S}^\mathsf{T}_\mathbb{P} ; \iota_{\wp;\mathbb{T}}) ; \mathfrak{S}) ; \mathbb{A} ; \iota^\mathsf{T}_{\mathbb{E}_6}\\
&= (\Phi \oslash \Psi) ; \mathbb{A} ; \iota^\mathsf{T}_{\mathbb{E}_6} \quad \text{where}\\
&\qquad \Phi = ([p^\rightarrow]^\mathsf{T} \oslash [q^\rightarrow]^\mathsf{T}) ; \iota^\mathsf{T}_\times ; \iota_\times ; (\mathbb{S}^\mathsf{T}_\mathbb{P} \otimes \mathbb{I}_\mathbb{P}) ; \iota^\mathsf{T}_\times ; \mathbb{A}_\times ; \iota_{\mathbb{E}_6}\\
&\qquad \Psi = ([p^\rightarrow]^\mathsf{T} \oslash [q^\rightarrow]^\mathsf{T}) ; (\mathbb{I} \cap v_\leq) ; \pi_\mathbb{P} ; (\iota_{\wp;\mathbb{T}} \oslash \mathbb{S}^\mathsf{T}_\mathbb{P} ; \iota_{\wp;\mathbb{T}}) ; \mathfrak{S}\\
&= (\Phi \oslash \Psi) ; \mathbb{A} ; \iota^\mathsf{T}_{\mathbb{E}_6} \quad \text{where}\\
&\qquad \Phi = ([p^\rightarrow]^\mathsf{T} \oslash [q^\rightarrow]^\mathsf{T}) ; (\mathbb{I} \cap v_\leq) ; (\mathbb{S}^\mathsf{T}_\mathbb{P} \otimes \mathbb{I}_\mathbb{P}) ; \iota^\mathsf{T}_\times ; \iota_\times ; \mathbb{A}_\mathbb{P} ; \iota_{\mathbb{E}_6} \quad \text{expanded}\\
&\qquad \Psi = ([p^\rightarrow]^\mathsf{T} \oslash [q^\rightarrow]^\mathsf{T}) ; (\mathbb{I} \cap v_\leq) ; \pi_\mathbb{P} ; (\iota_{\wp;\mathbb{T}} \oslash \mathbb{S}^\mathsf{T}_\mathbb{P} ; \iota_{\wp;\mathbb{T}}) ; \mathfrak{S}\\
&= (\Phi \oslash \Psi) ; \mathbb{A} ; \iota^\mathsf{T}_{\mathbb{E}_6} \quad \text{where}\\
&\qquad \Phi = ([p^\rightarrow]^\mathsf{T} \oslash [q^\rightarrow]^\mathsf{T}) ; (\mathbb{S}^\mathsf{T}_\mathbb{P} \otimes \mathbb{I}_\mathbb{P}) ; (\mathbb{I} \cap v_\leq) ; \mathbb{A}_\mathbb{P} ; \iota_{\mathbb{E}_6}\\
&\qquad \Psi = ([p^\rightarrow]^\mathsf{T} \oslash [q^\rightarrow]^\mathsf{T}) ; \pi_\mathbb{P} ; (\iota_{\wp;\mathbb{T}} \oslash \mathbb{S}^\mathsf{T}_\mathbb{P} ; \iota_{\wp;\mathbb{T}}) ; \mathfrak{S} \quad \text{because } p \leq q\\
&= (\Phi \oslash \Psi) ; \mathbb{A} ; \iota^\mathsf{T}_{\mathbb{E}_6} \quad \text{where}\\
&\qquad \Phi = ([p^\rightarrow]^\mathsf{T} ; \mathbb{S}^\mathsf{T}_\mathbb{P} \oslash [q^\rightarrow]^\mathsf{T}) ; \mathbb{A}_\mathbb{P} ; \iota_{\mathbb{E}_6} \quad \text{since indeed (prime predecessor of } p) < p \leq q\\
&\qquad \Psi = [p^\rightarrow]^\mathsf{T} ; (\iota_{\wp;\mathbb{T}} \oslash \mathbb{S}^\mathsf{T}_\mathbb{P} ; \iota_{\wp;\mathbb{T}}) ; \mathfrak{S} = ([p^\rightarrow]^\mathsf{T} ; \iota_{\wp;\mathbb{T}} \oslash [p^\rightarrow]^\mathsf{T} ; \mathbb{S}^\mathsf{T}_\mathbb{P} ; \iota_{\wp;\mathbb{T}}) ; \mathfrak{S} \quad \text{since } [p^\rightarrow] \text{ is a point}\\
&= (\Phi \oslash \Psi) ; \mathbb{A} ; \iota^\mathsf{T}_{\mathbb{E}_6} \quad \text{where}\\
&\qquad \Phi = \mathbb{T} ; z^\mathsf{T}_{\mathbb{E}_6} ; \mathbb{S}_{\mathbb{E}_6}{}^{\frac{(p-d)+q-6}{2}} ; \iota_{\mathbb{E}_6} = \mathbb{T} ; z^\mathsf{T} ; \mathbb{S}^{(p-d)+q} \quad \text{see Prop. 10.2.2; obviously even} \geq 6\\
&\qquad \Psi = (\mathbb{T} ; z^\mathsf{T} ; \mathbb{S}^p \oslash \mathbb{T} ; z^\mathsf{T} ; \mathbb{S}^{(p-d)}) ; \mathfrak{S} = \mathbb{T} ; z^\mathsf{T} ; \mathbb{S}^{p-(p-d)} = \mathbb{T} ; z^\mathsf{T} ; \mathbb{S}^d\\
&= (\mathbb{T} ; z^\mathsf{T} ; \mathbb{S}^{(p-d)+q} \oslash \mathbb{T} ; z^\mathsf{T} ; \mathbb{S}^d) ; \mathbb{A} ; \iota^\mathsf{T}_{\mathbb{E}_6} = \mathbb{T} ; z^\mathsf{T} ; \mathbb{S}^{[(p-d)+q]+d} ; \iota^\mathsf{T}_{\mathbb{E}_6} = \mathbb{T} ; z^\mathsf{T} ; \mathbb{S}^{p+q} ; \iota^\mathsf{T}_{\mathbb{E}_6}
\end{aligned}$$

But also

$$\begin{aligned}
&\Big[\iota_\times ; ([p^\rightarrow] \oslash [q^\rightarrow])\Big]^\mathsf{T} ; \mathbb{A}_\times\\
&= ([p^\rightarrow]^\mathsf{T} \oslash [q^\rightarrow]^\mathsf{T}) ; \iota^\mathsf{T}_\times ; \iota_\times ; \mathbb{A}_\mathbb{P} = ([p^\rightarrow]^\mathsf{T} \oslash [q^\rightarrow]^\mathsf{T}) ; (\mathbb{I} \cap v_\leq) ; \mathbb{A}_\mathbb{P}\\
&= ([p^\rightarrow]^\mathsf{T} \oslash [q^\rightarrow]^\mathsf{T}) ; \mathbb{A}_\mathbb{P} \quad \text{since } p \leq q \text{ has been assumed}\\
&= \mathbb{T} ; z^\mathsf{T} ; \mathbb{S}^{p+q} ; \iota^\mathsf{T}_{\mathbb{E}_6} \quad \text{due to Prop. 10.2.2}
\end{aligned}$$

Finally $\left[\iota_{\times}{;}\,([p^{\rightarrow}] \circledS [q^{\rightarrow}])\right]^{\mathsf{T}}{;}\,\mathbb{A}_{\times 0} \subseteq \left[\iota_{\times}{;}\,([p^{\rightarrow}] \circledS [q^{\rightarrow}])\right]^{\mathsf{T}}{;}\,\mathbb{A}_{\times}$ has to be shown. This follows more or less directly the pattern of the proofs above. □

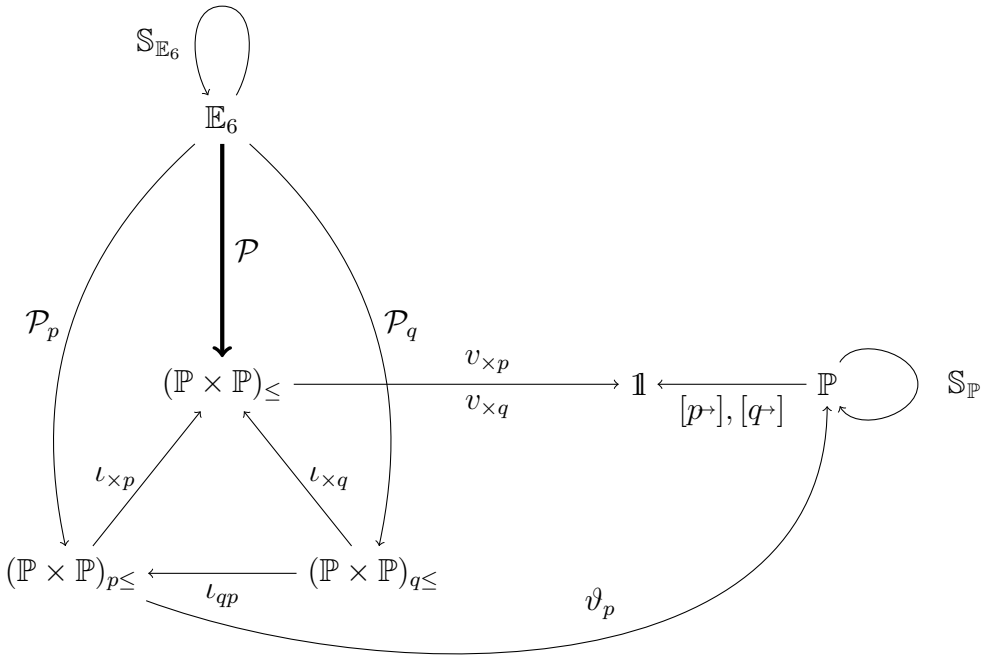


Fig. 10.3.5 Further typing for primes

Observe that the sum of two odd primes is even and ≥ 6.

10.3.8 Definition (*Relations concerning single primes*)**.**

$v_p := \pi_{\mathbb{P}}{;}\,[p^{\rightarrow}] : \mathbb{P}\times\mathbb{P} \longrightarrow \mathbb{1}$, vector describing prime pairs with p as first component
$v_{\times p} := \iota_{\times}{;}\,v_p : (\mathbb{P}\times\mathbb{P})_{\leq} \longrightarrow \mathbb{1}$, lexicographic prime pairs with p as first component
$\iota_{\times p} := \mathtt{Inject}(v_{\times p}) : (\mathbb{P}\times\mathbb{P})_{p\leq} \longrightarrow \mathbb{P}\times\mathbb{P}$, thus *defining* $(\mathbb{P}\times\mathbb{P})_{p\leq}$
$\vartheta_p := \iota_{\times p}{;}\,\iota_{\times}{;}\,([p^{\rightarrow}] \circledS \mathbb{I}_{\mathbb{P}}) : (\mathbb{P}\times\mathbb{P})_{p\leq} \longrightarrow \mathbb{P}$, mapping prime pairs (p,q) with $p \leq q$ on q
$\iota_{qp} := \iota_{\times q}{;}\,\mathfrak{F}_{\times}^{\mathsf{T}}{;}\,\iota_{\times p}^{\mathsf{T}} : (\mathbb{P}\times\mathbb{P})_{q\leq} \longrightarrow (\mathbb{P}\times\mathbb{P})_{p\leq}$, relating consecutive primes $p < q$ □

Since in the definition of ι_{qp} occurs $\mathfrak{F}_{\times}$, this q is indeed the next prime after p.

10.3.9 Proposition. i) $\mathtt{sup}_{p\geq 3,\ \mathrm{prime}}\,[p^{\rightarrow}] = \mathbb{T}_{\mathbb{P}}$ $\qquad$ $\mathtt{sup}_{p\geq 3,\ \mathrm{prime}}\,v_p{;}\,\mathbb{T} = \mathbb{T}_{(\mathbb{P}\times\mathbb{P})\mathbb{1}}$

ii) $\mathtt{sup}_{p\geq 3,\ \mathrm{prime}}\,v_{\times p} = \mathbb{T}_{(\mathbb{P}\times\mathbb{P})_{\leq}}$

iii) Given any odd prime p, the relation $\iota_{\times p}$ is an injective map.

iv) $\mathbb{I}_{(\mathbb{P}\times\mathbb{P})_{\leq}} = \mathtt{sup}_{p\geq 3,\ \mathrm{prime}}\,\iota_{\times p}^{\mathsf{T}}{;}\,\iota_{\times p}$

v) $\iota_{\times p}{;}\,\iota_{\times}{;}\,\pi_{\mathbb{P}} = [p^{\rightarrow}]^{\mathsf{T}}$ $\qquad$ $\iota_{\times p}{;}\,\iota_{\times}{;}\,\pi_{\mathbb{P}}{;}\,[p^{\rightarrow}] = \mathbb{T}$ $\qquad$ $\iota_{\times}^{\mathsf{T}}{;}\,\iota_{\times p}^{\mathsf{T}}{;}\,\mathbb{T} \subseteq \pi_{\mathbb{P}}{;}\,[p^{\rightarrow}]$ $\qquad$ $\iota_{\times p}{;}\,\iota_{\times}{;}\,\rho_{\mathbb{P}} = \vartheta_p$

vi) For all odd primes p the construct ϑ_p is an injective map.

vii) $\iota_{qp} = \vartheta_q ; \vartheta_p^\intercal$ if $p \leq q$ are two *consecutive* odd primes

viii) $\iota_{\times p}^\intercal ; \vartheta_p = \iota_\times ; ([p^\rightarrow] \circledcirc \mathbb{I}_\mathbb{P})$

ix) $\iota_{\times p}^\intercal ; \iota_{\times p} ; \iota_\times = \iota_\times ; (\mathbb{I} \cap v_p ; \mathbb{T})$

x) ϑ_p is an injective mapping with $\mathbb{T} ; \vartheta_p = [p^\rightarrow]^\intercal ; \mathbb{S}_\mathbb{P}^*$

Proof: i) $\mathtt{sup}_{p\geq 3,\text{ prime}}\, [p^\rightarrow] = \mathtt{sup}_{p\geq 3,\text{ prime}}\, \iota_{\wp\mathbb{T}} ; \mathbb{S}^{\intercal p} ; z ; \mathbb{T} = \iota_{\wp\mathbb{T}} ; \mathtt{sup}_{p\geq 3,\text{ prime}}\, \mathbb{S}^{\intercal p} ; z ; \mathbb{T}$
$= \iota_{\wp\mathbb{T}} ; \wp ; \mathbb{T} = \mathbb{T}_\mathbb{P}$

$\mathtt{sup}_{p\geq 3,\text{ prime}}\, v_p ; \mathbb{T} = \mathtt{sup}_{p\geq 3,\text{ prime}}\, \pi_\mathbb{P} ; [p^\rightarrow] = \mathtt{sup}_{p\geq 3,\text{ prime}}\, \pi_\mathbb{P} ; \iota_{\wp\mathbb{T}} ; \mathbb{S}^{\intercal p} ; z ; \mathbb{T}$
$= \pi_\mathbb{P} ; \iota_{\wp\mathbb{T}} ; \mathtt{sup}_{p\geq 3,\text{ prime}}\, \mathbb{S}^{\intercal p} ; z ; \mathbb{T}$
$= \pi_\mathbb{P} ; \iota_{\wp\mathbb{T}} ; \wp ; \mathbb{T}$
$= \pi_\mathbb{P} ; \iota_{\wp\mathbb{T}} ; \iota_{\wp\mathbb{T}}^\intercal ; \mathbb{T}_\mathbb{P}$ because $\iota_{\wp\mathbb{T}}$ extrudes $\wp ; \mathbb{T}$
$= \pi_\mathbb{P} ; \mathbb{T}_\mathbb{P}$ because an extrusion is always an injective map
$= \mathbb{T}_{(\mathbb{P}\times\mathbb{P})}$

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ii) is a simple consequence of (i)

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iii) This is true by the generic construction of the extrusion of a subset.

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iv) $\mathtt{sup}_{p\geq 3,\text{ prime}}\, \iota_{\times p}^\intercal ; \iota_{\times p} = \mathtt{sup}_{p\geq 3,\text{ prime}} (\mathbb{I} \cap v_{\times p} ; \mathbb{T})$ since $\iota_{\times p}$ is extrusion of $v_{\times p} ; \mathbb{T}$
$= \mathtt{sup}_{p\geq 3,\text{ prime}} (\mathbb{I} \cap \iota_\times ; v_p ; \mathbb{T}) = \mathtt{sup}_{p\geq 3,\text{ prime}} (\mathbb{I} \cap \iota_\times ; \pi_\mathbb{P} ; [p^\rightarrow])$ expanded
$= \mathbb{I} \cap \iota_\times ; \pi_\mathbb{P} ; \mathtt{sup}_{p\geq 3,\text{ prime}}\, [p^\rightarrow]$
$= \mathbb{I} \cap \iota_\times ; \pi_\mathbb{P} ; \mathbb{T}_\mathbb{P}$ see (i)
$= \mathbb{I} \cap \iota_\times ; \mathbb{T}_{\mathbb{P}\times\mathbb{P}}$
$= \mathbb{I} \cap \mathbb{T}_{(\mathbb{P}\times\mathbb{P})_\leq} = \mathbb{I}_{(\mathbb{P}\times\mathbb{P})_\leq}$

---

v) Both sides are obviously mappings, so it suffices to prove containment $\iota_{\times p} ; \iota_\times ; \pi_\mathbb{P} \subseteq [p^\rightarrow]^\intercal$:
$\iff$ $[p^\rightarrow] ; \iota_{\times p} ; \iota_\times \subseteq \pi_\mathbb{P}^\intercal$
Indeed, we get using that $[p^\rightarrow]$ is a point and that $\iota_\times$ is univalent
$[p^\rightarrow] ; \mathbb{T} ; \iota_{\times p} ; \iota_\times = [p^\rightarrow] ; \mathbb{T} ; v_{\times p}^\intercal ; \iota_\times = [p^\rightarrow] ; \mathbb{T} ; v_p^\intercal ; \iota_\times^\intercal ; \iota_\times = [p^\rightarrow] ; [p^\rightarrow]^\intercal ; \pi_\mathbb{P}^\intercal ; \iota_\times^\intercal ; \iota_\times \subseteq \pi_\mathbb{P}^\intercal$

$\iota_{\times p} ; \iota_\times ; \pi_\mathbb{P} ; [p^\rightarrow] = [p^\rightarrow]^\intercal ; [p^\rightarrow] = \mathbb{T}$ with the result shown before

---

$\iota_\times^\intercal ; \iota_{\times p}^\intercal \mathbb{T} = \iota_\times^\intercal ; v_{\times p} ; \mathbb{T} = \iota_\times^\intercal ; \iota_\times ; v_p = \iota_\times^\intercal ; \iota_\times ; \pi_\mathbb{P} ; [p^\rightarrow] \subseteq \pi_\mathbb{P} ; [p^\rightarrow]$

$\vartheta_p = \iota_{\times p} ; \iota_\times ; ([p^\rightarrow] \circledcirc \mathbb{I}_\mathbb{P}) = \iota_{\times p} ; \iota_\times ; \left[\pi_\mathbb{P} ; [p^\rightarrow] \cap \rho_\mathbb{P}\right]$
$= \iota_{\times p} ; \iota_\times ; \pi_\mathbb{P} ; [p^\rightarrow] \cap \iota_{\times p} ; \iota_\times ; \rho_\mathbb{P}$
$= [p^\rightarrow]^\intercal ; [p^\rightarrow] \cap \iota_{\times p} ; \iota_\times ; \rho_\mathbb{P}$ employing the first of these four results
$= \mathbb{T}_{(\mathbb{P}\times\mathbb{P})_\leq \times \mathbb{P}} \cap \iota_{\times p} ; \iota_\times ; \rho_\mathbb{P}$

---

vi) univalent:

$([p^{\rightarrow}]^{\mathsf{T}} \oslash \mathbb{I}_{\mathbb{P}}) ; \iota_{\times}^{\mathsf{T}} ; \iota_{\times p}^{\mathsf{T}} ; \iota_{\times p} ; \iota_{\times} ; ([p^{\rightarrow}] \oslash \mathbb{I}_{\mathbb{P}}) \subseteq ([p^{\rightarrow}]^{\mathsf{T}} \oslash \mathbb{I}_{\mathbb{P}}) ; \iota_{\times}^{\mathsf{T}} ; \iota_{\times} ; ([p^{\rightarrow}] \oslash \mathbb{I}_{\mathbb{P}})$ since $\iota_{\times p}$ is univalent
$\subseteq ([p^{\rightarrow}]^{\mathsf{T}} \oslash \mathbb{I}_{\mathbb{P}}) ; ([p^{\rightarrow}] \oslash \mathbb{I}_{\mathbb{P}})$ since $\iota_{\times}$ is univalent
$= [p^{\rightarrow}]^{\mathsf{T}} ; [p^{\rightarrow}] \cap \mathbb{I}_{\mathbb{P}} \subseteq \mathbb{I}_{\mathbb{P}}$ since $[p^{\rightarrow}]$ is a point

injective and total:

$\iota_{\times p} ; \iota_{\times} ; ([p^{\rightarrow}] \oslash \mathbb{I}_{\mathbb{P}}) ; ([p^{\rightarrow}]^{\mathsf{T}} \oslash \mathbb{I}_{\mathbb{P}}) ; \iota_{\times}^{\mathsf{T}} ; \iota_{\times p}^{\mathsf{T}} = \iota_{\times p} ; \iota_{\times} ; ([p^{\rightarrow}] ; [p^{\rightarrow}]^{\mathsf{T}} \otimes \mathbb{I}_{\mathbb{P}}) ; \iota_{\times}^{\mathsf{T}} ; \iota_{\times p}^{\mathsf{T}}$ Prop. 2.1.1.ii
$= \iota_{\times p} ; \iota_{\times} ; \left[\pi_{\mathbb{P}} ; [p^{\rightarrow}] ; [p^{\rightarrow}]^{\mathsf{T}} ; \pi_{\mathbb{P}}^{\mathsf{T}} \cap \rho_{\mathbb{P}} ; \rho_{\mathbb{P}}^{\mathsf{T}}\right] ; \iota_{\times}^{\mathsf{T}} ; \iota_{\times p}^{\mathsf{T}}$
$= \iota_{\times p} ; \iota_{\times} ; \pi_{\mathbb{P}} ; [p^{\rightarrow}] ; [p^{\rightarrow}]^{\mathsf{T}} ; \pi_{\mathbb{P}}^{\mathsf{T}} ; \iota_{\times}^{\mathsf{T}} ; \iota_{\times p}^{\mathsf{T}} \cap \iota_{\times p} ; \iota_{\times} ; \rho_{\mathbb{P}} ; \rho_{\mathbb{P}}^{\mathsf{T}} ; \iota_{\times}^{\mathsf{T}} ; \iota_{\times p}^{\mathsf{T}}$

Now for "total" using (v)

$= \mathbb{T} ; \mathbb{T} \cap \iota_{\times p} ; \iota_{\times} ; \rho_{\mathbb{P}} ; \rho_{\mathbb{P}}^{\mathsf{T}} ; \iota_{\times}^{\mathsf{T}} ; \iota_{\times p}^{\mathsf{T}} \supseteq \iota_{\times p} ; \iota_{\times} ; \mathbb{I} ; \iota_{\times}^{\mathsf{T}} ; \iota_{\times p}^{\mathsf{T}}$ since $\rho_{\mathbb{P}}$ is total
$= \mathbb{I}$ since both are injective maps

and for "injective":

$\subseteq \iota_{\times p} ; \iota_{\times} ; \pi_{\mathbb{P}} ; \mathbb{I} ; \pi_{\mathbb{P}}^{\mathsf{T}} ; \iota_{\times}^{\mathsf{T}} ; \iota_{\times p}^{\mathsf{T}} \cap \iota_{\times p} ; \iota_{\times} ; \rho_{\mathbb{P}} ; \rho_{\mathbb{P}}^{\mathsf{T}} ; \iota_{\times}^{\mathsf{T}} ; \iota_{\times p}^{\mathsf{T}}$ since a point is injective
$= \iota_{\times p} ; \iota_{\times} ; \left[\pi_{\mathbb{P}} ; \pi_{\mathbb{P}}^{\mathsf{T}} \cap \rho_{\mathbb{P}} ; \rho_{\mathbb{P}}^{\mathsf{T}}\right] ; \iota_{\times}^{\mathsf{T}} ; \iota_{\times p}^{\mathsf{T}} = \iota_{\times p} ; \iota_{\times} ; \mathbb{I} ; \iota_{\times}^{\mathsf{T}} ; \iota_{\times p}^{\mathsf{T}}$
$\subseteq \mathbb{I}$ since both maps are injective

---

vii) Direction "$\supseteq$":

$\vartheta_q ; \vartheta_p^{\mathsf{T}} = \iota_{\times q} ; \iota_{\times} ; ([q^{\rightarrow}] \oslash \mathbb{I}_{\mathbb{P}}) ; ([p^{\rightarrow}]^{\mathsf{T}} \oslash \mathbb{I}_{\mathbb{P}}) ; \iota_{\times}^{\mathsf{T}} ; \iota_{\times p}^{\mathsf{T}}$
$= \iota_{\times q} ; \iota_{\times} ; ([q^{\rightarrow}] ; [p^{\rightarrow}]^{\mathsf{T}} \otimes \mathbb{I}_{\mathbb{P}}) ; \iota_{\times}^{\mathsf{T}} ; \iota_{\times p}^{\mathsf{T}}$ Prop. 2.1.1.ii
$\subseteq \iota_{\times q} ; \iota_{\times} ; (\mathbb{S}_{\mathbb{P}}^{\mathsf{T}} \otimes \mathbb{I}_{\mathbb{P}}) ; \iota_{\times}^{\mathsf{T}} ; \iota_{\times p}^{\mathsf{T}}$ since $\mathbb{S}_{\mathbb{P}} ; [q^{\rightarrow}] = [p^{\rightarrow}]$ and the point $[q^{\rightarrow}]$ is injective
$= \iota_{\times q} ; \mathfrak{F}_{\times}^{\mathsf{T}} ; \iota_{\times p}^{\mathsf{T}} = \iota_{qp}$

Direction "$\subseteq$":

$\iota_{qp} \subseteq \vartheta_q ; \vartheta_p^{\mathsf{T}} \iff \vartheta_q^{\mathsf{T}} ; \iota_{qp} ; \vartheta_p \subseteq \mathbb{I}$ shunting twice
$\vartheta_q^{\mathsf{T}} ; \iota_{qp} ; \vartheta_p = ([q^{\rightarrow}]^{\mathsf{T}} \oslash \mathbb{I}_{\mathbb{P}}) ; \iota_{\times}^{\mathsf{T}} ; \iota_{\times q}^{\mathsf{T}} ; \iota_{\times q} ; \mathfrak{F}_{\times}^{\mathsf{T}} ; \iota_{\times p}^{\mathsf{T}} ; \iota_{\times p} ; \iota_{\times} ; ([p^{\rightarrow}] \oslash \mathbb{I}_{\mathbb{P}})$ multiply expanded
$\subseteq ([q^{\rightarrow}]^{\mathsf{T}} \oslash \mathbb{I}_{\mathbb{P}}) ; \iota_{\times}^{\mathsf{T}} ; \mathfrak{F}_{\times}^{\mathsf{T}} ; \iota_{\times} ; ([p^{\rightarrow}] \oslash \mathbb{I}_{\mathbb{P}})$ since $\iota_{\times q}$ and $\iota_{\times p}$ are maps
$= ([q^{\rightarrow}]^{\mathsf{T}} \oslash \mathbb{I}_{\mathbb{P}}) ; \iota_{\times}^{\mathsf{T}} ; \iota_{\times} ; (\mathbb{S}_{\mathbb{P}}^{\mathsf{T}} \otimes \mathbb{I}_{\mathbb{P}}) ; \iota_{\times}^{\mathsf{T}} ; \iota_{\times} ; ([p^{\rightarrow}] \oslash \mathbb{I}_{\mathbb{P}})$ expanding $\mathfrak{F}_{\times}$
$\subseteq ([q^{\rightarrow}]^{\mathsf{T}} \oslash \mathbb{I}_{\mathbb{P}}) ; (\mathbb{S}_{\mathbb{P}}^{\mathsf{T}} \otimes \mathbb{I}_{\mathbb{P}}) ; ([p^{\rightarrow}] \oslash \mathbb{I}_{\mathbb{P}})$ map property of $\iota_{\times}$
$= ([q^{\rightarrow}]^{\mathsf{T}} \oslash \mathbb{I}_{\mathbb{P}}) ; (\mathbb{S}_{\mathbb{P}}^{\mathsf{T}} ; [p^{\rightarrow}] \oslash \mathbb{I}_{\mathbb{P}})$
$= ([q^{\rightarrow}]^{\mathsf{T}} \oslash \mathbb{I}_{\mathbb{P}}) ; ([q^{\rightarrow}] \oslash \mathbb{I}_{\mathbb{P}}) = [q^{\rightarrow}]^{\mathsf{T}} ; [q^{\rightarrow}] \cap \mathbb{I}_{\mathbb{P}} \subseteq \mathbb{T}_{\mathbb{P}} \cap \mathbb{I}_{\mathbb{P}} = \mathbb{I}_{\mathbb{P}}$

---

viii) $\iota_{\times p}^{\mathsf{T}} ; \vartheta_p = \iota_{\times p}^{\mathsf{T}} ; \iota_{\times p} ; \iota_{\times} ; ([p^{\rightarrow}] \oslash \mathbb{I}_{\mathbb{P}})$
$= (\mathbb{I} \cap v_{\times p} ; \mathbb{T}) ; \iota_{\times} ; ([p^{\rightarrow}] \oslash \mathbb{I}_{\mathbb{P}})$
$= (\mathbb{I} \cap \iota_{\times} ; v_p ; \mathbb{T}) ; \iota_{\times} ; ([p^{\rightarrow}] \oslash \mathbb{I}_{\mathbb{P}})$
$= (\iota_{\times} \cap \iota_{\times} ; v_p ; \mathbb{T}) ; ([p^{\rightarrow}] \oslash \mathbb{I}_{\mathbb{P}})$ masking
$= \iota_{\times} ; (\mathbb{I} \cap v_p ; \mathbb{T}) ; ([p^{\rightarrow}] \oslash \mathbb{I}_{\mathbb{P}}) = \iota_{\times} ; (\mathbb{I} \cap \pi_{\mathbb{P}} ; [p^{\rightarrow}]) ; ([p^{\rightarrow}] \oslash \mathbb{I}_{\mathbb{P}})$
$= \iota_{\times} ; (([p^{\rightarrow}] \oslash \mathbb{I}_{\mathbb{P}}) \cap \pi_{\mathbb{P}} ; [p^{\rightarrow}])$ masking again
$= \iota_{\times} ; (\pi_{\mathbb{P}} ; [p^{\rightarrow}] \cap \rho_{\mathbb{P}} \cap \pi_{\mathbb{P}} ; [p^{\rightarrow}])$
$= \iota_{\times} ; (\pi_{\mathbb{P}} ; [p^{\rightarrow}] \cap \rho_{\mathbb{P}}) = \iota_{\times} ; ([p^{\rightarrow}] \oslash \mathbb{I}_{\mathbb{P}})$

---

ix) $\iota_{\times p}^{\mathsf{T}} ; \iota_{\times p} ; \iota_{\times} = (\mathbb{I} \cap v_{\times p} ; \mathbb{T}) ; \iota_{\times}$ since $\iota_{\times p}$ extrudes $v_{\times p} ; \mathbb{T}$
$= (\mathbb{I} \cap \iota_{\times} ; v_p ; \mathbb{T}) ; \iota_{\times}$
$= \iota_{\times} \cap \iota_{\times} ; v_p ; \mathbb{T}$ masking
$= \iota_{\times} ; (\mathbb{I} \cap v_p ; \mathbb{T})$

---

x) Univalency is obvious and so is injectivity with $([p^{\rightarrow}] \circledcirc \mathbb{I}_{\mathbb{P}})\fatsemi([p^{\rightarrow}] \circledcirc \mathbb{I}_{\mathbb{P}})^{\mathsf{T}} \subseteq \mathbb{I}$. Totality is obtained using (v):

$\vartheta_p\fatsemi\mathbb{T} = \iota_{\times p}\fatsemi\iota_{\times}\fatsemi\rho_{\mathbb{P}}\fatsemi\mathbb{T} = \mathbb{T}$

We start with the first part of the term

$\mathbb{T}\fatsemi\iota_{\times p}\fatsemi\iota_{\times} = \mathbb{T}\fatsemi v^{\mathsf{T}}_{\times p}\fatsemi\iota_{\times} = \mathbb{T}\fatsemi v^{\mathsf{T}}_{p}\fatsemi\iota^{\mathsf{T}}_{\times}\fatsemi\iota_{\times} = [p^{\rightarrow}]^{\mathsf{T}}\fatsemi\pi^{\mathsf{T}}_{\mathbb{P}}\fatsemi\iota^{\mathsf{T}}_{\times}\fatsemi\iota_{\times}$
$= [p^{\rightarrow}]^{\mathsf{T}}\fatsemi\pi^{\mathsf{T}}_{\mathbb{P}}\fatsemi(\mathbb{I}_{\mathbb{P}\times\mathbb{P}} \cap (\mathbb{S}^{*}_{\mathbb{P}} \circledcirc \mathbb{I}_{\mathbb{P}})\fatsemi\mathbb{T})$
$= [p^{\rightarrow}]^{\mathsf{T}}\fatsemi\pi^{\mathsf{T}}_{\mathbb{P}}\fatsemi(\mathbb{I}_{\mathbb{P}\times\mathbb{P}} \cap \mathbb{T}\fatsemi(\mathbb{S}^{\mathsf{T}*}_{\mathbb{P}} \circledcirc \mathbb{I}_{\mathbb{P}}))$ transposing a partial identity
$= [p^{\rightarrow}]^{\mathsf{T}}\fatsemi\pi^{\mathsf{T}}_{\mathbb{P}} \cap \mathbb{T}\fatsemi(\mathbb{S}^{\mathsf{T}*}_{\mathbb{P}} \circledcirc \mathbb{I}_{\mathbb{P}})$ masking

Now the total term:

$\mathbb{T}\fatsemi\vartheta_p = \mathbb{T}\fatsemi\iota_{\times p}\fatsemi\iota_{\times}\fatsemi([p^{\rightarrow}] \circledcirc \mathbb{I}_{\mathbb{P}}) = \Big[\, [p^{\rightarrow}]^{\mathsf{T}}\fatsemi\pi^{\mathsf{T}}_{\mathbb{P}} \cap \mathbb{T}\fatsemi(\mathbb{S}^{\mathsf{T}*}_{\mathbb{P}} \circledcirc \mathbb{I}_{\mathbb{P}}) \,\Big]\fatsemi([p^{\rightarrow}] \circledcirc \mathbb{I}_{\mathbb{P}})$
$= [p^{\rightarrow}]^{\mathsf{T}}\fatsemi\pi^{\mathsf{T}}_{\mathbb{P}}\fatsemi([p^{\rightarrow}] \circledcirc \mathbb{I}_{\mathbb{P}}) \cap \mathbb{T}\fatsemi(\mathbb{S}^{\mathsf{T}*}_{\mathbb{P}} \circledcirc \mathbb{I}_{\mathbb{P}})\fatsemi([p^{\rightarrow}] \circledcirc \mathbb{I}_{\mathbb{P}})$ since the right join term is injective
$= [p^{\rightarrow}]^{\mathsf{T}}\fatsemi[p^{\rightarrow}] \cap \mathbb{T}\fatsemi\big[\mathbb{S}^{\mathsf{T}*}_{\mathbb{P}}\fatsemi[p^{\rightarrow}] \cap \mathbb{I}_{\mathbb{P}}\big]$
$= \mathbb{T} \cap \mathbb{T}\fatsemi\big[\, [p^{\rightarrow}]^{\mathsf{T}}\fatsemi\mathbb{S}^{*}_{\mathbb{P}} \cap \mathbb{I}_{\mathbb{P}}\big]$ transposing a partial identity
$= [p^{\rightarrow}]^{\mathsf{T}}\fatsemi\mathbb{S}^{*}_{\mathbb{P}} \cap \mathbb{T}\fatsemi\mathbb{I}_{\mathbb{P}}$ masking □

The claim (iv) is visualized with Fig. 10.3.8.

$$
\begin{array}{r}
 \\
(3{\rightarrow},3{\rightarrow}){\rightarrow}{\rightarrow} \\
(3{\rightarrow},5{\rightarrow}){\rightarrow}{\rightarrow} \\
(3{\rightarrow},7{\rightarrow}){\rightarrow}{\rightarrow} \\
(3{\rightarrow},11{\rightarrow}){\rightarrow}{\rightarrow} \\
(3{\rightarrow},13{\rightarrow}){\rightarrow}{\rightarrow} \\
(3{\rightarrow},17{\rightarrow}){\rightarrow}{\rightarrow}
\end{array}
\begin{array}{c}
\begin{array}{cccccc} 3{\rightarrow} & 5{\rightarrow} & 7{\rightarrow} & 11{\rightarrow} & 13{\rightarrow} & 17{\rightarrow} \end{array} \\
\begin{pmatrix}
\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}
\end{pmatrix}
\end{array}
$$

$$
\begin{array}{r}
 \\
(5{\rightarrow},5{\rightarrow}){\rightarrow}{\rightarrow} \\
(5{\rightarrow},7{\rightarrow}){\rightarrow}{\rightarrow} \\
(5{\rightarrow},11{\rightarrow}){\rightarrow}{\rightarrow} \\
(5{\rightarrow},13{\rightarrow}){\rightarrow}{\rightarrow} \\
(5{\rightarrow},17{\rightarrow}){\rightarrow}{\rightarrow}
\end{array}
\begin{array}{c}
\begin{array}{cccccc} 3{\rightarrow} & 5{\rightarrow} & 7{\rightarrow} & 11{\rightarrow} & 13{\rightarrow} & 17{\rightarrow} \end{array} \\
\begin{pmatrix}
\mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}
\end{pmatrix}
\end{array}
$$

$$
\begin{array}{r}
 \\
(7{\rightarrow},7{\rightarrow}){\rightarrow}{\rightarrow} \\
(7{\rightarrow},11{\rightarrow}){\rightarrow}{\rightarrow} \\
(7{\rightarrow},13{\rightarrow}){\rightarrow}{\rightarrow} \\
(7{\rightarrow},17{\rightarrow}){\rightarrow}{\rightarrow}
\end{array}
\begin{array}{c}
\begin{array}{cccccc} 3{\rightarrow} & 5{\rightarrow} & 7{\rightarrow} & 11{\rightarrow} & 13{\rightarrow} & 17{\rightarrow} \end{array} \\
\begin{pmatrix}
\mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}
\end{pmatrix}
\end{array}
$$

$$
\begin{array}{r}
 \\
(11{\rightarrow},11{\rightarrow}){\rightarrow}{\rightarrow} \\
(11{\rightarrow},13{\rightarrow}){\rightarrow}{\rightarrow} \\
(11{\rightarrow},17{\rightarrow}){\rightarrow}{\rightarrow}
\end{array}
\begin{array}{c}
\begin{array}{cccccc} 3{\rightarrow} & 5{\rightarrow} & 7{\rightarrow} & 11{\rightarrow} & 13{\rightarrow} & 17{\rightarrow} \end{array} \\
\begin{pmatrix}
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}
\end{pmatrix}
\end{array}
$$

Fig. 10.3.6 Instances of $\vartheta_p = \iota_{\times p}\fatsemi\iota_{\times}\fatsemi([p^{\rightarrow}] \circledcirc \mathbb{I}_{\mathbb{P}})$ for $p = 3, 5, 7, 11$

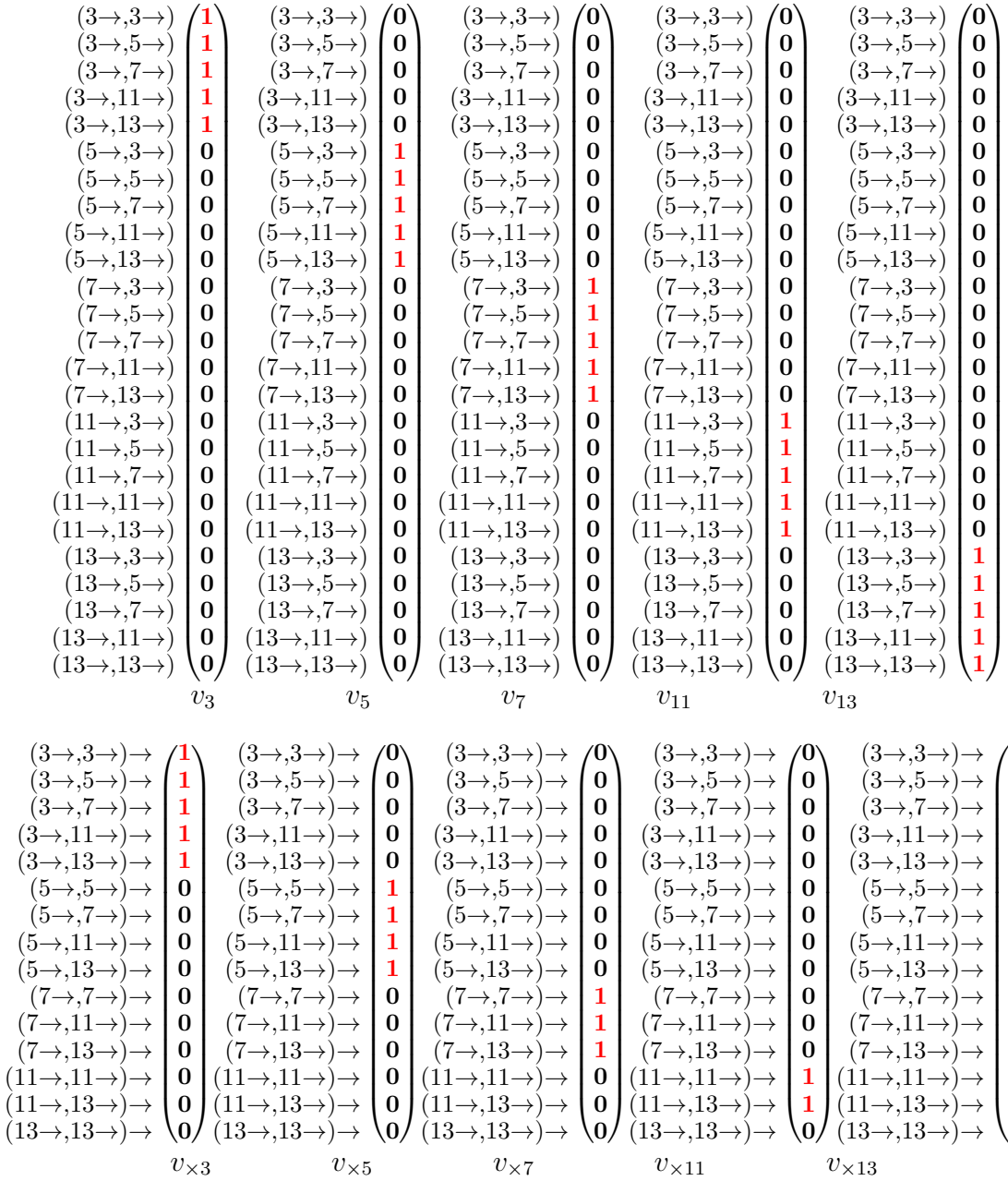


Fig. 10.3.7 v_p $v_{\times p}$

$$\iota_{\times 3} = \begin{array}{c} \\ (3\to,3\to)\to\to \\ (3\to,5\to)\to\to \\ (3\to,7\to)\to\to \\ (3\to,11\to)\to\to \\ (3\to,13\to)\to\to \end{array} \begin{array}{c} \begin{matrix} (3\to,3\to)\uparrow\!\uparrow & (3\to,5\to)\uparrow\!\uparrow & (3\to,7\to)\uparrow\!\uparrow & (3\to,11\to)\uparrow\!\uparrow & (3\to,13\to)\uparrow\!\uparrow \end{matrix} \\ \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} \end{pmatrix} \end{array}$$

$$\iota_{\times 5} = \begin{array}{c} \\ (5\to,5\to)\to\to \\ (5\to,7\to)\to\to \\ (5\to,11\to)\to\to \\ (5\to,13\to)\to\to \end{array} \begin{array}{c} \begin{matrix} (5\to,5\to)\uparrow\!\uparrow & (5\to,7\to)\uparrow\!\uparrow & (5\to,11\to)\uparrow\!\uparrow & (5\to,13\to)\uparrow\!\uparrow \end{matrix} \\ \begin{pmatrix} 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{pmatrix} \end{array}$$

$$\iota_{\times 7} = \begin{array}{c} \\ (7\to,7\to)\to\to \\ (7\to,11\to)\to\to \\ (7\to,13\to)\to\to \end{array} \begin{array}{c} \begin{matrix} (7\to,7\to)\uparrow\!\uparrow & (7\to,11\to)\uparrow\!\uparrow & (7\to,13\to)\uparrow\!\uparrow \end{matrix} \\ \begin{pmatrix} 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & \mathbf{1} & 0 \end{pmatrix} \end{array}$$

$$\iota_{\times 11} = \begin{array}{c} \\ (11\to,11\to)\to\to \\ (11\to,13\to)\to\to \end{array} \begin{array}{c} \begin{matrix} (11\to,11\to)\uparrow\!\uparrow & (11\to,13\to)\uparrow\!\uparrow \end{matrix} \\ \begin{pmatrix} 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} \end{pmatrix} \end{array}$$

$$\iota_{\times 13} = (13\to,13\to)\to\to \begin{array}{c} \begin{matrix} (13\to,13\to)\uparrow\!\uparrow \end{matrix} \\ \begin{pmatrix} 0 & \mathbf{1} \end{pmatrix} \end{array}$$

$$\iota_{53} = \begin{array}{c} \\ (5\to,5\to)\to\to \\ (5\to,7\to)\to\to \\ (5\to,11\to)\to\to \\ (5\to,13\to)\to\to \end{array} \begin{array}{c} \begin{matrix} (3\to,3\to)\uparrow\!\uparrow & (3\to,5\to)\uparrow\!\uparrow & (3\to,7\to)\uparrow\!\uparrow & (3\to,11\to)\uparrow\!\uparrow & (3\to,13\to)\uparrow\!\uparrow \end{matrix} \\ \begin{pmatrix} 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} \end{pmatrix} \end{array}$$

$$\iota_{75} = \begin{array}{c} \\ (7\to,7\to)\to\to \\ (7\to,11\to)\to\to \\ (7\to,13\to)\to\to \end{array} \begin{array}{c} \begin{matrix} (5\to,5\to)\uparrow\!\uparrow & (5\to,7\to)\uparrow\!\uparrow & (5\to,11\to)\uparrow\!\uparrow & (5\to,13\to)\uparrow\!\uparrow \end{matrix} \\ \begin{pmatrix} 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{pmatrix} \end{array}$$

$$\iota_{117} = \begin{array}{c} \\ (11\to,11\to)\to\to \\ (11\to,13\to)\to\to \end{array} \begin{array}{c} \begin{matrix} (7\to,7\to)\uparrow\!\uparrow & (7\to,11\to)\uparrow\!\uparrow & (7\to,13\to)\uparrow\!\uparrow \end{matrix} \\ \begin{pmatrix} 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} \end{pmatrix} \end{array}$$

$$\iota_{1311} = (13\to,13\to)\to\to \begin{array}{c} \begin{matrix} (11\to,11\to)\uparrow\!\uparrow & (11\to,13\to)\uparrow\!\uparrow \end{matrix} \\ \begin{pmatrix} 0 & \mathbf{1} \end{pmatrix} \end{array}$$

Fig. 10.3.8 Examples of relations $\iota_{\times p}$ and ι_{qp}

10.3.10 Proposition. The following will hold if p, q are two *consecutive* odd primes:

i) $\iota^{\mathsf{T}}_{\times p} ; \iota^{\mathsf{T}}_{qp} ; \iota_{\times q} \subseteq \mathfrak{F}_\times$

ii) The relation ι_{qp} is an injective map.

Proof: i) $\iota^{\mathsf{T}}_{\times p} ; \iota^{\mathsf{T}}_{qp} ; \iota_{\times q} = \iota^{\mathsf{T}}_{\times p} ; \iota_{\times p} ; \mathfrak{F}_\times ; \iota^{\mathsf{T}}_{\times q} ; \iota_{\times q}$ expanding ι_{qp}
$= (\mathbb{I} \cap v_{\times p} ; \mathbb{T}) ; \mathfrak{F}_\times ; (\mathbb{I} \cap v_{\times q} ; \mathbb{T})$ since, e.g., $\iota_{\times p}$ extrudes $v_{\times p}$
$\subseteq \mathfrak{F}_\times$

---

ii) Univalency, injectivity and totality are shown separately:

$\iota^{\mathsf{T}}_{qp} ; \iota_{qp} = \iota_{\times p} ; \mathfrak{F}_\times ; \iota^{\mathsf{T}}_{\times q} ; \iota_{\times q} ; \mathfrak{F}^{\mathsf{T}}_\times ; \iota^{\mathsf{T}}_{\times p} \subseteq \iota_{\times p} ; \mathfrak{F}_\times ; \mathfrak{F}^{\mathsf{T}}_\times ; \iota^{\mathsf{T}}_{\times p}$ since $\iota_{\times q}$ is univalent
$\subseteq \iota_{\times p} ; \iota^{\mathsf{T}}_{\times p}$ because $\mathfrak{F}_\times$ is injective, Prop. 10.3.3
$= \mathbb{I}$ with $\iota_{\times p}$ an injective map

$\iota_{qp} ; \iota^{\mathsf{T}}_{qp} = \iota_{\times q} ; \mathfrak{F}^{\mathsf{T}}_\times ; \iota^{\mathsf{T}}_{\times p} ; \iota_{\times p} ; \mathfrak{F}_\times ; \iota^{\mathsf{T}}_{\times q} \subseteq \iota_{\times q} ; \mathfrak{F}^{\mathsf{T}}_\times ; \mathfrak{F}_\times ; \iota^{\mathsf{T}}_{\times q}$ since $\iota_{\times p}$ is univalent
$\subseteq \iota_{\times q} ; \iota^{\mathsf{T}}_{\times q}$ because $\mathfrak{F}_\times$ is univalent, again Prop. 10.3.3
$= \mathbb{I}$ with $\iota_{\times q}$ an injective map

Now we prove totality:

$$\begin{aligned}
&\iota_{qp}\fatsemi\mathbb{T} = \iota_{\times q}\fatsemi\mathfrak{F}_\times^\intercal\fatsemi\iota_{\times p}^\intercal\fatsemi\mathbb{T}\\
&= \iota_{\times q}\fatsemi\iota_\times\fatsemi(\mathbb{S}_\mathbb{P}^\intercal\otimes\mathbb{I}_\mathbb{P})\fatsemi\iota_\times^\intercal\fatsemi\iota_{\times p}^\intercal\fatsemi\mathbb{T} \quad \text{by definition of } \mathfrak{F}_\times\\
&\supseteq \iota_{\times q}\fatsemi\iota_\times\fatsemi([q\!\rightarrow]\fatsemi[p\!\rightarrow]^\intercal\otimes\mathbb{I}_\mathbb{P})\fatsemi\iota_\times^\intercal\fatsemi\iota_{\times p}^\intercal\fatsemi\mathbb{T} \quad \text{since } p,q \text{ are consecutive primes}\\
&= \iota_{\times q}\fatsemi\iota_\times\fatsemi([q\!\rightarrow]\oslash\mathbb{I}_\mathbb{P})\fatsemi([p\!\rightarrow]^\intercal\oslash\mathbb{I}_\mathbb{P})\fatsemi\iota_\times^\intercal\fatsemi\iota_{\times p}^\intercal\fatsemi\mathbb{T} \quad \text{due to Prop. 2.1.1.ii}\\
&= \iota_{\times q}\fatsemi\iota_\times\fatsemi(\pi_\mathbb{P}\fatsemi[q\!\rightarrow]\cap\rho_\mathbb{P})\fatsemi([p\!\rightarrow]^\intercal\fatsemi\pi_\mathbb{P}^\intercal\cap\rho_\mathbb{P}^\intercal)\fatsemi\iota_\times^\intercal\fatsemi\iota_{\times p}^\intercal\fatsemi\mathbb{T}\\
&= (\iota_{\times q}\fatsemi\iota_\times\fatsemi\pi_\mathbb{P}\fatsemi[q\!\rightarrow]\cap\iota_{\times q}\fatsemi\iota_\times\fatsemi\rho_\mathbb{P})\fatsemi([p\!\rightarrow]^\intercal\fatsemi\pi_\mathbb{P}^\intercal\fatsemi\iota_\times^\intercal\fatsemi\iota_{\times p}^\intercal\cap\rho_\mathbb{P}^\intercal\fatsemi\iota_\times^\intercal\fatsemi\iota_{\times p}^\intercal)\fatsemi\mathbb{T}\\
&= (\mathbb{T}\cap\iota_{\times q}\fatsemi\iota_\times\fatsemi\rho_\mathbb{P})\fatsemi(\mathbb{T}\cap\rho_\mathbb{P}^\intercal\fatsemi\iota_\times^\intercal\fatsemi\iota_{\times p}^\intercal)\fatsemi\mathbb{T} \quad \text{following Prop. 10.3.9.v}\\
&= \iota_{\times q}\fatsemi\iota_\times\fatsemi\rho_\mathbb{P}\fatsemi\rho_\mathbb{P}^\intercal\fatsemi\iota_\times^\intercal\fatsemi\iota_{\times p}^\intercal\fatsemi\mathbb{T}\\
&\supseteq \mathbb{T} \quad \text{see what follows}
\end{aligned}$$

$$\begin{aligned}
&\iff \quad \rho_\mathbb{P}^\intercal\fatsemi\iota_\times^\intercal\fatsemi\iota_{\times q}^\intercal\fatsemi\mathbb{T}\subseteq\rho_\mathbb{P}^\intercal\fatsemi\iota_\times^\intercal\fatsemi\iota_{\times p}^\intercal\fatsemi\mathbb{T} \quad \text{maps } \iota_{\times q},\iota_\times \text{ and } \rho_\mathbb{P} \text{ shunted}\\
&\iff \quad \rho_\mathbb{P}^\intercal\fatsemi\iota_\times^\intercal\fatsemi v_{\times q}\fatsemi\mathbb{T}\subseteq\rho_\mathbb{P}^\intercal\fatsemi\iota_\times^\intercal\fatsemi v_{\times p}\fatsemi\mathbb{T} \quad \text{by definition of both these extrusions}
\end{aligned}$$

Finally, we evaluate the left side:

$$\begin{aligned}
&\rho_\mathbb{P}^\intercal\fatsemi\iota_\times^\intercal\fatsemi v_{\times q}\fatsemi\mathbb{T} = \rho_\mathbb{P}^\intercal\fatsemi\iota_\times^\intercal\fatsemi\iota_\times\fatsemi v_q\fatsemi\mathbb{T}\\
&= \rho_\mathbb{P}^\intercal\fatsemi(\mathbb{I}\cap v_\leq\fatsemi\mathbb{T})\fatsemi v_q\fatsemi\mathbb{T} \quad \text{by definition of the extrusion } \iota_\times\\
&= \rho_\mathbb{P}^\intercal\fatsemi(\mathbb{I}\cap\mathtt{vec}(\mathbb{S}_\mathbb{P}^*))\fatsemi\pi_\mathbb{P}\fatsemi[q\!\rightarrow] \quad \text{further expanded}\\
&= \big[\mathtt{rel}(\mathtt{vec}(\mathbb{S}_\mathbb{P}^*))\big]^\intercal\fatsemi[q\!\rightarrow] \quad \text{see the conditions of Prop. 2.2.5.iv}\\
&= \mathbb{S}_\mathbb{P}^{\intercal *}\fatsemi\mathbb{S}_\mathbb{P}^\intercal\fatsemi[p\!\rightarrow] \quad \text{condition that } p,q \text{ are consecutive primes}\\
&= \mathbb{S}_\mathbb{P}^{\intercal +}\fatsemi[p\!\rightarrow]
\end{aligned}$$

Evaluating the right side correspondingly, we get $\mathbb{S}_\mathbb{P}^{\intercal *}\fatsemi[p\!\rightarrow]$, which exceeds the left result. □

$$\iota_{0705}^\intercal\fatsemi\iota_{0705} = \begin{array}{r} \\ (5\!\rightarrow\!,5\!\rightarrow)\!\rightarrow\!\rightarrow \\ (5\!\rightarrow\!,7\!\rightarrow)\!\rightarrow\!\rightarrow \\ (5\!\rightarrow\!,11\!\rightarrow)\!\rightarrow\!\rightarrow \\ (5\!\rightarrow\!,13\!\rightarrow)\!\rightarrow\!\rightarrow \end{array}\!\!\begin{array}{c} \begin{array}{cccc}(5\!\rightarrow\!,5\!\rightarrow)\!\rightarrow\!\rightarrow & (5\!\rightarrow\!,7\!\rightarrow)\!\rightarrow\!\rightarrow & (5\!\rightarrow\!,11\!\rightarrow)\!\rightarrow\!\rightarrow & (5\!\rightarrow\!,13\!\rightarrow)\!\rightarrow\!\rightarrow\end{array} \\ \begin{pmatrix}\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}\end{pmatrix}\end{array}$$

$$\iota_{1107}^\intercal\fatsemi\iota_{1107} = \begin{array}{r} \\ (7\!\rightarrow\!,7\!\rightarrow)\!\rightarrow\!\rightarrow \\ (7\!\rightarrow\!,11\!\rightarrow)\!\rightarrow\!\rightarrow \\ (7\!\rightarrow\!,13\!\rightarrow)\!\rightarrow\!\rightarrow \end{array}\!\!\begin{array}{c} \begin{array}{ccc}(7\!\rightarrow\!,7\!\rightarrow)\!\rightarrow\!\rightarrow & (7\!\rightarrow\!,11\!\rightarrow)\!\rightarrow\!\rightarrow & (7\!\rightarrow\!,13\!\rightarrow)\!\rightarrow\!\rightarrow\end{array} \\ \begin{pmatrix}\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{1}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{1}\end{pmatrix}\end{array}$$

$$\iota_{0503}^\intercal\fatsemi\iota_{0705}^\intercal\fatsemi\iota_{1107}^\intercal = \begin{array}{r} \\ (3\!\rightarrow\!,3\!\rightarrow)\!\rightarrow\!\rightarrow \\ (3\!\rightarrow\!,5\!\rightarrow)\!\rightarrow\!\rightarrow \\ (3\!\rightarrow\!,7\!\rightarrow)\!\rightarrow\!\rightarrow \\ (3\!\rightarrow\!,11\!\rightarrow)\!\rightarrow\!\rightarrow \\ (3\!\rightarrow\!,13\!\rightarrow)\!\rightarrow\!\rightarrow \\ (3\!\rightarrow\!,17\!\rightarrow)\!\rightarrow\!\rightarrow \\ (3\!\rightarrow\!,19\!\rightarrow)\!\rightarrow\!\rightarrow \\ (3\!\rightarrow\!,23\!\rightarrow)\!\rightarrow\!\rightarrow \\ (3\!\rightarrow\!,29\!\rightarrow)\!\rightarrow\!\rightarrow \end{array}\!\!\begin{array}{c} \begin{array}{cccccc}(11\!\rightarrow\!,11\!\rightarrow)\!\rightarrow\!\rightarrow & (11\!\rightarrow\!,13\!\rightarrow)\!\rightarrow\!\rightarrow & (11\!\rightarrow\!,17\!\rightarrow)\!\rightarrow\!\rightarrow & (11\!\rightarrow\!,19\!\rightarrow)\!\rightarrow\!\rightarrow & (11\!\rightarrow\!,23\!\rightarrow)\!\rightarrow\!\rightarrow & (11\!\rightarrow\!,29\!\rightarrow)\!\rightarrow\!\rightarrow\end{array} \\ \begin{pmatrix}\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}\end{pmatrix}\end{array}$$

10.4 Selfsimilar prime pair patterns — 3 approaches

Using the constituents developed so far, we are in a position to describe part of the situation as follows.

10.4.1 Definition. We denote the relation sending even numbers ≥ 6 to lexicographically ordered pairs of primes, they are the sum of, as $\mathcal{P} : \mathbb{E}_6 \longrightarrow (\mathbb{P} \times \mathbb{P})_{\leq}$, calling it the **Goldbach pattern relation**

$\mathcal{P} := \mathbb{A}^{\intercal}_{\times}$.

Given any prime p, we will speak of its **Goldbach function**

$\mathcal{P}_p := \mathcal{P} ; \iota^{\intercal}_{\times p} : \mathbb{E}_6 \longrightarrow (\mathbb{P} \times \mathbb{P})_{p\leq}$. □

A Goldbach function is certainly univalent and injective. It is not a mapping as demonstrated by the rows corresponding to 12 of Fig. 10.4.3, e.g. Looking at Fig. 10.4.1, we decide to speak of right-hanging patterns.

10.4.2 Proposition. $\mathcal{P}$ may be written as a disjoint sum $\mathcal{P} = \mathtt{sup}_{p \text{ odd prime}} \mathcal{P}_p ; \iota_{\times p}$

Proof: Disjointness is obvious.

$\mathtt{sup}_{p \text{ odd prime}} \mathcal{P}_p ; \iota_{\times p}$
$= \mathtt{sup}_{p \text{ odd prime}} \mathcal{P} ; \iota^{\intercal}_{\times p} ; \iota_{\times p}$
$= \mathtt{sup}_{p \text{ odd prime}} \mathcal{P} ; (\mathbb{I} \cap v_{\times p} ; \mathbb{T})$ since $\iota_{\times p}$ extrudes $v_{\times p} ; \mathbb{T}$
$= \mathcal{P} ; \mathtt{sup}_{p \text{ odd prime}} (\mathbb{I} \cap v_{\times p} ; \mathbb{T})$
$= \mathcal{P} ; (\mathbb{I} \cap \mathtt{sup}_{p \text{ odd prime}} v_{\times p} ; \mathbb{T}) = \mathcal{P} ; (\mathbb{I} \cap \mathbb{T}) = \mathcal{P}$ see Prop. 10.3.9.ii □

	(3→,3→)	(3→,5→)	(3→,7→)	(3→,11→)	(3→,13→)	(3→,17→)	(3→,19→)	(3→,23→)	(3→,29→)	(5→,5→)	(5→,7→)	(5→,11→)	(5→,13→)	(5→,17→)	(5→,19→)	(5→,23→)	(5→,29→)	(7→,7→)	(7→,11→)	(7→,13→)	(7→,17→)	(7→,19→)	(7→,23→)	(7→,29→)	(11→,11→)	(11→,13→)	(11→,17→)	(11→,19→)	(11→,23→)	(11→,29→)	(13→,13→)	(13→,17→)	(13→,19→)	(13→,23→)	(13→,29→)	(17→,17→)	(17→,19→)	(17→,23→)	(17→,29→)	(19→,19→)	(19→,23→)	(19→,29→)	(23→,23→)	(23→,29→)	(29→,29→)
6→	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8→	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10→	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12→	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14→	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16→	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
18→	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
20→	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
22→	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
24→	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
26→	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
28→	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Fig. 10.4.1 $\mathcal{P}$ decomposes evens 6 ... 28 out of 0 ... 29 into pairs of odd primes (p, q) with $p \leq q$

The difficult point is that the right-hanging patterns are moved downwards by different distances. When we here see 1- or 2-steps, this is misleading. We will soon find longer gaps. Recalling basic knowledge of number theory, this will soon become worse: It has long been proved that density of primes less or equal to n comes close to $\frac{n}{\ln n}$. Nevertheless, we try to get hold of this movement of the right-hanging patterns.

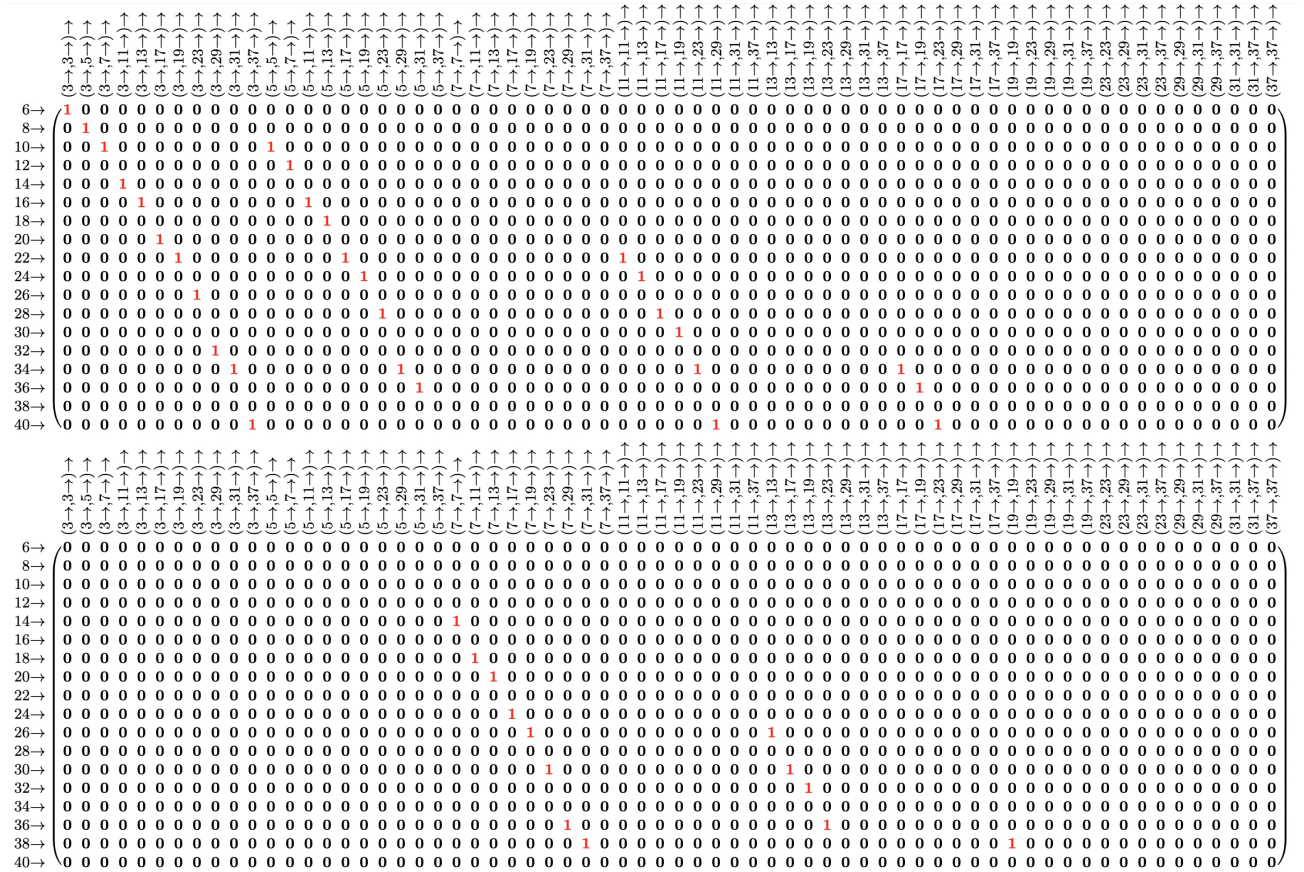


Fig. 10.4.2 Splitting the pattern as to prime distances $g = 2$ and $g = 4$ of the first prime

That these patterns reproduce themselves needs a proof; it is, however, fairly obvious: Looking at row 12 in $\mathcal{P}_3$, we find as the reason for the line being empty that the number 9 is not prime. When we proceed to $\mathcal{P}_5$ with prime distance 2 between 3 and 5 and look at row 14, it is again empty because 9 is not prime. This situation will correspondingly appear for other prime distances; e.g. for distance 4 between 7 and 11: look at row 22 in $\mathcal{P}_7$ and row 26 in $\mathcal{P}_{11}$.

The following relation shows the evens above 6 out of 0 ... 29 decomposed into a sum of two primes with the second prime not less than the first:

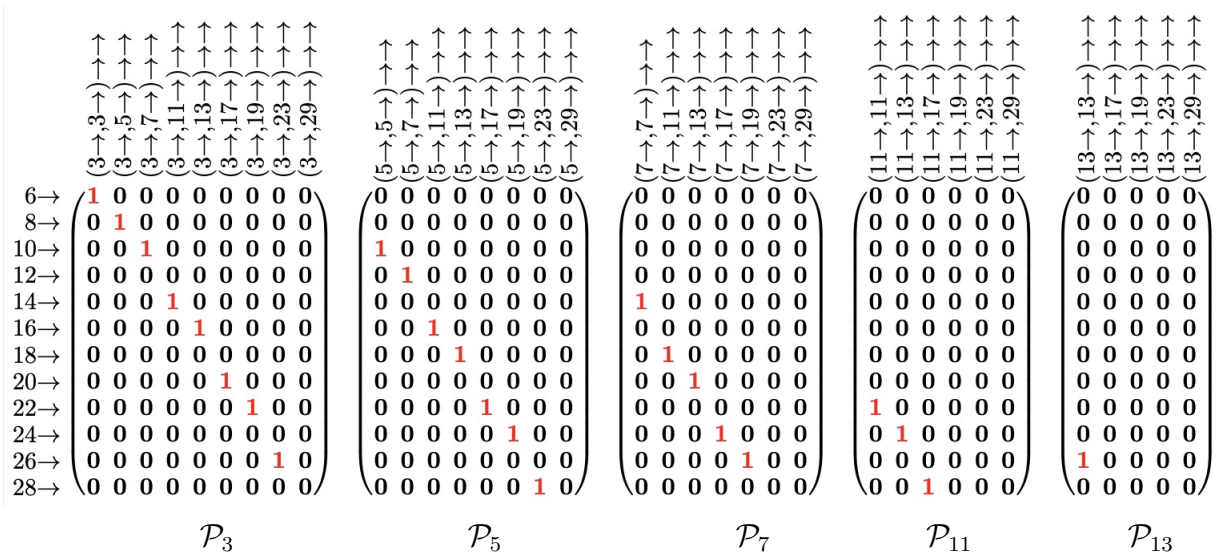


Fig. 10.4.3 The non-empty Goldbach functions when naturals are limited by 29

The last ones $\mathcal{P}_{17}, \mathcal{P}_{19}, \mathcal{P}_{23}, \mathcal{P}_{29}$, all $= \perp\!\!\!\perp$ in their respective typing, will become smaller and smaller. Compared with Fig. 10.4.1, there is an additional arrow in the column inscriptions that documents the injection into pairs with the respective first component.

The preceding tables seem indeed to offer some regularity. Considering the right-hanging geometrical pattern belonging to the odd primes $3, 5, 7, 11, 13, 17, 19, 23$ in this sequence, they always loose an element at the upper left edge and are identical — up to their being cut off at the lower right end. Is it possible to capture this idea formally so as to obtain some sort of a sophisticated recursion?

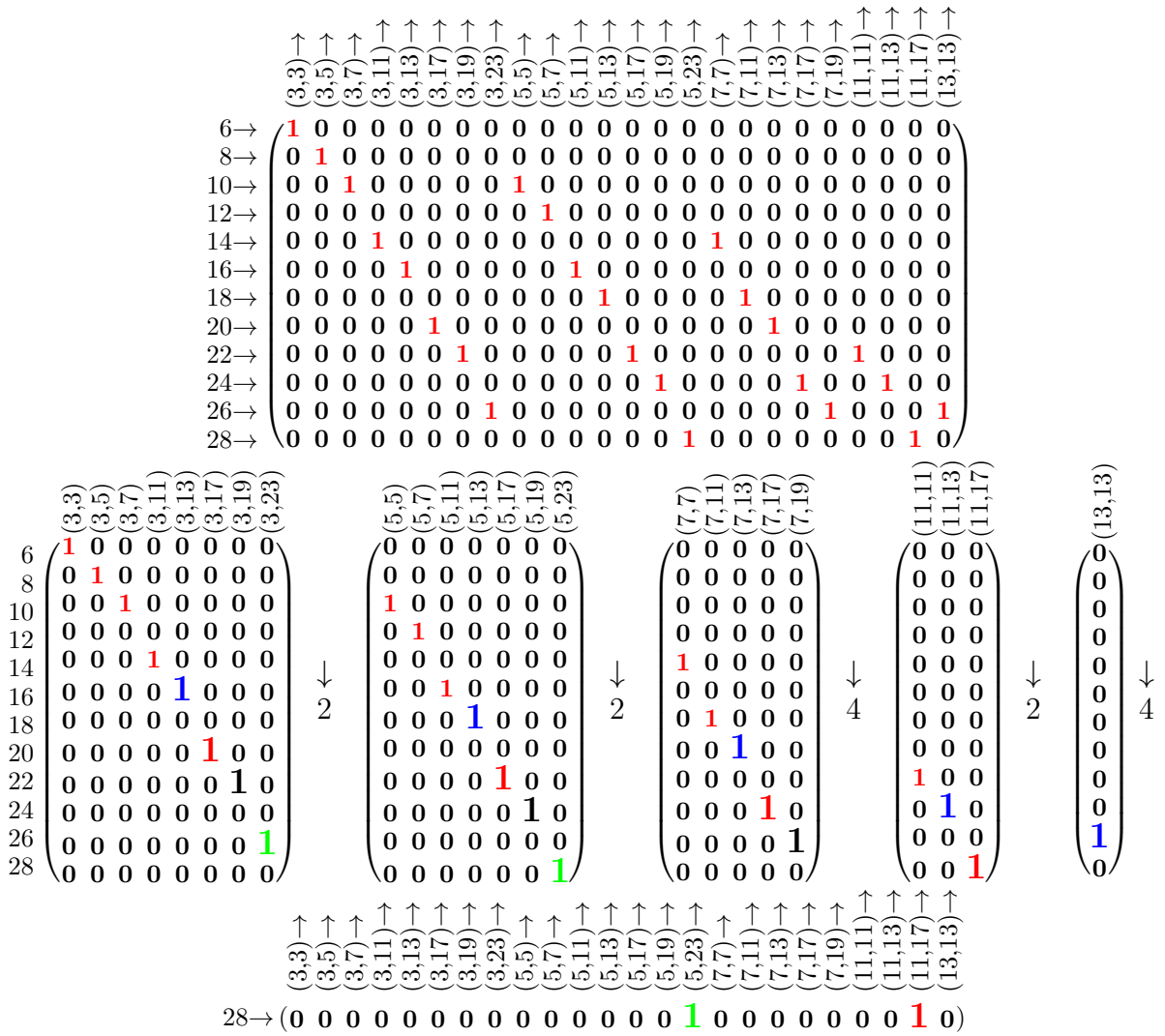
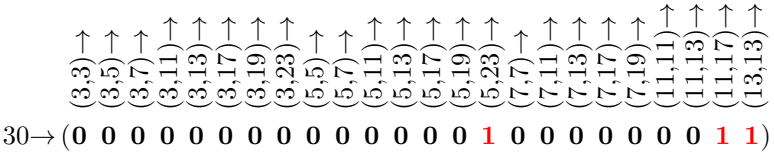


Fig. 10.4.4 Computing line 28 from the preceding bottom lines — that all stem from the $\mathcal{P}_3$ pattern

Remark.[7] Even the line for 30 may more or less be obtained, however just in slightly incomplete form: One has to look for the prime distances over which lowering has to take place. Lowering by 2 would lead from 28 to 30 outside the limit currently chosen for our naturals, and this concerns the entries $(5, 23)\rightarrow$ to be sent to $(7, 23)\rightarrow$ as well as $(11, 17)\rightarrow$ to be sent to $(13, 17)\rightarrow$. There are, however, also lowerings by 4 leading from 26 to 30, which concerns $(7, 19)\rightarrow$ to be sent to $(11, 19)\rightarrow$. Observe that there is another entry on line 26, namely $(13, 13)\rightarrow$, which to send to $(17, 13)\rightarrow$ would bring us outside our lexicographic ordering.



□

[7] Compare with Fig. 10.4.3; there the limit is 29; here it is 28. Furthermore: the present figure has far earlier been produced and it lacks several extrusion arrows. It is however too difficult to renew all the modifications.

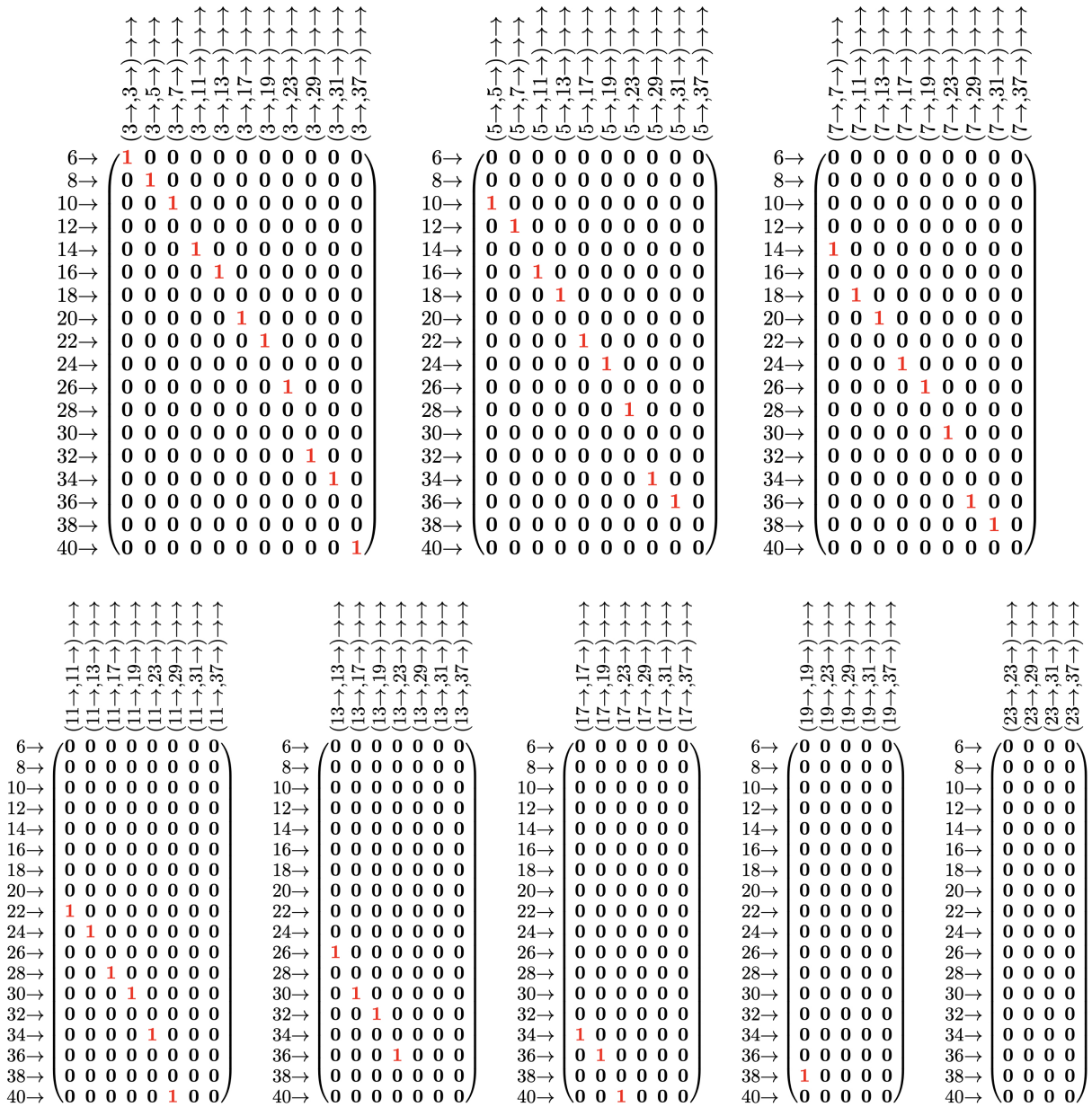


Fig. 10.4.5 Geometric patterns for naturals limited at 40

$\mathcal{P}_3$ $\mathcal{P}_5$ $\mathcal{P}_7$

$\mathcal{P}_{11}$ $\mathcal{P}_{13}$ $\mathcal{P}_{17}$ $\mathcal{P}_{19}$ $\mathcal{P}_{23}$

On the side of even numbers, this step is reciprocated by a downshift. It may be by just 2, when the primes are at distance 2. It will be 4, when the primes have distance 4, etc.

Anticipating the selfsimilarity of the right-hanging pattern, we may compute the row for 30 already when only the naturals up to 28 are given! It is immaterial that the new prime 29 occurs: It cannot contribute in a sum of primes $30 = p_1 + p_2$ — since we work with $\wp : \mathbb{T}$, i.e. primes ≥ 3.

Only relatively late, the author became aware that $\mathcal{P}_p$ with its target or "breadth" depending on the odd prime p has a counterpart, that may be easier to handle. Even later, he found

yet another term in this direction. We here introduce these variant definitions, study their interdependency and hope to obtain further intuition starting from the triple figures of pages 321–324.

10.4.3 Definition. We assume given an odd prime p.

i) $\mathcal{Q}_p := \mathbb{A}^{\mathsf{T}}_{\mathbb{P}} \big[([p^{\rightarrow}] \circledcirc \mathbb{I}_{\mathbb{P}}) \cap v_{\leq} \mathbb{T} \big]$

ii) $\mathcal{Q} := \mathtt{sup}_{p \text{ odd prime}} \mathcal{Q}_p$ □

10.4.4 Definition. We assume given an odd prime p.

i) $\mathcal{R}_p := \iota_{\mathbb{E}_6} \mathbb{S}^{\mathsf{T}p} \iota^{\mathsf{T}}_{\wp\mathbb{T}}$

ii) $\mathcal{R} := \mathtt{sup}_{p \text{ odd prime}} \mathcal{R}_p$ □

Obviously, $\mathcal{R}_p = \perp\!\!\!\perp$ in case p should not be an odd prime, except for $\mathcal{R}_1$.

We face, thus, the problem that we should on one side study the interdependency of $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ and on the other in which way self-similarity may be formulated with one or the other of them.

10.4.5 Proposition. $\mathcal{Q}_p \vartheta^{\mathsf{T}}_p = \mathcal{P}_p$ for an odd prime p

Proof: $\mathcal{Q}_p \vartheta^{\mathsf{T}}_p = \mathbb{A}^{\mathsf{T}}_{\mathbb{P}} \big[([p^{\rightarrow}] \circledcirc \mathbb{I}_{\mathbb{P}}) \cap v_{\leq} \mathbb{T} \big] \rho^{\mathsf{T}}_{\mathbb{P}} \iota^{\mathsf{T}}_{\times} \iota^{\mathsf{T}}_{\times p}$ expanded by Def. 10.4.3; Prop. 10.3.9.v

$= \mathbb{A}^{\mathsf{T}}_{\mathbb{P}} \big[([p^{\rightarrow}] \circledcirc \mathbb{I}_{\mathbb{P}}) \rho^{\mathsf{T}}_{\mathbb{P}} \cap v_{\leq} \mathbb{T} \big] \iota^{\mathsf{T}}_{\times} \iota^{\mathsf{T}}_{\times p}$ masking

$= \mathbb{A}^{\mathsf{T}}_{\mathbb{P}} \big[(\pi_{\mathbb{P}} [p^{\rightarrow}] \cap \rho_{\mathbb{P}}) \rho^{\mathsf{T}}_{\mathbb{P}} \cap v_{\leq} \mathbb{T} \big] \iota^{\mathsf{T}}_{\times} \iota^{\mathsf{T}}_{\times p}$

$= \mathbb{A}^{\mathsf{T}}_{\mathbb{P}} \big[(\pi_{\mathbb{P}} [p^{\rightarrow}] \cap \mathbb{I}) \cap v_{\leq} \mathbb{T} \big] \iota^{\mathsf{T}}_{\times} \iota^{\mathsf{T}}_{\times p}$ destroy and append; $\rho_{\mathbb{P}}$ is total

$= \mathbb{A}^{\mathsf{T}}_{\mathbb{P}} \big[(\pi_{\mathbb{P}} [p^{\rightarrow}] \cap \mathbb{I}) \iota^{\mathsf{T}}_{\times} \cap v_{\leq} \mathbb{T} \big] \iota^{\mathsf{T}}_{\times p}$ since $\iota_{\times}$ is total and univalent

$= \mathbb{A}^{\mathsf{T}}_{\mathbb{P}} \big[\pi_{\mathbb{P}} [p^{\rightarrow}] \cap \iota^{\mathsf{T}}_{\times} \cap v_{\leq} \mathbb{T} \big] \iota^{\mathsf{T}}_{\times p}$ again: since $\iota_{\times}$ is total and univalent

$= \mathbb{A}^{\mathsf{T}}_{\mathbb{P}} \big[\pi_{\mathbb{P}} [p^{\rightarrow}] \cap \iota^{\mathsf{T}}_{\times} \big] \iota^{\mathsf{T}}_{\times p}$ because $\iota_{\times}$ extrudes $v_{\leq} \mathbb{T}$, i.e. $\iota^{\mathsf{T}}_{\times} \mathbb{T} = v_{\leq} \mathbb{T}$

$= \mathbb{A}^{\mathsf{T}}_{\mathbb{P}} \iota^{\mathsf{T}}_{\times} \iota^{\mathsf{T}}_{\times p}$ see below

$= \mathbb{A}^{\mathsf{T}}_{\times} \iota^{\mathsf{T}}_{\times p} = \mathcal{P} \iota^{\mathsf{T}}_{\times p} = \mathcal{P}_p$

It has yet to be shown that $\iota^{\mathsf{T}}_{\times} \iota^{\mathsf{T}}_{\times p} \subseteq \big[\pi_{\mathbb{P}} [p^{\rightarrow}] \cap \iota^{\mathsf{T}}_{\times} \big] \iota^{\mathsf{T}}_{\times p}$

$\iff \quad \iota^{\mathsf{T}}_{\times p} \subseteq \iota_{\times} \big[\pi_{\mathbb{P}} [p^{\rightarrow}] \cap \iota^{\mathsf{T}}_{\times} \big] \iota^{\mathsf{T}}_{\times p}$

$= \big[\iota_{\times} \pi_{\mathbb{P}} [p^{\rightarrow}] \cap \iota_{\times} \iota^{\mathsf{T}}_{\times} \big] \iota^{\mathsf{T}}_{\times p} = \big[v_{\times p} \mathbb{T} \cap \mathbb{I} \big] \iota^{\mathsf{T}}_{\times p}$ due to Def. 10.3.8; $\iota_{\times}$ injective map

$= \big[\mathbb{T} v^{\mathsf{T}}_{\times p} \cap \mathbb{I} \big] \iota^{\mathsf{T}}_{\times p} = \big[\mathbb{T} \iota_{\times p} \cap \mathbb{I} \big] \iota^{\mathsf{T}}_{\times p}$ transposing a partial identity; extrusion

$= \mathbb{T} \iota_{\times p} \iota^{\mathsf{T}}_{\times p} \cap \mathbb{I} \iota^{\mathsf{T}}_{\times p}$ since $\iota_{\times p}$ is univalent

$= \mathbb{T} \mathbb{I} \cap \iota^{\mathsf{T}}_{\times p}$ □

The transition from $\mathcal{R}$ to $\mathcal{Q}$ seems fairly obvious when considering the matrices that follow.

10.4.6 Proposition. For an odd prime p always $\mathcal{Q}_p = \mathcal{R}_p \cap [p^{\rightarrow}]^{\mathsf{T}} \mathbb{S}^*_{\mathbb{P}}$

Proof: We prove this using points $x = \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{a}{}_{;}z_{\mathbb{E}_6}{}_{;}\mathbb{T}$ and $y := \mathbb{S}^{\mathsf{T}^b}_{\mathbb{P}}{}_{;}z_{\mathbb{P}}{}_{;}\mathbb{T} = [\mathtt{SeqNtoPri}(b)\!\!\rightarrow]$. It should hold

$$x{}_{;}y^{\mathsf{T}} \subseteq \mathcal{Q}_p \quad \Longleftrightarrow \quad x{}_{;}y^{\mathsf{T}} \subseteq \mathcal{R}_p \cap [p\!\!\rightarrow]^{\mathsf{T}}{}_{;}\mathbb{S}^*_{\mathbb{P}}$$

We begin with the left containment.

$$\begin{aligned}
\Longleftrightarrow \quad & x \subseteq \mathcal{Q}_p{}_{;}y = \mathbb{A}^{\mathsf{T}}_{\mathbb{P}}{}_{;}\big[\,([p\!\!\rightarrow] \circledcirc \mathbb{I}_{\mathbb{P}}) \cap v_{\leq}{}_{;}\mathbb{T}\big]{}_{;}y \quad \text{due to Def. 10.4.3.i} \\
& = \mathbb{A}^{\mathsf{T}}_{\mathbb{P}}{}_{;}\big[\,([p\!\!\rightarrow] \circledcirc y) \cap v_{\leq}{}_{;}\mathbb{T}\big] \quad \text{observing that } y \text{ is a point and thus } \neq \perp\!\!\!\perp \text{ and surjective} \\
& = \mathbb{A}^{\mathsf{T}}_{\mathbb{P}}{}_{;}\big[\,([p\!\!\rightarrow] \circledcirc [\mathtt{SeqNtoPri}(b)\!\!\rightarrow]) \cap v_{\leq}{}_{;}\mathbb{T}\big]
\end{aligned}$$

Now we have a point x on the left side, so that the right one must not be $= \perp\!\!\!\perp$. This implies $p \leq \mathtt{SeqNtoPri}(b)$ first. In addition we may compute the sum and arrive at this point

$$= \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{\frac{p+\mathtt{SeqNtoPri}(b)-6}{2}}{}_{;}z_{\mathbb{E}_6}{}_{;}\mathbb{T} \quad \text{according to Prop. 10.2.2.i}$$

leaving us with the second condition

$$a = \tfrac{p+\mathtt{SeqNtoPri}(b)-6}{2}$$

Now the right side:

$$\begin{aligned}
\Longleftrightarrow \quad & x \subseteq \big[\mathcal{R}_p \cap [p\!\!\rightarrow]^{\mathsf{T}}{}_{;}\mathbb{S}^*_{\mathbb{P}}\big]{}_{;}y = \iota_{\mathbb{E}_6}{}_{;}\mathbb{S}^{\mathsf{T}p}{}_{;}\iota^{\mathsf{T}}_{\wp;\mathbb{T}}{}_{;}y \cap [p\!\!\rightarrow]^{\mathsf{T}}{}_{;}\mathbb{S}^*_{\mathbb{P}}{}_{;}y \quad \text{due to Def. 10.4.4.i} \\
\Longleftrightarrow \quad & x \subseteq \iota_{\mathbb{E}_6}{}_{;}\mathbb{S}^{\mathsf{T}p}{}_{;}\iota^{\mathsf{T}}_{\wp;\mathbb{T}}{}_{;}y \qquad \wedge \qquad x \subseteq [p\!\!\rightarrow]^{\mathsf{T}}{}_{;}\mathbb{S}^*_{\mathbb{P}}{}_{;}y \\
\Longleftrightarrow \quad & x \subseteq \iota_{\mathbb{E}_6}{}_{;}\mathbb{S}^{\mathsf{T}p}{}_{;}\iota^{\mathsf{T}}_{\wp;\mathbb{T}}{}_{;}\mathbb{S}^{\mathsf{T}^b}_{\mathbb{P}}{}_{;}z_{\mathbb{P}}{}_{;}\mathbb{T} \qquad \wedge \qquad [p\!\!\rightarrow]{}_{;}x = [p\!\!\rightarrow] \subseteq \mathbb{S}^*_{\mathbb{P}}{}_{;}y
\end{aligned}$$

First part alone

$$\begin{aligned}
x \subseteq \iota_{\mathbb{E}_6}{}_{;}\mathbb{S}^{\mathsf{T}p}{}_{;}\iota^{\mathsf{T}}_{\wp;\mathbb{T}}{}_{;}\mathbb{S}^{\mathsf{T}^b}_{\mathbb{P}}{}_{;}z_{\mathbb{P}}{}_{;}\mathbb{T} &= \iota_{\mathbb{E}_6}{}_{;}\mathbb{S}^{\mathsf{T}p}{}_{;}\iota^{\mathsf{T}}_{\wp;\mathbb{T}}{}_{;}[\mathtt{SeqNtoPri}(b)\!\!\rightarrow] \quad \text{see above} \\
&= \iota_{\mathbb{E}_6}{}_{;}\mathbb{S}^{\mathsf{T}p}{}_{;}\mathbb{S}^{\mathsf{T}\mathtt{SeqNtoPri}(b)}{}_{;}z{}_{;}\mathbb{T} \quad \text{due to Prop. 9.4.4.i} \\
&= \iota_{\mathbb{E}_6}{}_{;}\mathbb{S}^{\mathsf{T}p+\mathtt{SeqNtoPri}(b)}{}_{;}z{}_{;}\mathbb{T} \\
&= \mathbb{S}^{\frac{p+\mathtt{SeqNtoPri}(b)-6}{2}}_{\mathbb{E}_6}{}_{;}z_{\mathbb{E}_6}{}_{;}\mathbb{T} \quad \text{with Prop. 10.1.3.vii}
\end{aligned}$$

□

We see again that it is difficult to handle the different appearances of a prime:

— as a normal number in $\mathbb{N}$, or

— as an element in $\mathbb{P}$

with its sequence number pointing to the prime in $\mathbb{N}$.

Looking at the following triples $\mathcal{P}_p, \mathcal{Q}_p, \mathcal{R}_p$, shown for $p = 3, 5, 7, 11, 13, 17, 19$ the "equivalence" seems obvious — but has still to be proved. This will make it much easier to handle the selfsimilarity of the patterns $\mathcal{P}_p$, $\mathcal{Q}_p$ and $\mathcal{R}_p$.

$\mathcal{P}_3$

	(3→,3→)→→	(3→,5→)→→	(3→,7→)→→	(3→,11→)→→	(3→,13→)→→	(3→,17→)→→	(3→,19→)→→	(3→,23→)→→	(3→,29→)→→	(3→,31→)→→	(3→,37→)→→
6→	1	0	0	0	0	0	0	0	0	0	0
8→	0	1	0	0	0	0	0	0	0	0	0
10→	0	0	1	0	0	0	0	0	0	0	0
12→	0	0	0	0	0	0	0	0	0	0	0
14→	0	0	0	1	0	0	0	0	0	0	0
16→	0	0	0	0	1	0	0	0	0	0	0
18→	0	0	0	0	0	0	0	0	0	0	0
20→	0	0	0	0	0	1	0	0	0	0	0
22→	0	0	0	0	0	0	1	0	0	0	0
24→	0	0	0	0	0	0	0	0	0	0	0
26→	0	0	0	0	0	0	0	1	0	0	0
28→	0	0	0	0	0	0	0	0	0	0	0
30→	0	0	0	0	0	0	0	0	0	0	0
32→	0	0	0	0	0	0	0	0	1	0	0
34→	0	0	0	0	0	0	0	0	0	1	0
36→	0	0	0	0	0	0	0	0	0	0	0
38→	0	0	0	0	0	0	0	0	0	0	0
40→	0	0	0	0	0	0	0	0	0	0	1

$\mathcal{Q}_3$

	3→	5→	7→	11→	13→	17→	19→	23→	29→	31→	37→
6→	1	0	0	0	0	0	0	0	0	0	0
8→	0	1	0	0	0	0	0	0	0	0	0
10→	0	0	1	0	0	0	0	0	0	0	0
12→	0	0	0	0	0	0	0	0	0	0	0
14→	0	0	0	1	0	0	0	0	0	0	0
16→	0	0	0	0	1	0	0	0	0	0	0
18→	0	0	0	0	0	0	0	0	0	0	0
20→	0	0	0	0	0	1	0	0	0	0	0
22→	0	0	0	0	0	0	1	0	0	0	0
24→	0	0	0	0	0	0	0	0	0	0	0
26→	0	0	0	0	0	0	0	1	0	0	0
28→	0	0	0	0	0	0	0	0	0	0	0
30→	0	0	0	0	0	0	0	0	0	0	0
32→	0	0	0	0	0	0	0	0	1	0	0
34→	0	0	0	0	0	0	0	0	0	1	0
36→	0	0	0	0	0	0	0	0	0	0	0
38→	0	0	0	0	0	0	0	0	0	0	0
40→	0	0	0	0	0	0	0	0	0	0	1

$\mathcal{R}_3$

	3→	5→	7→	11→	13→	17→	19→	23→	29→	31→	37→
6→	1	0	0	0	0	0	0	0	0	0	0
8→	0	1	0	0	0	0	0	0	0	0	0
10→	0	0	1	0	0	0	0	0	0	0	0
12→	0	0	0	0	0	0	0	0	0	0	0
14→	0	0	0	1	0	0	0	0	0	0	0
16→	0	0	0	0	1	0	0	0	0	0	0
18→	0	0	0	0	0	0	0	0	0	0	0
20→	0	0	0	0	0	1	0	0	0	0	0
22→	0	0	0	0	0	0	1	0	0	0	0
24→	0	0	0	0	0	0	0	0	0	0	0
26→	0	0	0	0	0	0	0	1	0	0	0
28→	0	0	0	0	0	0	0	0	0	0	0
30→	0	0	0	0	0	0	0	0	0	0	0
32→	0	0	0	0	0	0	0	0	1	0	0
34→	0	0	0	0	0	0	0	0	0	1	0
36→	0	0	0	0	0	0	0	0	0	0	0
38→	0	0	0	0	0	0	0	0	0	0	0
40→	0	0	0	0	0	0	0	0	0	0	1

$\mathcal{P}_5$

	(5→,5→)→→	(5→,7→)→→	(5→,11→)→→	(5→,13→)→→	(5→,17→)→→	(5→,19→)→→	(5→,23→)→→	(5→,29→)→→	(5→,31→)→→	(5→,37→)→→
6→	0	0	0	0	0	0	0	0	0	0
8→	0	0	0	0	0	0	0	0	0	0
10→	1	0	0	0	0	0	0	0	0	0
12→	0	1	0	0	0	0	0	0	0	0
14→	0	0	0	0	0	0	0	0	0	0
16→	0	0	1	0	0	0	0	0	0	0
18→	0	0	0	1	0	0	0	0	0	0
20→	0	0	0	0	0	0	0	0	0	0
22→	0	0	0	0	1	0	0	0	0	0
24→	0	0	0	0	0	1	0	0	0	0
26→	0	0	0	0	0	0	0	0	0	0
28→	0	0	0	0	0	0	1	0	0	0
30→	0	0	0	0	0	0	0	0	0	0
32→	0	0	0	0	0	0	0	0	0	0
34→	0	0	0	0	0	0	0	1	0	0
36→	0	0	0	0	0	0	0	0	1	0
38→	0	0	0	0	0	0	0	0	0	0
40→	0	0	0	0	0	0	0	0	0	0

$\mathcal{Q}_5$

	3→	5→	7→	11→	13→	17→	19→	23→	29→	31→	37→
6→	0	0	0	0	0	0	0	0	0	0	0
8→	0	0	0	0	0	0	0	0	0	0	0
10→	0	1	0	0	0	0	0	0	0	0	0
12→	0	0	1	0	0	0	0	0	0	0	0
14→	0	0	0	0	0	0	0	0	0	0	0
16→	0	0	0	1	0	0	0	0	0	0	0
18→	0	0	0	0	1	0	0	0	0	0	0
20→	0	0	0	0	0	0	0	0	0	0	0
22→	0	0	0	0	0	1	0	0	0	0	0
24→	0	0	0	0	0	0	1	0	0	0	0
26→	0	0	0	0	0	0	0	0	0	0	0
28→	0	0	0	0	0	0	0	1	0	0	0
30→	0	0	0	0	0	0	0	0	0	0	0
32→	0	0	0	0	0	0	0	0	0	0	0
34→	0	0	0	0	0	0	0	0	1	0	0
36→	0	0	0	0	0	0	0	0	0	1	0
38→	0	0	0	0	0	0	0	0	0	0	0
40→	0	0	0	0	0	0	0	0	0	0	0

$\mathcal{R}_5$

	3→	5→	7→	11→	13→	17→	19→	23→	29→	31→	37→
6→	0	0	0	0	0	0	0	0	0	0	0
8→	1	0	0	0	0	0	0	0	0	0	0
10→	0	1	0	0	0	0	0	0	0	0	0
12→	0	0	1	0	0	0	0	0	0	0	0
14→	0	0	0	0	0	0	0	0	0	0	0
16→	0	0	0	1	0	0	0	0	0	0	0
18→	0	0	0	0	1	0	0	0	0	0	0
20→	0	0	0	0	0	0	0	0	0	0	0
22→	0	0	0	0	0	1	0	0	0	0	0
24→	0	0	0	0	0	0	1	0	0	0	0
26→	0	0	0	0	0	0	0	0	0	0	0
28→	0	0	0	0	0	0	0	1	0	0	0
30→	0	0	0	0	0	0	0	0	0	0	0
32→	0	0	0	0	0	0	0	0	0	0	0
34→	0	0	0	0	0	0	0	0	1	0	0
36→	0	0	0	0	0	0	0	0	0	1	0
38→	0	0	0	0	0	0	0	0	0	0	0
40→	0	0	0	0	0	0	0	0	0	0	0

$\mathcal{P}_7$

	(7→,7→)→→	(7→,11→)→→	(7→,13→)→→	(7→,17→)→→	(7→,19→)→→	(7→,23→)→→	(7→,29→)→→	(7→,31→)→→	(7→,37→)→→
6→	0	0	0	0	0	0	0	0	0
8→	0	0	0	0	0	0	0	0	0
10→	0	0	0	0	0	0	0	0	0
12→	0	0	0	0	0	0	0	0	0
14→	1	0	0	0	0	0	0	0	0
16→	0	0	0	0	0	0	0	0	0
18→	0	1	0	0	0	0	0	0	0
20→	0	0	1	0	0	0	0	0	0
22→	0	0	0	0	0	0	0	0	0
24→	0	0	0	1	0	0	0	0	0
26→	0	0	0	0	1	0	0	0	0
28→	0	0	0	0	0	0	0	0	0
30→	0	0	0	0	0	1	0	0	0
32→	0	0	0	0	0	0	0	0	0
34→	0	0	0	0	0	0	0	0	0
36→	0	0	0	0	0	0	1	0	0
38→	0	0	0	0	0	0	0	1	0
40→	0	0	0	0	0	0	0	0	0

$\mathcal{Q}_7$

	3→	5→	7→	11→	13→	17→	19→	23→	29→	31→	37→
6→	0	0	0	0	0	0	0	0	0	0	0
8→	0	0	0	0	0	0	0	0	0	0	0
10→	0	0	0	0	0	0	0	0	0	0	0
12→	0	0	0	0	0	0	0	0	0	0	0
14→	0	0	1	0	0	0	0	0	0	0	0
16→	0	0	0	0	0	0	0	0	0	0	0
18→	0	0	0	1	0	0	0	0	0	0	0
20→	0	0	0	0	1	0	0	0	0	0	0
22→	0	0	0	0	0	0	0	0	0	0	0
24→	0	0	0	0	0	1	0	0	0	0	0
26→	0	0	0	0	0	0	1	0	0	0	0
28→	0	0	0	0	0	0	0	0	0	0	0
30→	0	0	0	0	0	0	0	1	0	0	0
32→	0	0	0	0	0	0	0	0	0	0	0
34→	0	0	0	0	0	0	0	0	0	0	0
36→	0	0	0	0	0	0	0	0	1	0	0
38→	0	0	0	0	0	0	0	0	0	1	0
40→	0	0	0	0	0	0	0	0	0	0	0

$\mathcal{R}_7$

	3→	5→	7→	11→	13→	17→	19→	23→	29→	31→	37→
6→	0	0	0	0	0	0	0	0	0	0	0
8→	0	0	0	0	0	0	0	0	0	0	0
10→	1	0	0	0	0	0	0	0	0	0	0
12→	0	1	0	0	0	0	0	0	0	0	0
14→	0	0	1	0	0	0	0	0	0	0	0
16→	0	0	0	0	0	0	0	0	0	0	0
18→	0	0	0	1	0	0	0	0	0	0	0
20→	0	0	0	0	1	0	0	0	0	0	0
22→	0	0	0	0	0	0	0	0	0	0	0
24→	0	0	0	0	0	1	0	0	0	0	0
26→	0	0	0	0	0	0	1	0	0	0	0
28→	0	0	0	0	0	0	0	0	0	0	0
30→	0	0	0	0	0	0	0	1	0	0	0
32→	0	0	0	0	0	0	0	0	0	0	0
34→	0	0	0	0	0	0	0	0	0	0	0
36→	0	0	0	0	0	0	0	0	1	0	0
38→	0	0	0	0	0	0	0	0	0	1	0
40→	0	0	0	0	0	0	0	0	0	0	0

$\mathcal{P}_{11}$

	(11→,11→)→→	(11→,13→)→→	(11→,17→)→→	(11→,19→)→→	(11→,23→)→→	(11→,29→)→→	(11→,31→)→→	(11→,37→)→→
6→	0	0	0	0	0	0	0	0
8→	0	0	0	0	0	0	0	0
10→	0	0	0	0	0	0	0	0
12→	0	0	0	0	0	0	0	0
14→	0	0	0	0	0	0	0	0
16→	0	0	0	0	0	0	0	0
18→	0	0	0	0	0	0	0	0
20→	0	0	0	0	0	0	0	0
22→	1	0	0	0	0	0	0	0
24→	0	1	0	0	0	0	0	0
26→	0	0	0	0	0	0	0	0
28→	0	0	1	0	0	0	0	0
30→	0	0	0	1	0	0	0	0
32→	0	0	0	0	0	0	0	0
34→	0	0	0	0	1	0	0	0
36→	0	0	0	0	0	0	0	0
38→	0	0	0	0	0	0	0	0
40→	0	0	0	0	0	1	0	0

$\mathcal{Q}_{11}$

	3→	5→	7→	11→	13→	17→	19→	23→	29→	31→	37→
6→	0	0	0	0	0	0	0	0	0	0	0
8→	0	0	0	0	0	0	0	0	0	0	0
10→	0	0	0	0	0	0	0	0	0	0	0
12→	0	0	0	0	0	0	0	0	0	0	0
14→	0	0	0	0	0	0	0	0	0	0	0
16→	0	0	0	0	0	0	0	0	0	0	0
18→	0	0	0	0	0	0	0	0	0	0	0
20→	0	0	0	0	0	0	0	0	0	0	0
22→	0	0	0	1	0	0	0	0	0	0	0
24→	0	0	0	0	1	0	0	0	0	0	0
26→	0	0	0	0	0	0	0	0	0	0	0
28→	0	0	0	0	0	1	0	0	0	0	0
30→	0	0	0	0	0	0	1	0	0	0	0
32→	0	0	0	0	0	0	0	0	0	0	0
34→	0	0	0	0	0	0	0	1	0	0	0
36→	0	0	0	0	0	0	0	0	0	0	0
38→	0	0	0	0	0	0	0	0	0	0	0
40→	0	0	0	0	0	0	0	0	1	0	0

$\mathcal{R}_{11}$

	3→	5→	7→	11→	13→	17→	19→	23→	29→	31→	37→
6→	0	0	0	0	0	0	0	0	0	0	0
8→	0	0	0	0	0	0	0	0	0	0	0
10→	0	0	0	0	0	0	0	0	0	0	0
12→	0	0	0	0	0	0	0	0	0	0	0
14→	1	0	0	0	0	0	0	0	0	0	0
16→	0	1	0	0	0	0	0	0	0	0	0
18→	0	0	1	0	0	0	0	0	0	0	0
20→	0	0	0	0	0	0	0	0	0	0	0
22→	0	0	0	1	0	0	0	0	0	0	0
24→	0	0	0	0	1	0	0	0	0	0	0
26→	0	0	0	0	0	0	0	0	0	0	0
28→	0	0	0	0	0	1	0	0	0	0	0
30→	0	0	0	0	0	0	1	0	0	0	0
32→	0	0	0	0	0	0	0	0	0	0	0
34→	0	0	0	0	0	0	0	1	0	0	0
36→	0	0	0	0	0	0	0	0	0	0	0
38→	0	0	0	0	0	0	0	0	0	0	0
40→	0	0	0	0	0	0	0	0	1	0	0

	(13→,13→)→→	(13→,17→)→→	(13→,19→)→→	(13→,23→)→→	(13→,29→)→→	(13→,31→)→→	(13→,37→)→→
6→	0	0	0	0	0	0	0
8→	0	0	0	0	0	0	0
10→	0	0	0	0	0	0	0
12→	0	0	0	0	0	0	0
14→	0	0	0	0	0	0	0
16→	0	0	0	0	0	0	0
18→	0	0	0	0	0	0	0
20→	0	0	0	0	0	0	0
22→	0	0	0	0	0	0	0
24→	0	0	0	0	0	0	0
26→	1	0	0	0	0	0	0
28→	0	0	0	0	0	0	0
30→	0	1	0	0	0	0	0
32→	0	0	1	0	0	0	0
34→	0	0	0	0	0	0	0
36→	0	0	0	1	0	0	0
38→	0	0	0	0	0	0	0
40→	0	0	0	0	0	0	0

$\mathcal{P}_{13}$

	3→	5→	7→	11→	13→	17→	19→	23→	29→	31→	37→
6→	0	0	0	0	0	0	0	0	0	0	0
8→	0	0	0	0	0	0	0	0	0	0	0
10→	0	0	0	0	0	0	0	0	0	0	0
12→	0	0	0	0	0	0	0	0	0	0	0
14→	0	0	0	0	0	0	0	0	0	0	0
16→	0	0	0	0	0	0	0	0	0	0	0
18→	0	0	0	0	0	0	0	0	0	0	0
20→	0	0	0	0	0	0	0	0	0	0	0
22→	0	0	0	0	0	0	0	0	0	0	0
24→	0	0	0	0	0	0	0	0	0	0	0
26→	0	0	0	0	1	0	0	0	0	0	0
28→	0	0	0	0	0	0	0	0	0	0	0
30→	0	0	0	0	0	1	0	0	0	0	0
32→	0	0	0	0	0	0	1	0	0	0	0
34→	0	0	0	0	0	0	0	0	0	0	0
36→	0	0	0	0	0	0	0	1	0	0	0
38→	0	0	0	0	0	0	0	0	0	0	0
40→	0	0	0	0	0	0	0	0	0	0	0

$\mathcal{Q}_{13}$

	3→	5→	7→	11→	13→	17→	19→	23→	29→	31→	37→
6→	0	0	0	0	0	0	0	0	0	0	0
8→	0	0	0	0	0	0	0	0	0	0	0
10→	0	0	0	0	0	0	0	0	0	0	0
12→	0	0	0	0	0	0	0	0	0	0	0
14→	0	0	0	0	0	0	0	0	0	0	0
16→	1	0	0	0	0	0	0	0	0	0	0
18→	0	1	0	0	0	0	0	0	0	0	0
20→	0	0	1	0	0	0	0	0	0	0	0
22→	0	0	0	0	0	0	0	0	0	0	0
24→	0	0	0	1	0	0	0	0	0	0	0
26→	0	0	0	0	1	0	0	0	0	0	0
28→	0	0	0	0	0	0	0	0	0	0	0
30→	0	0	0	0	0	1	0	0	0	0	0
32→	0	0	0	0	0	0	1	0	0	0	0
34→	0	0	0	0	0	0	0	0	0	0	0
36→	0	0	0	0	0	0	0	1	0	0	0
38→	0	0	0	0	0	0	0	0	0	0	0
40→	0	0	0	0	0	0	0	0	0	0	0

$\mathcal{R}_{13}$

	(17→,17→)→→	(17→,19→)→→	(17→,23→)→→	(17→,29→)→→	(17→,31→)→→	(17→,37→)→→
6→	0	0	0	0	0	0
8→	0	0	0	0	0	0
10→	0	0	0	0	0	0
12→	0	0	0	0	0	0
14→	0	0	0	0	0	0
16→	0	0	0	0	0	0
18→	0	0	0	0	0	0
20→	0	0	0	0	0	0
22→	0	0	0	0	0	0
24→	0	0	0	0	0	0
26→	0	0	0	0	0	0
28→	0	0	0	0	0	0
30→	0	0	0	0	0	0
32→	0	0	0	0	0	0
34→	1	0	0	0	0	0
36→	0	1	0	0	0	0
38→	0	0	0	0	0	0
40→	0	0	1	0	0	0

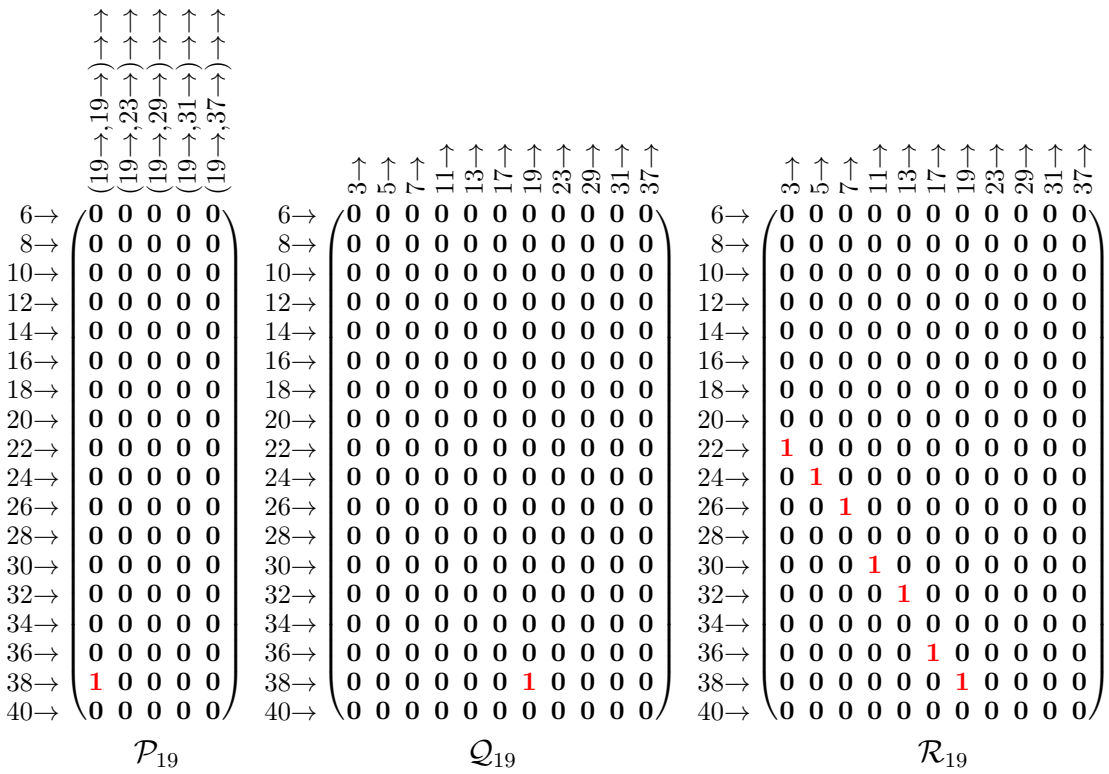
$\mathcal{P}_{17}$

	3→	5→	7→	11→	13→	17→	19→	23→	29→	31→	37→
6→	0	0	0	0	0	0	0	0	0	0	0
8→	0	0	0	0	0	0	0	0	0	0	0
10→	0	0	0	0	0	0	0	0	0	0	0
12→	0	0	0	0	0	0	0	0	0	0	0
14→	0	0	0	0	0	0	0	0	0	0	0
16→	0	0	0	0	0	0	0	0	0	0	0
18→	0	0	0	0	0	0	0	0	0	0	0
20→	0	0	0	0	0	0	0	0	0	0	0
22→	0	0	0	0	0	0	0	0	0	0	0
24→	0	0	0	0	0	0	0	0	0	0	0
26→	0	0	0	0	0	0	0	0	0	0	0
28→	0	0	0	0	0	0	0	0	0	0	0
30→	0	0	0	0	0	0	0	0	0	0	0
32→	0	0	0	0	0	0	0	0	0	0	0
34→	0	0	0	0	0	1	0	0	0	0	0
36→	0	0	0	0	0	0	1	0	0	0	0
38→	0	0	0	0	0	0	0	0	0	0	0
40→	0	0	0	0	0	0	0	1	0	0	0

$\mathcal{Q}_{17}$

	3→	5→	7→	11→	13→	17→	19→	23→	29→	31→	37→
6→	0	0	0	0	0	0	0	0	0	0	0
8→	0	0	0	0	0	0	0	0	0	0	0
10→	0	0	0	0	0	0	0	0	0	0	0
12→	0	0	0	0	0	0	0	0	0	0	0
14→	0	0	0	0	0	0	0	0	0	0	0
16→	0	0	0	0	0	0	0	0	0	0	0
18→	0	0	0	0	0	0	0	0	0	0	0
20→	1	0	0	0	0	0	0	0	0	0	0
22→	0	1	0	0	0	0	0	0	0	0	0
24→	0	0	1	0	0	0	0	0	0	0	0
26→	0	0	0	0	0	0	0	0	0	0	0
28→	0	0	0	1	0	0	0	0	0	0	0
30→	0	0	0	0	1	0	0	0	0	0	0
32→	0	0	0	0	0	0	0	0	0	0	0
34→	0	0	0	0	0	1	0	0	0	0	0
36→	0	0	0	0	0	0	1	0	0	0	0
38→	0	0	0	0	0	0	0	0	0	0	0
40→	0	0	0	0	0	0	0	1	0	0	0

$\mathcal{R}_{17}$



The following hight-hanging patterns shall serve illustration. They show $\mathcal{P}_3 \ldots \mathcal{P}_{13}$ for numbers limited by 182.[8]

Observe the increasing size of horizontal gaps up to, e.g. 128–138. Recalling traditional number theory, the primes will soon prove to be an extremely sparse population with density in $\mathbb{N}$ increasing only logarithmically. This means in particular that such zones of $\mathbf{0}$-rows will occur ever and ever more.

The gap 118–128 in $\mathcal{P}_3$ is lowered to 120–130 in $\mathcal{P}_5$ since 3 has prime distance 2. It is then lowered to 122–132 in $\mathcal{P}_7$ since 5 also has prime distance 2. When proceeding to $\mathcal{P}_{11}$ it is moved to 126–136 due to prime distance 4 of 7.

[8] These are early outputs; some of them without injection arrows in case of extruded elements.

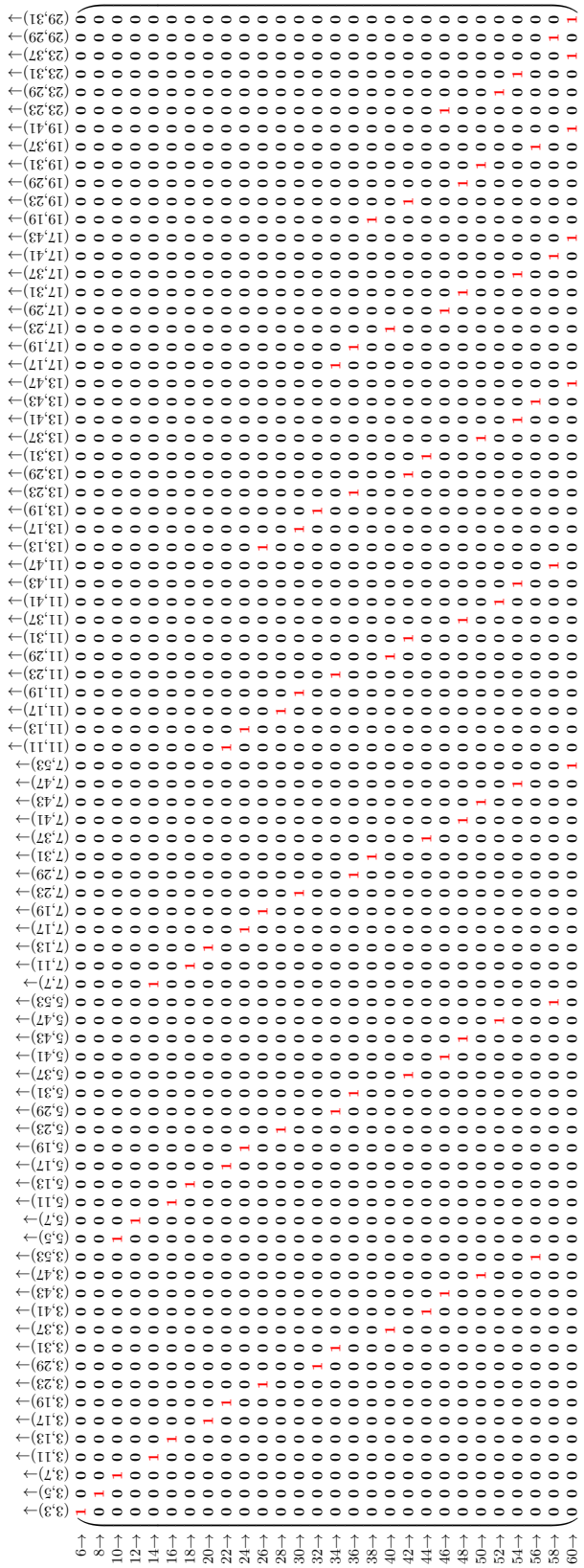


Fig. 10.4.6 Right-hanging patterns of all ordered prime pairs adding to sums up to 60

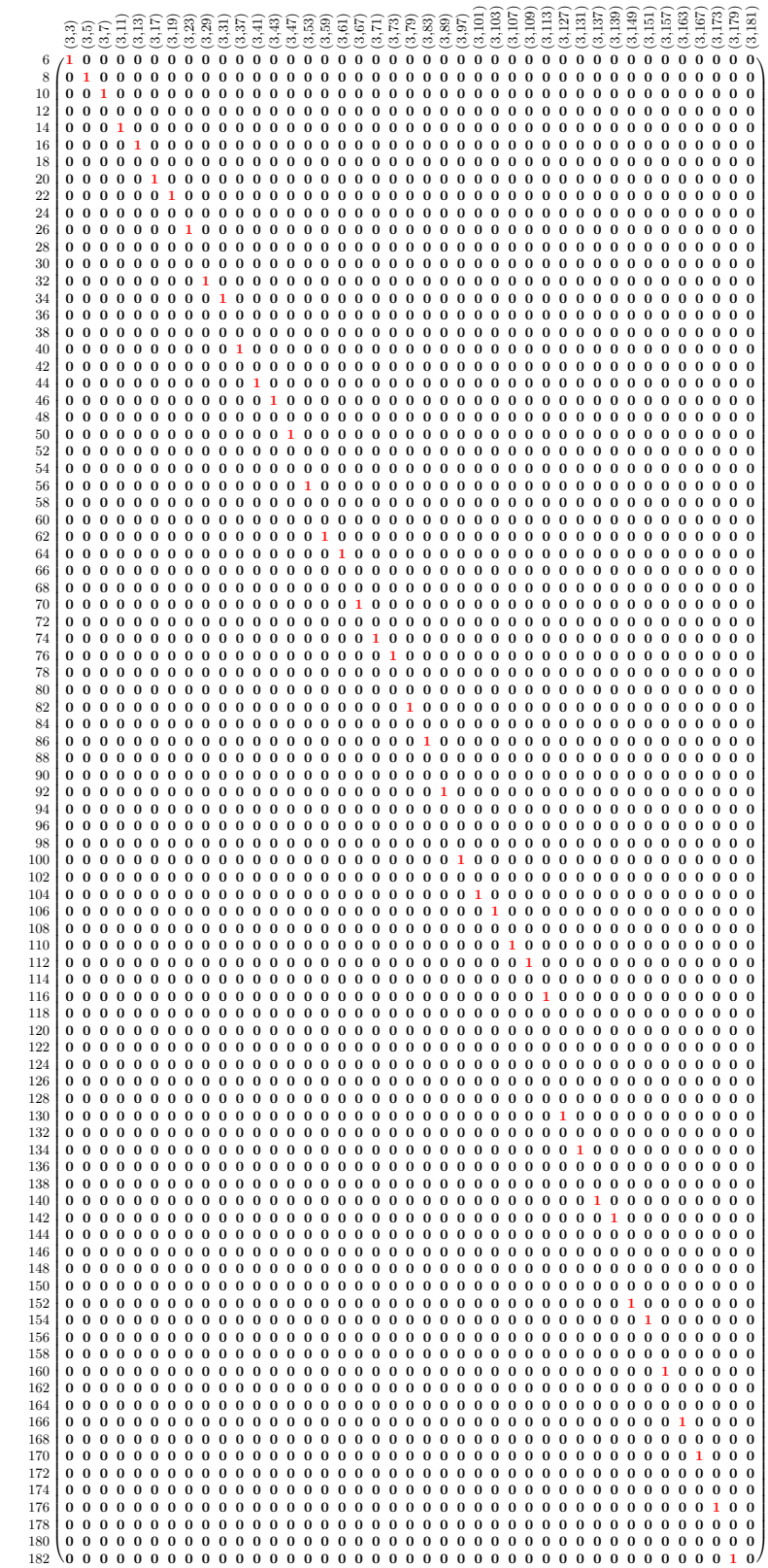


Fig. 10.4.7 Right-hanging pattern $\mathcal{P}_3$ of prime pairs $\text{rowNr} \mapsto (3, \text{rowNr} - 3)$

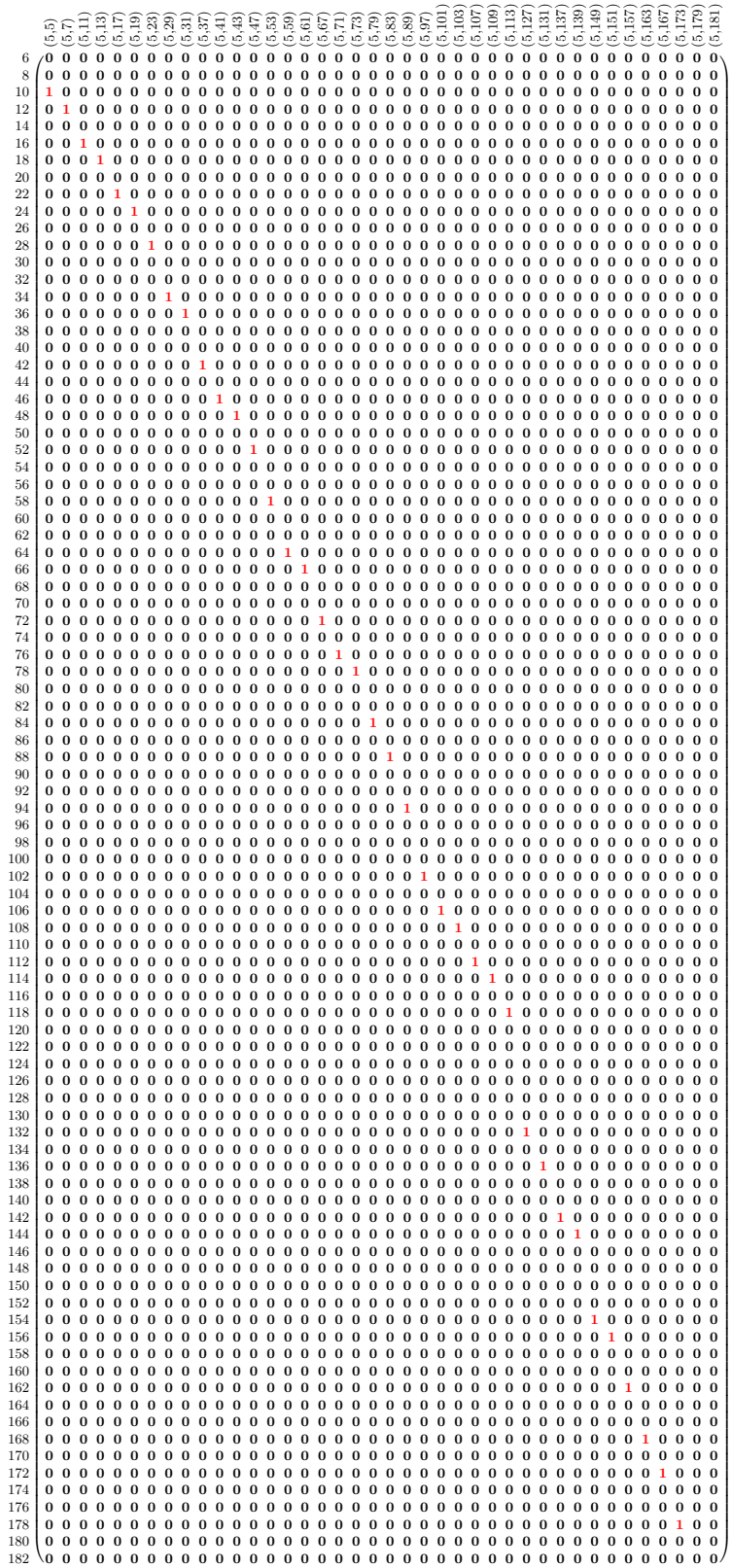


Fig. 10.4.8 Right-hanging pattern $\mathcal{P}_5$ of prime pairs rowNr $\mapsto (5, \text{rowNr} - 5)$

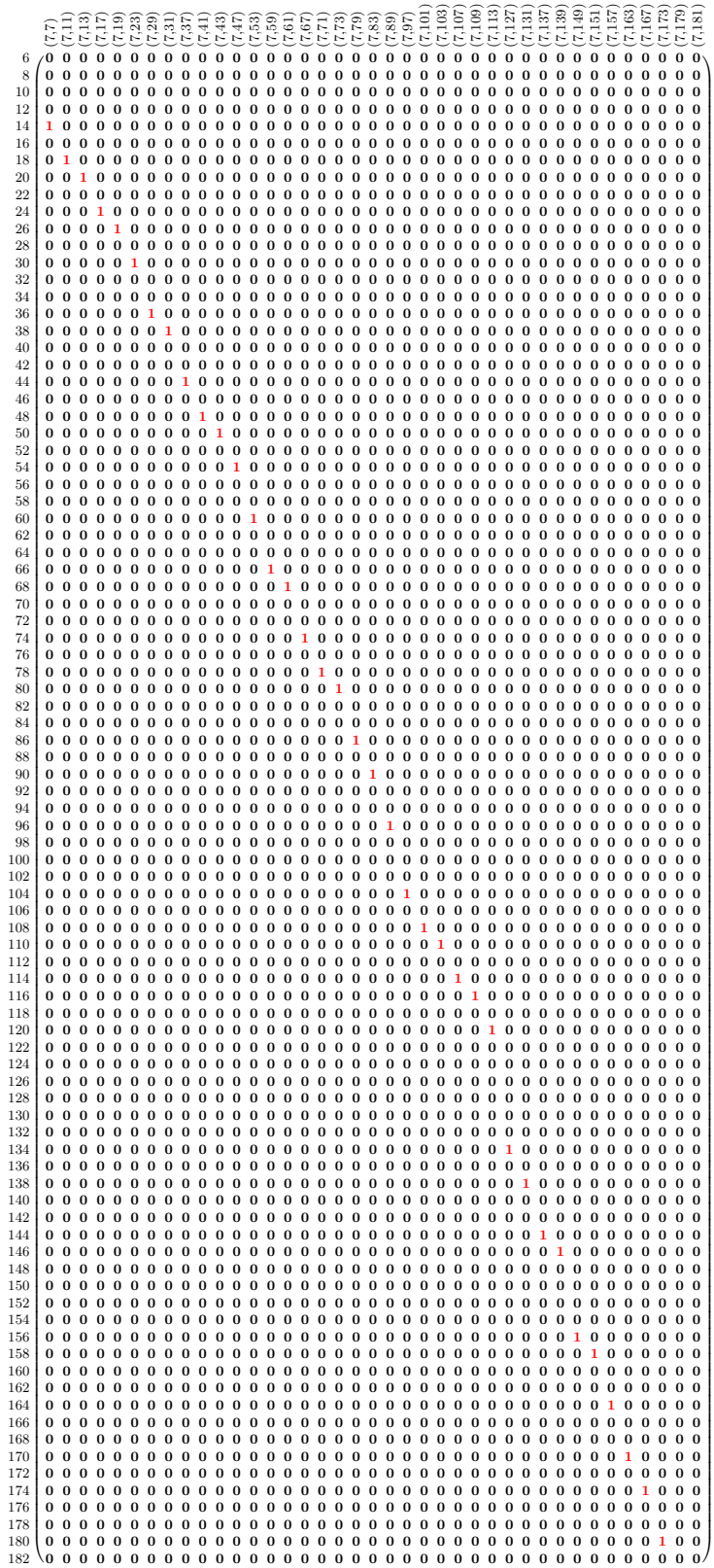


Fig. 10.4.9 Right-hanging pattern $\mathcal{P}_7$ of prime pairs rowNr $\mapsto (7, \text{rowNr} - 7)$

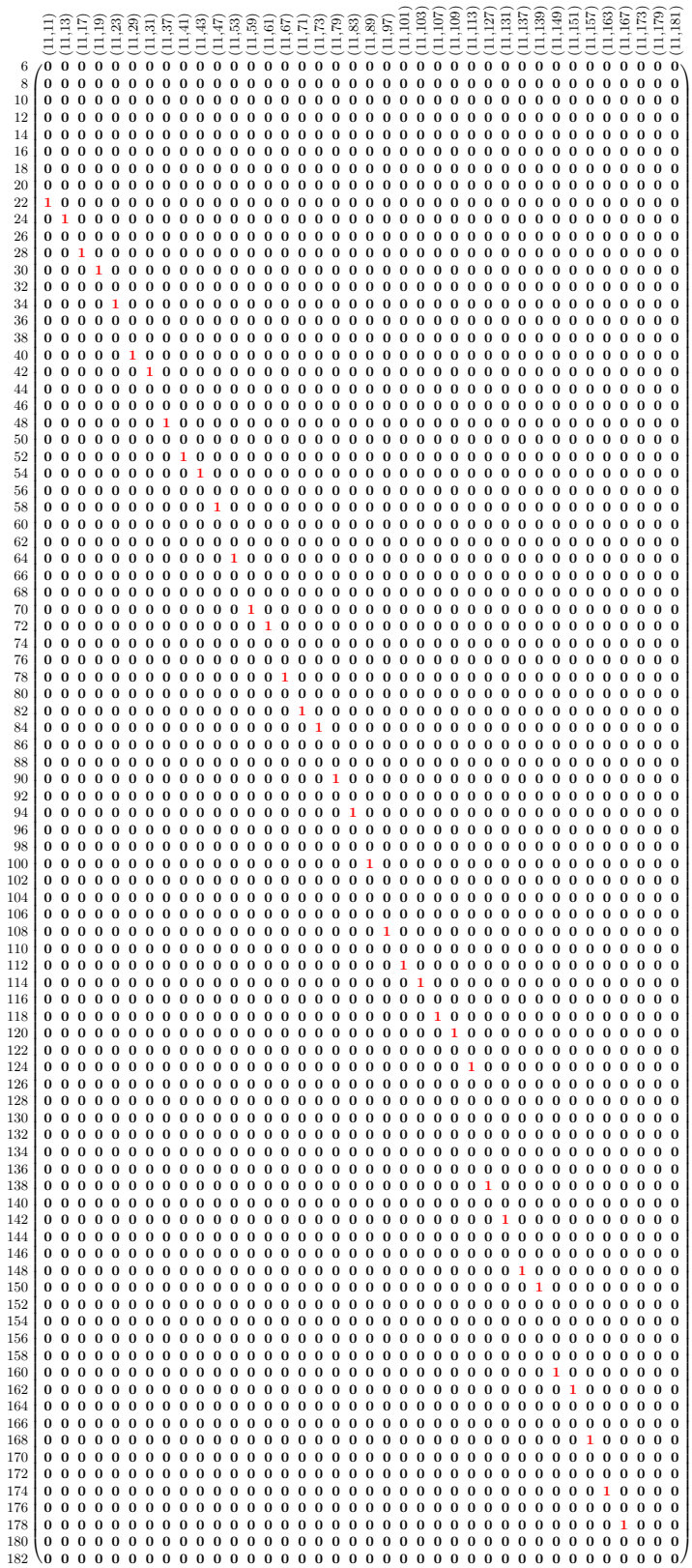


Fig. 10.4.10 Right-hanging pattern $\mathcal{P}_{11}$ of prime pairs rowNr $\mapsto (11, \text{rowNr} - 11)$

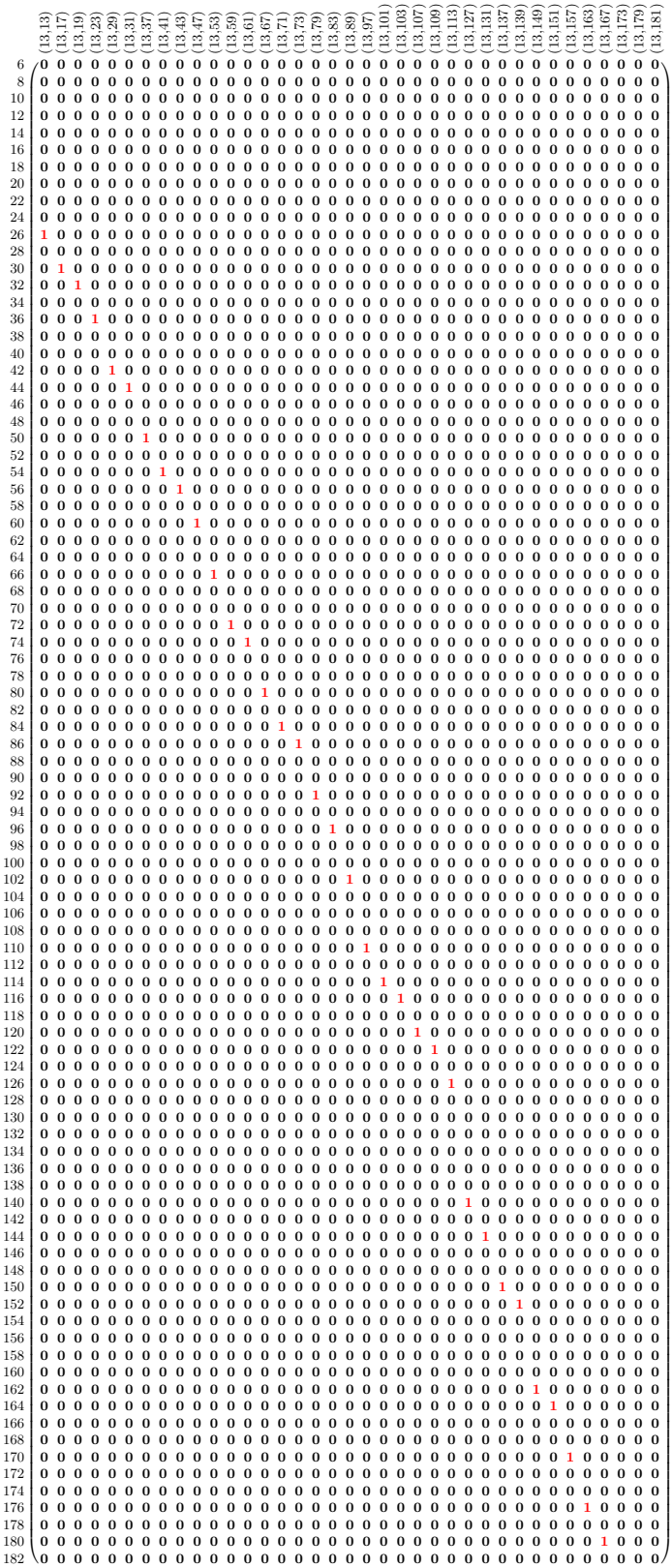


Fig. 10.4.11 Right-hanging pattern $\mathcal{P}_{13}$ of prime pairs rowNr $\mapsto (13, \text{rowNr} - 13)$

10.5 Yet another approach to selfsimilarity

When handling the three presented approaches to selfsimilarity, not least checking them as binary relational operations and inspecting also their inverses, another idea appeared. So far we had relations of these types

— $\mathcal{P}_p \subseteq \mathcal{P} : \mathbb{E}_6 \longrightarrow (\mathbb{P} \times \mathbb{P})_{\leq}$
— $\mathcal{Q}_p \subseteq \mathcal{Q} : \mathbb{E}_6 \longrightarrow \mathbb{P}$
— $\mathcal{R}_p \subseteq \mathcal{R} : \mathbb{E}_6 \longrightarrow \mathbb{P}$

The idea came up that an iterative situation, from one prime to the next, might be helpful in making the relations small; i.e. their row as well as column types. This seems to have been achieved via the following highly detailed but low — sub university — level propositions. Observe that all is presented with the prime p as the only parameter.

10.5.1 Lemma. Let be given two *consecutive* odd primes $p < q$.

i) Any two odd primes $p' \leq q'$ with sum $2 * p < p' + q' < p + q$, will necessarily satisfy

$$3 \leq p' < p \qquad \text{and} \qquad q \leq q' < p + q - 3.$$

ii) Any two odd primes $p' \leq q'$ with sum $p + q < p' + q' < 2 * q$, will necessarily satisfy

$$3 \leq p' \leq p \qquad \text{and} \qquad q < q' < 2 * q - 3.$$

iii) In such case, these three intervals are of equal length $d := q - p$

$$p < e < q \qquad 2 * p < e < p + q \qquad p + q < e < 2 * q$$

and the latter two contain the same number of evens e, namely $\frac{d}{2} - 1$.

iv) In case p is the lower sibling of a prime twin, the intervals of evens will both be empty.

Proof: In (i) and (ii), the statement $3 \leq p'$ is obvious since 2 is the only even prime.

i) Assume to the contrary that $p' \geq p$, meaning either $p' > p$ or $p' = p$. In the first case follows $q' \geq p' \geq q$ from consecutivity $p < q$, so that $p' + q' \geq 2 * q > p + q$; a contradiction. In the second case $p' = p$, so that $p + q' = p' + q' < p + q$ implies $q' < q$ and thus $q' \leq p$ by consecutivity. This brings the contradiction $p' + q' \leq 2 * p$.

Concerning the other statement assume to the contrary $q' < q$. Via consecutivity we get $q' \leq p$ with contradiction $p' + q' \leq 2 * p$. The final statement $q' < p + q - 3$ follows because $p' + q' < p + q$ by assumption and $3 \leq p'$.

---

ii) Concerning the first statement assume to the contrary $p' > p$. Via consecutivity we get $q' \geq p' \geq q$ with contradiction $p' + q' \geq 2 * q$.

Assume to the contrary that $q' \leq q$, meaning either $q' < q$ or $q' = q$. In the first case, in addition $p' \leq q' \leq p < q$ since p, q are consecutive, so that $p' + q' < p + q$; a contradiction.

Considering the second case, we have $p' \leq q' = q$ and distinguish anew two cases, namely $p' < q'$ and $p' = q'$. In the latter case, obviously $p' + q' = 2 * q$ — a contradiction. In the former, we obtain a contradiction again, since p, q are consecutive and, thus $p' \leq p$ results in $p' + q' \leq p + q$.

The final statement $q' < 2 * q - 3$ follows because $p' + q' < 2 * q$ by assumption and $3 \leq p'$.

---

iii) The equal length statement is trivial because $q - p = (p + q) - (2 * p) = (2 * q) - (p + q)$. With $d := q - p$, the number of evens contained in the first interval is certainly $\frac{d}{2}$. The latter two, delimited by two even numbers, contain strictly in between just $\frac{d}{2} - 1$.

---

iv) This is a trivial consequence of $d = 2$ in (iii). □

This tiny Lemma seems to offer the chance of solving the Goldbach-problem by a — slightly unusual — induction that is intended to work as follows:

- Induction over the successor relation $\mathbb{S}_{\mathbb{P}}$ on primes $\mathbb{P}$, leading from any prime p to the next consecutive one, $[q\!\rightarrow] := \mathbb{S}^{\mathrm{I}}_{\mathbb{P}}[p\!\rightarrow]$.
- Assuming always that the Goldbach decompositions are known to exist for all the evens[9] $< 2 * p$
- – Proving that then all the evens $< p + q$ are Goldbach decomposable.
  – Showing afterwards — using the intermediate result just mentioned — that all the evens $< 2 * q$ are Goldbach decomposable.
- The induction is started with $p = 3$ where the assumption trivially holds.

The idea for the induction planned is to use the size restrictions developed in the preceding proposition in order to find a typing that allows a relational approach. We will namely define two zones of even numbers, the lower and the upper zone, corresponding to Lemma 10.5.1.iii. Then we define source and target, i.e. row and column type, of the relations that are involved in bringing two primes to their sum. Notational indicators are R/C for row/column and also L/U for lower/upper zone.

10.5.2 Definition. i) Following Lemma 10.5.1.i,ii, we define two **zones of even numbers**

$$\mathtt{LZ} := \mathtt{LZ}_p := \{e \mid e \text{ even} \quad \wedge \quad 2 * p \leq e \leq p + q\}$$
$$\mathtt{UZ} := \mathtt{UZ}_p := \{e \mid e \text{ even} \quad \wedge \quad p + q \leq e \leq 2 * q\},$$

ii) First for the lower zone LZ, we define source and target according to Lemma 10.5.1.i,ii

$$P_{RL} := P_{RL,p} := \{r \mid r \text{ odd prime} \ \wedge \ r_{L1} \leq r \leq r_{L2}\}$$
$$P_{CL} := P_{CL,p} := \{c \mid c \text{ odd prime} \ \wedge \ c_{L1} \leq c \leq c_{L2}\},$$

where

$$r_{L1} = 3 \qquad r_{L2} = p$$

[9] . . . and by construction also for $2 * p, p + q$ and $2 * q$

$c_{L1} = p$ $c_{L2} = g_1 =$ greatest prime below $p + q - 2$.

The thus introduced subsets of primes will need injections into the odd primes $\mathbb{P}$

$\iota_{RL} : P_{RL} \longrightarrow \mathbb{P}$ $\iota_{CL} : P_{CL} \longrightarrow \mathbb{P}$.

iii) Source and target so defined will be used for these relations

$ML_e : P_{RL} \longrightarrow P_{CL}$ $ML_e := \iota_{RL} ; \mathtt{rel}(\mathbb{A}_{\mathbb{P}} ; \iota_{\mathbb{E}_6} ; \mathbb{S}^{\mathsf{T}^e} ; z ; \mathbb{T}) ; \iota^{\mathsf{T}}_{CL}$,

that are involved in bringing two primes to the given sum $e \in \mathtt{LZ}$.

iv) For the upper zone UZ, we define source and target according to Lemma 10.5.1.i,ii

$P_{RU} := P_{RU,p} := \{r \mid r \text{ odd prime } \wedge \ r_{U1} \leq r \leq r_{U2}\}$
$P_{CU} := P_{CU,p} := \{c \mid c \text{ odd prime } \wedge \ c_{U1} \leq c \leq g_2\}$,

where

$r_{U1} = 3$ $r_{U2} = q$
$c_{U1} = q$ $g_2 =$ greatest prime below $2 * q - 2$.

The thus introduced subsets of primes will need injections into the odd primes $\mathbb{P}$

$\iota_{RU} : P_{RU} \longrightarrow \mathbb{P}$ $\iota_{CU} : P_{CU} \longrightarrow \mathbb{P}$.

v) Source and target so defined will be used for these relations

$MU_e : P_{RU} \longrightarrow P_{CU}$ $MU_e := \iota_{RU} ; \mathtt{rel}(\mathbb{A}_{\mathbb{P}} ; \iota_{\mathbb{E}_6} ; \mathbb{S}^{\mathsf{T}^e} ; z ; \mathbb{T}) ; \iota^{\mathsf{T}}_{CU}$

that are involved in bringing two primes to the given sum $e \in \mathtt{UZ}$. □

Less formally, we might express the relations intended also thus:

$$ML_e(r,c) := \begin{cases} \mathbf{1} & \text{if } r + c = e \\ \mathbf{0} & \text{if } r + c \neq e \end{cases} \qquad MU_e(r,c) := \begin{cases} \mathbf{1} & \text{if } r + c = e \\ \mathbf{0} & \text{if } r + c \neq e \end{cases}$$

We also introduce further notation for the primes according to how they are extruded. Primes r, s may in addition to the two versions

$\mathbb{S}^{\mathsf{T}^r} ; z ; \mathbb{T}$ and $[r\!\mapsto] = \iota_{\wp;\mathbb{T}} ; \mathbb{S}^{\mathsf{T}^r} ; z ; \mathbb{T}$
$\mathbb{S}^{\mathsf{T}^c} ; z ; \mathbb{T}$ and $[c\!\mapsto] = \iota_{\wp;\mathbb{T}} ; \mathbb{S}^{\mathsf{T}^c} ; z ; \mathbb{T}$

now also appear as

$[r\rangle := [r\rangle_{RL} := \iota_{RL} ; [r\!\mapsto]$ $[c\rangle := [c\rangle_{CL} := \iota_{CL} ; [c\!\mapsto]$,

$[r\rangle := [r\rangle_{RU} := \iota_{RU} ; [r\!\mapsto]$ $[c\rangle := [c\rangle_{CU} := \iota_{CU} ; [c\!\mapsto]$.

In order not to overcrowd notation, we will mostly avoid to mention p, RL, CL, RU, CU.

Between the odd primes, we now introduce distances as follows. One will easily find out that — based on the first — the next four definitions are nearly identical.

10.5.3 Definition. We define several **prime distance relations** for an even $d \geq 0$:

i) inside the coherent interval P_{RL} of $\mathbb{P}$, injected by $\iota_{RL} : P_{RL} \to \mathbb{P}$ as follows

${}^d\mathtt{D}_{RL} := \iota_{RL} ; {}^d\mathtt{D} ; \iota^{\mathsf{T}}_{RL} = \mathtt{sup}\{[r\rangle ; [r'\rangle^{\mathsf{T}} \mid r, r' \in \mathbb{N} \ \wedge \ r' - r = d\} : P_{RL} \longrightarrow P_{RL}$.

ii) inside the coherent interval P_{CL} of $\mathbb{P}$, injected by $\iota_{CL} : P_{CL} \to \mathbb{P}$ as follows

$${}^{d}\mathtt{D}_{CL} := \iota_{CL};{}^{d}\mathtt{D};\iota_{CL}^{\mathsf{T}} = \mathtt{sup}\{[c\rangle;[c'\rangle^{\mathsf{T}} \mid c,c' \in \mathbb{N} \quad \wedge \quad c'-c=d\} : P_{CL} \longrightarrow P_{CL}.$$

iii) inside the coherent interval P_{RU} of $\mathbb{P}$, injected by $\iota_{RU} : P_{RU} \to \mathbb{P}$ as follows

$${}^{d}\mathtt{D}_{RU} := \iota_{RU};{}^{d}\mathtt{D};\iota_{RU}^{\mathsf{T}} = \mathtt{sup}\{[r\rangle;[r'\rangle^{\mathsf{T}} \mid r,r' \in \mathbb{N} \quad \wedge \quad r'-r=d\} : P_{RU} \longrightarrow P_{RU}.$$

iv) inside the coherent interval P_{CU} of $\mathbb{P}$, injected by $\iota_{CU} : P_{CU} \to \mathbb{P}$ as follows

$${}^{d}\mathtt{D}_{CU} := \iota_{CU};{}^{d}\mathtt{D};\iota_{CU}^{\mathsf{T}} = \mathtt{sup}\{[c\rangle;[c'\rangle^{\mathsf{T}} \mid c,c' \in \mathbb{N} \quad \wedge \quad c'-c=d\} : P_{CU} \longrightarrow P_{CU}. \qquad \square$$

Their appearance in case of the interval $P_{RL} := [3..19] \subseteq \mathbb{P}$ is shown with Fig. 10.5.1, and correspondingly for the other prime areas P_{RL}, P_{CU}, P_{RU}.

$${}^{0}\mathtt{D}_{RL}$$

	3→	5→	7→	11→	13→	17→	19→
3→	**1**	0	0	0	0	0	0
5→	0	**1**	0	0	0	0	0
7→	0	0	**1**	0	0	0	0
11→	0	0	0	**1**	0	0	0
13→	0	0	0	0	**1**	0	0
17→	0	0	0	0	0	**1**	0
19→	0	0	0	0	0	0	**1**

${}^{2}\mathtt{D}_{RL}$

	3→	5→	7→	11→	13→	17→	19→
3→	0	**1**	0	0	0	0	0
5→	0	0	**1**	0	0	0	0
7→	0	0	0	0	0	0	0
11→	0	0	0	0	**1**	0	0
13→	0	0	0	0	0	0	0
17→	0	0	0	0	0	0	**1**
19→	0	0	0	0	0	0	0

${}^{4}\mathtt{D}_{RL}$

	3→	5→	7→	11→	13→	17→	19→
3→	0	0	**1**	0	0	0	0
5→	0	0	0	0	0	0	0
7→	0	0	0	**1**	0	0	0
11→	0	0	0	0	0	0	0
13→	0	0	0	0	0	**1**	0
17→	0	0	0	0	0	0	0
19→	0	0	0	0	0	0	0

${}^{6}\mathtt{D}_{RL}$

	3→	5→	7→	11→	13→	17→	19→
3→	0	0	0	0	0	0	0
5→	0	0	0	**1**	0	0	0
7→	0	0	0	0	**1**	0	0
11→	0	0	0	0	0	**1**	0
13→	0	0	0	0	0	0	**1**
17→	0	0	0	0	0	0	0
19→	0	0	0	0	0	0	0

${}^{8}\mathtt{D}_{RL}$

	3→	5→	7→	11→	13→	17→	19→
3→	0	0	0	**1**	0	0	0
5→	0	0	0	0	**1**	0	0
7→	0	0	0	0	0	0	0
11→	0	0	0	0	0	0	**1**
13→	0	0	0	0	0	0	0
17→	0	0	0	0	0	0	0
19→	0	0	0	0	0	0	0

${}^{10}\mathtt{D}_{RL}$

	3→	5→	7→	11→	13→	17→	19→
3→	0	0	0	0	**1**	0	0
5→	0	0	0	0	0	0	0
7→	0	0	0	0	0	**1**	0
11→	0	0	0	0	0	0	0
13→	0	0	0	0	0	0	0
17→	0	0	0	0	0	0	0
19→	0	0	0	0	0	0	0

${}^{12}\mathtt{D}_{RL}$

	3→	5→	7→	11→	13→	17→	19→
3→	0	0	0	0	0	0	0
5→	0	0	0	0	0	**1**	0
7→	0	0	0	0	0	0	**1**
11→	0	0	0	0	0	0	0
13→	0	0	0	0	0	0	0
17→	0	0	0	0	0	0	0
19→	0	0	0	0	0	0	0

${}^{14}\mathtt{D}_{RL}$

	3→	5→	7→	11→	13→	17→	19→
3→	0	0	0	0	0	**1**	0
5→	0	0	0	0	0	0	**1**
7→	0	0	0	0	0	0	0
11→	0	0	0	0	0	0	0
13→	0	0	0	0	0	0	0
17→	0	0	0	0	0	0	0
19→	0	0	0	0	0	0	0

${}^{16}\mathtt{D}_{RL}$

	3→	5→	7→	11→	13→	17→	19→
3→	0	0	0	0	0	0	**1**
5→	0	0	0	0	0	0	0
7→	0	0	0	0	0	0	0
11→	0	0	0	0	0	0	0
13→	0	0	0	0	0	0	0
17→	0	0	0	0	0	0	0
19→	0	0	0	0	0	0	0

Fig. 10.5.1 All ${}^{d}\mathtt{D}_{RL}$ exhausting E_{RL} in case $p = 19$

The result Prop. 9.6.9.iii is now transferred into this context. We define

$$E_{RL} := \iota_{RL};\mathbb{S}_{\mathbb{P}}^{*};\iota_{RL}^{\mathsf{T}} \qquad C_{RL} := \iota_{RL};\mathbb{S}_{\mathbb{P}}^{+};\iota_{RL}^{\mathsf{T}}$$

and correspondingly for $E_{CL}, C_{CL}, E_{RU}, C_{RU}, E_{CU}, C_{CU}$.

10.5.4 Proposition. $\mathtt{sup}_{d\geq 0}\,{}^{d}\mathtt{D}_{RL} = E_{RL} \qquad \mathtt{sup}_{d>0}\,{}^{d}\mathtt{D}_{RL} = C_{RL}$

and correspondingly for $E_{CL}, C_{CL}, E_{RU}, C_{RU}, E_{CU}, C_{CU}$.

Proof: The common point in all these proofs is this:

$$\mathtt{sup}_{d\geq 0}\,{}^{d}\mathtt{D}_{RL} = \mathtt{sup}_{d\geq 0}\,\iota_{RL};{}^{d}\mathtt{D};\iota^{\mathsf{T}}_{RL} = \iota_{RL};\big[\,\mathtt{sup}_{d\geq 0}\,{}^{d}\mathtt{D}\big];\iota^{\mathsf{T}}_{RL} = \iota_{RL};\mathbb{S}^{*}_{\mathbb{P}};\iota^{\mathsf{T}}_{RL} \qquad \text{Prop. 9.6.9.iii} \qquad \square$$

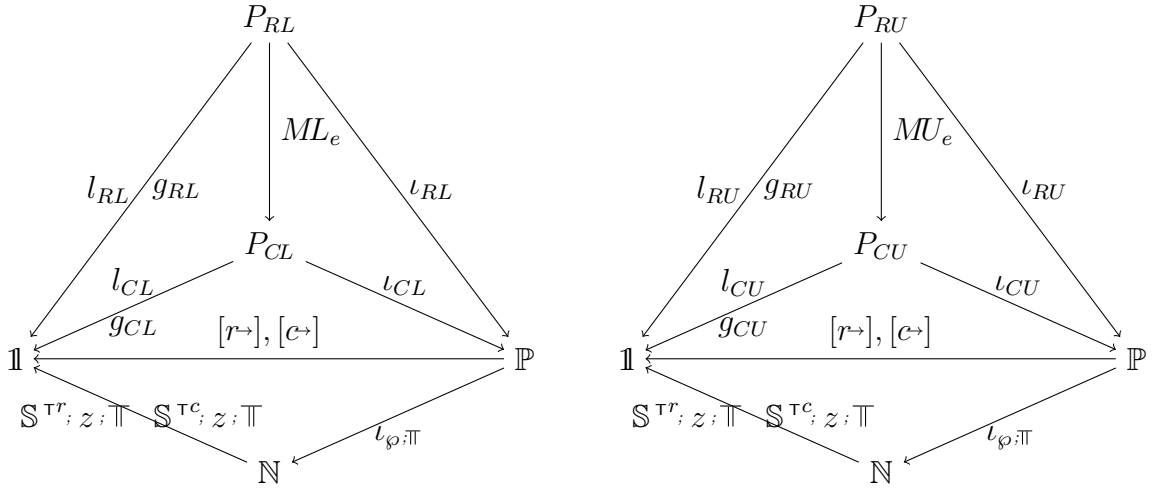


Fig. 10.5.2 Typing while defining ML_e, MU_e in an attempt to prove the Goldbach conjecture

The points $l_{RL}, g_{RL}, l_{CL}, g_{CL}, l_{RU}, g_{RU}, l_{CU}, g_{CU}$ etc. mean the least, resp. greatest of the respective set of primes.

Now follows a proposition that expresses the fact that we may proceed in this fashion, because everything will happen inside the region so restricted — or "shrink-wrapped".

10.5.5 Lemma. i) In the present context it is impossible to find two primes $r \leq c$, with sum limited

$$2 * p < r + c < p + q$$

that do **not** satisfy both, $\quad r_{L1} \leq r \leq r_{L2} = p \quad$ and $\quad p = c_{L1} \leq c \leq c_{L2}$.

ii) In the present context it is impossible to find two primes $r \leq c$, with sum limited

$$p + q < r + c < 2 * q$$

that do **not** satisfy both, $\quad r_{U1} \leq r \leq r_{U2} = q \quad$ and $\quad q = c_{U1} \leq c \leq g_2$.

Proof: i) Negating the final two statements of (i), we arrive at

$$r < r_{L1} = 3 \qquad \text{or} \quad r > r_{L2} = p \qquad \text{or} \qquad c < c_{L1} = p \qquad \text{or} \qquad c > c_{L2},$$

for which we show that any of these four leads to a contradiction.

When starting from the right-most one, we identify a prime c greater than c_{L2}, greatest prime below $p + q - 2$. As the even $p + q - 2$ is certainly not prime, this means $r + c > c > p + q - 2$ or else $r + c \geq p + q$ and, thus, provides us with the first. Having $c < c_{L1} = p$, and thus also $r \leq c < p$ leads to $r + c < 2 * p$, a second contradiction. Now assume $r > r_{L2} = p$ from which follows via consecutivity $c \geq r \geq q > p$ with consequence $r + c \geq 2 * q$, the third one. Finally, $r < r_{L1}$ would constitute an odd prime $r < r_{L1} = 3$, which is impossible.

---

ii) Negating the final two statements of (ii), we arrive at

$$r < r_{U1} = 3 \qquad \text{or} \qquad r > r_{U2} = q \qquad \text{or} \qquad c < c_{U1} = q \qquad \text{or} \qquad c > g_2,$$

for which we show that any of these four leads to a contradiction.

When starting from the right-most one, we identify a prime c greater than g_2, greatest prime below $2 * q - 2$. As $2 * q - 2$ is certainly not prime, this means $r + c > c > 2 * q - 2$, or else $r + c \geq 2 * q$ and, thus, provides us with the first. Having $c < c_{U1} = q$, and thus also $r \leq c \leq p < q$ due to consecutivity, we obtain $r + c < p + q$, a second contradiction. Now assume $r > r_{U2} = q$ from which follows $c \geq r > q$ with consequence $r + c > 2 * q$, the third one. Finally let us start with $r < r_{U1}$ meaning an odd prime $r < r_{U1} = 3$ which does not exist. □

The intention of this is not easily understood: We have managed to describe Goldbach decompositions relationally. We may, namely, define them all via relations

$$ML_e = \mathtt{sup}\{\langle r]_{\vdots}\langle c]^{\mathsf{T}} \mid r \in \mathbb{N} \quad \wedge \quad c \in \mathbb{N} \quad \wedge \quad r + c = e\}$$
$$MU_e = \mathtt{sup}\{\langle r]_{\vdots}\langle c]^{\mathsf{T}} \mid r \in \mathbb{N} \quad \wedge \quad c \in \mathbb{N} \quad \wedge \quad r + c = e\}$$

that assign to an element $e \in \mathtt{LZ}$, or $e \in \mathtt{UZ}$ respectively, the relation showing all of its Goldbach decompositions. Thus, either the Goldbach decompositions may be considered as to Lemma 10.5.1.i or as to Lemma 10.5.1.ii. The typing is then different.

We show this for the example of $p = 53$ leading to

$$\mathtt{LZ} = \{108, 110\} \qquad P_{RL} = \{3, 5, \ldots 53\} \qquad P_{CL} = \{53, 59, \ldots 109\}$$

Similarly

$$\mathtt{UZ} = \{114, 116\} \qquad P_{RU} = \{3, 5, \ldots 59\} \qquad P_{CU} = \{59, 61, \ldots 113\}$$

See Fig. 10.5.3.

$ML_{108} =$

	53	59	61	67	71	73	79	83	89	97	101	103	107	109
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	1	0	0
7	0	0	0	0	0	0	0	0	0	0	1	0	0	0
11	0	0	0	0	0	0	0	0	0	1	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	1	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0	0	0	0	0	0
29	0	0	0	0	0	0	1	0	0	0	0	0	0	0
31	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37	0	0	0	0	1	0	0	0	0	0	0	0	0	0
41	0	0	0	1	0	0	0	0	0	0	0	0	0	0
43	0	0	0	0	0	0	0	0	0	0	0	0	0	0
47	0	0	1	0	0	0	0	0	0	0	0	0	0	0
53	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$= ML_{110}$

	53	59	61	67	71	73	79	83	89	97	101	103	107	109
3	0	0	0	0	0	0	0	0	0	0	0	0	1	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0	1	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	1	0	0	0
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0	0	0	0	0	0
29	0	0	0	0	0	0	0	0	0	0	0	0	0	0
31	0	0	0	0	0	0	1	0	0	0	0	0	0	0
37	0	0	0	0	0	1	0	0	0	0	0	0	0	0
41	0	0	0	0	0	0	0	0	0	0	0	0	0	0
43	0	0	0	1	0	0	0	0	0	0	0	0	0	0
47	0	0	0	0	0	0	0	0	0	0	0	0	0	0
53	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$MU_{114} =$

	59	61	67	71	73	79	83	89	97	101	103	107	109	113
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	1	0	
7	0	0	0	0	0	0	0	0	0	0	1	0	0	
11	0	0	0	0	0	0	0	0	0	1	0	0	0	
13	0	0	0	0	0	0	0	0	1	0	0	0	0	
17	0	0	0	0	0	0	0	0	1	0	0	0	0	
19	0	0	0	0	0	0	0	0	0	0	0	0	0	
23	0	0	0	0	0	0	0	0	0	0	0	0	0	
29	0	0	0	0	0	0	0	0	0	0	0	0	0	
31	0	0	0	0	0	0	1	0	0	0	0	0	0	
37	0	0	0	0	0	0	0	0	0	0	0	0	0	
41	0	0	0	0	1	0	0	0	0	0	0	0	0	
43	0	0	0	1	0	0	0	0	0	0	0	0	0	
47	0	0	1	0	0	0	0	0	0	0	0	0	0	
53	0	1	0	0	0	0	0	0	0	0	0	0	0	
59	0	0	0	0	0	0	0	0	0	0	0	0	0	

$= MU_{116}$

	59	61	67	71	73	79	83	89	97	101	103	107	109	113
3	0	0	0	0	0	0	0	0	0	0	0	0	0	1
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0	0	1	0
11	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	1	0	0	0
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0	1	0	0	0	0
23	0	0	0	0	0	0	0	0	0	0	0	0	0	0
29	0	0	0	0	0	0	0	0	0	0	0	0	0	0
31	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37	0	0	0	0	0	1	0	0	0	0	0	0	0	0
41	0	0	0	0	0	0	0	0	0	0	0	0	0	0
43	0	0	0	0	1	0	0	0	0	0	0	0	0	0
47	0	0	0	0	0	0	0	0	0	0	0	0	0	0
53	0	0	0	0	0	0	0	0	0	0	0	0	0	0
59	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Fig. 10.5.3 Examples $ML_{108}, ML_{110}, MU_{114}, MU_{116}$

It may further be visualized — this time for the lower zone $\texttt{LZ} = [346, 352]$ only — for the prime $p = 173$ with consecutive one $q = 179$. There is $p + q = 352$ with 349 as the greatest prime below this even number. With this Lemma, we have, thus, managed that all the Goldbach decompositions of an $e \in \texttt{LZ}$ show up as $e = r + c$, restricted as shown above. So we have for any e indeed a relation $ML_e : P_{RL} \longrightarrow P_{CL}$. The following examples show ML_{346} and ML_{352}, followed by $ML_{346} \ldots ML_{352}$ for $p = 173$:

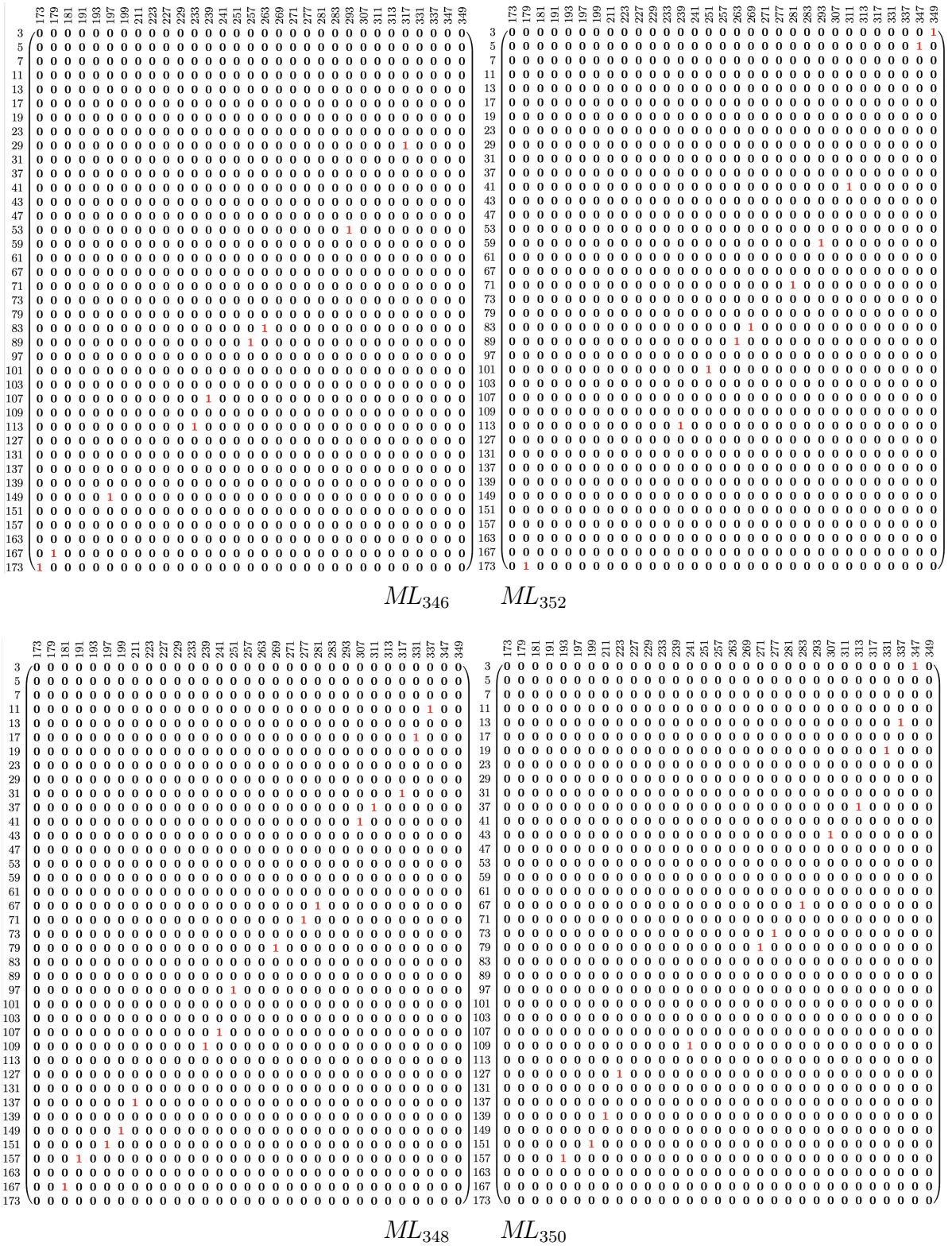


Fig. 10.5.4 Examples $ML_{346}, ML_{352}, ML_{348}, ML_{350}$

Here follows the union $\texttt{sup}_{176 \le e \le 346}\, ML_e$ over all the evens $e \le 2 * p$:

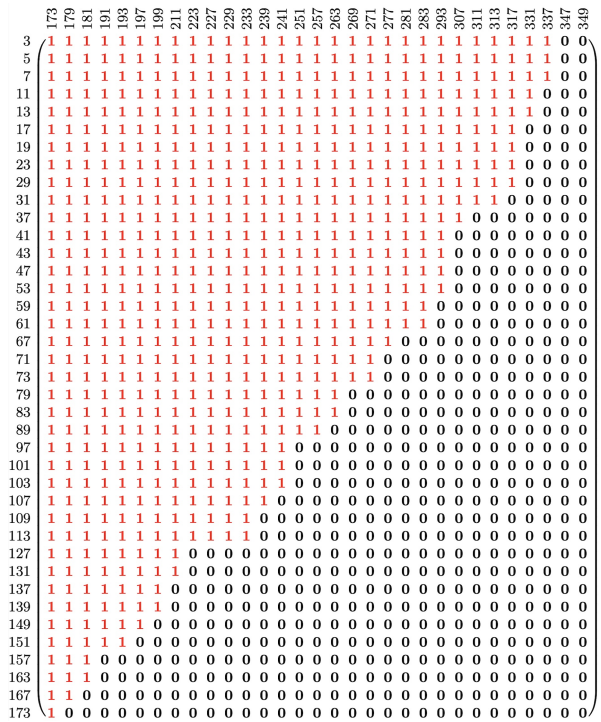


Fig. 10.5.5 $\mathtt{sup}_{176\leq e\leq 346}\, ML_e$

In total, there is a big "triangle", including the "borderline" ML_{346} already mentioned. Only between the triangle and the borderline ML_{352} mentioned above, the question of the possibility of Goldbach decompositions arises, i.e. for 348 and 350 — a rather tiny region.

At this point, we investigate self-similarity of the relations involved with the following proposition:

10.5.6 Proposition. i) Using these prime distance relations, we have for all $e \in \mathtt{LZ}$

$${}^{d}\mathtt{D}^{\mathsf{T}}_{RL:}\, ML_{e-d} \subseteq ML_e \qquad \text{as well as} \qquad ML_{e-d:}\, {}^{d}\mathtt{D}_{CL} \subseteq ML_e$$

ii) and for all $e \in \mathtt{UZ}$:

$${}^{d}\mathtt{D}^{\mathsf{T}}_{RU:}\, MU_{e-d} \subseteq MU_e \qquad \text{as well as} \qquad MU_{e-d:}\, {}^{d}\mathtt{D}_{CU} \subseteq MU_e$$

Proof: i) Predicate-logic version with proof; first for the lower zone

$$\forall a \in P_{RL}: \forall b \in P_{CL}: \big(\exists x \in P_{RL}: a_{:} x^{\mathsf{T}} \subseteq {}^{d}\mathtt{D}^{\mathsf{T}}_{RL} \ \wedge\ x_{:} b^{\mathsf{T}} \subseteq ML_{e-d}\big) \ \rightarrow\ a_{:} b^{\mathsf{T}} \subseteq ML_e$$
$$a - x = d \qquad x + b = e - d \qquad a + b = e$$

$$\forall a \in P_{RL}: \forall b \in P_{CL}: \big(\exists x \in P_{CL}: a_{:} x^{\mathsf{T}} \subseteq ML_{e-d} \ \wedge\ x_{:} b^{\mathsf{T}} \subseteq {}^{d}\mathtt{D}_{CL}\big) \ \rightarrow\ a_{:} b^{\mathsf{T}} \subseteq ML_e$$
$$a + x = e - d \qquad b - x = d \qquad a + b = e$$

ii) and now for the upper:

$$\forall a \in P_{RU}: \forall b \in P_{CU}: \big(\exists x \in P_{RU}: a_{:} x^{\mathsf{T}} \subseteq {}^{d}\mathtt{D}^{\mathsf{T}}_{RU} \ \wedge\ x_{:} b^{\mathsf{T}} \subseteq MU_{e-d}\big) \ \rightarrow\ a_{:} b^{\mathsf{T}} \subseteq MU_e$$
$$a - x = d \qquad x + b = e - d \qquad a + b = e$$

$$\forall a \in P_{RU} : \forall b \in P_{CU} : \big(\exists x \in P_{CU} : a \fatsemi x^{\intercal} \subseteq MU_{e-d} \ \wedge \ x \fatsemi b^{\intercal} \subseteq {}^{d}\mathtt{D}_{CU}\big) \ \to \ a \fatsemi b^{\intercal} \subseteq MU_e$$
$$a + x = e - d \qquad b - x = d \qquad a + b = e \qquad \square$$

The following concepts are visualized via Fig. 10.5.6. The complex term may be understood as follows: The start of $\mathbb{A}_{\mathbb{P}}$ is formed by two arbitrary odd primes to be added. This ends in $\mathbb{E}_6$, is increased arbitrarily by $\mathbb{S}^*_{\mathbb{E}_6}$, sent into $\mathbb{N}$ and accepted only if it hits the number $\mathbb{S}^{\intercal e} \fatsemi z \fatsemi \mathbb{T}$. The set of odd primes thus characterized is converted in a relation and extruded via ι_{RL}, ι_{CL} so as to lead from the lower to the greater of the two primes.

10.5.7 Lemma. We consider the odd prime p and have, thus, also available the extrusions ι_{RL} and ι_{CL}. Assuming Goldbach decompositions to be possible up to $p + p$ and considering $e \in \mathtt{LZ}$, i.e. for $p + p \leq e \leq p + q$, we define these relations:

$TL_{\leq e} := \iota_{RL} \fatsemi \mathtt{rel}(\, \mathbb{A}_{\mathbb{P}} \fatsemi \mathbb{S}^*_{\mathbb{E}_6} \fatsemi \iota_{\mathbb{E}_6} \fatsemi \mathbb{S}^{\intercal e} \fatsemi z \fatsemi \mathbb{T}) \fatsemi \iota^{\intercal}_{CL} \qquad TL_{\geq e} := \iota_{RL} \fatsemi \mathtt{rel}(\, \mathbb{A}_{\mathbb{P}} \fatsemi \mathbb{S}^{\intercal}_{\mathbb{E}_6}{}^* \fatsemi \iota_{\mathbb{E}_6} \fatsemi \mathbb{S}^{\intercal e} \fatsemi z \fatsemi \mathbb{T}) \fatsemi \iota^{\intercal}_{CL}$

and prove the following results:

i) $TL_{\leq e} = \mathtt{sup}_{f \leq e} \, ML_f$ as a disjoint union

ii) $TL_{\leq e} \neq \perp\!\!\!\perp \qquad TL_{\geq e} \neq \perp\!\!\!\perp$

iii) $ML_e = TL_{\leq e} \cap TL_{\geq e}$

Proof: i) We start with

$\mathtt{sup}_{f \leq e} \, ML_f = \mathtt{sup}_{f \leq e} \, \iota_{RL} \fatsemi \mathtt{rel}(\, \mathbb{A}_{\mathbb{P}} \fatsemi \iota_{\mathbb{E}_6} \fatsemi \mathbb{S}^{\intercal f} \fatsemi z \fatsemi \mathbb{T}) \fatsemi \iota^{\intercal}_{CL}$ Def. 10.5.2.iii
$= \iota_{RL} \fatsemi \Big[\mathtt{sup}_{f \leq e} \, \mathtt{rel}(\, \mathbb{A}_{\mathbb{P}} \fatsemi \iota_{\mathbb{E}_6} \fatsemi \mathbb{S}^{\intercal f} \fatsemi z \fatsemi \mathbb{T}) \Big] \fatsemi \iota^{\intercal}_{CL}$
$= \iota_{RL} \fatsemi \mathtt{rel}(\mathtt{sup}_{f \leq e} \, \mathbb{A}_{\mathbb{P}} \fatsemi \iota_{\mathbb{E}_6} \fatsemi \mathbb{S}^{\intercal f} \fatsemi z \fatsemi \mathbb{T}) \fatsemi \iota^{\intercal}_{CL}$ since the transition $\mathtt{rel}$ is additive; page 15
$= \iota_{RL} \fatsemi \mathtt{rel}(\, \mathbb{A}_{\mathbb{P}} \fatsemi \iota_{\mathbb{E}_6} \fatsemi \mathtt{sup}_{f \leq e} \, \mathbb{S}^{\intercal f} \fatsemi z \fatsemi \mathbb{T}) \fatsemi \iota^{\intercal}_{CL} = \iota_{RL} \fatsemi \mathtt{rel}(\, \mathbb{A}_{\mathbb{P}} \fatsemi \iota_{\mathbb{E}_6} \fatsemi \big[\mathtt{sup}_{f \leq e} \, \mathbb{S}^{\intercal f} \big] \fatsemi z \fatsemi \mathbb{T}) \fatsemi \iota^{\intercal}_{CL}$
$= \iota_{RL} \fatsemi \mathtt{rel}(\, \mathbb{A}_{\mathbb{P}} \fatsemi \iota_{\mathbb{E}_6} \fatsemi \mathbb{S}^* \fatsemi \mathbb{S}^{\intercal e} \fatsemi z \fatsemi \mathbb{T}) \fatsemi \iota^{\intercal}_{CL}$
$= \iota_{RL} \fatsemi \mathtt{rel}(\, \mathbb{A}_{\mathbb{P}} \fatsemi \mathbb{S}^*_{\mathbb{E}_6} \fatsemi \iota_{\mathbb{E}_6} \fatsemi \mathbb{S}^{\intercal e} \fatsemi z \fatsemi \mathbb{T}) \fatsemi \iota^{\intercal}_{CL}$
$= TL_{\leq e}$

In the last major step it has been used that e is an even number in

$\iota_{\mathbb{E}_6} \fatsemi \mathbb{S}^* \fatsemi \mathbb{S}^{\intercal e} \fatsemi z \fatsemi \mathbb{T} = \iota_{\mathbb{E}_6} \fatsemi \big[(\mathbb{S}^2)^* \cup \mathtt{sup}_{i \geq 0} \, \mathbb{S}^{2i+1} \big] \fatsemi \mathbb{S}^{\intercal e} \fatsemi z \fatsemi \mathbb{T}$
$= \iota_{\mathbb{E}_6} \fatsemi (\mathbb{S}^2)^* \fatsemi \mathbb{S}^{\intercal e} \fatsemi z \fatsemi \mathbb{T} \cup \iota_{\mathbb{E}_6} \fatsemi \mathtt{sup}_{i \geq 0} \, \mathbb{S}^{2i+1} \fatsemi \mathbb{S}^{\intercal e} \fatsemi z \fatsemi \mathbb{T}$
$= \iota_{\mathbb{E}_6} \fatsemi (\mathbb{S}^2)^* \fatsemi \mathbb{S}^{\intercal e} \fatsemi z \fatsemi \mathbb{T} \cup \perp\!\!\!\perp$ since $\iota_{\mathbb{E}_6}$ always ends even and ≥ 6 and e is even
$= \iota_{\mathbb{E}_6} \fatsemi \mathtt{sup}_{m \geq 0} (\mathbb{S}^2)^m \fatsemi \mathbb{S}^{\intercal e} \fatsemi z \fatsemi \mathbb{T}$
$= \mathtt{sup}_{m \geq 0} \, \mathbb{S}^m_{\mathbb{E}_6} \fatsemi \iota_{\mathbb{E}_6} \fatsemi \mathbb{S}^{\intercal e} \fatsemi z \fatsemi \mathbb{T}$ due to Prop. 10.1.3.viii
$= \mathbb{S}^*_{\mathbb{E}_6} \fatsemi \iota_{\mathbb{E}_6} \fatsemi \mathbb{S}^{\intercal e} \fatsemi z \fatsemi \mathbb{T}$

Disjointness $ML_{e_1} \cap ML_{e_2} = \perp\!\!\!\perp$ is obvious in case $e_1 \neq e_2$.

---

ii) For $e \in \mathtt{LZ}$ we have certainly $\perp\!\!\!\perp \neq ML_{p+p} \subseteq TL_{\leq e}$, so that, thus, in a corresponding way $TL_{\geq e} \supseteq ML_{p+q} \neq \perp\!\!\!\perp$.

---

iii) is trivial. $\square$

We recall here the relational standard definitions[10] of concepts for an ordering $E : X \longrightarrow X$ in connection with a set U, namely upper bounds, greatest elements, as well as lower bounds and least elements:

$$\mathtt{ubd}_E(U) = \overline{\overline{E}^{\mathsf{T}};U} \qquad \mathtt{gre}_E(U) = U \cap \mathtt{ubd}_E(U)$$
$$\mathtt{lbd}_E(U) = \overline{\overline{E};U} \qquad \mathtt{lea}_E(U) = U \cap \mathtt{lbd}_E(U)$$

So far it has been assumed that the constructs are taken column-wise. Should they be used row-wise, the notations are correspondingly $\mathtt{ubdR}, \mathtt{greR}, \mathtt{lbdR}, \mathtt{leaR}$.

This shall now be applied to RL, CL:

$$\mathtt{gre}_{CL}(TL_{\leq e}) = TL_{\leq e} \cap \overline{\overline{E_{CL}}^{\mathsf{T}};TL_{\leq e}} \qquad \mathtt{lea}_{CL}(TL_{\geq e}) = TL_{\geq e} \cap \overline{\overline{E_{CL}};TL_{\geq e}}$$

The situation is visualized for $p = 53$ — with $q = 59$, $p+p = 106$, $p+q = 112$ and $q+q = 118$:

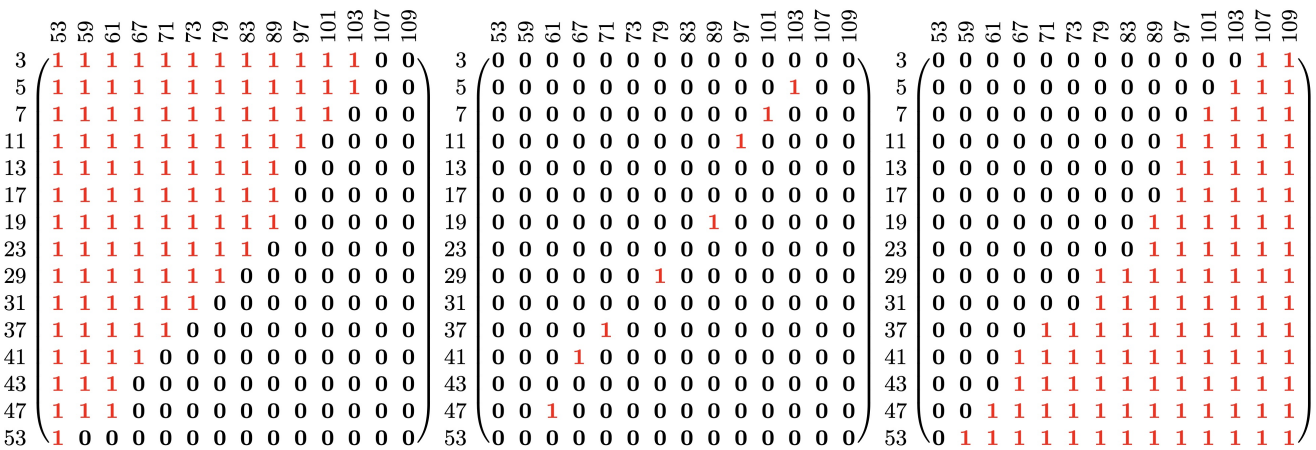


Fig. 10.5.6 $TL_{\leq 108}$ $\mathtt{greR}(TL_{\leq 108}) \supsetneqq ML_{108} \subsetneqq \mathtt{leaR}(TL_{\geq 108})$ $TL_{\geq 108}$

Obviously $TL_{\leq 108} \cap TL_{\geq 108} = ML_{108}$ and probably

$$ML_{108} = \mathtt{greR}(TL_{\leq 108}) \cap \mathtt{leaR}(TL_{\geq 108})$$

The Goldbach problem amounts to the question whether $ML_e \neq \perp\!\!\!\perp$ for all evens e. So we are going to investigate properties of evens e for which this condition fails, i.e. when $ML_e = \perp\!\!\!\perp$. To this end, we make use of the limitations developed, namely that either $e \in \mathtt{LZ}$ or $e \in \mathtt{UZ}$ for some odd prime p. We finitely aim at proving this result: There can be no first/smallest such even number e.

[10] See [11, Def. 3.3.2, Def. 3.3.5], [12, Def. 3.3.2, Def. 3.3.5] and [9, v, Def. 9.6]

10.5.8 Lemma. Let e be the smallest even number $e \in \mathtt{LZ}$ with the property that the relation $ML_e : P_{RL} \longrightarrow P_{CL}$ satisfies $ML_e = \perp\!\!\!\perp$. This will lead to a contradiction.

Proof: We start from the Prop. 10.5.6.i, and recall that for all $d \geq 0$

$${}^{d}\mathtt{D}^{\intercal}_{RL} ; ML_{e-d} \subseteq ML_e \qquad \textbf{and} \qquad ML_{e-d} ; {}^{d}\mathtt{D}_{CL} \subseteq ML_e .$$

By relation-algebraic computation via the Schröder rule, we obtain as equivalent conditions that always

$$ML_{e-d} \subseteq \overline{{}^{d}\mathtt{D}_{RL} ; \overline{ML_e}} \qquad \textbf{and} \qquad ML_{e-d} \subseteq \overline{\overline{ML_e} ; {}^{d}\mathtt{D}^{\intercal}_{CL}} .$$

In the situation assumed, namely $ML_e = \perp\!\!\!\perp$ and e minimal, this would mean that

$$ML_{e-d} \subseteq \overline{{}^{d}\mathtt{D}_{RL} ; \mathbb{T}} \qquad \textbf{and} \qquad ML_{e-d} \subseteq \overline{\mathbb{T} ; {}^{d}\mathtt{D}^{\intercal}_{CL}}$$

and $ML_{e-d} \neq \perp\!\!\!\perp$ for an arbitrary $d > 0$. So there will exist for every such d an entry

$$[r_d\rangle ; [c_d\rangle^{\intercal} \subseteq ML_{e-d},$$

expressed by two points of the respective types, namely $[r_d\rangle : P_{RL} \longrightarrow \mathbb{1}$ and $[c_d\rangle : P_{CL} \longrightarrow \mathbb{1}$. Thus, always will hold

$$[r_d\rangle ; [c_d\rangle^{\intercal} \subseteq \overline{{}^{d}\mathtt{D}_{RL} ; \mathbb{T}} \qquad \textbf{and} \qquad [r_d\rangle ; [c_d\rangle^{\intercal} \subseteq \overline{\mathbb{T} ; {}^{d}\mathtt{D}^{\intercal}_{CL}} \qquad \textbf{and} \qquad r_d + c_d = e - d.$$

Applying the Schröder equivalence again, this ends for all $d > 0$ in

$${}^{d}\mathtt{D}_{RL} ; \mathbb{T} ; [c_d\rangle = {}^{d}\mathtt{D}_{RL} ; \mathbb{T} \subseteq \overline{[r_d\rangle} \qquad \textbf{and} \qquad [r_d\rangle^{\intercal} ; \mathbb{T} ; {}^{d}\mathtt{D}^{\intercal}_{CL} = \mathbb{T} ; {}^{d}\mathtt{D}^{\intercal}_{CL} \subseteq \overline{[c_d\rangle}^{\intercal} .$$

It is indicated in which way we get rid of the respective second prime, because for an arbitrary point $\perp\!\!\!\perp \neq x = x ; \mathbb{T}$ always $\mathbb{T} ; x ; \mathbb{T} = \mathbb{T}$; see Def. 2.2.1. We have, thus, again via the Schröder equivalence

$$\forall d > 0 : \ {}^{d}\mathtt{D}^{\intercal}_{RL} ; [r_d\rangle \subseteq \perp\!\!\!\perp \qquad \textbf{and} \qquad [c_d\rangle^{\intercal} ; {}^{d}\mathtt{D}_{CL} \subseteq \perp\!\!\!\perp \qquad \textbf{and} \qquad r_d + c_d = e - d.$$

Proof so far ok, but still unfinished — finishing means: proof of the Goldbach conjecture. □

10.6 Selfsimilarity expressed via $\mathcal{P}$

As may be observed in Fig. 10.4.3, the following result is remarkably evident. For the proof, one has to work on both sides of a pattern at the same time. For, e.g. $\mathcal{P}_7$, one has to lower 18 to 22 plus shifting left by 1 in order to cut off the first column and arrive at the pattern $\mathcal{P}_{11}$.

The following is accompanied by analogical Props. 10.7.1 and 10.8.2 for $\mathcal{Q} < \mathcal{R}$.

10.6.1 Proposition. Given two *consecutive* odd primes p, q, i.e. two points satisfying

$$\mathbb{S}^{\intercal p} ; z ; \mathbb{T} = \mathtt{NP} ; \mathbb{S}^{\intercal q} ; z ; \mathbb{T} \text{ , or equivalently } \ \mathbb{S}^{\intercal}_{\mathbb{P}} ; [p^{\rightarrow}] = [q^{\rightarrow}],$$

we have, abbreviating $d := f'(p)$[11], always

i) $\mathcal{P}_q = \mathbb{S}^{\intercal}_{\mathbb{E}_6} ; {}^{d}\mathcal{P}_p ; \iota^{\intercal}_{qp}$.

ii) $\mathbb{S}^{d}_{\mathbb{E}_6} ; \mathcal{P}_q ; \iota_{qp} \subseteq \mathcal{P}_p$

Proof: i) A proof attempt using the points

$$u : \mathbb{E}_6 \longrightarrow \mathbb{1} \quad \text{and} \quad w : (\mathbb{P} \times \mathbb{P})_{\leq q} \longrightarrow \mathbb{1}$$

[11] see Def. 9.4.7

means the goal to show

$$u ; w^\intercal \subseteq \mathcal{P}_q \qquad \Longleftrightarrow \qquad u ; w^\intercal \subseteq \mathbb{S}_{\mathbb{E}_6}^\intercal{}^{d} ; \mathcal{P}_p ; \iota_{qp}^\intercal .$$

Every point $u : \mathbb{E}_6 \longrightarrow \mathbb{1}$ has the obvious representation $u = \mathbb{S}_{\mathbb{E}_6}^\intercal{}^{a} ; z_{\mathbb{E}_6} ; \mathbb{T}$. It is more difficult to find a representation for an arbitrary point

$$\begin{array}{ll} & w : (\mathbb{P}\times\mathbb{P})_{\leq q} \longrightarrow \mathbb{1} \\ \Longleftrightarrow & \iota_{\times q}^\intercal ; w : (\mathbb{P}\times\mathbb{P})_{\leq} \longrightarrow \mathbb{1} \\ \Longleftrightarrow & \iota_{\times}^\intercal ; \iota_{\times q}^\intercal ; w : \mathbb{P}\times\mathbb{P} \longrightarrow \mathbb{1} \end{array}$$

In general, a point in $\mathbb{P}\times\mathbb{P}$ has the representation $([b\rightarrow] \oslash [c\rightarrow])$ with $[b\rightarrow], [c\rightarrow]$ points in $\mathbb{P}$, or else $b, c \in \mathbb{N}$ odd primes. Now the following construct is obviously a point and satisfies

$$\begin{array}{l} \iota_{\times}^\intercal ; \iota_{\times q}^\intercal ; w = \iota_{\times}^\intercal ; \iota_{\times q}^\intercal ; \iota_{\times q} ; \iota_{\times q}^\intercal ; w \quad \text{since } \iota_{\times q} \text{ is an injective map} \\ = \iota_{\times}^\intercal ; (\mathbb{I} \cap v_{\times q}) ; \iota_{\times q}^\intercal ; w = \iota_{\times}^\intercal ; (\mathbb{I} \cap \iota_{\times} ; v_q) ; \iota_{\times q}^\intercal ; w \quad \text{Def. 10.3.8} \\ = (\iota_{\times}^\intercal \cap v_q) ; \iota_{\times q}^\intercal ; w \quad \text{destroy and append} \\ = (\iota_{\times}^\intercal \cap \pi_{\mathbb{P}} ; [q\rightarrow]) ; \iota_{\times q}^\intercal ; w = \iota_{\times}^\intercal ; \iota_{\times q}^\intercal ; w \cap \pi_{\mathbb{P}} ; [q\rightarrow] \quad \text{expanded; masking} \end{array}$$

so that necessarily b turns out to be equal to q. As a consequence we have

$$\iota_{\times}^\intercal ; \iota_{\times q}^\intercal ; w = ([q\rightarrow] \oslash [c\rightarrow]) \quad \text{with } q \leq c \text{ and also } c \text{ a prime,}$$

and therefore work from now on with the point

$$w = \iota_{\times q} ; \iota_{\times} ; ([q\rightarrow] \oslash [c\rightarrow]) .$$

Our goal has, thus, been modified to proving

$$\begin{array}{ll} & u ; w^\intercal = \mathbb{S}_{\mathbb{E}_6}^\intercal{}^{a} ; z_{\mathbb{E}_6} ; \mathbb{T} ; ([q\rightarrow]^\intercal \oslash [c\rightarrow]^\intercal) ; \iota_{\times}^\intercal ; \iota_{\times q}^\intercal \subseteq \mathcal{P}_q \\ \Longleftrightarrow & \mathbb{S}_{\mathbb{E}_6}^\intercal{}^{a} ; z_{\mathbb{E}_6} ; \mathbb{T} ; ([q\rightarrow]^\intercal \oslash [c\rightarrow]^\intercal) ; \iota_{\times}^\intercal ; \iota_{\times q}^\intercal \subseteq \mathbb{S}_{\mathbb{E}_6}^\intercal{}^{d} ; \mathcal{P}_p ; \iota_{qp}^\intercal . \end{array}$$

We follow equivalences starting from the latter containment:

$$\begin{array}{ll} \Longleftrightarrow & \mathbb{S}_{\mathbb{E}_6}^\intercal{}^{a} ; z_{\mathbb{E}_6} ; \mathbb{T} ; ([q\rightarrow]^\intercal \oslash [c\rightarrow]^\intercal) ; \iota_{\times}^\intercal ; \iota_{\times q}^\intercal ; \iota_{qp} \subseteq \mathbb{S}_{\mathbb{E}_6}^\intercal{}^{d} ; \mathcal{P}_p \quad \text{with } \iota_{qp} \text{ a map} \\ \Longleftrightarrow & ([q\rightarrow]^\intercal \oslash [c\rightarrow]^\intercal) ; \iota_{\times}^\intercal ; \iota_{\times q}^\intercal ; \iota_{qp} \subseteq \mathbb{T} ; z_{\mathbb{E}_6}^\intercal ; \mathbb{S}_{\mathbb{E}_6}^{a} ; \mathbb{S}_{\mathbb{E}_6}^\intercal{}^{d} ; \mathcal{P}_p \quad \text{with } \mathbb{S}_{\mathbb{E}_6}^\intercal{}^{a} ; z_{\mathbb{E}_6} ; \mathbb{T} \text{ a point} \\ \Longleftrightarrow & ([q\rightarrow]^\intercal \oslash [c\rightarrow]^\intercal) ; \iota_{\times}^\intercal ; \iota_{\times q}^\intercal ; \iota_{qp} \subseteq \mathbb{T} ; z_{\mathbb{E}_6}^\intercal ; \mathbb{S}_{\mathbb{E}_6}^{a} ; \mathbb{S}_{\mathbb{E}_6}^\intercal{}^{d} ; \mathcal{P} ; \iota_{\times p}^\intercal \quad \text{expanding } \mathcal{P}_p \\ \Longleftrightarrow & ([q\rightarrow]^\intercal \oslash [c\rightarrow]^\intercal) ; \iota_{\times}^\intercal ; \iota_{\times q}^\intercal ; \iota_{qp} ; \iota_{\times p} \subseteq \mathbb{T} ; z_{\mathbb{E}_6}^\intercal ; \mathbb{S}_{\mathbb{E}_6}^{a} ; \mathbb{S}_{\mathbb{E}_6}^\intercal{}^{d} ; \mathcal{P} \quad \text{since } \iota_{\times p} \text{ is a map} \\ \Longleftrightarrow & ([q\rightarrow]^\intercal \oslash [c\rightarrow]^\intercal) ; \iota_{\times}^\intercal ; \iota_{\times q}^\intercal ; \iota_{qp} ; \iota_{\times p} \subseteq \mathbb{T} ; z_{\mathbb{E}_6}^\intercal ; \mathbb{S}_{\mathbb{E}_6}^{a} ; \mathbb{S}_{\mathbb{E}_6}^\intercal{}^{d} ; \mathbb{A}_{\mathbb{P}}^\intercal ; \iota_{\times}^\intercal \quad \text{expanding } \mathcal{P} \\ \Longleftrightarrow & ([q\rightarrow]^\intercal \oslash [c\rightarrow]^\intercal) ; \iota_{\times}^\intercal ; \iota_{\times q}^\intercal ; \iota_{qp} ; \iota_{\times p} ; \iota_{\times} \subseteq \mathbb{T} ; z_{\mathbb{E}_6}^\intercal ; \mathbb{S}_{\mathbb{E}_6}^{a} ; \mathbb{S}_{\mathbb{E}_6}^\intercal{}^{d} ; \mathbb{A}_{\mathbb{P}}^\intercal \quad \text{with } \iota_{\times} \text{ a map} \qquad (*) \end{array}$$

Now we study the most intimidating part of (*) separately:

$$\begin{array}{l} \iota_{\times}^\intercal ; \iota_{\times q}^\intercal ; \iota_{qp} ; \iota_{\times p} ; \iota_{\times} = \iota_{\times}^\intercal ; \iota_{\times q}^\intercal ; \iota_{\times q} ; \mathfrak{F}_{\times}^\intercal ; \iota_{\times p}^\intercal ; \iota_{\times p} ; \iota_{\times} \quad \text{expanded according to Def. 10.3.8} \\ = \iota_{\times}^\intercal ; (\mathbb{I} \cap v_{\times q}) ; \mathfrak{F}_{\times}^\intercal ; (\mathbb{I} \cap v_{\times p}^\intercal) ; \iota_{\times} = \iota_{\times}^\intercal ; (\mathbb{I} \cap \iota_{\times} ; v_q) ; \mathfrak{F}_{\times}^\intercal ; (\mathbb{I} \cap v_p^\intercal ; \iota_{\times}^\intercal) ; \iota_{\times} \quad \text{by definition} \\ = (\iota_{\times}^\intercal \cap v_q) ; \mathfrak{F}_{\times}^\intercal ; (\iota_{\times} \cap v_p^\intercal) \quad \text{destroy and append twice} \\ = (\iota_{\times}^\intercal \cap v_q) ; \iota_{\times} ; (\mathbb{S}_{\mathbb{P}}^\intercal \otimes \mathbb{I}_{\mathbb{P}}) ; \iota_{\times}^\intercal ; (\iota_{\times} \cap v_p^\intercal) \quad \text{Def. 10.3.2} \\ = (\mathbb{I} \cap v_q ; \mathbb{T} ; \iota_{\times}) ; (\mathbb{S}_{\mathbb{P}}^\intercal \otimes \mathbb{I}_{\mathbb{P}}) ; (\mathbb{I} \cap \iota_{\times}^\intercal ; \mathbb{T} ; v_p^\intercal) \quad \text{destroy and append twice} \\ = (\mathbb{I} \cap v_q ; \mathbb{T} \cap \mathbb{T} ; \iota_{\times}) ; (\mathbb{S}_{\mathbb{P}}^\intercal \otimes \mathbb{I}_{\mathbb{P}}) ; (\mathbb{I} \cap \iota_{\times}^\intercal ; \mathbb{T} \cap \mathbb{T} ; v_p^\intercal) \quad \text{masking} \\ = (\mathbb{I} \cap \mathbb{T} ; \iota_{\times}) ; (\mathbb{I} \cap v_q ; \mathbb{T}) ; (\mathbb{S}_{\mathbb{P}}^\intercal \otimes \mathbb{I}_{\mathbb{P}}) ; (\mathbb{I} \cap \mathbb{T} ; v_p^\intercal) ; (\mathbb{I} \cap \iota_{\times}^\intercal ; \mathbb{T}) \quad \text{partial identities} \\ = \iota_{\times}^\intercal ; \iota_{\times} ; (\mathbb{I}_{\mathbb{P}\times\mathbb{P}} \cap \pi_{\mathbb{P}} ; [q\rightarrow]) ; (\mathbb{S}_{\mathbb{P}}^\intercal \otimes \mathbb{I}_{\mathbb{P}}) ; (\mathbb{I}_{\mathbb{P}\times\mathbb{P}} \cap [p\rightarrow]^\intercal ; \pi_{\mathbb{P}}^\intercal) ; \iota_{\times}^\intercal ; \iota_{\times} \\ = \iota_{\times}^\intercal ; \iota_{\times} ; (\mathbb{S}_{\mathbb{P}}^\intercal \cap [q\rightarrow] \cap [p\rightarrow]^\intercal \otimes \mathbb{I}_{\mathbb{P}}) ; \iota_{\times}^\intercal ; \iota_{\times} \\ = \iota_{\times}^\intercal ; \iota_{\times} ; ([q\rightarrow] \cap [p\rightarrow]^\intercal \otimes \mathbb{I}_{\mathbb{P}}) ; \iota_{\times}^\intercal ; \iota_{\times} \quad \text{since } [q\rightarrow] ; [p\rightarrow]^\intercal \subseteq \mathbb{S}_{\mathbb{P}}^\intercal \text{, because } p < q \text{ are consecutive} \\ = (\mathbb{I} \cap \iota_{\times}^\intercal ; \mathbb{T}) ; ([q\rightarrow] ; [p\rightarrow]^\intercal \otimes \mathbb{I}_{\mathbb{P}}) ; (\mathbb{I} \cap \mathbb{T} ; \iota_{\times}) \quad \text{masking} \\ = ([q\rightarrow] ; [p\rightarrow]^\intercal \otimes \mathbb{I}_{\mathbb{P}}) \cap \mathbb{T} ; \iota_{\times} \cap \iota_{\times}^\intercal ; \mathbb{T} \quad \text{masking again} \end{array}$$

We may, thus, continue the modified (*) with

$$
\begin{aligned}
&([q^{\rightarrow}]^{\intercal} \oslash [c^{\rightarrow}]^{\intercal}) ; \Big[([q^{\rightarrow}] ; [p^{\rightarrow}]^{\intercal} \otimes \mathbb{I}_{\mathbb{P}}) \cap \mathbb{T} ; \iota_{\times} \cap \iota_{\times}^{\intercal} ; \mathbb{T} \Big] \\
&= ([q^{\rightarrow}]^{\intercal} \oslash [c^{\rightarrow}]^{\intercal}) ; \Big[([q^{\rightarrow}] ; [p^{\rightarrow}]^{\intercal} \otimes \mathbb{I}_{\mathbb{P}}) \cap \iota_{\times}^{\intercal} ; \mathbb{T} \Big] \cap \mathbb{T} ; \iota_{\times} \quad \text{masking} \\
&= ([q^{\rightarrow}]^{\intercal} \oslash [c^{\rightarrow}]^{\intercal}) ; ([q^{\rightarrow}] ; [p^{\rightarrow}]^{\intercal} \otimes \mathbb{I}_{\mathbb{P}}) \cap ([q^{\rightarrow}]^{\intercal} \oslash [c^{\rightarrow}]^{\intercal}) ; \iota_{\times}^{\intercal} ; \mathbb{T} \cap \mathbb{T} ; \iota_{\times} \\
&= ([q^{\rightarrow}]^{\intercal} ; [q^{\rightarrow}] ; [p^{\rightarrow}]^{\intercal} \oslash [c^{\rightarrow}]^{\intercal}) \cap ([q^{\rightarrow}]^{\intercal} \oslash [c^{\rightarrow}]^{\intercal}) ; \iota_{\times}^{\intercal} ; \mathbb{T} \cap \mathbb{T} ; \iota_{\times} \\
&= ([p^{\rightarrow}]^{\intercal} \oslash [c^{\rightarrow}]^{\intercal}) \cap ([q^{\rightarrow}]^{\intercal} \oslash [c^{\rightarrow}]^{\intercal}) ; \iota_{\times}^{\intercal} ; \mathbb{T} \cap \mathbb{T} ; \iota_{\times} \\
&= ([p^{\rightarrow}]^{\intercal} \oslash [c^{\rightarrow}]^{\intercal}) \cap \mathbb{T} ; \iota_{\times} \quad \text{since } q \leq c \\
&= ([p^{\rightarrow}]^{\intercal} \oslash [c^{\rightarrow}]^{\intercal}) \subseteq \mathbb{T} ; z_{\mathbb{E}_6}^{\intercal} ; \mathbb{S}_{\mathbb{E}_6}^{a} ; \mathbb{S}_{\mathbb{E}_6}^{\intercal\, d} ; \mathbb{A}_{\mathbb{P}}^{\intercal} \quad \text{since } p < q \leq c \\
&\iff \quad \mathbb{I} \subseteq \mathbb{T} ; z_{\mathbb{E}_6}^{\intercal} ; \mathbb{S}_{\mathbb{E}_6}^{a} ; \mathbb{S}_{\mathbb{E}_6}^{\intercal\, d} ; \mathbb{A}_{\mathbb{P}}^{\intercal} ; ([p^{\rightarrow}] \oslash [c^{\rightarrow}]) \quad \text{because } ([p^{\rightarrow}] \oslash [c^{\rightarrow}]) \text{ is a point} \\
&\qquad = \mathbb{T} ; z_{\mathbb{E}_6}^{\intercal} ; \mathbb{S}_{\mathbb{E}_6}^{a} ; \mathbb{S}_{\mathbb{E}_6}^{\intercal\, d} ; \mathbb{S}_{\mathbb{E}_6}^{\intercal\, \frac{p+c-6}{2}} ; z_{\mathbb{E}_6} ; \mathbb{T} \quad \text{Prop. 10.2.2.i} \\
&\iff \quad \mathbb{S}_{\mathbb{E}_6}^{a} ; z_{\mathbb{E}_6} ; \mathbb{T} \subseteq \mathbb{S}_{\mathbb{E}_6}^{\intercal\, d} ; \mathbb{S}_{\mathbb{E}_6}^{\intercal\, \frac{p+c-6}{2}} ; z_{\mathbb{E}_6} ; \mathbb{T} \quad \text{shunting a point} \\
&\iff \quad \mathbb{S}_{\mathbb{E}_6}^{a} ; z_{\mathbb{E}_6} ; \mathbb{T} \subseteq \mathbb{S}_{\mathbb{E}_6}^{\intercal\, d + \frac{p+c-6}{2}} ; z_{\mathbb{E}_6} ; \mathbb{T}
\end{aligned}
$$

Containment of one point in another means, thus, $a = d + \frac{p+c-6}{2}$.

Now the other containment:

$$
\begin{aligned}
&u ; w^{\intercal} = \mathbb{S}_{\mathbb{E}_6}^{\intercal\, a} ; z_{\mathbb{E}_6} ; \mathbb{T} ; ([q^{\rightarrow}]^{\intercal} \oslash [c^{\rightarrow}]^{\intercal}) ; \iota_{\times}^{\intercal} ; \iota_{\times q}^{\intercal} \subseteq \mathcal{P}_q \\
&\iff \quad \mathbb{S}_{\mathbb{E}_6}^{\intercal\, a} ; z_{\mathbb{E}_6} ; \mathbb{T} ; ([q^{\rightarrow}]^{\intercal} \oslash [c^{\rightarrow}]^{\intercal}) ; \iota_{\times}^{\intercal} ; \iota_{\times q}^{\intercal} \subseteq \mathcal{P} ; \iota_{\times q}^{\intercal} \\
&\iff \quad \mathbb{S}_{\mathbb{E}_6}^{\intercal\, a} ; z_{\mathbb{E}_6} ; \mathbb{T} ; ([q^{\rightarrow}]^{\intercal} \oslash [c^{\rightarrow}]^{\intercal}) ; \iota_{\times}^{\intercal} ; \iota_{\times q}^{\intercal} ; \iota_{\times q} \subseteq \mathcal{P} \\
&\iff \quad \mathbb{S}_{\mathbb{E}_6}^{\intercal\, a} ; z_{\mathbb{E}_6} ; \mathbb{T} ; ([q^{\rightarrow}]^{\intercal} \oslash [c^{\rightarrow}]^{\intercal}) ; \iota_{\times}^{\intercal} ; \iota_{\times q}^{\intercal} ; \iota_{\times q} \subseteq \mathbb{A}_{\mathbb{P}}^{\intercal} ; \iota_{\times}^{\intercal} \\
&\iff \quad \mathbb{S}_{\mathbb{E}_6}^{\intercal\, a} ; z_{\mathbb{E}_6} ; \mathbb{T} ; ([q^{\rightarrow}]^{\intercal} \oslash [c^{\rightarrow}]^{\intercal}) ; \iota_{\times}^{\intercal} ; \iota_{\times q}^{\intercal} ; \iota_{\times q} ; \iota_{\times} \subseteq \mathbb{A}_{\mathbb{P}}^{\intercal}
\end{aligned}
$$

Again we study the corresponding part separately

$$
\begin{aligned}
&\iota_{\times}^{\intercal} ; \iota_{\times q}^{\intercal} ; \iota_{\times q} ; \iota_{\times} \\
&= \iota_{\times}^{\intercal} ; (\mathbb{I} \cap v_{\times q}) ; \iota_{\times} = \iota_{\times}^{\intercal} ; (\mathbb{I} \cap \iota_{\times} ; v_q) ; \iota_{\times} \quad \text{by definition} \\
&= (\iota_{\times}^{\intercal} \cap v_q) ; \iota_{\times} \quad \text{destroy and append} \\
&= \mathbb{I} \cap v_q ; \mathbb{T} ; \iota_{\times} \quad \text{destroy and append} \\
&= \mathbb{I} \cap v_q ; \mathbb{T} \cap \mathbb{T} ; \iota_{\times} \quad \text{masking} \\
&= (\mathbb{I} \cap \mathbb{T} ; \iota_{\times}) ; (\mathbb{I} \cap v_q ; \mathbb{T}) \quad \text{partial identities} \\
&= \iota_{\times}^{\intercal} ; \iota_{\times} ; (\mathbb{I} \cap \pi_{\mathbb{P}} ; [q^{\rightarrow}])
\end{aligned}
$$

---

and may, thus, continue with

$$
\begin{aligned}
&\iff \quad \mathbb{S}_{\mathbb{E}_6}^{\intercal\, a} ; z_{\mathbb{E}_6} ; \mathbb{T} ; ([q^{\rightarrow}]^{\intercal} \oslash [c^{\rightarrow}]^{\intercal}) ; \iota_{\times}^{\intercal} ; \iota_{\times} ; (\mathbb{I} \cap \pi_{\mathbb{P}} ; [q^{\rightarrow}]) \subseteq \mathbb{A}_{\mathbb{P}}^{\intercal} \\
&\iff \quad \mathbb{S}_{\mathbb{E}_6}^{\intercal\, a} ; z_{\mathbb{E}_6} ; \mathbb{T} ; ([q^{\rightarrow}]^{\intercal} \oslash [c^{\rightarrow}]^{\intercal}) ; (\mathbb{I} \cap [q^{\rightarrow}]^{\intercal} ; \pi_{\mathbb{P}}^{\intercal}) \subseteq \mathbb{A}_{\mathbb{P}}^{\intercal} \quad \text{as } q \leq c\text{; transposed partial identity} \\
&\iff \quad \mathbb{S}_{\mathbb{E}_6}^{\intercal\, a} ; z_{\mathbb{E}_6} ; \mathbb{T} ; ([q^{\rightarrow}]^{\intercal} \cap [q^{\rightarrow}]^{\intercal} \oslash [c^{\rightarrow}]^{\intercal}) \subseteq \mathbb{A}_{\mathbb{P}}^{\intercal} \\
&\iff \quad \mathbb{S}_{\mathbb{E}_6}^{\intercal\, a} ; z_{\mathbb{E}_6} ; \mathbb{T} ; ([q^{\rightarrow}]^{\intercal} \oslash [c^{\rightarrow}]^{\intercal}) \subseteq \mathbb{A}_{\mathbb{P}}^{\intercal} \\
&\iff \quad \mathbb{S}_{\mathbb{E}_6}^{\intercal\, a} ; z_{\mathbb{E}_6} ; \mathbb{T} \subseteq \mathbb{A}_{\mathbb{P}}^{\intercal} ; ([q^{\rightarrow}] \oslash [c^{\rightarrow}]) \\
&\iff \quad \mathbb{S}_{\mathbb{E}_6}^{\intercal\, a} ; z_{\mathbb{E}_6} ; \mathbb{T} \subseteq \mathbb{S}_{\mathbb{E}_6}^{\intercal\, \frac{q+c-6}{2}} ; z_{\mathbb{E}_6} ; \mathbb{T} \quad \text{due to Prop. 10.2.2.i}
\end{aligned}
$$

in order to obtain with $a = d + \frac{p+c-6}{2} = \frac{q+c-6}{2}$ the same result — having in mind the prime distance $d' = q - p$ in $\mathbb{N}$ according to Prop. 9.4.7, corresponding to $d = \frac{q-p}{2}$ in $\mathbb{E}_6$.

---

ii) Using (i), this proof is now fairly simple:

$$\mathbb{S}_{\mathbb{E}_6}^{d} ; \mathcal{P}_q ; \iota_{qp} = \mathbb{S}_{\mathbb{E}_6}^{d} ; \mathbb{S}_{\mathbb{E}_6}^{\intercal\, d} ; \mathcal{P}_p ; \iota_{qp}^{\intercal} ; \iota_{qp} \subseteq \mathcal{P}_p \quad \text{since } \mathbb{S}_{\mathbb{E}_6} \text{ is injective and } \iota_{qp} \text{ is univalent} \qquad \square$$

Propositions 10.7.4 will later show corresponding results for $\mathcal{Q}$ and $\mathcal{R}$.

10.6.2 Proposition. For any given odd prime p the following holds:
$\mathtt{sup}_{3\le r\le p,\, r \text{ prime}}\, \mathcal{P}_r ; \mathbb{T} \supseteq \mathbb{S}^*_{\mathbb{E}_6} ; \mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\ p-3} ; z_{\mathbb{E}_6} ; \mathbb{T}$

Proof: We attempt a proof by induction over the odd primes p, i.e. starting with $p = 3$:

$\mathtt{sup}_{3\le r\le 3,\, r \text{ prime}}\, \mathcal{P}_r ; \mathbb{T} = \mathcal{P}_3 ; \mathbb{T}$
$= \mathcal{P} ; \iota^{\mathsf{T}}_{\times 3} ; \mathbb{T} = \mathcal{P} ; v_{\times 3} ; \mathbb{T}$ since $\iota_{\times 3}$ extrudes $v_{\times 3} ; \mathbb{T}$
$= \mathbb{A}^{\mathsf{T}}_{\times} ; \iota_{\times} ; v_3 ; \mathbb{T}$ by the definitions
$= \mathbb{A}^{\mathsf{T}}_{\mathbb{P}} ; \iota^{\mathsf{T}}_{\times} ; \iota_{\times} ; \pi_{\mathbb{P}} ; [3\!\rightarrow] \quad$ again by definitions
$= \mathbb{A}^{\mathsf{T}}_{\mathbb{P}} ; (\mathbb{I} \cap v_{\le} ; \mathbb{T}) ; \pi_{\mathbb{P}} ; [3\!\rightarrow] \quad$ since $\iota_{\times}$ extrudes $v_{\le} ; \mathbb{T}$
$= \mathbb{A}^{\mathsf{T}}_{\mathbb{P}} ; \big[\pi_{\mathbb{P}} ; [3\!\rightarrow] \cap (\mathbb{S}^*_{\mathbb{P}} \circledcirc \mathbb{IP} \times \mathbb{P}) ; \mathbb{T}\big] \quad$ masking and expanding $v_{\le} ; \mathbb{T}$
$= \mathbb{A}^{\mathsf{T}}_{\mathbb{P}} ; \big[\pi_{\mathbb{P}} ; [3\!\rightarrow] \cap (\mathbb{S}^*_{\mathbb{P}} \circledcirc \mathbb{I})\big] ; \mathbb{T} \quad$ masking
$= \mathbb{A}^{\mathsf{T}}_{\mathbb{P}} ; \big[\pi_{\mathbb{P}} ; [3\!\rightarrow] \cap \pi_{\mathbb{P}} ; \mathbb{S}^*_{\mathbb{P}} \cap \rho_{\mathbb{P}}\big] ; \mathbb{T}$
$= \mathbb{A}^{\mathsf{T}}_{\mathbb{P}} ; ([3\!\rightarrow] \cap \mathbb{S}^*_{\mathbb{P}} \circledcirc \mathbb{I}) ; \mathbb{T}$
$= \mathbb{A}^{\mathsf{T}}_{\mathbb{P}} ; ([3\!\rightarrow] \circledcirc \mathbb{I}_{\mathbb{P}}) ; \mathbb{T} \quad$ due to Prop. 9.4.4.iii,iv we have $[3\!\rightarrow] \subseteq \mathbb{S}^*_{\mathbb{P}}$
$= \iota_{\mathbb{E}_6} ; \mathbb{A}^{\mathsf{T}} ; (\iota^{\mathsf{T}}_{\wp ; \mathbb{T}} \otimes \iota^{\mathsf{T}}_{\wp ; \mathbb{T}}) ; ([3\!\rightarrow] \circledcirc \mathbb{I}) ; \mathbb{T} \quad$ by definition of $\mathbb{A}_{\mathbb{P}}$
$= \iota_{\mathbb{E}_6} ; \mathbb{A}^{\mathsf{T}} ; (\iota^{\mathsf{T}}_{\wp ; \mathbb{T}} ; [3\!\rightarrow] \circledcirc \iota^{\mathsf{T}}_{\wp ; \mathbb{T}}) ; \mathbb{T}$
$= \iota_{\mathbb{E}_6} ; \mathbb{A}^{\mathsf{T}} ; (\mathbb{S}^{\mathsf{T}3} ; z ; \mathbb{T} \circledcirc \iota^{\mathsf{T}}_{\wp ; \mathbb{T}}) ; \mathbb{T} \quad$ Prop. 9.6.4.ii
$= \iota_{\mathbb{E}_6} ; \mathbb{A}^{\mathsf{T}} ; (\mathbb{S}^{\mathsf{T}3} ; z ; \mathbb{T} \circledcirc \mathbb{I}) ; \iota^{\mathsf{T}}_{\wp ; \mathbb{T}} ; \mathbb{T} \quad$ since $\iota_{\wp ; \mathbb{T}}$ is total and univalent
$\supseteq \iota_{\mathbb{E}_6} ; \mathbb{A}^{\mathsf{T}} ; (\mathbb{S}^{\mathsf{T}3} ; z ; \mathbb{T} \circledcirc \mathbb{I}) ; \mathbb{S}^{\mathsf{T}3} ; z ; \mathbb{T}$
$= \iota_{\mathbb{E}_6} ; \mathbb{A}^{\mathsf{T}} ; (\mathbb{S}^{\mathsf{T}3} ; z ; \mathbb{T} \circledcirc \mathbb{S}^{\mathsf{T}3} ; z ; \mathbb{T}) \quad$ with $\mathbb{S}^{\mathsf{T}3} ; z ; \mathbb{T}$ a point
$= \iota_{\mathbb{E}_6} ; \mathbb{S}^{\mathsf{T}6} ; z ; \mathbb{T} = \iota_{\mathbb{E}_6} ; \iota^{\mathsf{T}}_{\mathbb{E}_6} ; z_{\mathbb{E}_6} ; \mathbb{T} \quad$ following Prop. 10.1.3.v
$= z_{\mathbb{E}_6} ; \mathbb{T}$
$= \mathbb{S}^*_{\mathbb{E}_6} ; z_{\mathbb{E}_6} ; \mathbb{T} \quad$ because for $n > 0$ always $\mathbb{S}^n_{\mathbb{E}_6} ; z_{\mathbb{E}_6} = \mathbb{\perp}$
$= \mathbb{S}^*_{\mathbb{E}_6} ; \mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\ 3-3} ; z_{\mathbb{E}_6} ; \mathbb{T}$

Now the induction step $p \mapsto q$, or better for $[p\!\rightarrow] \mapsto [q\!\rightarrow] = \mathbb{S}^{\mathsf{T}}_{\mathbb{P}} ; [p\!\rightarrow]$, i.e. to the next consecutive prime. We have to prove

$\mathtt{sup}_{3\le r\le q,\, r \text{ prime}}\, \mathcal{P}_r ; \mathbb{T} \supseteq \mathbb{S}^*_{\mathbb{E}_6} ; \mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\ q-3} ; z_{\mathbb{E}_6} ; \mathbb{T}.$

Certainly, these two statements will hold:

$\mathtt{sup}_{3\le r\le q,\, r \text{ prime}}\, \mathcal{P}_r ; \mathbb{T} \supseteq \mathbb{S}^*_{\mathbb{E}_6} ; \mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\ p-3} ; z_{\mathbb{E}_6} ; \mathbb{T} \quad$ since $p < q$, hypothesis
$\mathtt{sup}_{3\le r\le q,\, r \text{ prime}}\, \mathcal{P}_r ; \mathbb{T} \supseteq \quad\ \mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\ q-3} ; z_{\mathbb{E}_6} ; \mathbb{T} \quad$ next doubled prime; proved rigorously as follows

---

$\mathtt{sup}_{3\le r\le q,\, q \text{ prime}}\, \mathcal{P}_r ; \mathbb{T} \supseteq \mathcal{P}_q ; \mathbb{T}$
$= \mathcal{P} ; \iota^{\mathsf{T}}_{\times q} ; \mathbb{T} = \mathcal{P} ; v_{\times q} ; \mathbb{T} \quad$ since $\iota_{\times q}$ extrudes $v_{\times q} ; \mathbb{T}$
$= \mathbb{A}^{\mathsf{T}}_{\times} ; \iota_{\times} ; v_q ; \mathbb{T} \quad$ by the definitions
$= \mathbb{A}^{\mathsf{T}}_{\mathbb{P}} ; \iota^{\mathsf{T}}_{\times} ; \iota_{\times} ; \pi_{\mathbb{P}} ; [q\!\rightarrow] \quad$ again by definitions
$= \mathbb{A}^{\mathsf{T}}_{\mathbb{P}} ; (\mathbb{I} \cap v_{\le} ; \mathbb{T}) ; \pi_{\mathbb{P}} ; [q\!\rightarrow] \quad$ since $\iota_{\times}$ extrudes $v_{\le} ; \mathbb{T}$
$= \mathbb{A}^{\mathsf{T}}_{\mathbb{P}} ; (\mathbb{I} \cap (\mathbb{S}^*_{\mathbb{P}} \circledcirc \mathbb{I}_{\mathbb{P}}) ; \mathbb{T}) ; \pi_{\mathbb{P}} ; [q\!\rightarrow] \quad$ by Def. 10.3.1
$= \mathbb{A}^{\mathsf{T}}_{\mathbb{P}} ; \big[\pi_{\mathbb{P}} ; [q\!\rightarrow] \cap (\mathbb{S}^*_{\mathbb{P}} \circledcirc \mathbb{I}_{\mathbb{P}}) ; \mathbb{T}\big] \quad$ masking
$= \mathbb{A}^{\mathsf{T}}_{\mathbb{P}} ; \big[\pi_{\mathbb{P}} ; [q\!\rightarrow] \cap (\mathbb{S}^*_{\mathbb{P}} \circledcirc \mathbb{I}_{\mathbb{P}})\big] ; \mathbb{T} \quad$ masking again
$= \mathbb{A}^{\mathsf{T}}_{\mathbb{P}} ; ([q\!\rightarrow] \cap \mathbb{S}^*_{\mathbb{P}} \circledcirc \mathbb{I}_{\mathbb{P}}) ; \mathbb{T}$
$\supseteq \mathbb{A}^{\mathsf{T}}_{\mathbb{P}} ; ([q\!\rightarrow] ; [q\!\rightarrow]^{\mathsf{T}} \circledcirc \mathbb{I}_{\mathbb{P}}) ; \mathbb{T} \quad$ because $[q\!\rightarrow]$ is injective, i.e. $[q\!\rightarrow] ; [q\!\rightarrow]^{\mathsf{T}} \subseteq \mathbb{I} \subseteq \mathbb{S}^*_{\mathbb{P}}$
$\supseteq \mathbb{A}^{\mathsf{T}}_{\mathbb{P}} ; ([q\!\rightarrow] ; [q\!\rightarrow]^{\mathsf{T}} \circledcirc \mathbb{I}_{\mathbb{P}}) ; [q\!\rightarrow]$

$= \mathbb{A}_{\mathbb{P}}^{\intercal};([q^{\rightarrow}];[q^{\rightarrow}]^{\intercal};[q^{\rightarrow}] \oslash [q^{\rightarrow}])$
$= \mathbb{A}_{\mathbb{P}}^{\intercal};([q^{\rightarrow}] \oslash [q^{\rightarrow}])$ because for a point p always $p^{\intercal};p = \mathbb{T}$
$= \mathbb{S}_{\mathbb{E}_6}^{\intercal\ q-3};z_{\mathbb{E}_6};\mathbb{T}$ following Prop. 10.2.2.i

---

So we have to care for being contained only for the vector

$$\mathbb{S}^{+}_{\mathbb{E}_6};\mathbb{S}_{\mathbb{E}_6}^{\intercal\ q-3};z_{\mathbb{E}_6};\mathbb{T} \cap \overline{\mathbb{S}^{*}_{\mathbb{E}_6};\mathbb{S}_{\mathbb{E}_6}^{\intercal\ p-3};z_{\mathbb{E}_6};\mathbb{T}} \subseteq \mathtt{sup}_{3\leq r\leq q,\ r\text{ prime}}\,\mathcal{P}_r;\mathbb{T},$$

thus ignoring all from p and below as well as the biggest one by omitting the reflexive part of the transitive closure. We are going to consider this point-wise for an arbitrary point

$$a \subseteq \mathbb{S}^{+}_{\mathbb{E}_6};\mathbb{S}_{\mathbb{E}_6}^{\intercal\ q-3};z_{\mathbb{E}_6};\mathbb{T} \cap \overline{\mathbb{S}^{*}_{\mathbb{E}_6};\mathbb{S}_{\mathbb{E}_6}^{\intercal\ p-3};z_{\mathbb{E}_6};\mathbb{T}}.$$

Such a point a has necessarily the structure $a = \mathbb{S}_{\mathbb{E}_6}^{\intercal\ s-3};z_{\mathbb{E}_6};\mathbb{T}$ with $p < s < q$. Since a is a point, it will according to Prop. 2.2.2.ii be fully contained in at least one item of the union of the $\mathcal{P}_r;\mathbb{T}$ with $3 \leq r \leq q$, r prime. The task is, thus, to find for an arbitrary s an r as follows:

$\forall s : p < s < q :$
$\quad \exists r : 3 \leq r \leq q$ and r prime: $\qquad \mathbb{S}_{\mathbb{E}_6}^{\intercal\ s-3};z_{\mathbb{E}_6};\mathbb{T} \subseteq \mathcal{P}_r;\mathbb{T}$

Our observation in Prop. 10.6.1 had been that all $\mathcal{P}_r$'s (with r an odd prime) may always be expressed via the next lower prime and, thus, finally by $\mathcal{P}_3$. Therefore this means that at least one of all these has to hold:

$$\mathbb{S}_{\mathbb{E}_6}^{\intercal\ s-3};z_{\mathbb{E}_6};\mathbb{T} \subseteq \mathcal{P}_3;\mathbb{T}$$
$$\mathbb{S}_{\mathbb{E}_6}^{\intercal\ s-3};z_{\mathbb{E}_6};\mathbb{T} \subseteq \mathcal{P}_5;\mathbb{T} = \mathbb{S}_{\mathbb{E}_6}^{\intercal};\mathcal{P}_3;\iota_{0503}^{\intercal};\mathbb{T}$$
$$\mathbb{S}_{\mathbb{E}_6}^{\intercal\ s-3};z_{\mathbb{E}_6};\mathbb{T} \subseteq \mathcal{P}_7;\mathbb{T} = \mathbb{S}_{\mathbb{E}_6}^{\intercal};\mathbb{S}_{\mathbb{E}_6}^{\intercal};\mathcal{P}_3;\iota_{0503}^{\intercal};\iota_{0705}^{\intercal};\mathbb{T}$$
$$\mathbb{S}_{\mathbb{E}_6}^{\intercal\ s-3};z_{\mathbb{E}_6};\mathbb{T} \subseteq \mathcal{P}_{11};\mathbb{T} = \mathbb{S}_{\mathbb{E}_6}^{\intercal\ 2};\mathbb{S}_{\mathbb{E}_6}^{\intercal};\mathbb{S}_{\mathbb{E}_6}^{\intercal};\mathcal{P}_3;\iota_{0503}^{\intercal};\iota_{0705}^{\intercal};\iota_{1107}^{\intercal};\mathbb{T}$$
$$\mathbb{S}_{\mathbb{E}_6}^{\intercal\ s-3};z_{\mathbb{E}_6};\mathbb{T} \subseteq \mathcal{P}_{13};\mathbb{T} = \mathbb{S}_{\mathbb{E}_6}^{\intercal};\mathbb{S}_{\mathbb{E}_6}^{\intercal\ 2};\mathbb{S}_{\mathbb{E}_6}^{\intercal};\mathbb{S}_{\mathbb{E}_6}^{\intercal};\mathcal{P}_3;\iota_{0503}^{\intercal};\iota_{0705}^{\intercal};\iota_{1107}^{\intercal};\iota_{1311}^{\intercal};\mathbb{T}$$
$$\mathbb{S}_{\mathbb{E}_6}^{\intercal\ s-3};z_{\mathbb{E}_6};\mathbb{T} \subseteq \mathcal{P}_{17};\mathbb{T} = \mathbb{S}_{\mathbb{E}_6}^{\intercal\ 2};\mathbb{S}_{\mathbb{E}_6}^{\intercal};\mathbb{S}_{\mathbb{E}_6}^{\intercal\ 2};\mathbb{S}_{\mathbb{E}_6}^{\intercal};\mathbb{S}_{\mathbb{E}_6}^{\intercal};\mathcal{P}_3;\iota_{0503}^{\intercal};\iota_{0705}^{\intercal};\iota_{1107}^{\intercal};\iota_{1311}^{\intercal};\iota_{1713}^{\intercal};\mathbb{T}$$

etc. i.e.

$$\mathbb{S}_{\mathbb{E}_6}^{\intercal\ s-3};z_{\mathbb{E}_6};\mathbb{T} \subseteq \mathcal{P}_r;\mathbb{T} = \overline{\mathbb{S}_{\mathbb{E}_6}^{\intercal\ \frac{r-3}{2}};\mathcal{P}_3;\iota_{\dots}^{\intercal}\dots\iota_{\dots}^{\intercal};\mathbb{T}}$$
$$\iff \quad \mathbb{S}_{\mathbb{E}_6}^{s-3};\overline{\mathbb{S}_{\mathbb{E}_6}^{\intercal\ \frac{r-3}{2}};\mathcal{P}_3;\iota_{\dots}^{\intercal}\dots\iota_{\dots}^{\intercal};\mathbb{T}} \subseteq \overline{z_{\mathbb{E}_6};\mathbb{T}} = \mathbb{S}_{\mathbb{E}_6}^{\intercal};\mathbb{T}$$
$$\iff \quad \mathbb{S}_{\mathbb{E}_6}^{s-3};\left[\mathbb{S}_{\mathbb{E}_6}^{\intercal\ \frac{r-3}{2}};\overline{\mathcal{P}_3;\iota_{\dots}^{\intercal}\dots\iota_{\dots}^{\intercal};\mathbb{T}} \cup \overline{\mathbb{S}_{\mathbb{E}_6}^{\intercal\ \frac{r-3}{2}};\mathbb{T}}\right] \subseteq \mathbb{S}_{\mathbb{E}_6}^{\intercal};\mathbb{T} \quad \text{since } \mathbb{S}_{\mathbb{E}_6}^{\intercal\ \frac{r-3}{2}} \text{ is univalent}$$
$$\iff \quad \mathbb{S}_{\mathbb{E}_6}^{s-3};\mathbb{S}_{\mathbb{E}_6}^{\intercal\ \frac{r-3}{2}};\overline{\mathcal{P}_3;\iota_{\dots}^{\intercal}\dots\iota_{\dots}^{\intercal};\mathbb{T}} \subseteq \mathbb{S}_{\mathbb{E}_6}^{\intercal};\mathbb{T} \quad \text{and} \quad \mathbb{S}_{\mathbb{E}_6}^{s-3};\overline{\mathbb{S}_{\mathbb{E}_6}^{\intercal\ \frac{r-3}{2}};\mathbb{T}} \subseteq \mathbb{S}_{\mathbb{E}_6}^{\intercal};\mathbb{T} = \overline{z_{\mathbb{E}_6};\mathbb{T}}$$

The right one of these containments is interesting, because it may with the Schröder rule be transformed in the equivalent

$$\mathbb{S}_{\mathbb{E}_6}^{\intercal\ s-3};z_{\mathbb{E}_6};\mathbb{T} \subseteq \mathbb{S}_{\mathbb{E}_6}^{\intercal\ \frac{r-3}{2}};\mathbb{T}.$$

The left side is a point. We then recall how $\mathbb{S}_{\mathbb{E}_6}$ and $\mathbb{S}_{\mathbb{E}_6}^{\intercal};\mathbb{T}$ look like in principle,

$$\mathbb{S}_{\mathbb{E}_6} = \begin{array}{c} \\ 6\rightarrow \\ 8\rightarrow \\ 10\rightarrow \\ 12\rightarrow \\ 14\rightarrow \\ 16\rightarrow \end{array} \begin{array}{c} \begin{array}{cccccc} 6\uparrow & 8\uparrow & 10\uparrow & 12\uparrow & 14\uparrow & 16\uparrow \end{array} \\ \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{array} \qquad \mathbb{S}_{\mathbb{E}_6}^{\intercal};\mathbb{T} = \begin{array}{c} \\ 6\rightarrow \\ 8\rightarrow \\ 10\rightarrow \\ 12\rightarrow \\ 14\rightarrow \\ 16\rightarrow \end{array} \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \end{pmatrix} \qquad \mathbb{S}_{\mathbb{E}_6}^{\intercal\ 2};\mathbb{T} = \begin{array}{c} \\ 6\rightarrow \\ 8\rightarrow \\ 10\rightarrow \\ 12\rightarrow \\ 14\rightarrow \\ 16\rightarrow \end{array} \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \end{pmatrix},$$

and find out that the necessary condition is

$$s - 3 \geq \tfrac{r-3}{2} \quad \text{or equivalently} \quad r \leq 2 * s - 3.$$

Proof so far ok, but still unfinished — finishing means: proof of the Goldbach conjecture. □

An attempt of a proof:

We refer to Fig. 10.4.5, start with $p < s < q$, and try to prove that $2 * s$ is decomposable. Certainly

$2 * p < 2 * s < 2 * q$.

Since p, q are assumed to be consecutive, s cannot be prime. The pattern $\mathcal{P}_q$ will not contribute, but $\mathcal{P}_p$ may do so. Confusingly, the even number $2 * s$ has several representations

by row-inscription	$(2 * s)\rightarrow$	in $\mathcal{P}_p$ — be this row empty or not,
by row-number	$1 + (s - 3)$	in the matrix representing $\mathcal{P}_p$,
as point	$\mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{s-3}{;}z_{\mathbb{E}_6}{;}\mathbb{T}$	in $\mathbb{E}_6$, the source of relation $\mathcal{P}_p$.

For an example assume $p = 13$ together with

$s = 14$, generating row inscription $28\rightarrow$, row number 12, and $\mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{11}{;}z_{\mathbb{E}_6}{;}\mathbb{T}$ with empty row,
$s = 15$, generating row inscription $30\rightarrow$, row number 13, and $\mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{12}{;}z_{\mathbb{E}_6}{;}\mathbb{T}$,
$s = 16$, generating row inscription $32\rightarrow$, row number 14, and $\mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{13}{;}z_{\mathbb{E}_6}{;}\mathbb{T}$.

The crucial point is now to look at $\mathcal{P}_3$ which has gradually been lowered to $\mathcal{P}_p$ — thereby cutting off the left column in every step. First we identify which entry corresponds to the left uppermost entry $((2 * p)\rightarrow, (p\rightarrow, p\rightarrow))\rightarrow\rightarrow$ in $\mathcal{P}_p$:

by row-inscription	$(p + 3)\rightarrow$	in $\mathcal{P}_3$
by row-number	$1 + \frac{p-3}{2}$	in the matrix representing $\mathcal{P}_3$
as point	$\mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{\frac{p-3}{2}}{;}z_{\mathbb{E}_6}{;}\mathbb{T}$	in $\mathbb{E}_6$, the source of relation $\mathcal{P}_3$.

For an example assume again $p = 13$ and obtain in the various representations

by row-inscription	$16\rightarrow$	in $\mathcal{P}_3$
by row-number	6	in the matrix representing $\mathcal{P}_3$
as point	$\mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{5}{;}z_{\mathbb{E}_6}{;}\mathbb{T}$	in $\mathbb{E}_6$, the source of relation $\mathcal{P}_3$.

Obviously, $\mathcal{P}_3$ is univalent and we focus on the corresponding column, again in three versions:

by column-inscription	$(3\rightarrow, p\rightarrow)\rightarrow\rightarrow$	in $\mathcal{P}_3$
by column-number	$\mathtt{psn}(p)$	in the matrix representing $\mathcal{P}_3$
as point	$\iota_{\times p;}\iota_{\times;}\,([3\rightarrow] \circledcirc [p\rightarrow])$	in $(\mathbb{P} \times \mathbb{P})_{\leq 3}$, the target of relation $\mathcal{P}_3$.

Now comes a deep coherence: Should the row inscripted $2 * s$ in $\mathcal{P}_p$ be an empty row, so will be the row inscripted $2 * s - (p - 3)$ in $\mathcal{P}_3$.

As an example take $s = 14$ look at row $28\rightarrow = (2 * 14)\rightarrow$ in $\mathcal{P}_{13}$, which stems via lowering from row $18\rightarrow = (28 - (13 - 3))\rightarrow = (2 * 14 - (13 - 3))\rightarrow$ in $\mathcal{P}_3$. This row $18\rightarrow$, however, is by induction assumption filled by a row in some $\mathcal{P}_r$ with $r < p$.

10.7 Selfsimilarity expressed via $\mathcal{Q}$

Another way to study selfsimilarity is via $\mathcal{Q}$ defined in Def. 10.4.3.

10.7.1 Proposition. Let p be an odd prime.

i) $\mathcal{Q}_p = \iota_{\mathbb{E}_6}; \mathbb{A}^{\mathsf{T}}; (\mathbb{I} \cap \mathtt{vec}(\mathbb{S}^*); \mathbb{T}); (\mathbb{S}^{\mathsf{T}p}; z; \mathbb{T} \circledcirc \mathbb{I}); \iota^{\mathsf{T}}_{\wp;\mathbb{T}}$

$\mathcal{Q}_p = \mathbb{A}^{\mathsf{T}}_{\mathbb{P}}; ([p\!\rightarrow] \circledcirc \mathbb{I}_{\mathbb{P}} \cap \mathbb{S}^{\mathsf{T}*}_{\mathbb{P}}; [p\!\rightarrow])$

$\mathcal{Q}_3 = \mathbb{A}^{\mathsf{T}}_{\mathbb{P}}; ([3\!\rightarrow] \circledcirc \mathbb{I}_{\mathbb{P}})$

ii) $\mathcal{Q} = \mathbb{A}^{\mathsf{T}}_{\mathbb{P}}; \left[\rho_{\mathbb{P}} \cap v_{\leq}; \mathbb{T}\right]$

iii) $\iota_{\mathbb{E}_6}; \mathtt{GR}; \iota^{\mathsf{T}}_{\wp;\mathbb{T}} = \mathcal{Q} \cup \mathbb{A}^{\mathsf{T}}_{\mathbb{P}}; \left[\pi_{\mathbb{P}} \cap v_{\leq}; \mathbb{T}\right]$

iv) $\mathcal{Q}_p \subseteq \mathbb{T}; \mathbb{S}^{\xi(p)}_{\mathbb{P}}$

Proof: i) $\iota_{\mathbb{E}_6}; \mathbb{A}^{\mathsf{T}}; (\mathbb{I} \cap \mathtt{vec}(\mathbb{S}^*); \mathbb{T}); (\mathbb{S}^{\mathsf{T}p}; z; \mathbb{T} \circledcirc \mathbb{I}); \iota^{\mathsf{T}}_{\wp;\mathbb{T}}$

$= \iota_{\mathbb{E}_6}; \mathbb{A}^{\mathsf{T}}; (\mathbb{I} \cap \mathtt{vec}(\mathbb{S}^*); \mathbb{T}); (\iota^{\mathsf{T}}_{\wp;\mathbb{T}}; \iota_{\wp;\mathbb{T}}; \mathbb{S}^{\mathsf{T}p}; z; \mathbb{T} \circledcirc \mathbb{I}); \iota^{\mathsf{T}}_{\wp;\mathbb{T}}$ by assumption p is an odd prime

$= \iota_{\mathbb{E}_6}; \mathbb{A}^{\mathsf{T}}; (\mathbb{I} \cap \mathtt{vec}(\mathbb{S}^*); \mathbb{T}); (\iota^{\mathsf{T}}_{\wp;\mathbb{T}}; \iota_{\wp;\mathbb{T}}; \mathbb{S}^{\mathsf{T}p}; z; \mathbb{T} \circledcirc \iota^{\mathsf{T}}_{\wp;\mathbb{T}})$ because $\iota^{\mathsf{T}}_{\wp;\mathbb{T}}$ is a transposed map

$= \iota_{\mathbb{E}_6}; \mathbb{A}^{\mathsf{T}}; (\mathbb{I} \cap \mathbb{T}; \mathtt{vec}(\mathbb{S}^*)^{\mathsf{T}}); (\iota^{\mathsf{T}}_{\wp;\mathbb{T}} \otimes \iota^{\mathsf{T}}_{\wp;\mathbb{T}}); (\iota_{\wp;\mathbb{T}}; \mathbb{S}^{\mathsf{T}p}; z; \mathbb{T} \circledcirc \mathbb{I})$ transposed partial identity

$= \iota_{\mathbb{E}_6}; \mathbb{A}^{\mathsf{T}}; \left[(\iota^{\mathsf{T}}_{\wp;\mathbb{T}} \otimes \iota^{\mathsf{T}}_{\wp;\mathbb{T}}) \cap \mathbb{T}; \mathtt{vec}(\mathbb{S}^*)^{\mathsf{T}}; (\iota^{\mathsf{T}}_{\wp;\mathbb{T}} \otimes \iota^{\mathsf{T}}_{\wp;\mathbb{T}}) \right]; ([p\!\rightarrow] \circledcirc \mathbb{I})$ Def. 9.4.2

$= \iota_{\mathbb{E}_6}; \mathbb{A}^{\mathsf{T}}; \left[(\iota^{\mathsf{T}}_{\wp;\mathbb{T}} \otimes \iota^{\mathsf{T}}_{\wp;\mathbb{T}}) \cap \mathbb{T}; v^{\mathsf{T}}_{\leq} \right]; ([p\!\rightarrow] \circledcirc \mathbb{I})$ Prop. 10.3.3.iii

$= \left[\iota_{\mathbb{E}_6}; \mathbb{A}^{\mathsf{T}}; (\iota^{\mathsf{T}}_{\wp;\mathbb{T}} \otimes \iota^{\mathsf{T}}_{\wp;\mathbb{T}}) \cap \mathbb{T}; v^{\mathsf{T}}_{\leq}\right]; ([p\!\rightarrow] \circledcirc \mathbb{I})$ masking

$= \left[\mathbb{A}^{\mathsf{T}}_{\mathbb{P}} \cap \mathbb{T}; v^{\mathsf{T}}_{\leq}\right]; ([p\!\rightarrow] \circledcirc \mathbb{I})$ definition of $\mathbb{A}_{\mathbb{P}}$ in Def. 10.2.1

$= \mathbb{A}^{\mathsf{T}}_{\mathbb{P}}; \left[([p\!\rightarrow] \circledcirc \mathbb{I}) \cap v_{\leq}; \mathbb{T}\right] = \mathcal{Q}_p$ Def. 10.4.3.i and mask shifting

---

The question of the second claim is whether

$\mathcal{Q}_p = \mathbb{A}^{\mathsf{T}}_{\mathbb{P}}; \left[([p\!\rightarrow] \circledcirc \mathbb{I}_{\mathbb{P}}) \cap v_{\leq}; \mathbb{T}\right] = \mathbb{A}^{\mathsf{T}}_{\mathbb{P}}; ([p\!\rightarrow] \circledcirc \mathbb{I}_{\mathbb{P}} \cap \mathbb{S}^{\mathsf{T}*}_{\mathbb{P}}; [p\!\rightarrow])$ by Def. 10.4.3

We omit the factor $\mathbb{A}^{\mathsf{T}}_{\mathbb{P}}$ and prove what remains:

$([p\!\rightarrow] \circledcirc \mathbb{I}_{\mathbb{P}}) \cap v_{\leq}; \mathbb{T} = ([p\!\rightarrow] \circledcirc \mathbb{I}_{\mathbb{P}} \cap \mathbb{S}^{\mathsf{T}*}_{\mathbb{P}}; [p\!\rightarrow])\,,$

starting from the left side:

$([p\!\rightarrow] \circledcirc \mathbb{I}_{\mathbb{P}}) \cap v_{\leq}; \mathbb{T} = ([p\!\rightarrow] \circledcirc \mathbb{I}_{\mathbb{P}}) \cap (\mathbb{I}_{\mathbb{P}} \circledcirc \mathbb{S}^{\mathsf{T}*}_{\mathbb{P}}); \mathbb{T}$ Def. 10.3.1

$= \pi_{\mathbb{P}}; [p\!\rightarrow] \cap \rho_{\mathbb{P}} \cap \left[\pi_{\mathbb{P}} \cap \rho_{\mathbb{P}}; \mathbb{S}^{\mathsf{T}*}_{\mathbb{P}}\right]; \mathbb{T} = \rho_{\mathbb{P}} \cap \left[\pi_{\mathbb{P}}; [p\!\rightarrow] \cap \pi_{\mathbb{P}} \cap \rho_{\mathbb{P}}; \mathbb{S}^{\mathsf{T}*}_{\mathbb{P}}\right]; \mathbb{T}$ masking

$= \rho_{\mathbb{P}} \cap \left[\pi_{\mathbb{P}}; ([p\!\rightarrow] \cap \mathbb{I}_{\mathbb{P}}) \cap \rho_{\mathbb{P}}; \mathbb{S}^{\mathsf{T}*}_{\mathbb{P}}\right]; \mathbb{T} = \rho_{\mathbb{P}} \cap \left[\pi_{\mathbb{P}}; [p\!\rightarrow]; [p\!\rightarrow]^{\mathsf{T}} \cap \rho_{\mathbb{P}}; \mathbb{S}^{\mathsf{T}*}_{\mathbb{P}}\right]; \mathbb{T}$

$= \rho_{\mathbb{P}} \cap \left[\pi_{\mathbb{P}}; [p\!\rightarrow] \cap [p\!\rightarrow]^{\mathsf{T}} \cap \rho_{\mathbb{P}}; \mathbb{S}^{\mathsf{T}*}_{\mathbb{P}}\right]; \mathbb{T} = \rho_{\mathbb{P}} \cap \left[\pi_{\mathbb{P}}; [p\!\rightarrow] \cap \rho_{\mathbb{P}}; \mathbb{S}^{\mathsf{T}*}_{\mathbb{P}}\right]; \left\{\mathbb{T} \cap [p\!\rightarrow]\right\}$ mask shifting

$= \rho_{\mathbb{P}} \cap \rho_{\mathbb{P}}; \mathbb{S}^{\mathsf{T}*}_{\mathbb{P}}; [p\!\rightarrow] \cap \pi_{\mathbb{P}}; [p\!\rightarrow]$ masking

$= ([p\!\rightarrow] \circledcirc \mathbb{I}_{\mathbb{P}} \cap \mathbb{S}^{\mathsf{T}*}_{\mathbb{P}}; [p\!\rightarrow])$

This result is trivial because $[3\!\rightarrow]$ is least and, thus, $v_{\leq}; \mathbb{T}$ always satisfied.

---

ii) $\mathcal{Q} = \mathtt{sup}_{p \text{ an odd prime}}\, \mathcal{Q}_p$ by Def. 10.4.3.ii

$= \mathtt{sup}_p\, \mathbb{A}^{\mathsf{T}}_{\mathbb{P}}; \left[([p\!\rightarrow] \circledcirc \mathbb{I}) \cap v_{\leq}; \mathbb{T}\right]$ by Def. 10.4.3.i

$= \mathbb{A}^{\mathsf{T}}_{\mathbb{P}}; \mathtt{sup}_p \left[([p\!\rightarrow] \circledcirc \mathbb{I}) \cap v_{\leq}; \mathbb{T}\right]$

$= \mathbb{A}^{\mathsf{T}}_{\mathbb{P}}; \left[\{\mathtt{sup}_p\, ([p\!\rightarrow] \circledcirc \mathbb{I})\} \cap v_{\leq}; \mathbb{T}\right] = \mathbb{A}^{\mathsf{T}}_{\mathbb{P}}; \left[(\mathtt{sup}_p\, [p\!\rightarrow] \circledcirc \mathbb{I}) \cap v_{\leq}; \mathbb{T}\right]$

$= \mathbb{A}^{\mathsf{T}}_{\mathbb{P}}; \left[(\mathbb{T}_{\mathbb{P}} \circledcirc \mathbb{I}) \cap v_{\leq}; \mathbb{T}\right] = \mathbb{A}^{\mathsf{T}}_{\mathbb{P}}; \left[\rho_{\mathbb{P}} \cap v_{\leq}; \mathbb{T}\right]$ in analogy to Prop. 5.2.1.ii.a

---

iii) Using $\overline{\mathbb{S}^*} = \mathbb{S}^{\mathsf{T}+}$, we have $v_{\leq}\,;\mathbb{T} = \mathtt{vec}(\mathbb{S}^*) = \overline{\mathtt{vec}(\overline{\mathbb{S}^*})} = \overline{\mathtt{vec}(\mathbb{S}^{\mathsf{T}+})}$; see page 15. Now, we recall Prop. 10.2.2.iii: $\mathbb{A}_{\mathbb{P}}^{\mathsf{T}}\,;\pi_{\mathbb{P}} = \iota_{\mathbb{E}_6}\,;\mathtt{GR}\,;\iota_{\wp;\mathbb{T}}^{\mathsf{T}} = \mathbb{A}_{\mathbb{P}}^{\mathsf{T}}\,;\rho_{\mathbb{P}}$, insert an identity and split it in two — with regard to $\mathbb{I} = \mathbb{S}^* \cap \mathbb{S}^{\mathsf{T}*}$ — non-disjoint parts:

$$\begin{aligned}
&\mathbb{A}_{\mathbb{P}}^{\mathsf{T}}\,;\mathbb{I}_{\mathbb{P}\times\mathbb{P}}\,;\rho_{\mathbb{P}} = \mathbb{A}_{\mathbb{P}}^{\mathsf{T}}\,;\Big[\big(\mathbb{I}\cap\mathtt{vec}(\mathbb{S}^*)\big)\cup\big(\mathbb{I}\cap\mathtt{vec}(\mathbb{S}^{\mathsf{T}*})\big)\Big]\,;\rho_{\mathbb{P}}\\
&= \mathbb{A}_{\mathbb{P}}^{\mathsf{T}}\,;(\mathbb{I}\cap v_{\leq}\,;\mathbb{T})\,;\rho_{\mathbb{P}}\cup\mathbb{A}_{\mathbb{P}}^{\mathsf{T}}\,;(\mathbb{I}\cap\mathtt{vec}(\mathbb{S}^{\mathsf{T}*}))\,;\rho_{\mathbb{P}}\\
&= \mathbb{A}_{\mathbb{P}}^{\mathsf{T}}\,;(\rho_{\mathbb{P}}\cap v_{\leq}\,;\mathbb{T})\cup\mathbb{A}_{\mathbb{P}}^{\mathsf{T}}\,;(\mathbb{I}\cap\mathtt{vec}(\mathbb{S}^{\mathsf{T}*}))\,;\rho_{\mathbb{P}}\quad\text{masking with } \rho_{\mathbb{P}} \text{ surjective}\\
&= \mathcal{Q}\cup\mathbb{A}_{\mathbb{P}}^{\mathsf{T}}\,;(\mathbb{I}\cap\mathtt{vec}(\mathbb{S}^{\mathsf{T}*}))\,;\rho_{\mathbb{P}}
\end{aligned}$$

The final step is now

$$\begin{aligned}
&\mathbb{A}_{\mathbb{P}}^{\mathsf{T}}\,;(\mathbb{I}\cap\mathtt{vec}(\mathbb{S}^{\mathsf{T}*}))\,;\rho_{\mathbb{P}} = \mathbb{A}_{\mathbb{P}}^{\mathsf{T}}\,;\mathfrak{P}\,;\mathfrak{P}\,;(\mathbb{I}\cap\mathtt{vec}(\mathbb{S}^{\mathsf{T}*}))\,;\rho_{\mathbb{P}}\\
&= \mathbb{A}_{\mathbb{P}}^{\mathsf{T}}\,;(\mathfrak{P}\cap\mathfrak{P}\,;\mathtt{vec}(\mathbb{S}^{\mathsf{T}*}))\,;\rho_{\mathbb{P}}\quad\text{because } \mathbb{A}_{\mathbb{P}} \text{ is commutative}\\
&= \mathbb{A}_{\mathbb{P}}^{\mathsf{T}}\,;(\mathfrak{P}\cap\mathtt{vec}(\mathbb{S}^*))\,;\rho_{\mathbb{P}}\quad\text{Prop. 2.2.6.ii}\\
&= \mathbb{A}_{\mathbb{P}}^{\mathsf{T}}\,;\big[\mathfrak{P}\,;\rho_{\mathbb{P}}\cap\mathtt{vec}(\mathbb{S}^*)\big] = \mathbb{A}_{\mathbb{P}}^{\mathsf{T}}\,;\big[\pi_{\mathbb{P}}\cap\mathtt{vec}(\mathbb{S}^*)\big] = \mathbb{A}_{\mathbb{P}}^{\mathsf{T}}\,;\big[\pi_{\mathbb{P}}\cap v_{\leq}\,;\mathbb{T}\big]
\end{aligned}$$

---

iv) $\mathcal{Q}_p = \mathbb{A}_{\mathbb{P}}^{\mathsf{T}}\,;\big([p\!\rightarrowtail]\circledcirc\mathbb{I}_{\mathbb{P}}\cap\mathbb{S}_{\mathbb{P}}^{\mathsf{T}*}\,;[p\!\rightarrowtail]\big)$ see (i)
$= \mathbb{A}_{\mathbb{P}}^{\mathsf{T}}\,;\big([p\!\rightarrowtail]\circledcirc\mathbb{I}_{\mathbb{P}}\cap[p\!\rightarrowtail]^{\mathsf{T}}\,;\mathbb{S}_{\mathbb{P}}^*\big)$ transposed partial identity
$\subseteq \mathbb{T}\,;\big(\mathbb{T}\circledcirc[p\!\rightarrowtail]^{\mathsf{T}}\,;\mathbb{S}_{\mathbb{P}}^*\big)$
$= \mathbb{T}\,;\rho\,;[p\!\rightarrowtail]^{\mathsf{T}}\,;\mathbb{S}_{\mathbb{P}}^* \subseteq \mathbb{T}\,;z_{\mathbb{P}}^{\mathsf{T}}\,;\mathbb{S}_{\mathbb{P}}^{\xi(p)}\,;\mathbb{S}_{\mathbb{P}}^* = \mathbb{T}\,;z_{\mathbb{P}}^{\mathsf{T}}\,;\mathbb{S}_{\mathbb{P}}^*\,;\mathbb{S}_{\mathbb{P}}^{\xi(p)} \subseteq \mathbb{T}\,;\mathbb{S}_{\mathbb{P}}^{\xi(p)}$ □

A visualization is given via Fig. 10.7.1.

The following proposition is an analogy to Prop. 10.6.1. Observe that in (iii) it is just one entry of the relation that makes the difference.

10.7.2 Proposition. Given any two *consecutive* odd primes p, q as two points satisfying
$\mathbb{S}^{\mathsf{T}p}\,;z\,;\mathbb{T} = \mathtt{NP}\,;\mathbb{S}^{\mathsf{T}q}\,;z\,;\mathbb{T}$, or equivalently $\mathbb{S}_{\mathbb{P}}^{\mathsf{T}}\,;[p\!\rightarrowtail] = [q\!\rightarrowtail]$,
we have, abbreviating $d := \frac{q-p}{2}$, these assertions:

i) $\mathcal{Q}_q = \mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\,d}\,;\mathcal{Q}_p\cap[q\!\rightarrowtail]^{\mathsf{T}}\,;\mathbb{S}_{\mathbb{P}}^* = \mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\,d}\,;\mathcal{Q}_p\cap\mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\,q-3}\,;\mathbb{T}$

ii) $\mathcal{Q}_p = \mathbb{S}_{\mathbb{E}_6}^{d}\,;\mathcal{Q}_q\cup\mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\,p-3}\,;z_{\mathbb{E}_6}\,;\mathbb{T}\,;[p\!\rightarrowtail]^{\mathsf{T}}$ holds as a disjoint union with right term an entry

iii) For arbitrary odd primes $p < r$ always

$\mathcal{Q}_r = \mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\,\frac{r-p}{2}}\,;\mathcal{Q}_p\cap[r\!\rightarrowtail]^{\mathsf{T}}\,;\mathbb{S}_{\mathbb{P}}{}^*$ in particular: $\mathcal{Q}_r = \mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\,\frac{r-3}{2}}\,;\mathcal{Q}_3\cap[r\!\rightarrowtail]^{\mathsf{T}}\,;\mathbb{S}_{\mathbb{P}}{}^*$

$\mathcal{Q}_r = \mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\,\frac{r-p}{2}}\,;\mathcal{Q}_p\cap\mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\,r-3}\,;\mathbb{T}$ in particular: $\mathcal{Q}_r = \mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\,\frac{r-3}{2}}\,;\mathcal{Q}_3\cap\mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\,r-3}\,;\mathbb{T}$

Proof: i) As may be observed in the figures above, this result is remarkably evident. A proof attempt using the points

$u : \mathbb{E}_6 \longrightarrow \mathbb{1}$ and $w : \mathbb{P} \longrightarrow \mathbb{1}$

means the goal to show

$$u\,;w^{\mathsf{T}} \subseteq \mathcal{Q}_q \qquad\Longleftrightarrow\qquad u\,;w^{\mathsf{T}} \subseteq \mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\,d}\,;\mathcal{Q}_p\cap[q\!\rightarrowtail]^{\mathsf{T}}\,;\mathbb{S}_{\mathbb{P}}^*.$$

Every point $u : \mathbb{E}_6 \longrightarrow \mathbb{1}$ has the obvious representation $u = \mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\,a}\,;z_{\mathbb{E}_6}\,;\mathbb{T}$. A representation for an arbitrary point $w : \mathbb{P} \longrightarrow \mathbb{1}$ is via the prime sequence exponent $\mathtt{psn} : \mathbb{P} \longrightarrow \mathbb{N}$ according to Prop. 9.6.1.

$w = \mathbb{S}_{\mathbb{P}}^{\mathsf{T}b}\,;z_{\mathbb{P}}\,;\mathbb{T} = \iota_{\wp;\mathbb{T}}\,;\mathbb{S}^{\mathsf{T}c}\,;z\,;\mathbb{T} = [c\!\rightarrowtail]$,

$$v_{\leq} = \begin{array}{c} \\ (3\rightarrow,3\rightarrow) \\ (3\rightarrow,5\rightarrow) \\ (3\rightarrow,7\rightarrow) \\ (3\rightarrow,11\rightarrow) \\ (3\rightarrow,13\rightarrow) \\ (3\rightarrow,17\rightarrow) \\ (3\rightarrow,19\rightarrow) \\ (3\rightarrow,23\rightarrow) \\ (3\rightarrow,29\rightarrow) \\ (5\rightarrow,3\rightarrow) \\ (5\rightarrow,5\rightarrow) \\ (5\rightarrow,7\rightarrow) \\ (5\rightarrow,11\rightarrow) \\ (5\rightarrow,13\rightarrow) \\ (5\rightarrow,17\rightarrow) \\ (5\rightarrow,19\rightarrow) \\ (5\rightarrow,23\rightarrow) \\ (5\rightarrow,29\rightarrow) \\ (7\rightarrow,3\rightarrow) \\ (7\rightarrow,5\rightarrow) \\ (7\rightarrow,7\rightarrow) \\ (7\rightarrow,11\rightarrow) \\ (7\rightarrow,13\rightarrow) \\ (7\rightarrow,17\rightarrow) \\ (7\rightarrow,19\rightarrow) \\ (7\rightarrow,23\rightarrow) \\ (7\rightarrow,29\rightarrow) \\ (11\rightarrow,3\rightarrow) \\ (11\rightarrow,5\rightarrow) \\ (11\rightarrow,7\rightarrow) \\ (11\rightarrow,11\rightarrow) \\ (11\rightarrow,13\rightarrow) \\ (11\rightarrow,17\rightarrow) \\ (11\rightarrow,19\rightarrow) \\ (11\rightarrow,23\rightarrow) \\ (11\rightarrow,29\rightarrow) \\ (13\rightarrow,3\rightarrow) \\ (13\rightarrow,5\rightarrow) \\ (13\rightarrow,7\rightarrow) \\ (13\rightarrow,11\rightarrow) \\ (13\rightarrow,13\rightarrow) \\ (13\rightarrow,17\rightarrow) \\ (13\rightarrow,19\rightarrow) \\ (13\rightarrow,23\rightarrow) \\ (13\rightarrow,29\rightarrow) \\ (17\rightarrow,3\rightarrow) \\ (17\rightarrow,5\rightarrow) \\ (17\rightarrow,7\rightarrow) \\ (17\rightarrow,11\rightarrow) \\ (17\rightarrow,13\rightarrow) \\ (17\rightarrow,17\rightarrow) \\ (17\rightarrow,19\rightarrow) \\ (17\rightarrow,23\rightarrow) \\ (17\rightarrow,29\rightarrow) \\ (19\rightarrow,3\rightarrow) \\ (19\rightarrow,5\rightarrow) \\ (19\rightarrow,7\rightarrow) \\ (19\rightarrow,11\rightarrow) \\ (19\rightarrow,13\rightarrow) \\ (19\rightarrow,17\rightarrow) \\ (19\rightarrow,19\rightarrow) \\ (19\rightarrow,23\rightarrow) \\ (19\rightarrow,29\rightarrow) \\ (23\rightarrow,3\rightarrow) \\ (23\rightarrow,5\rightarrow) \\ (23\rightarrow,7\rightarrow) \\ (23\rightarrow,11\rightarrow) \\ (23\rightarrow,13\rightarrow) \\ (23\rightarrow,17\rightarrow) \\ (23\rightarrow,19\rightarrow) \\ (23\rightarrow,23\rightarrow) \\ (23\rightarrow,29\rightarrow) \\ (29\rightarrow,3\rightarrow) \\ (29\rightarrow,5\rightarrow) \\ (29\rightarrow,7\rightarrow) \\ (29\rightarrow,11\rightarrow) \\ (29\rightarrow,13\rightarrow) \\ (29\rightarrow,17\rightarrow) \\ (29\rightarrow,19\rightarrow) \\ (29\rightarrow,23\rightarrow) \\ (29\rightarrow,29\rightarrow) \end{array} \begin{array}{c} \begin{array}{ccccccccc} 3\rightarrow & 5\rightarrow & 7\rightarrow & 11\rightarrow & 13\rightarrow & 17\rightarrow & 19\rightarrow & 23\rightarrow & 29\rightarrow \end{array} \\ \begin{pmatrix} 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \end{pmatrix} \end{array}$$

$$\begin{array}{c} \\ 6\rightarrow \\ 8\rightarrow \\ 10\rightarrow \\ 12\rightarrow \\ 14\rightarrow \\ 16\rightarrow \\ 18\rightarrow \\ 20\rightarrow \\ 22\rightarrow \\ 24\rightarrow \\ 26\rightarrow \\ 28\rightarrow \end{array} \begin{array}{c} \begin{array}{ccccccccc} 3\rightarrow & 5\rightarrow & 7\rightarrow & 11\rightarrow & 13\rightarrow & 17\rightarrow & 19\rightarrow & 23\rightarrow & 29\rightarrow \end{array} \\ \begin{pmatrix} 1&0&0&0&0&0&0&0&0 \\ 1&1&0&0&0&0&0&0&0 \\ 1&1&1&0&0&0&0&0&0 \\ 0&1&1&0&0&0&0&0&0 \\ 1&0&1&1&0&0&0&0&0 \\ 1&1&0&1&1&0&0&0&0 \\ 0&1&1&1&1&0&0&0&0 \\ 1&0&1&0&1&1&0&0&0 \\ 1&1&0&1&0&1&1&0&0 \\ 0&1&1&1&1&1&1&0&0 \\ 1&0&1&0&1&0&1&1&0 \\ 0&1&0&1&0&1&0&1&0 \end{pmatrix} \end{array} = \iota_{\mathbb{E}_6 ; \mathtt{GR} ; \iota^{\intercal}_{\wp ; \mathbb{T}}}$$

$$\begin{array}{c} \\ 6\rightarrow \\ 8\rightarrow \\ 10\rightarrow \\ 12\rightarrow \\ 14\rightarrow \\ 16\rightarrow \\ 18\rightarrow \\ 20\rightarrow \\ 22\rightarrow \\ 24\rightarrow \\ 26\rightarrow \\ 28\rightarrow \end{array} \begin{array}{c} \begin{array}{ccccccccc} 3\rightarrow & 5\rightarrow & 7\rightarrow & 11\rightarrow & 13\rightarrow & 17\rightarrow & 19\rightarrow & 23\rightarrow & 29\rightarrow \end{array} \\ \begin{pmatrix} 1&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0 \\ 0&1&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&1&1&0&0&0&0&0 \\ 0&0&0&1&1&0&0&0&0 \\ 0&0&0&1&1&0&0&0&0 \\ 0&0&0&0&1&1&0&0&0 \\ 0&0&0&1&0&1&1&0&0 \\ 0&0&0&0&1&1&1&0&0 \\ 0&0&0&0&1&0&1&1&0 \\ 0&0&0&0&0&1&0&1&0 \end{pmatrix} \end{array} = \mathcal{Q}$$

$$\begin{array}{c} \\ 6\rightarrow \\ 8\rightarrow \\ 10\rightarrow \\ 12\rightarrow \\ 14\rightarrow \\ 16\rightarrow \\ 18\rightarrow \\ 20\rightarrow \\ 22\rightarrow \\ 24\rightarrow \\ 26\rightarrow \\ 28\rightarrow \end{array} \begin{array}{c} \begin{array}{ccccccccc} 3\rightarrow & 5\rightarrow & 7\rightarrow & 11\rightarrow & 13\rightarrow & 17\rightarrow & 19\rightarrow & 23\rightarrow & 29\rightarrow \end{array} \\ \begin{pmatrix} 1&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0 \\ 1&1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0 \\ 1&0&1&0&0&0&0&0&0 \\ 1&1&0&0&0&0&0&0&0 \\ 0&1&1&0&0&0&0&0&0 \\ 1&0&1&0&0&0&0&0&0 \\ 1&1&0&1&0&0&0&0&0 \\ 0&1&1&1&0&0&0&0&0 \\ 1&0&1&0&1&0&0&0&0 \\ 0&1&0&1&0&0&0&0&0 \end{pmatrix} \end{array} = \mathbb{A}^{\intercal}_{\mathbb{P}^i}\left[(\pi_{\mathbb{P}} \cup \rho_{\mathbb{P}}) \cap v_{\leq^i}\mathbb{T}\right]$$

$$\begin{array}{c} \\ 6\rightarrow \\ 8\rightarrow \\ 10\rightarrow \\ 12\rightarrow \\ 14\rightarrow \\ 16\rightarrow \\ 18\rightarrow \\ 20\rightarrow \\ 22\rightarrow \\ 24\rightarrow \\ 26\rightarrow \\ 28\rightarrow \end{array} \begin{array}{c} \begin{array}{ccccccccc} 3\rightarrow & 5\rightarrow & 7\rightarrow & 11\rightarrow & 13\rightarrow & 17\rightarrow & 19\rightarrow & 23\rightarrow & 29\rightarrow \end{array} \\ \begin{pmatrix} 0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0 \\ 1&1&0&0&0&0&0&0&0 \\ 0&1&1&0&0&0&0&0&0 \\ 1&0&1&0&0&0&0&0&0 \\ 1&1&0&0&0&0&0&0&0 \\ 0&1&1&1&0&0&0&0&0 \\ 1&0&1&0&0&0&0&0&0 \\ 0&1&0&1&0&0&0&0&0 \end{pmatrix} \end{array} = \mathbb{A}^{\intercal}_{\mathbb{P}^i}\left[\pi_{\mathbb{P}} \cap v_{\leq^i}\mathbb{T}\right]$$

Fig. 10.7.1 Illustrating $\mathcal{Q}$ etc.

which leads to defining $c := \xi^{-1}(b)$ — an odd prime.

Our goal has, with these primes, been modified slightly to proving

$$\mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{a}; z_{\mathbb{E}_6}; \mathbb{T}; [c\rightarrow]^{\mathsf{T}} \subseteq \mathcal{Q}_q \qquad \Longleftrightarrow \qquad \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{a}; z_{\mathbb{E}_6}; \mathbb{T}; [c\rightarrow]^{\mathsf{T}} \subseteq \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{d}; \mathcal{Q}_p \cap [q\rightarrow]^{\mathsf{T}}; \mathbb{S}^{*}_{\mathbb{P}},$$

or, after shunting twice on both sides and using that points are injective,

$$\underset{LC}{\mathbb{T} \subseteq \mathbb{T}; z^{\mathsf{T}}_{\mathbb{E}_6}; \mathbb{S}^{a}_{\mathbb{E}_6}; \mathcal{Q}_q; [c\rightarrow]} \quad \Longleftrightarrow \quad \underset{RC_1}{\mathbb{T} \subseteq \mathbb{T}; z^{\mathsf{T}}_{\mathbb{E}_6}; \mathbb{S}^{a}_{\mathbb{E}_6}; \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{d}; \mathcal{Q}_p; [c\rightarrow]} \;,\quad \underset{RC_2}{\mathbb{T} \subseteq \mathbb{T}; z^{\mathsf{T}}_{\mathbb{E}_6}; \mathbb{S}^{a}_{\mathbb{E}_6}; [q\rightarrow]^{\mathsf{T}}; \mathbb{S}^{*}_{\mathbb{P}}; [c\rightarrow]}$$

For LC on the left, we evaluate the final product separately:

$\mathcal{Q}_q; [c\rightarrow] = \mathbb{A}^{\mathsf{T}}_{\mathbb{P}}; \big[([q\rightarrow] \circledcirc \mathbb{I}) \cap v_{\leq}; \mathbb{T} \big]; [c\rightarrow]$ expanding $\mathcal{Q}_q$ via Def. 10.4.3.i
$= \mathbb{A}^{\mathsf{T}}_{\mathbb{P}}; \big[([q\rightarrow] \circledcirc [c\rightarrow]) \cap v_{\leq}; \mathbb{T} \big]$ since $[c\rightarrow]$ is a point: injective and surjective
$= \mathbb{A}^{\mathsf{T}}_{\mathbb{P}}; ([q\rightarrow] \circledcirc [c\rightarrow])$ in order not to have $\mathbb{T} \subseteq \mathbb{1}$ via $v_{\leq}$, $q \leq c$ is implied
$= \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{\frac{q+c-6}{2}}; z_{\mathbb{E}_6}; \mathbb{T}$ due to Prop. 10.2.2.i

The left containment LC, thus, reads in full

$$\mathbb{T} \subseteq \mathbb{T}; z^{\mathsf{T}}_{\mathbb{E}_6}; \mathbb{S}^{a}_{\mathbb{E}_6}; \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{\frac{q+c-6}{2}}; z_{\mathbb{E}_6}; \mathbb{T}$$

and holds precisely if $q \leq c$ and $a = \frac{q+c-6}{2}$.

The containment RC_1 on the right holds — in analogy with the evaluation of LC just presented — precisely if $p \leq c$ and $a = d + \frac{p+c-6}{2}$.

The containment RC_2 on the right simplifies since $\mathbb{T}; z^{\mathsf{T}}_{\mathbb{E}_6}; \mathbb{S}^{a}_{\mathbb{E}_6}; [q\rightarrow]^{\mathsf{T}} = \mathbb{T}; z^{\mathsf{T}}_{\mathbb{E}_6}; \mathbb{S}^{a}_{\mathbb{E}_6}; \mathbb{T}; [q\rightarrow]^{\mathsf{T}} = [q\rightarrow]^{\mathsf{T}}$ and holds exactly if

$$\mathbb{T} \subseteq [q\rightarrow]^{\mathsf{T}}; \mathbb{S}^{*}_{\mathbb{P}}; [c\rightarrow] \quad \Longleftrightarrow \quad [q\rightarrow] \subseteq \mathbb{S}^{*}_{\mathbb{P}}; [c\rightarrow]$$

when shunted, which means $q \leq c$.

In total, we have to show $LC = RC_1 \wedge RC_2$, i.e. that

$$q \leq c \quad \wedge \quad a = \tfrac{q+c-6}{2} \quad \Longleftrightarrow \quad p \leq c \quad \wedge \quad a = d + \tfrac{p+c-6}{2} \quad \wedge \quad q \leq c.$$

Recalling that $p < q$ are consecutive primes by assumption, c is some odd prime, and $d = \frac{q-p}{2}$, both sides are indeed equivalent.

Finally the second equality claimed: Concerning LC and RC_1 everything stays the same. Just RC_2 has been changed from the former $\mathbb{T} \subseteq \mathbb{T}; z^{\mathsf{T}}_{\mathbb{E}_6}; \mathbb{S}^{a}_{\mathbb{E}_6}; [q\rightarrow]^{\mathsf{T}}; \mathbb{S}^{*}_{\mathbb{P}}; [c\rightarrow]$ to now

$$\mathbb{T} \subseteq \mathbb{T}; z^{\mathsf{T}}_{\mathbb{E}_6}; \mathbb{S}^{a}_{\mathbb{E}_6}; \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{q-3}; \mathbb{T}.$$

This time $LC = RC_1 \wedge RC_2$ reads

$$q \leq c \quad \wedge \quad a = \tfrac{q+c-6}{2} \quad \Longleftrightarrow \quad p \leq c \quad \wedge \quad a = d + \tfrac{p+c-6}{2} \quad \wedge \quad a \geq q - 3.$$

At this point, we might say "which obviously holds"; however we are slightly more explicit:
"$\Longrightarrow$"

$p < q \leq c$
$d + \frac{p+c-6}{2} = \frac{q-p}{2} + \frac{p+c-6}{2} = \frac{q+c-6}{2} = a$
$a = \frac{q+c-6}{2} \geq \frac{2*q-6}{2} = q - 3$

"$\Longleftarrow$"

$\frac{q+c-6}{2} = \frac{p+2*d+c-6}{2} = d + \frac{p+c-6}{2} = a$

$$q \le a+3 = \left[d + \tfrac{p+c-6}{2}\right] + 3 = d + \tfrac{p+c}{2} = \tfrac{q-p}{2} + \tfrac{p+c}{2} = \tfrac{q+c}{2} \quad \iff \quad q \le c$$

---

ii) Using (i), we get

$$\mathbb{S}^{d}_{\mathbb{E}_6} ; \mathcal{Q}_q = \mathbb{S}^{d}_{\mathbb{E}_6} ; \left[\mathbb{S}^{\intercal}_{\mathbb{E}_6}{}^{d} ; \mathcal{Q}_p \cap [q^{\rightarrow}]^{\intercal} ; \mathbb{S}^{*}_{\mathbb{P}}\right] \subseteq \mathbb{S}^{d}_{\mathbb{E}_6} ; \mathbb{S}^{\intercal}_{\mathbb{E}_6}{}^{d} ; \mathcal{Q}_p \subseteq \mathcal{Q}_p$$

We restrict to showing that $\mathbb{S}^{\intercal}_{\mathbb{E}_6}{}^{p-3} ; z_{\mathbb{E}_6} ; \mathbb{T} ; [p^{\rightarrow}]^{\intercal} \subseteq \mathcal{Q}_p$. Via shunting the points on both sides, we achieve this proof as follows:

$$\begin{aligned}
&\mathbb{T} ; z^{\intercal}_{\mathbb{E}_6} ; \mathbb{S}^{p-3}_{\mathbb{E}_6} ; \mathcal{Q}_p ; [p^{\rightarrow}] = \mathbb{T} ; z^{\intercal}_{\mathbb{E}_6} ; \mathbb{S}^{p-3}_{\mathbb{E}_6} ; \mathbb{A}^{\intercal}_{\mathbb{P}} ; \left[([p^{\rightarrow}] \oslash \mathbb{I}) \cap v_{\le} ; \mathbb{T}\right] ; [p^{\rightarrow}] \\
&= \mathbb{T} ; z^{\intercal}_{\mathbb{E}_6} ; \mathbb{S}^{p-3}_{\mathbb{E}_6} ; \mathbb{A}^{\intercal}_{\mathbb{P}} ; \left[([p^{\rightarrow}] \oslash [p^{\rightarrow}]) \cap v_{\le} ; \mathbb{T}\right] \quad \text{because } [p^{\rightarrow}] \text{ is a point} \\
&= \mathbb{T} ; z^{\intercal}_{\mathbb{E}_6} ; \mathbb{S}^{p-3}_{\mathbb{E}_6} ; \mathbb{A}^{\intercal}_{\mathbb{P}} ; ([p^{\rightarrow}] \oslash [p^{\rightarrow}]) \quad \text{since } p \le p \\
&= \mathbb{T} ; z^{\intercal}_{\mathbb{E}_6} ; \mathbb{S}^{p-3}_{\mathbb{E}_6} ; \mathbb{S}^{\intercal}_{\mathbb{E}_6}{}^{\frac{2*p-6}{2}} ; z_{\mathbb{E}_6} ; \mathbb{T} = \mathbb{T} \quad \text{Prop. 10.2.2.i}
\end{aligned}$$

still missing: $\mathcal{Q}_p \subseteq \mathbb{S}^{d}_{\mathbb{E}_6} ; \mathcal{Q}_q \cup \mathbb{S}^{\intercal}_{\mathbb{E}_6}{}^{p-3} ; z_{\mathbb{E}_6} ; \mathbb{T} ; [p^{\rightarrow}]^{\intercal}$

Disjointness:

$$\begin{aligned}
&\mathbb{S}^{d}_{\mathbb{E}_6} ; \mathcal{Q}_q \cap \mathbb{S}^{\intercal}_{\mathbb{E}_6}{}^{p-3} ; z_{\mathbb{E}_6} ; \mathbb{T} ; [p^{\rightarrow}]^{\intercal} = \mathbb{S}^{d}_{\mathbb{E}_6} ; \mathbb{A}^{\intercal}_{\mathbb{P}} ; \left[([q^{\rightarrow}] \oslash \mathbb{I}) \cap v_{\le} ; \mathbb{T}\right] \cap \mathbb{S}^{\intercal}_{\mathbb{E}_6}{}^{p-3} ; z_{\mathbb{E}_6} ; \mathbb{T} ; [p^{\rightarrow}]^{\intercal} \\
&\subseteq (\mathbb{S}^{\intercal}_{\mathbb{E}_6}{}^{p-3} ; z_{\mathbb{E}_6} ; \mathbb{T} \cap \mathbb{S}^{d}_{\mathbb{E}_6} ; \mathbb{A}^{\intercal}_{\mathbb{P}} ; \left[([q^{\rightarrow}] \oslash \mathbb{I}) \cap v_{\le} ; \mathbb{T}\right] ; [p^{\rightarrow}]) ; (\ldots \cap \ldots) \quad \text{Dedekind rule}
\end{aligned}$$

and now

$$\left[([q^{\rightarrow}] \oslash \mathbb{I}) \cap v_{\le} ; \mathbb{T}\right] ; [p^{\rightarrow}] = ([q^{\rightarrow}] \oslash [p^{\rightarrow}]) \cap v_{\le} ; \mathbb{T} \subseteq \perp\!\!\!\perp \quad \text{since } [p^{\rightarrow}] \text{ is a point and } p < q$$

---

iii) This is proved by induction with hypothesis $\mathcal{Q}_{p_i} = \mathbb{S}^{\intercal}_{\mathbb{E}_6}{}^{\frac{p_i - p_0}{2}} ; \mathcal{Q}_{p_0} \cap [p_i^{\rightarrow}]^{\intercal} ; \mathbb{S}_{\mathbb{P}}{}^{*}$ and starting with the full sequence of primes $p =: p_0 < p_1 < \ldots < r$.

$$\begin{aligned}
&\mathcal{Q}_{p_1} = \mathbb{S}^{\intercal}_{\mathbb{E}_6}{}^{\frac{p_1 - p_0}{2}} ; \mathcal{Q}_{p_0} \cap [p_1{}^{\rightarrow}]^{\intercal} ; \mathbb{S}_{\mathbb{P}}{}^{*} \quad \text{following (i)} \\
&\mathcal{Q}_{p_{i+1}} = \mathbb{S}^{\intercal}_{\mathbb{E}_6}{}^{\frac{p_{i+1} - p_i}{2}} ; \mathcal{Q}_{p_i} \cap [p_{i+1}{}^{\rightarrow}]^{\intercal} ; \mathbb{S}_{\mathbb{P}}{}^{*} \quad \text{using (i) again} \\
&= \mathbb{S}^{\intercal}_{\mathbb{E}_6}{}^{\frac{p_{i+1} - p_i}{2}} ; \left[\mathbb{S}^{\intercal}_{\mathbb{E}_6}{}^{\frac{p_i - p_0}{2}} ; \mathcal{Q}_{p_0} \cap [p_i{}^{\rightarrow}]^{\intercal} ; \mathbb{S}_{\mathbb{P}}{}^{*}\right] \cap [p_{i+1}{}^{\rightarrow}]^{\intercal} ; \mathbb{S}_{\mathbb{P}}{}^{*} \quad \text{hypothesis} \\
&= \mathbb{S}^{\intercal}_{\mathbb{E}_6}{}^{\frac{p_{i+1} - p_i}{2}} ; \mathbb{S}^{\intercal}_{\mathbb{E}_6}{}^{\frac{p_i - p_0}{2}} ; \mathcal{Q}_{p_0} \cap [p_i{}^{\rightarrow}]^{\intercal} ; \mathbb{S}_{\mathbb{P}}{}^{*} \cap [p_{i+1}{}^{\rightarrow}]^{\intercal} ; \mathbb{S}_{\mathbb{P}}{}^{*} \quad \text{masking columns} \\
&= \mathbb{S}^{\intercal}_{\mathbb{E}_6}{}^{\frac{p_{i+1} - p_0}{2}} ; \mathcal{Q}_{p_0} \cap [p_{i+1}{}^{\rightarrow}]^{\intercal} ; \mathbb{S}_{\mathbb{P}}{}^{*} \quad \text{using that } [p_i{}^{\rightarrow}]^{\intercal} ; \mathbb{S}_{\mathbb{P}}{}^{*} \supseteq [p_{i+1}{}^{\rightarrow}]^{\intercal} ; \mathbb{S}_{\mathbb{P}}{}^{*}
\end{aligned}$$

The second result follows in quite the same manner with $\mathbb{S}^{\intercal}_{\mathbb{E}_6}{}^{(i+1)-3} ; \mathbb{T} \subseteq \mathbb{S}^{\intercal}_{\mathbb{E}_6}{}^{i-3} ; \mathbb{T}$. □

For the following, we start with the observation that $[p^{\rightarrow}]^{\intercal} ; \mathbb{S}^{+}_{\mathbb{P}} = [q^{\rightarrow}]^{\intercal} ; \mathbb{S}^{*}_{\mathbb{P}}$ for two arbitrary consecutive primes $p < q$.

10.7.3 Proposition. For any given odd prime p the following hold:

i) $\mathcal{Q}_p ; \mathbb{T} = \mathbb{S}^{\intercal}_{\mathbb{E}_6}{}^{\frac{p-3}{2}} ; \mathcal{Q}_3 ; \mathbb{S}^{\intercal *}_{\mathbb{P}} ; [p^{\rightarrow}] = [\mathbb{S}^{\intercal}_{\mathbb{E}_6}{}^{\frac{p-3}{2}} \cap \mathbb{S}^{\intercal}_{\mathbb{E}_6}{}^{p-3} ; \mathbb{T}] ; \mathcal{Q}_3 ; \mathbb{T}$

ii) $\mathcal{Q}_p ; \mathbb{T} \supseteq \mathbb{S}^{\intercal}_{\mathbb{E}_6}{}^{p-3} ; z_{\mathbb{E}_6} ; \mathbb{T}$

Proof: i) $\mathcal{Q}_p ; \mathbb{T} = \left[\mathbb{S}^{\intercal}_{\mathbb{E}_6}{}^{\frac{p-3}{2}} ; \mathcal{Q}_3 \cap [p^{\rightarrow}]^{\intercal} ; \mathbb{S}^{*}_{\mathbb{P}}\right] ; \mathbb{T}$ Prop. 10.7.2.iii

$$= \mathbb{S}^{\intercal}_{\mathbb{E}_6}{}^{\frac{p-3}{2}} ; \mathcal{Q}_3 \left[\mathbb{T} \cap \mathbb{S}^{\intercal *}_{\mathbb{P}} ; [p^{\rightarrow}]\right] \quad \text{mask shifting}$$

$$= \mathbb{S}^{\intercal}_{\mathbb{E}_6}{}^{\frac{p-3}{2}} ; \mathcal{Q}_3 ; \mathbb{S}^{\intercal *}_{\mathbb{P}} ; [p^{\rightarrow}]$$

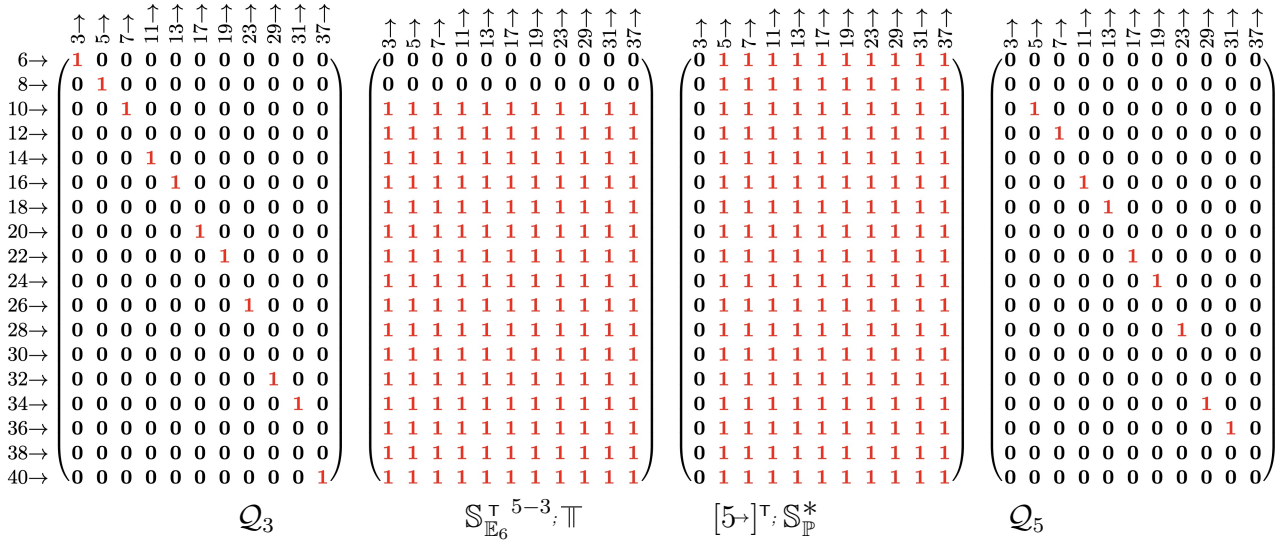
$\mathcal{Q}_p ; \mathbb{T} = \left[\mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\ \frac{p-3}{2}} ; \mathcal{Q}_3 \cap \mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\ p-3} ; \mathbb{T} \right] ; \mathbb{T} = \mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\ \frac{p-3}{2}} ; \mathcal{Q}_3 ; \mathbb{T} \cap \mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\ p-3} ; \mathbb{T}$ masking; Prop. 10.7.2.iii
$= \left[\mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\ \frac{p-3}{2}} \cap \mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\ p-3} ; \mathbb{T} \right] ; \mathcal{Q}_3 ; \mathbb{T}$ masking again

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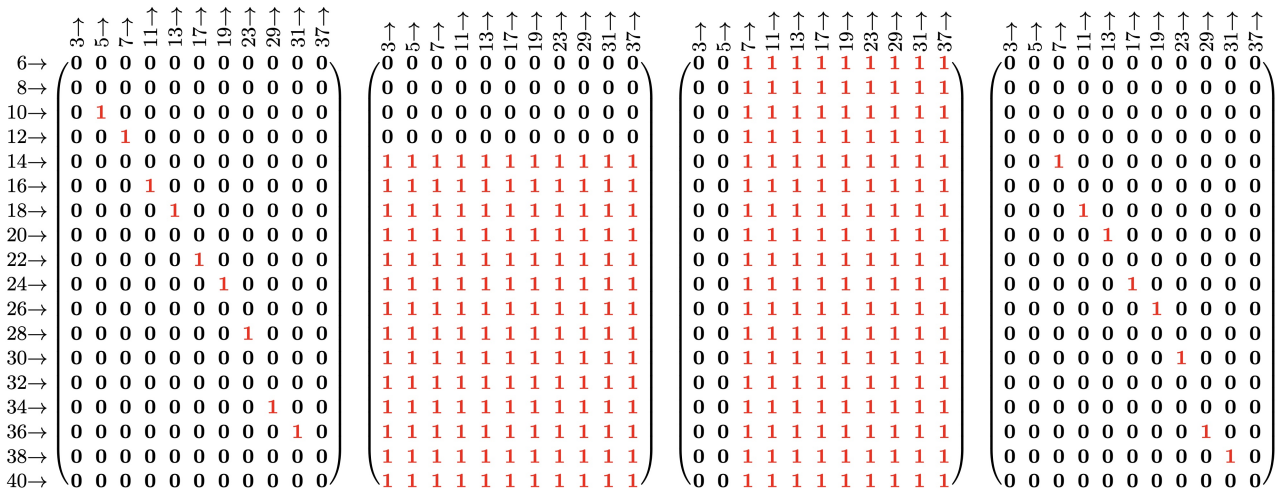
ii) $\mathcal{Q}_p ; \mathbb{T} \supseteq \mathbb{A}_{\mathbb{P}}^{\mathsf{T}} ; \left[\left([p\rightarrow] \circledcirc \mathbb{I}_{\mathbb{P}} \right) \cap v_{\leq} ; \mathbb{T} \right] ; [p\rightarrow]$ expanded according to Def. 10.4.3 with $\mathbb{T} \supseteq [p\rightarrow]$
$= \mathbb{A}_{\mathbb{P}}^{\mathsf{T}} ; \left[\left([p\rightarrow] \circledcirc \mathbb{I}_{\mathbb{P}} \right) ; [p\rightarrow] \cap v_{\leq} ; \mathbb{T} \right]$ since $[p\rightarrow]$ is a point, i.e. injective and surjective
$= \mathbb{A}_{\mathbb{P}}^{\mathsf{T}} ; \left[\left([p\rightarrow] \circledcirc [p\rightarrow] \right) \cap v_{\leq} ; \mathbb{T} \right]$ since $[p\rightarrow]$ is a point, i.e. injective and surjective
$= \mathbb{A}_{\mathbb{P}}^{\mathsf{T}} ; \left([p\rightarrow] \circledcirc [p\rightarrow] \right)$ because $p \leq p$
$= \mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\ p-3} ; z_{\mathbb{E}_6} ; \mathbb{T}$ following Prop. 10.2.2.i □

The following figures show the effect of this proposition. When lowering $\mathcal{Q}_p$ to $\mathcal{Q}_q$ for $p < q$ consecutive primes, one will always have situations with just one entry removed in the upper left — and in addition possibly others in the lowest lines. To cut off this point, two methods seem applicable. The first restricts horizontally and the second vertically. It is not yet clear which method will be easier to handle.

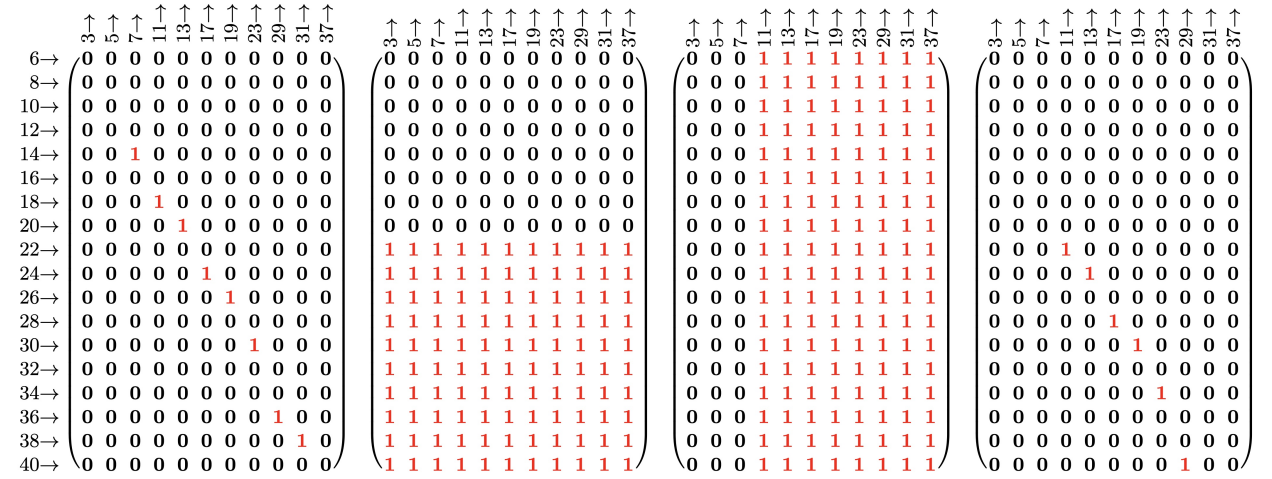
The entry to exclude is always found uniting the horizontal and the vertical mask in the lower right corner of the complement. As an example: Lowering $\mathcal{Q}_{11}$ by 2 will bring its entry $(22\rightarrow, 11\rightarrow)$ to position $(24\rightarrow, 11\rightarrow)$.



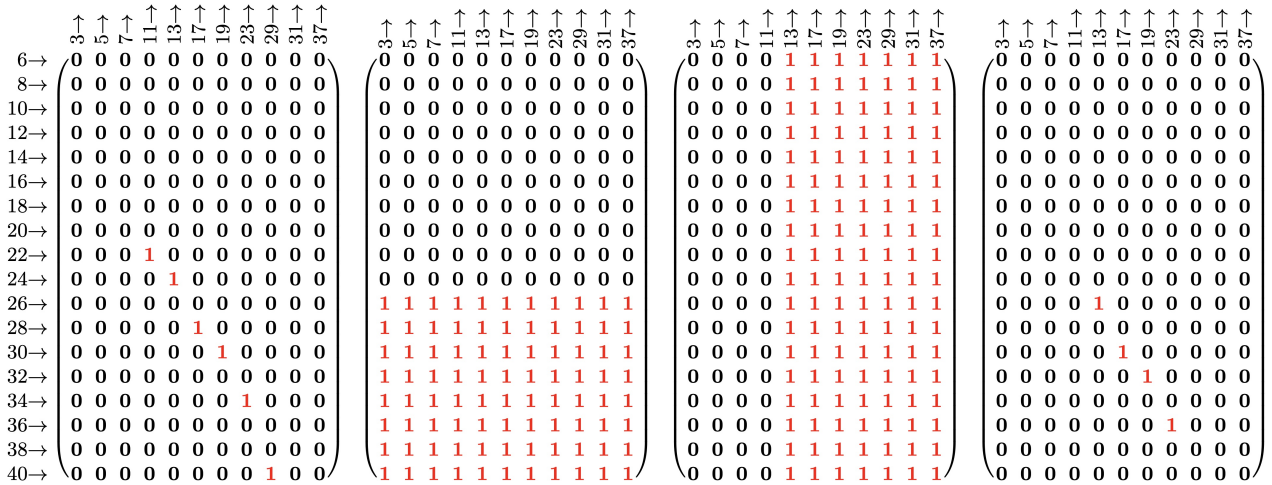
$\mathcal{Q}_3$ $\mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\ 5-3} ; \mathbb{T}$ $[5\rightarrow]^{\mathsf{T}} ; \mathbb{S}_{\mathbb{P}}^{*}$ $\mathcal{Q}_5$



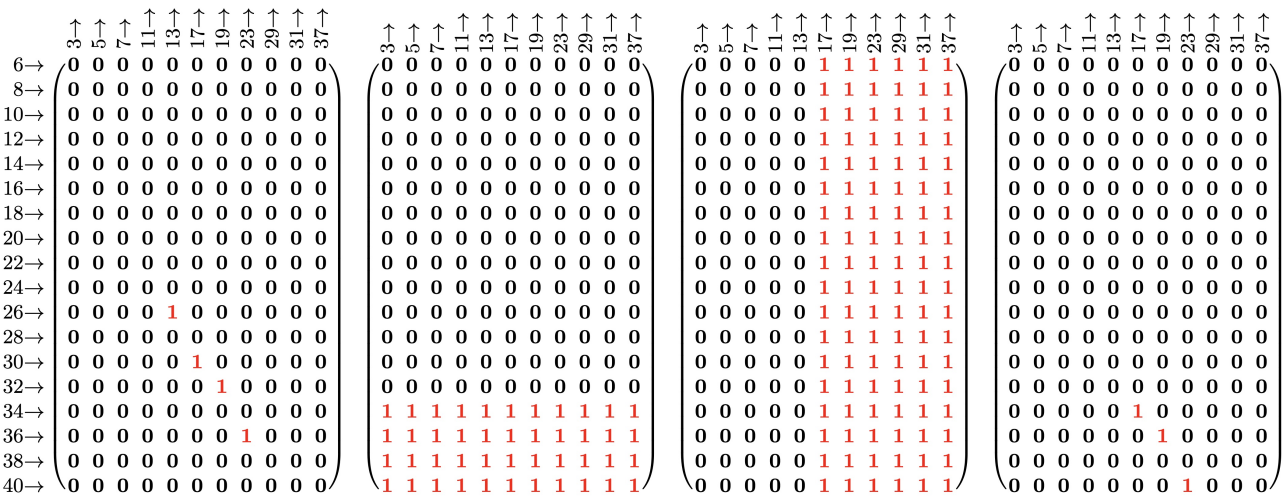
$\mathcal{Q}_5$ $\mathbb{S}_{\mathbb{E}_6}^{\intercal\ 7-3;\mathbb{T}}$ $[7\!\rightarrow]^{\intercal};\mathbb{S}_{\mathbb{P}}^{*}$ $\mathcal{Q}_7$



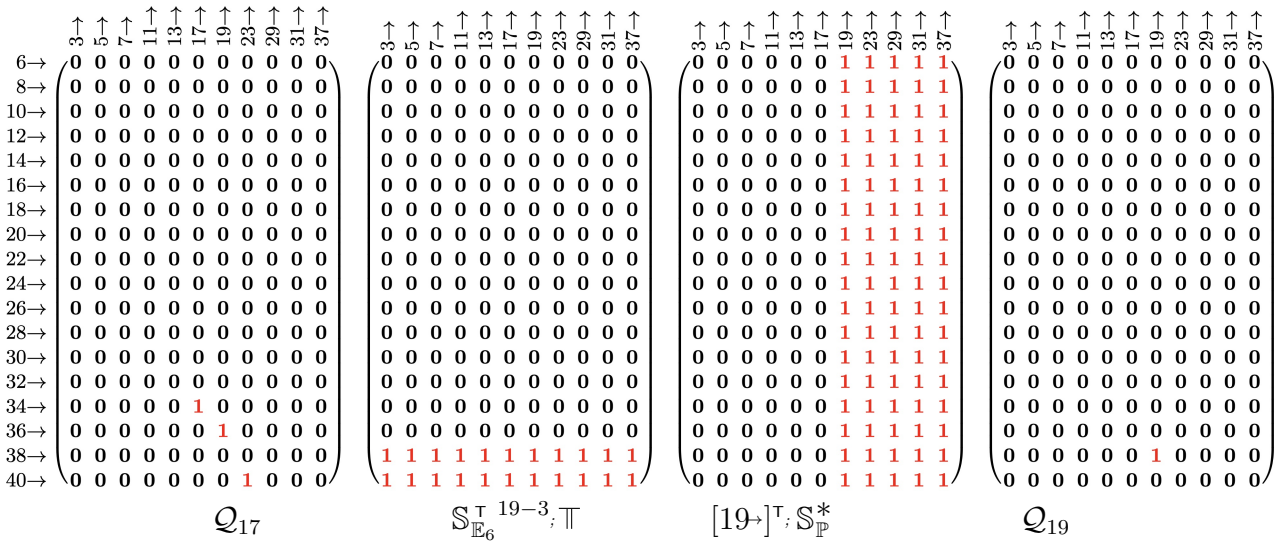
$\mathcal{Q}_7$ $\mathbb{S}_{\mathbb{E}_6}^{\intercal\ 11-3;\mathbb{T}}$ $[11\!\rightarrow]^{\intercal};\mathbb{S}_{\mathbb{P}}^{*}$ $\mathcal{Q}_{11}$

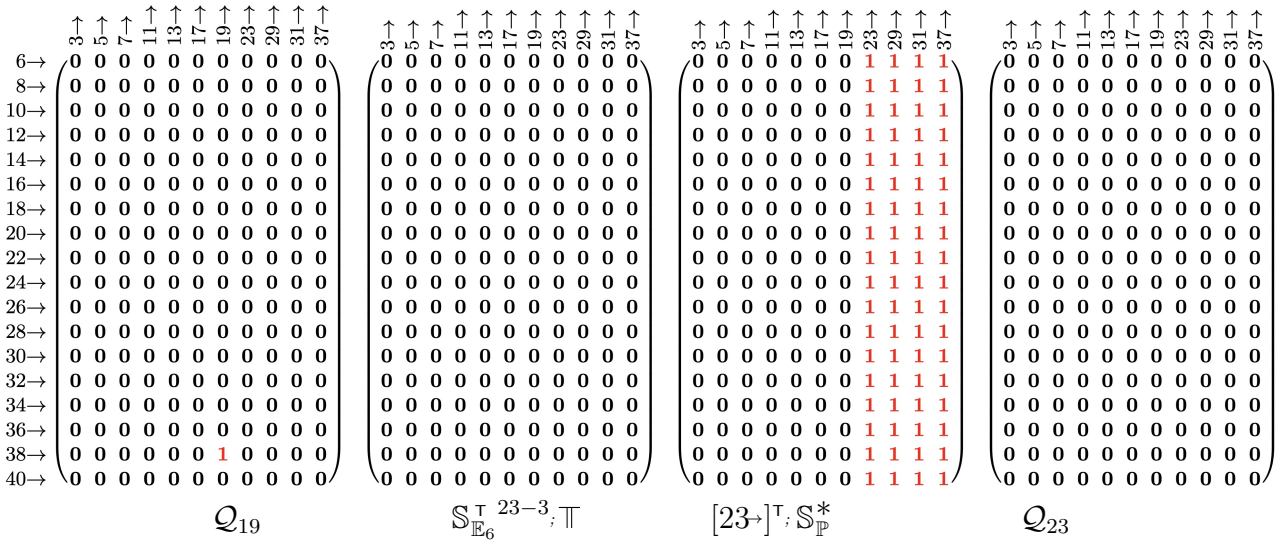


$\mathcal{Q}_{11}$ $\mathbb{S}_{\mathbb{E}_6}^{\intercal\ 13-3;\mathbb{T}}$ $[13\!\rightarrow]^{\intercal};\mathbb{S}_{\mathbb{P}}^{*}$ $\mathcal{Q}_{13}$



$\mathcal{Q}_{13}$ $\mathbb{S}_{\mathbb{E}_6}^{\intercal\ 17-3;\mathbb{T}}$ $[17\!\rightarrow]^{\intercal};\mathbb{S}_{\mathbb{P}}^{*}$ $\mathcal{Q}_{17}$





In view of Prop. 10.7.2, we now aggregate all these $\mathcal{Q}_3, \mathcal{Q}_5, \mathcal{Q}_7, \mathcal{Q}_{11} \ldots$ as a sequence of triples in the following way:

$(3, \frac{3-3}{2}, \mathcal{Q}_3), (5, \frac{5-3}{2}, \mathcal{Q}_5), (7, \frac{7-3}{2}, \mathcal{Q}_7), (11, \frac{11-3}{2}, \mathcal{Q}_{11}), (13, \frac{13-3}{2}, \mathcal{Q}_{13}), \ldots$
$(3, 0, \mathcal{Q}_3), (5, 1, \mathcal{Q}_5), (7, 2, \mathcal{Q}_7), (11, 4, \mathcal{Q}_{11}), (13, 5, \mathcal{Q}_{13}), \ldots$

The information concerning the lowering is redundant, thus, also included as a duplicate point. The second and the third component may, thus, replace each other — once $\mathcal{Q}_3$ is given.

The following is surrounded by analogical Props. 10.6.2 and another for $\mathcal{P}$ and $\mathcal{R}$.

10.7.4 Proposition. For any given odd prime p the following holds:

$$\mathtt{sup}_{3\leq r\leq p,\ r\text{ prime}}\ \mathcal{Q}_r{}_{;}\mathbb{T} \supseteq \mathbb{S}^{*}_{\mathbb{E}_6}{}_{;}\mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{p-3}{}_{;}z_{\mathbb{E}_6}{}_{;}\mathbb{T}$$

Proof: We attempt a proof by induction over the odd primes p, i.e. starting with $p = 3$:

$$\begin{aligned}
&\mathtt{sup}_{3\leq r\leq 3,\ r\text{ prime}}\ \mathcal{Q}_r{}_{;}\mathbb{T} = \mathcal{Q}_3{}_{;}\mathbb{T} = \mathbb{A}^{\mathsf{T}}_{\mathbb{P}}{}_{;}\big[\,([3\!\rightarrow] \circledcirc \mathbb{I}_{\mathbb{P}}) \cap v_{\leq}{}_{;}\mathbb{T}\big]{}_{;}\mathbb{T}\\
&= \mathbb{A}^{\mathsf{T}}_{\mathbb{P}}{}_{;}\big[\pi_{\mathbb{P}}{}_{;}[3\!\rightarrow] \cap \rho_{\mathbb{P}} \cap (\mathbb{S}^{*}_{\mathbb{P}} \circledcirc \mathbb{I}_{\mathbb{P}}){}_{;}\mathbb{T}\big]{}_{;}\mathbb{T} \quad \text{expanded}\\
&= \mathbb{A}^{\mathsf{T}}_{\mathbb{P}}{}_{;}\big[\{\pi_{\mathbb{P}}{}_{;}[3\!\rightarrow] \cap (\mathbb{S}^{*}_{\mathbb{P}} \circledcirc \mathbb{I}_{\mathbb{P}})\}{}_{;}\mathbb{T} \cap \rho_{\mathbb{P}}\big]{}_{;}\mathbb{T} \quad \text{shuffled and masked}
\end{aligned}$$

$= \mathbb{A}_{\mathbb{P}}^{\intercal}; \Big[([3\!\rightarrow] \cap \mathbb{S}_{\mathbb{P}}^{*} \circledcirc \mathbb{I}_{\mathbb{P}}); \mathbb{T} \cap \rho_{\mathbb{P}} \Big]; \mathbb{T}$
$= \mathbb{A}_{\mathbb{P}}^{\intercal}; \Big[([3\!\rightarrow] \circledcirc \mathbb{I}_{\mathbb{P}}); \mathbb{T} \cap \rho_{\mathbb{P}} \Big]; \mathbb{T}$ due to Prop. 9.4.4.i,ii we have $[3\!\rightarrow] \subseteq \mathbb{S}_{\mathbb{P}}^{*}$
$= \mathbb{A}_{\mathbb{P}}^{\intercal}; \Big[([3\!\rightarrow] \circledcirc \mathbb{I}_{\mathbb{P}}); \mathbb{T} \cap \rho_{\mathbb{P}}; \mathbb{T} \Big] = \mathbb{A}_{\mathbb{P}}^{\intercal}; ([3\!\rightarrow] \circledcirc \mathbb{I}_{\mathbb{P}}); \mathbb{T}$ masking and $\rho_{\mathbb{P}}$ is total
$= \iota_{\mathbb{E}_6}; \mathbb{A}^{\intercal}; (\iota_{\wp;\mathbb{T}}^{\intercal} \otimes \iota_{\wp;\mathbb{T}}^{\intercal}); ([3\!\rightarrow] \circledcirc \mathbb{I}_{\mathbb{P}}); \mathbb{T}$ by definition of $\mathbb{A}_{\mathbb{P}}$
$= \iota_{\mathbb{E}_6}; \mathbb{A}^{\intercal}; (\iota_{\wp;\mathbb{T}}^{\intercal}; [3\!\rightarrow] \circledcirc \iota_{\wp;\mathbb{T}}^{\intercal}); \mathbb{T}$
$= \iota_{\mathbb{E}_6}; \mathbb{A}^{\intercal}; (\mathbb{S}^{\intercal 3}; z; \mathbb{T} \circledcirc \iota_{\wp;\mathbb{T}}^{\intercal}); \mathbb{T}$ Prop. 9.6.4.ii
$= \iota_{\mathbb{E}_6}; \mathbb{A}^{\intercal}; (\mathbb{S}^{\intercal 3}; z; \mathbb{T} \circledcirc \mathbb{I}); \iota_{\wp;\mathbb{T}}^{\intercal}; \mathbb{T}$ since $\iota_{\wp;\mathbb{T}}$ is total and univalent
$\supseteq \iota_{\mathbb{E}_6}; \mathbb{A}^{\intercal}; (\mathbb{S}^{\intercal 3}; z; \mathbb{T} \circledcirc \mathbb{I}); \mathbb{S}^{\intercal 3}; z; \mathbb{T}$
$= \iota_{\mathbb{E}_6}; \mathbb{A}^{\intercal}; (\mathbb{S}^{\intercal 3}; z; \mathbb{T} \circledcirc \mathbb{S}^{\intercal 3}; z; \mathbb{T})$ with $\mathbb{S}^{\intercal 3}; z; \mathbb{T}$ a point
$= \iota_{\mathbb{E}_6}; \mathbb{S}^{\intercal 6}; z; \mathbb{T} = \iota_{\mathbb{E}_6}; \iota_{\mathbb{E}_6}^{\intercal}; z_{\mathbb{E}_6}; \mathbb{T}$ following Prop. 10.1.3.v
$= z_{\mathbb{E}_6}; \mathbb{T}$ since $\iota_{\mathbb{E}_6}$ is an injective map
$= \mathbb{S}_{\mathbb{E}_6}^{*}; z_{\mathbb{E}_6}; \mathbb{T}$ because for $n > 0$ always $\mathbb{S}_{\mathbb{E}_6}^{n}; z_{\mathbb{E}_6} = \perp\!\!\!\perp$
$= \mathbb{S}_{\mathbb{E}_6}^{*}; \mathbb{S}_{\mathbb{E}_6}^{\intercal\ 3-3}; z_{\mathbb{E}_6}; \mathbb{T}$ $(= \mathbb{S}_{\mathbb{E}_6}^{*}; \mathbb{I}; z_{\mathbb{E}_6}; \mathbb{T} = z_{\mathbb{E}_6}; \mathbb{T}$ not to be used here)

Now the induction step $p \mapsto q$, or better for $[p\!\rightarrow] \mapsto [q\!\rightarrow] = \mathbb{S}_{\mathbb{P}}^{\intercal}; [p\!\rightarrow]$, i.e. to the next consecutive prime. We have to prove

$\mathtt{sup}_{3 \leq r \leq q,\ r \text{ prime}}\ \mathcal{Q}_r; \mathbb{T} \supseteq \mathbb{S}_{\mathbb{E}_6}^{*}; \mathbb{S}_{\mathbb{E}_6}^{\intercal\ q-3}; z_{\mathbb{E}_6}; \mathbb{T}.$

Certainly, these two statements will hold:

$\mathtt{sup}_{3 \leq r \leq q,\ r \text{ prime}}\ \mathcal{Q}_r; \mathbb{T} \supseteq \mathtt{sup}_{3 \leq r \leq p,\ r \text{ prime}}\ \mathcal{Q}_r; \mathbb{T} \supseteq \mathbb{S}_{\mathbb{E}_6}^{*}; \mathbb{S}_{\mathbb{E}_6}^{\intercal\ p-3}; z_{\mathbb{E}_6}; \mathbb{T}$ since $p < q$, hypothesis
$\mathtt{sup}_{3 \leq r \leq q,\ r \text{ prime}}\ \mathcal{Q}_r; \mathbb{T} \supseteq \mathbb{S}_{\mathbb{E}_6}^{\intercal\ q-3}; z_{\mathbb{E}_6}; \mathbb{T}$ next doubled prime; already proved as Prop. 10.7.3.ii

---

To complete our reasoning, we have to care for being contained concerning the vector

$$\mathbb{S}_{\mathbb{E}_6}^{+}; \mathbb{S}_{\mathbb{E}_6}^{\intercal\ q-3}; z_{\mathbb{E}_6}; \mathbb{T} \cap \overline{\mathbb{S}_{\mathbb{E}_6}^{*}; \mathbb{S}_{\mathbb{E}_6}^{\intercal\ p-3}; z_{\mathbb{E}_6}; \mathbb{T}} \subseteq \mathtt{sup}_{3 \leq r \leq q,\ r \text{ prime}}\ \mathcal{Q}_r; \mathbb{T}, \tag{*}$$

thus ignoring all from p doubled and below as well as the biggest one by omitting the reflexive part of the transitive closure. We are going to consider this point-wise for an arbitrary point

$a \subseteq \mathbb{S}_{\mathbb{E}_6}^{+}; \mathbb{S}_{\mathbb{E}_6}^{\intercal\ q-3}; z_{\mathbb{E}_6}; \mathbb{T} \cap \overline{\mathbb{S}_{\mathbb{E}_6}^{*}; \mathbb{S}_{\mathbb{E}_6}^{\intercal\ p-3}; z_{\mathbb{E}_6}; \mathbb{T}}.$

Such a point a has necessarily the structure $a = \mathbb{S}_{\mathbb{E}_6}^{\intercal\ s}; z_{\mathbb{E}_6}; \mathbb{T}$ with $p - 3 < s < q - 3$. Since a is a point, it has according to Prop. 2.2.2.ii to be fully contained in at least one item of the union of the $\mathcal{Q}_r; \mathbb{T}$ with $3 \leq r \leq q$, r prime. The task expressed in predicate logic is, therefore, to prove that

$$\begin{aligned} &\forall s : p - 3 < s < q - 3 : \\ &\quad \exists r : 3 \leq r \leq q \text{ and } r \text{ prime: } \mathbb{S}_{\mathbb{E}_6}^{\intercal\ s}; z_{\mathbb{E}_6}; \mathbb{T} \subseteq \mathcal{Q}_r; \mathbb{T}, \end{aligned} \tag{**}$$

i.e., managing to find an r for every given such s.

Our observation in Props. 10.7.2 and 10.7.3 had been that $\mathcal{Q}_r$'s (with r an odd prime) may always be expressed via the next lower prime and, thus, finally by $\mathcal{Q}_3$. Therefore (**) means that at least one of all these shall hold:

$\mathbb{S}_{\mathbb{E}_6}^{\intercal\ s}; z_{\mathbb{E}_6}; \mathbb{T} \subseteq \mathcal{Q}_3; \mathbb{T}$
$\mathbb{S}_{\mathbb{E}_6}^{\intercal\ s}; z_{\mathbb{E}_6}; \mathbb{T} \subseteq \mathcal{Q}_5; \mathbb{T} = \mathbb{S}_{\mathbb{E}_6}^{\intercal\ 1}; \mathcal{Q}_3; \mathbb{S}_{\mathbb{P}}^{\intercal *}; [5\!\rightarrow]$ Prop. 10.7.3.i
$\mathbb{S}_{\mathbb{E}_6}^{\intercal\ s}; z_{\mathbb{E}_6}; \mathbb{T} \subseteq \mathcal{Q}_7; \mathbb{T} = \mathbb{S}_{\mathbb{E}_6}^{\intercal\ 2}; \mathcal{Q}_3; \mathbb{S}_{\mathbb{P}}^{\intercal *}; [7\!\rightarrow]$ again Prop. 10.7.3.i
$\mathbb{S}_{\mathbb{E}_6}^{\intercal\ s}; z_{\mathbb{E}_6}; \mathbb{T} \subseteq \mathcal{Q}_{11}; \mathbb{T} = \mathbb{S}_{\mathbb{E}_6}^{\intercal\ 4}; \mathcal{Q}_3; \mathbb{S}_{\mathbb{P}}^{\intercal *}; [11\!\rightarrow]$ again Prop. 10.7.3.i
$\ldots$

In general there should for any given $p-3 < s < q-3$ exist a prime r with $3 \leq r \leq q$ — or formulated via exponents $0 \leq \xi(r) \leq \xi(q)$ — satisfying

$$\mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{s} ; z_{\mathbb{E}_6} ; \mathbb{T} \subseteq \mathcal{Q}_r ; \mathbb{T} = \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{\frac{r-3}{2}} ; \mathcal{Q}_3 ; \mathbb{S}^{\mathsf{T}*}_{\mathbb{P}} ; [r\!\rightarrow]$$

This containment is equivalent with

$$\mathbb{S}^{s}_{\mathbb{E}_6} ; \overline{\mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{\frac{r-3}{2}} ; \mathcal{Q}_3 ; \mathbb{S}^{\mathsf{T}*}_{\mathbb{P}} ; [r\!\rightarrow]} \subseteq \overline{z_{\mathbb{E}_6} ; \mathbb{T}} = \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6} ; \mathbb{T} \quad \text{Schröder rule}$$

$$\iff \quad \mathbb{S}^{s}_{\mathbb{E}_6} ; \left[\mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{\frac{r-3}{2}} ; \overline{\mathcal{Q}_3 ; \mathbb{S}^{\mathsf{T}*}_{\mathbb{P}} ; [r\!\rightarrow]} \cup \overline{\mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{\frac{r-3}{2}} ; \mathbb{T}} \right] \subseteq \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6} ; \mathbb{T} \quad \text{since } \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{\frac{r-3}{2}} \text{ is univalent}$$

$$\iff \quad \mathbb{S}^{s}_{\mathbb{E}_6} ; \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{\frac{r-3}{2}} ; \overline{\mathcal{Q}_3 ; \mathbb{S}^{\mathsf{T}*}_{\mathbb{P}} ; [r\!\rightarrow]} \subseteq \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6} ; \mathbb{T} \quad \wedge \quad \mathbb{S}^{s}_{\mathbb{E}_6} ; \overline{\mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{\frac{r-3}{2}} ; \mathbb{T}} \subseteq \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6} ; \mathbb{T} = \overline{z_{\mathbb{E}_6} ; \mathbb{T}}$$

We might, however, also take the other version according to Prop. 10.7.3.i, so as to obtain

$$\mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{s} ; z_{\mathbb{E}_6} ; \mathbb{T} \subseteq \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{\frac{r-3}{2}} ; \mathcal{Q}_3 ; \mathbb{T} \cap \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{r-3} ; \mathbb{T}.$$

In total, we have these four criteria

(A) $\mathbb{S}^{s}_{\mathbb{E}_6} ; \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{\frac{r-3}{2}} ; \overline{\mathcal{Q}_3 ; \mathbb{S}^{\mathsf{T}*}_{\mathbb{P}} ; [r\!\rightarrow]} \subseteq \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6} ; \mathbb{T}$

(B) $\mathbb{S}^{s}_{\mathbb{E}_6} ; \overline{\mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{\frac{r-3}{2}} ; \mathbb{T}} \subseteq \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6} ; \mathbb{T} = \overline{z_{\mathbb{E}_6} ; \mathbb{T}} \quad \iff \quad \mathbb{S}^{\mathsf{T}}{}^{s}_{\mathbb{E}_6} ; z_{\mathbb{E}_6} ; \mathbb{T} \subseteq \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{\frac{r-3}{2}} ; \mathbb{T}$

(C) $\mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{s} ; z_{\mathbb{E}_6} ; \mathbb{T} \subseteq \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{\frac{r-3}{2}} ; \mathcal{Q}_3 ; \mathbb{T}$

(D) $\mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{s} ; z_{\mathbb{E}_6} ; \mathbb{T} \subseteq \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{r-3} ; \mathbb{T}$

from which — hopefully — the required r for every s may be determined. A first observation shows (D) $\Longrightarrow$ (B) since

$$\text{(B)} \quad \iff \quad \mathbb{T} \subseteq \mathbb{T} ; z^{\mathsf{T}}_{\mathbb{E}_6} ; \mathbb{S}^{s}_{\mathbb{E}_6} ; \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{\frac{r-3}{2}} ; \mathbb{T} \quad \iff \quad \tfrac{r-3}{2} \leq s \quad \iff \quad r \leq 2 * s + 3$$

$$\text{(D)} \quad \iff \quad \mathbb{T} \subseteq \mathbb{T} ; z^{\mathsf{T}}_{\mathbb{E}_6} ; \mathbb{S}^{s}_{\mathbb{E}_6} ; \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{r-3} ; \mathbb{T} \quad \iff \quad r - 3 \leq s \quad \iff \quad r \leq s + 3$$

so that (D) is the stronger condition and, thus, $r \leq s + 3$. The remaining conditions are

(A) $\mathbb{S}^{s}_{\mathbb{E}_6} ; \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{\frac{r-3}{2}} ; \overline{\mathcal{Q}_3 ; \mathbb{S}^{\mathsf{T}*}_{\mathbb{P}} ; [r\!\rightarrow]} \subseteq \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6} ; \mathbb{T}$

(C) $\mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{s} ; z_{\mathbb{E}_6} ; \mathbb{T} \subseteq \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{\frac{r-3}{2}} ; \mathcal{Q}_3 ; \mathbb{T}$

(D) means $r \leq s + 3$

Let us now observe that $\mathbb{S}^{\mathsf{T}*}_{\mathbb{P}} \cup \mathbb{S}^{+}_{\mathbb{P}} = \mathbb{T}$ since $\mathbb{S}^{+}_{\mathbb{P}}$ is a linear strictorder. The restriction of r means $[r\!\rightarrow] = \mathbb{S}^{\mathsf{T}^{\xi(r)}}_{\mathbb{P}} ; z_{\mathbb{P}} ; \mathbb{T} \subseteq v$, so that also

$$\mathbb{S}^{\mathsf{T}+}_{\mathbb{P}} ; [q\!\rightarrow] \subseteq \overline{v} \quad \text{and} \quad \mathbb{S}^{*}_{\mathbb{P}} ; [q\!\rightarrow] \subseteq v.$$

In total, we conclude that

$$(\mathcal{Q}_3 \cap u ; \mathbb{T}) ; \mathbb{S}^{\mathsf{T}*}_{\mathbb{P}} ; [r\!\rightarrow] = (\mathcal{Q}_3 \cap u ; \mathbb{T}) ; \mathbb{T},$$

so that the containment to prove mentioned above is equivalent with

$$\mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{s} ; z_{\mathbb{E}_6} ; \mathbb{T} \subseteq \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{\frac{r-3}{2}} ; \mathcal{Q}_3 ; \mathbb{T},$$

thus getting rid of one of the two occurrences of r!! After shunting the condition reads

$$\mathbb{T} \subseteq \mathbb{T} ; z^{\mathsf{T}}_{\mathbb{E}_6} ; \mathbb{S}^{s}_{\mathbb{E}_6} ; \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{\frac{r-3}{2}} ; \mathcal{Q}_3 ; \mathbb{T},$$

which can only hold in case $s \geq \frac{r-3}{2}$ or equivalently $r \leq 2 * s + 3$.

Proof so far ok, but still unfinished — finishing means: proof of the Goldbach conjecture. □

10.8 Selfsimilarity expressed via $\mathcal{R}$

Now the third version of possibly detecting selfsimilarity as defined earlier with Def. 10.4.4:

10.8.1 Proposition. $\pi_6^\mathsf{T};\, \mathbb{A}_\mathbb{P}^{\hookleftarrow} = \mathbb{A}_\mathbb{P}^\mathsf{T};\pi_\mathbb{P} = \mathbb{A}_\mathbb{P}^\mathsf{T};\rho_\mathbb{P} = \mathtt{sup}_{r \text{ odd prime}}\, \mathcal{R}_r = \mathcal{R} = \iota_{\mathbb{E}_6};\mathtt{GR};\, \iota_{\wp;\mathbb{T}}^\mathsf{T}$

Proof: We start with the first two equations and use Def. 3.2.1

$\pi_6^\mathsf{T};\, \mathbb{A}_\mathbb{P}^{\hookleftarrow} = \pi_6^\mathsf{T};\, (\,\mathbb{A}_\mathbb{P}^\mathsf{T} \circledcirc \rho_\mathbb{P}^\mathsf{T});\pi_\mathbb{P} = \pi_6^\mathsf{T};\big(\pi_6;\, \mathbb{A}_\mathbb{P}^\mathsf{T} \cap \rho_6;\rho_\mathbb{P}^\mathsf{T}\big);\, \pi_\mathbb{P}$ definition of the right inverse
$= \big(\,\mathbb{A}_\mathbb{P}^\mathsf{T} \cap \mathbb{T};\rho_\mathbb{P}^\mathsf{T}\big);\, \pi_\mathbb{P}$ destroy and append
$= \mathbb{A}_\mathbb{P}^\mathsf{T};\, \pi_\mathbb{P}$ because $\rho_\mathbb{P}$ is total
$= \mathbb{A}_\mathbb{P}^\mathsf{T};\, \rho_\mathbb{P}$ Prop. 6.2.1.ii.c, since $\mathbb{A}_\mathbb{P}$ has been defined in full analogy to $\mathbb{A}$

$\mathcal{R} = \mathtt{sup}_{r \text{ odd prime}}\, \mathcal{R}_r = \mathtt{sup}_{r \text{ odd prime}}\, \iota_{\mathbb{E}_6};\mathbb{S}^{\intercal r};\iota_{\wp;\mathbb{T}}^\mathsf{T} = \iota_{\mathbb{E}_6};\big[\,\mathtt{sup}_{r \text{ odd prime}}\, \mathbb{S}^{\intercal r}\big];\iota_{\wp;\mathbb{T}}^\mathsf{T}$
$= \iota_{\mathbb{E}_6};\big[\,\mathtt{sup}_{r \text{ odd prime}}\, \mathbb{A}^\intercal;\, (\mathbb{S}^{\intercal r};z;\mathbb{T} \circledcirc \mathbb{I})\,\big];\iota_{\wp;\mathbb{T}}^\mathsf{T}$ Prop. 6.2.1.ii.e
$= \iota_{\mathbb{E}_6};\, \mathbb{A}^\intercal;\, (\mathtt{sup}_{r \text{ odd prime}}\, \mathbb{S}^{\intercal r};z;\mathbb{T} \circledcirc \mathbb{I});\iota_{\wp;\mathbb{T}}^\mathsf{T}$
$= \iota_{\mathbb{E}_6};\, \mathbb{A}^\intercal;\, (\iota_{\wp;\mathbb{T}}^\mathsf{T};\mathbb{T} \circledcirc \mathbb{I});\iota_{\wp;\mathbb{T}}^\mathsf{T} = \iota_{\mathbb{E}_6};\, \mathbb{A}^\intercal;\, (\iota_{\wp;\mathbb{T}}^\mathsf{T};\mathbb{T};\, \iota_{\wp;\mathbb{T}}^\mathsf{T} \circledcirc \iota_{\wp;\mathbb{T}}^\mathsf{T})$
$= \iota_{\mathbb{E}_6};\, \mathbb{A}^\intercal;\, (\iota_{\wp;\mathbb{T}}^\mathsf{T};\mathbb{T} \circledcirc \iota_{\wp;\mathbb{T}}^\mathsf{T})$ since $\iota_{\wp;\mathbb{T}}$ is a map
$= \iota_{\mathbb{E}_6};\, \mathbb{A}^\intercal;\, (\wp;\mathbb{T} \circledcirc \wp;\iota_{\wp;\mathbb{T}}^\mathsf{T}) = \iota_{\mathbb{E}_6};\, \mathbb{A}^\intercal;\, (\wp \otimes \wp);\, (\mathbb{T}_{\mathbb{N}\times\mathbb{P}} \circledcirc \iota_{\wp;\mathbb{T}}^\mathsf{T})$
$= \iota_{\mathbb{E}_6};\, \mathbb{A}^\intercal;\, (\wp \otimes \wp);\rho;\iota_{\wp;\mathbb{T}}^\mathsf{T}$
meaning not least $= \iota_{\mathbb{E}_6};\, \mathbb{A}^\intercal;\, (\wp \otimes \wp);\rho;\iota_{\wp;\mathbb{T}}^\mathsf{T}$
$= \iota_{\mathbb{E}_6};\, \mathbb{A}^\intercal;\, (\wp \otimes \wp);\, (\iota_{\wp;\mathbb{T}}^\mathsf{T} \otimes \iota_{\wp;\mathbb{T}}^\mathsf{T});\rho_\mathbb{P} = \iota_{\mathbb{E}_6};\, \mathbb{A}^\intercal;\, (\iota_{\wp;\mathbb{T}}^\mathsf{T} \otimes \iota_{\wp;\mathbb{T}}^\mathsf{T});\rho_\mathbb{P} = \mathbb{A}_\mathbb{P}^\mathsf{T};\, \rho_\mathbb{P}$
$= \iota_{\mathbb{E}_6};\mathtt{GD};\, \rho;\, \iota_{\wp;\mathbb{T}}^\mathsf{T}$ with Def. 10.1.1.ii
$= \iota_{\mathbb{E}_6};\mathtt{GR};\, \iota_{\wp;\mathbb{T}}^\mathsf{T}$ according to Def. 10.1.1 □

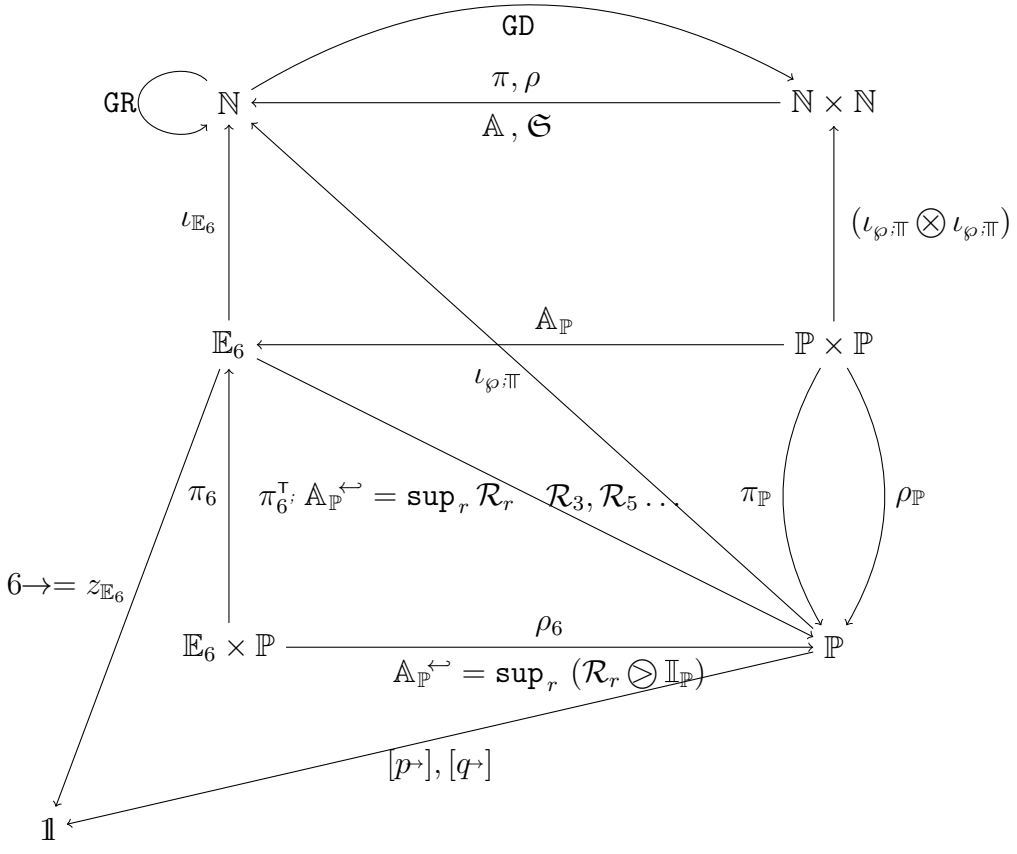


Fig. 10.8.1 Further typing for primes

The next idea is to consider the strictorder $\mathbb{S}^+_{\mathbb{P}}$ on $\mathbb{P}$ and use it in order to obtain row-wise the least prime of $\iota_{\mathbb{E}_6};\mathtt{GR};\iota^{\mathsf{T}}_{\wp;\mathbb{T}}$.

To understand the following, we should make clear that $\mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{m-3};z_{\mathbb{E}_6};\mathbb{T}$ means the even number ≥ 6 that is the sum $2*m$, i.e. a number doubled and thus contained in $\mathbb{E}_6$. In consequence, the following term $\mathbb{S}^*_{\mathbb{E}_6};\mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{m-3};z_{\mathbb{E}_6};\mathbb{T}$ means the subset of all evens below the doubled number p.

The following proposition follows the analogical Props. 10.6.1 and 10.7.2 for $\mathcal{P}$ and $\mathcal{Q}$.

10.8.2 Proposition. Given any two *consecutive* odd primes p, q, i.e. two points satisfying $\mathbb{S}^{\mathsf{T}p};z;\mathbb{T} = \mathtt{NP};\mathbb{S}^{\mathsf{T}q};z;\mathbb{T}$, or equivalently $\mathbb{S}^{\mathsf{T}}_{\mathbb{P}};[p\rightarrow] = [q\rightarrow]$,
we have, abbreviating $d := \frac{q-p}{2}$, these assertions:

i) $\mathcal{R}_q = \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{d};\mathcal{R}_p$

ii) $\mathcal{R}_p \supseteq \mathbb{S}^{d}_{\mathbb{E}_6};\mathcal{R}_q$

iii) For arbitrary odd primes $p < r$ always
$\mathcal{R}_r = \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{\frac{r-p}{2}};\mathcal{R}_p$ in particular: $\mathcal{R}_r = \mathbb{S}^{\mathsf{T}}_{\mathbb{E}_6}{}^{\frac{r-3}{2}};\mathcal{R}_3$

$$\pi^{\mathsf{T}}_6;\mathbb{A}_{\mathbb{P}}{}^{\hookleftarrow} = \begin{array}{c} \\ 6\rightarrow \\ 8\rightarrow \\ 10\rightarrow \\ 12\rightarrow \\ 14\rightarrow \\ 16\rightarrow \\ 18\rightarrow \\ 20\rightarrow \\ 22\rightarrow \\ 24\rightarrow \\ 26\rightarrow \\ 28\rightarrow \\ 30\rightarrow \\ 32\rightarrow \\ 34\rightarrow \\ 36\rightarrow \\ 38\rightarrow \\ 40\rightarrow \\ 42\rightarrow \\ 44\rightarrow \\ 46\rightarrow \\ 48\rightarrow \\ 50\rightarrow \\ 52\rightarrow \\ 54\rightarrow \\ 56\rightarrow \\ 58\rightarrow \\ 60\rightarrow \end{array} \begin{array}{c} \begin{array}{cccccccccccccccc} 3\rightarrow & 5\rightarrow & 7\rightarrow & 11\rightarrow & 13\rightarrow & 17\rightarrow & 19\rightarrow & 23\rightarrow & 29\rightarrow & 31\rightarrow & 37\rightarrow & 41\rightarrow & 43\rightarrow & 47\rightarrow & 53\rightarrow & 59\rightarrow \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array} = \mathcal{R}$$

Fig. 10.8.2 $\iota_{\mathbb{E}_6};\mathtt{GR};\iota^{\mathsf{T}}_{\wp;\mathbb{T}} = \mathcal{R}$ — recalling and enlarging Fig. 10.1.4

Proof: i) $\mathcal{R}_q = \iota_{\mathbb{E}_6};\mathbb{S}^{\mathsf{T}q};\iota^{\mathsf{T}}_{\wp;\mathbb{T}} = \iota_{\mathbb{E}_6};\mathbb{S}^{\mathsf{T}2*d};\mathbb{S}^{\mathsf{T}p};\iota^{\mathsf{T}}_{\wp;\mathbb{T}}$ by definition; now to be splitted

$= \iota_{\mathbb{E}_6};\Big[\Big(\mathbb{S}^{\mathsf{T}2*d} \cap \overline{\mathbb{T};z^{\mathsf{T}};\mathbb{S}^4 \cup \mathbb{T};z^{\mathsf{T}};\mathbb{S}^2 \cup \mathbb{T};z^{\mathsf{T}}}\Big) \cup \Big(\mathbb{S}^{\mathsf{T}2*d} \cap [\mathbb{T};z^{\mathsf{T}};\mathbb{S}^4 \cup \mathbb{T};z^{\mathsf{T}};\mathbb{S}^2 \cup \mathbb{T};z^{\mathsf{T}}]\Big)\Big];\mathbb{S}^{\mathsf{T}p};\iota^{\mathsf{T}}_{\wp;\mathbb{T}}$

$$= \left(\iota_{\mathbb{E}_6}; \mathbb{S}^{\intercal 2*d} \cap \overline{\mathbb{T}; z^\intercal; \mathbb{S}^4 \cup \mathbb{T}; z^\intercal; \mathbb{S}^2 \cup \mathbb{T}; z^\intercal}\right); \mathbb{S}^{\intercal p}; \iota^\intercal_{\wp;\mathbb{T}}$$
$$\cup \left(\iota_{\mathbb{E}_6}; \mathbb{S}^{\intercal 2*d} \cap [\mathbb{T}; z^\intercal; \mathbb{S}^4 \cup \mathbb{T}; z^\intercal; \mathbb{S}^2 \cup \mathbb{T}; z^\intercal]\right); \mathbb{S}^{\intercal p}; \iota^\intercal_{\wp;\mathbb{T}}$$
$$= \mathbb{S}^\intercal_{\mathbb{E}_6}{}^d; \iota_{\mathbb{E}_6}; \mathbb{S}^{\intercal p}; \iota^\intercal_{\wp;\mathbb{T}} \cup \perp\!\!\!\perp \quad \text{Prop. 10.1.3.vi and see below}$$
$$= \mathbb{S}^\intercal_{\mathbb{E}_6}{}^d; \mathcal{R}_p$$

The task remains to prove that in the second term for odd primes $p \geq 3$ always

$$\left(\iota_{\mathbb{E}_6}; \mathbb{S}^{\intercal 2*d} \cap [\mathbb{T}; z^\intercal; \mathbb{S}^4 \cup \mathbb{T}; z^\intercal; \mathbb{S}^2 \cup \mathbb{T}; z^\intercal]\right); \mathbb{S}^{\intercal p}; \iota^\intercal_{\wp;\mathbb{T}}$$
$$\subseteq [\mathbb{T}; z^\intercal; \mathbb{S}^4 \cup \mathbb{T}; z^\intercal; \mathbb{S}^2 \cup \mathbb{T}; z^\intercal]; \mathbb{S}^{\intercal p}; \iota^\intercal_{\wp;\mathbb{T}}$$
$$= \mathbb{T}; z^\intercal; \mathbb{S}^4; \mathbb{S}^{\intercal p}; \iota^\intercal_{\wp;\mathbb{T}} \cup \mathbb{T}; z^\intercal; \mathbb{S}^2; \mathbb{S}^{\intercal p}; \iota^\intercal_{\wp;\mathbb{T}} \cup \mathbb{T}; z^\intercal; \mathbb{S}^{\intercal p}; \iota^\intercal_{\wp;\mathbb{T}}$$
$$= \mathbb{T}; z^\intercal; \mathbb{S}^4; \mathbb{S}^{\intercal p}; \iota^\intercal_{\wp;\mathbb{T}} \cup \perp\!\!\!\perp \cup \perp\!\!\!\perp \quad \text{since } \mathbb{S}; z; \mathbb{T} = \perp\!\!\!\perp$$
$$= \begin{cases} \mathbb{T}; z^\intercal; \mathbb{S}; \iota^\intercal_{\wp;\mathbb{T}} = \mathbb{T}; e^\intercal; \iota^\intercal_{\wp;\mathbb{T}} & \text{if } p = 3 \\ \perp\!\!\!\perp & \text{if } p > 3 \end{cases}$$
$$= \perp\!\!\!\perp \quad \text{since } e; \mathbb{T} \not\subseteq \iota^\intercal_{\wp;\mathbb{T}}; \mathbb{T} \text{ is not an odd prime}$$

---

ii) $\mathbb{S}^d_{\mathbb{E}_6}; \mathcal{R}_q = (\iota_{\mathbb{E}_6}; \mathbb{S}^2; \iota^\intercal_{\mathbb{E}_6})^d; \iota_{\mathbb{E}_6}; \mathbb{S}^{\intercal q}; \iota^\intercal_{\wp;\mathbb{T}}$ expanding via Def. 10.1.2 and 10.4.4
$= \iota_{\mathbb{E}_6}; \mathbb{S}^{2*d}; \mathbb{S}^{\intercal q}; \iota^\intercal_{\wp;\mathbb{T}}$ Prop. 10.1.3.ii
$\subseteq \iota_{\mathbb{E}_6}; \mathbb{S}^{\intercal p}; \iota^\intercal_{\wp;\mathbb{T}}$ since $\mathbb{S}$ is injective and $q - 2 * d = p$
$= \mathcal{R}_p$ by definition

---

iii) This is proved by induction with hypothesis $\mathcal{R}_{p_i} = \mathbb{S}^\intercal_{\mathbb{E}_6}{}^{\frac{p_i - p_0}{2}}; \mathcal{R}_{p_0}$ and starting with the full sequence of primes $p =: p_0 < p_1 < \ldots < r$.

$\mathcal{R}_{p_1} = \mathbb{S}^\intercal_{\mathbb{E}_6}{}^{\frac{p_1 - p_0}{2}}; \mathcal{R}_{p_0}$ following (i)
$\mathcal{R}_{p_{i+1}} = \mathbb{S}^\intercal_{\mathbb{E}_6}{}^{\frac{p_{i+1} - p_i}{2}}; \mathcal{R}_{p_i}$ using (i) again
$= \mathbb{S}^\intercal_{\mathbb{E}_6}{}^{\frac{p_{i+1} - p_i}{2}}; \mathbb{S}^\intercal_{\mathbb{E}_6}{}^{\frac{p_i - p_0}{2}}; \mathcal{R}_{p_0}$ hypothesis
$= \mathbb{S}^\intercal_{\mathbb{E}_6}{}^{\frac{p_{i+1} - p_0}{2}}; \mathcal{R}_{p_0}$ □

$\mathcal{R}_{17}$

	3↑	5↑	7↑	11↑	13↑	17↑	19↑	23↑	29↑
6→	0	0	0	0	0	0	0	0	0
8→	0	0	0	0	0	0	0	0	0
10→	0	0	0	0	0	0	0	0	0
12→	0	0	0	0	0	0	0	0	0
14→	0	0	0	0	0	0	0	0	0
16→	0	0	0	0	0	0	0	0	0
18→	0	0	0	0	0	0	0	0	0
20→	1	0	0	0	0	0	0	0	0
22→	0	1	0	0	0	0	0	0	0
24→	0	0	1	0	0	0	0	0	0
26→	0	0	0	0	0	0	0	0	0
28→	0	0	0	1	0	0	0	0	0

$\mathcal{R}_{13}$

	3↑	5↑	7↑	11↑	13↑	17↑	19↑	23↑	29↑
6→	0	0	0	0	0	0	0	0	0
8→	0	0	0	0	0	0	0	0	0
10→	0	0	0	0	0	0	0	0	0
12→	0	0	0	0	0	0	0	0	0
14→	0	0	0	0	0	0	0	0	0
16→	1	0	0	0	0	0	0	0	0
18→	0	1	0	0	0	0	0	0	0
20→	0	0	1	0	0	0	0	0	0
22→	0	0	0	0	0	0	0	0	0
24→	0	0	0	1	0	0	0	0	0
26→	0	0	0	0	1	0	0	0	0
28→	0	0	0	0	0	0	0	0	0

$\mathcal{R}_{11}$

	3↑	5↑	7↑	11↑	13↑	17↑	19↑	23↑	29↑
6→	0	0	0	0	0	0	0	0	0
8→	0	0	0	0	0	0	0	0	0
10→	0	0	0	0	0	0	0	0	0
12→	0	0	0	0	0	0	0	0	0
14→	1	0	0	0	0	0	0	0	0
16→	0	1	0	0	0	0	0	0	0
18→	0	0	1	0	0	0	0	0	0
20→	0	0	0	0	0	0	0	0	0
22→	0	0	0	1	0	0	0	0	0
24→	0	0	0	0	1	0	0	0	0
26→	0	0	0	0	0	0	0	0	0
28→	0	0	0	0	0	1	0	0	0

Fig. 10.8.3 Illustrating $\mathbb{S}^\intercal_{\mathbb{E}_6}{}^{\frac{17-13}{2}}; \mathcal{R}_{17} \subseteq \mathcal{R}_{13} = \mathbb{S}^\intercal_{\mathbb{E}_6}{}^{\frac{13-11}{2}}; \mathcal{R}_{11}$

The following proposition is the analog of Prop. 10.7.3 for $\mathcal{Q}$.

10.8.3 Proposition. For any given odd prime p the following holds:

i) $\mathcal{R}_p ; \mathbb{T} = \mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\,\frac{p-3}{2}} ; \mathcal{R}_3 ; \mathbb{T}$

ii) $\mathcal{R}_p ; \mathbb{T} \supseteq \mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\,\frac{p-3}{2}} ; z_{\mathbb{E}_6} ; \mathbb{T}$

Proof: i) $\mathcal{R}_p ; \mathbb{T} = \mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\,\frac{p-3}{2}} ; \mathcal{R}_3 ; \mathbb{T}$ due to Prop. 10.8.2.iii

---

ii) We recall that $\mathcal{R}_p ; \mathbb{T} = \iota_{\mathbb{E}_6} ; \mathbb{S}^{\mathsf{T}p} ; \iota^{\mathsf{T}}_{\wp;\mathbb{T}} ; \mathbb{T}$ using Def. 10.4.4. Then the claim reads

$\mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\,\frac{p-3}{2}} ; z_{\mathbb{E}_6} ; \mathbb{T} \subseteq \iota_{\mathbb{E}_6} ; \mathbb{S}^{\mathsf{T}p} ; \iota^{\mathsf{T}}_{\wp;\mathbb{T}} ; \mathbb{T} = \iota_{\mathbb{E}_6} ; \mathbb{S}^{\mathsf{T}p} ; \wp ; \mathbb{T}$ since $\iota_{\wp;\mathbb{T}}$ extrudes $\wp ; \mathbb{T}$

$\Longleftarrow \quad \mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\,\frac{p-3}{2}} ; z_{\mathbb{E}_6} ; \mathbb{T} \subseteq \iota_{\mathbb{E}_6} ; \mathbb{S}^{\mathsf{T}p} ; \iota^{\mathsf{T}}_{\wp;\mathbb{T}} [3\!\rightarrow]$ specialized using $\iota^{\mathsf{T}}_{\wp;\mathbb{T}} [3\!\rightarrow] \subseteq \wp ; \mathbb{T}$

$\Longleftrightarrow \quad \mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\,\frac{p-3}{2}} ; z_{\mathbb{E}_6} ; \mathbb{T} \subseteq \iota_{\mathbb{E}_6} ; \mathbb{S}^{\mathsf{T}p} ; \mathbb{S}^{\mathsf{T}3} ; z ; \mathbb{T} = \iota_{\mathbb{E}_6} ; \mathbb{S}^{\mathsf{T}p+3} ; z ; \mathbb{T}$

$\Longleftrightarrow \quad \iota^{\mathsf{T}}_{\mathbb{E}_6} ; \mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\,\frac{p-3}{2}} ; z_{\mathbb{E}_6} ; \mathbb{T} \subseteq \mathbb{S}^{\mathsf{T}p+3} ; z ; \mathbb{T}$ via shunting

Now we evaluate the left term with Prop. 10.1.3.viii:

$\iota^{\mathsf{T}}_{\mathbb{E}_6} ; \mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\,\frac{p-3}{2}} ; z_{\mathbb{E}_6} ; \mathbb{T} = \mathbb{S}^{\mathsf{T}p-3} ; \iota^{\mathsf{T}}_{\mathbb{E}_6} ; z_{\mathbb{E}_6} ; \mathbb{T} = \mathbb{S}^{\mathsf{T}p-3} ; \mathbb{S}^{\mathsf{T}6} ; z ; \mathbb{T} = \mathbb{S}^{\mathsf{T}p+3} ; z ; \mathbb{T}$ □

The following proposition corresponds to Props. 10.6.2 and 10.7.4 for $\mathcal{P}$ and $\mathcal{Q}$.

10.8.4 Proposition. If we define for any given odd prime p,

$\mathtt{ROWUNION}_p := \mathtt{sup}_{3 \le r \le p,\ r \text{ prime}} \mathcal{R}_r ; \mathbb{T}$,

the following will hold

$\mathbb{S}^*_{\mathbb{E}_6} ; \mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\,p-3} ; z_{\mathbb{E}_6} ; \mathbb{T} \subseteq \mathtt{ROWUNION}_p$.

Proof: We attempt a proof by induction over the extruded odd primes $[p\!\rightarrow]$ of $\mathbb{P}$, starting with $p = 3$, resp. $[3\!\rightarrow]$:

$\mathbb{S}^*_{\mathbb{E}_6} ; \mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\,3-3} ; z_{\mathbb{E}_6} ; \mathbb{T} = \mathbb{S}^*_{\mathbb{E}_6} ; z_{\mathbb{E}_6} ; \mathbb{T} = z_{\mathbb{E}_6} ; \mathbb{T}$ since $\mathbb{S}_{\mathbb{E}_6} ; z_{\mathbb{E}_6} = \mathbb{\perp}$

$= \mathbb{S}^0_{\mathbb{E}_6} ; z_{\mathbb{E}_6} ; \mathbb{T}$

$= \iota_{\mathbb{E}_6} ; \mathbb{S}^{\mathsf{T}6} ; z ; \mathbb{T}$ Prop. 10.1.3.vii

$= \iota_{\mathbb{E}_6} ; \mathbb{S}^{\mathsf{T}3} ; \mathbb{S}^{\mathsf{T}3} ; z ; \mathbb{T} = \iota_{\mathbb{E}_6} ; \mathbb{S}^{\mathsf{T}3} ; \iota^{\mathsf{T}}_{\wp;\mathbb{T}} [3\!\rightarrow] = \mathcal{R}_3 ; [3\!\rightarrow]$ using Prop. 9.4.4.i and Def. 10.4.4

$\subseteq \mathcal{R}_3 ; \mathbb{T}$

$= \mathtt{ROWUNION}_3$

Now the induction step $p \mapsto q$, or better for $[p\!\rightarrow] \mapsto [q\!\rightarrow] = \mathbb{S}^{\mathsf{T}}_{\mathbb{P}} ; [p\!\rightarrow]$, i.e. to the next consecutive prime. We have to consider this area of numbers, mentioning distances $d := \frac{q-p}{2}$ below:

in $\mathbb{N}$: $\quad \ldots \quad 2*p \quad \ldots \quad p+q \quad \ldots \quad 2*q$

in $\mathbb{E}_6$: $\quad \ldots \quad \frac{2*p-6}{2} \quad \ldots \quad \frac{p+q-6}{2} \quad \ldots \quad \frac{2*q-6}{2}$, i.e.

$\mathbb{E}_6 \quad \supseteq \quad \mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\,p-3} ; z_{\mathbb{E}_6} ; \mathbb{T} \quad \mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\,\frac{p+q-6}{2}} ; z_{\mathbb{E}_6} ; \mathbb{T} \quad \mathbb{S}_{\mathbb{E}_6}^{\mathsf{T}\,q-3} ; z_{\mathbb{E}_6} ; \mathbb{T}$

$\ldots$ hypothesis; d to prove; d to prove; interval lengths in $\mathbb{E}_6$

Concerning these intervals remember Lemma 10.5.1. If p, q should happen to constitute a prime twin, everything is clear, because then $d = 1$, meaning that there cannot be an even number in the corresponding intervals.

As a second simpler case assume $d = 2$, meaning just one even number $2 * p < e < p + q$ as well as $p + q < e' < 2 * q$, namely $e = 2 * (p + 1) = 2 * (q - 3)$ and $e' = 2 * (p + 3) = 2 * (q - 1)$.

Proof so far ok, but still unfinished — finishing means: proof of the Goldbach conjecture. □

11 Further number concepts

With the methods developed so far, not least with canceling, we give an entry to rationals. This chapter is definitely open-ended and serves mainly the purpose of visualization.

11.1 Nonnegative rationals

We start from the canceling relation C as derived on page 194. Then $\Xi := C;C^\mathsf{T} \cup \overline{C;\mathbb{T}};\overline{C;\mathbb{T}}^\mathsf{T}$ is an equivalence that may be divided out with η_Ξ, the respective natural projection. In order to get rid of 0's in the denominator, we modify it, however, to

$\eta := \eta_\Xi \cap \overline{\rho;z;\mathbb{T}}$,

thus making η just a univalent and surjective relation instead of a map and obtaining

$\eta_\Xi =$

	[0,0]	[1,1]	[1,2]	[1,3]	[1,4]	[2,1]	[2,3]	[3,1]	[3,2]	[3,4]	[4,1]	[4,3]
(0,0)	1	0	0	0	0	0	0	0	0	0	0	0
(0,1)	1	0	0	0	0	0	0	0	0	0	0	0
(0,2)	1	0	0	0	0	0	0	0	0	0	0	0
(0,3)	1	0	0	0	0	0	0	0	0	0	0	0
(0,4)	1	0	0	0	0	0	0	0	0	0	0	0
(1,0)	1	0	0	0	0	0	0	0	0	0	0	0
(1,1)	0	1	0	0	0	0	0	0	0	0	0	0
(1,2)	0	0	1	0	0	0	0	0	0	0	0	0
(1,3)	0	0	0	1	0	0	0	0	0	0	0	0
(1,4)	0	0	0	0	1	0	0	0	0	0	0	0
(2,0)	1	0	0	0	0	0	0	0	0	0	0	0
(2,1)	0	0	0	0	0	1	0	0	0	0	0	0
(2,2)	0	1	0	0	0	0	0	0	0	0	0	0
(2,3)	0	0	0	0	0	0	1	0	0	0	0	0
(2,4)	0	0	1	0	0	0	0	0	0	0	0	0
(3,0)	1	0	0	0	0	0	0	0	0	0	0	0
(3,1)	0	0	0	0	0	0	0	1	0	0	0	0
(3,2)	0	0	0	0	0	0	0	0	1	0	0	0
(3,3)	0	1	0	0	0	0	0	0	0	0	0	0
(3,4)	0	0	0	0	0	0	0	0	0	1	0	0
(4,0)	1	0	0	0	0	0	0	0	0	0	0	0
(4,1)	0	0	0	0	0	0	0	0	0	0	1	0
(4,2)	0	0	0	0	0	1	0	0	0	0	0	0
(4,3)	0	0	0	0	0	0	0	0	0	0	0	1
(4,4)	0	1	0	0	0	0	0	0	0	0	0	0

$\eta =$

	[0,0]	[1,1]	[1,2]	[1,3]	[1,4]	[2,1]	[2,3]	[3,1]	[3,2]	[3,4]	[4,1]	[4,3]
(0,0)	0	0	0	0	0	0	0	0	0	0	0	0
(0,1)	1	0	0	0	0	0	0	0	0	0	0	0
(0,2)	1	0	0	0	0	0	0	0	0	0	0	0
(0,3)	1	0	0	0	0	0	0	0	0	0	0	0
(0,4)	1	0	0	0	0	0	0	0	0	0	0	0
(1,0)	0	0	0	0	0	0	0	0	0	0	0	0
(1,1)	0	1	0	0	0	0	0	0	0	0	0	0
(1,2)	0	0	1	0	0	0	0	0	0	0	0	0
(1,3)	0	0	0	1	0	0	0	0	0	0	0	0
(1,4)	0	0	0	0	1	0	0	0	0	0	0	0
(2,0)	0	0	0	0	0	0	0	0	0	0	0	0
(2,1)	0	0	0	0	0	1	0	0	0	0	0	0
(2,2)	0	1	0	0	0	0	0	0	0	0	0	0
(2,3)	0	0	0	0	0	0	1	0	0	0	0	0
(2,4)	0	0	1	0	0	0	0	0	0	0	0	0
(3,0)	0	0	0	0	0	0	0	0	0	0	0	0
(3,1)	0	0	0	0	0	0	0	1	0	0	0	0
(3,2)	0	0	0	0	0	0	0	0	1	0	0	0
(3,3)	0	1	0	0	0	0	0	0	0	0	0	0
(3,4)	0	0	0	0	0	0	0	0	0	1	0	0
(4,0)	0	0	0	0	0	0	0	0	0	0	0	0
(4,1)	0	0	0	0	0	0	0	0	0	0	1	0
(4,2)	0	0	0	0	0	1	0	0	0	0	0	0
(4,3)	0	0	0	0	0	0	0	0	0	0	0	1
(4,4)	0	1	0	0	0	0	0	0	0	0	0	0

Fig. 11.1.1 η and η_{Xi}

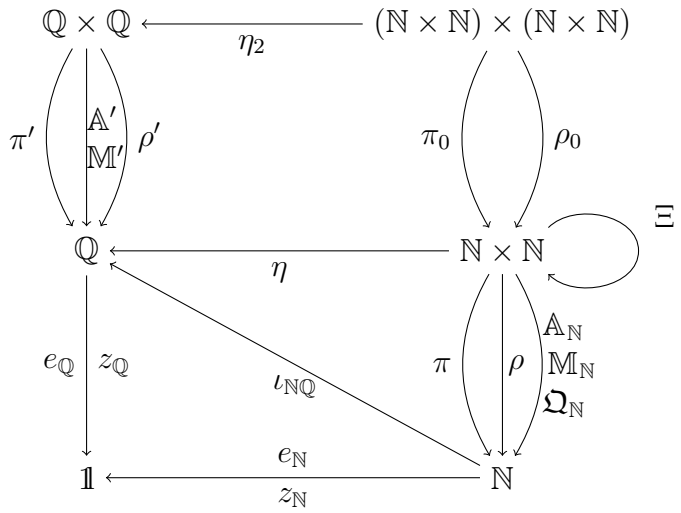


Fig. 11.1.2 Forming nonnegative rationals $\mathbb{Q}$ out of $\mathbb{N}$

Some of the necessary definitions are immediate:

injection of naturals into the rationals $\iota_{\mathbb{N}\mathbb{Q}} := (\mathbb{I} \oslash \mathbb{T} ; e_{\mathbb{N}}^{\mathsf{T}}) ; \eta$
z of rationals via z into the naturals $z_{\mathbb{Q}} := \iota_{\mathbb{N}\mathbb{Q}}^{\mathsf{T}} ; z_{\mathbb{N}}$
e of rationals via e into the naturals $e_{\mathbb{Q}} := \iota_{\mathbb{N}\mathbb{Q}}^{\mathsf{T}} ; e_{\mathbb{N}}$

$$\iota_{\mathbb{N}\mathbb{Q}} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccccccccc} [0,0] & [1,1] & [1,2] & [1,3] & [1,4] & [2,1] & [2,3] & [3,1] & [3,2] & [3,4] & [4,1] & [4,3] \end{array} \\ \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \end{pmatrix} \end{array}$$

Fig. 11.1.3 Injecting naturals $\mathbb{N}$ into the corresponding rationals $\mathbb{Q}$

Further, we have to define the traditional operations on $\mathbb{Q}$, corresponding to

$$\tfrac{a}{b} + \tfrac{c}{d} = \tfrac{ad+bc}{bd} \qquad \tfrac{a}{b} - \tfrac{c}{d} = \tfrac{ad-bc}{bd} \qquad \tfrac{a}{b} \cdot \tfrac{c}{d} = \tfrac{ac}{bd} \qquad \tfrac{a}{b} : \tfrac{c}{d} = \tfrac{ad}{bc}.$$

The construct $\eta_2 := (\eta \otimes \eta)$ will mainly be used in reversed direction to conceive a pair of rationals so as to work further with a pair of pairs of naturals when defining addition and other binary operations via those on the naturals.

From the above operations addition is very likely to run into problems when $ad + bc$ is out of limits — if we work with limited naturals. To facilitate our work, let us therefore start with multiplication. When unfolding the terms to trees (they are in fact DAGs, directed acyclic gaphs), they look as in Fig. 11.1.4.

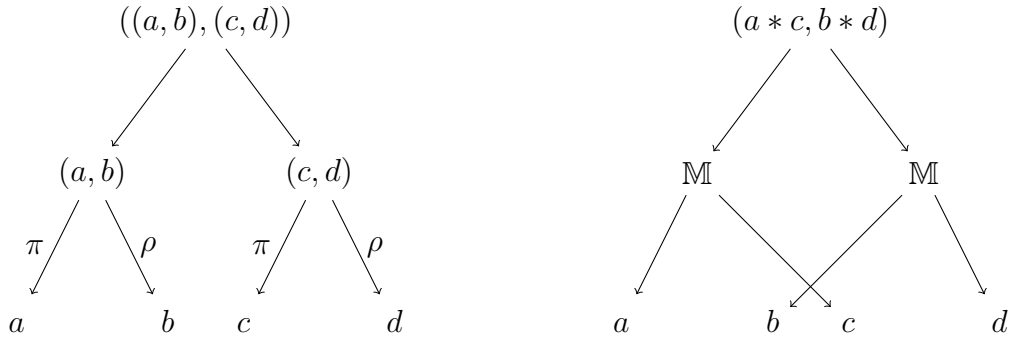


Fig. 11.1.4 Multiplying two rationals

Most naively, this is realized as

$\mathbb{M}_{\mathbb{Q}} := \eta_2^{\mathsf{T}} ; \big((\pi \otimes \pi) ; \mathbb{M} \oslash (\rho \otimes \rho) ; \mathbb{M} \big) ; \eta,$

but it fails to bring all conceivable products, restricting to those shown on the left of Fig. 11.1.6. Consider numbers limited by 4 and the task

$\frac{2}{3} * \frac{3}{2}$

resulting in $\frac{6}{6} = \frac{1}{1}$ inside limitation, however with intermediate results outside. The chance to cancel by 3 has not been used, which leads us to go from $((2,3),(3,2))$ flipping denominators to $((2,2),(3,3))$, so as to obtain $(1,1)$ — and to arrive on the right of Fig. 11.1.6.

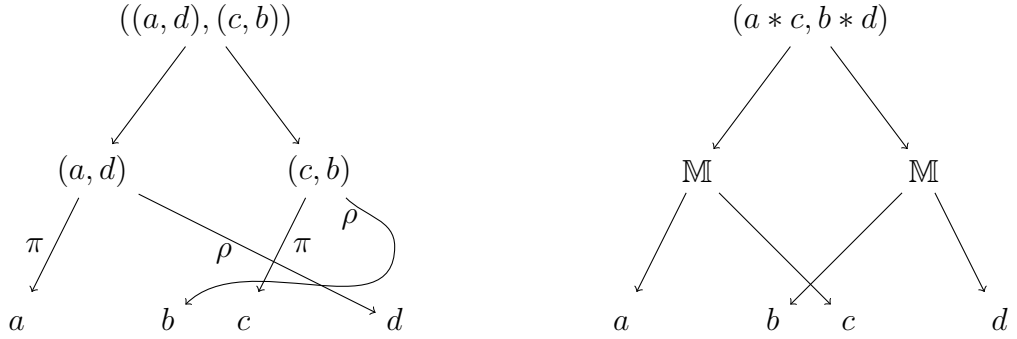


Fig. 11.1.5 Multiplying two rationals

$$\begin{aligned}\mathbb{M}_{\mathbb{Q}} &:= \eta_2^{\mathsf{T}} ; \big[\mathbb{I} \cup \big((\pi ; \pi^{\mathsf{T}} \oslash \rho ; \rho^{\mathsf{T}}) ; C \oslash (\rho ; \rho^{\mathsf{T}} \oslash \pi ; \pi^{\mathsf{T}}) ; C \big) \big] ; \big((\pi \otimes \pi) ; \mathbb{M} \oslash (\rho \otimes \rho) ; \mathbb{M} \big) ; \eta \\ &= \eta_2^{\mathsf{T}} \big[\mathbb{I} \cup \big((\pi ; \pi^{\mathsf{T}} \oslash \rho ; \rho^{\mathsf{T}}) \oslash (\rho ; \rho^{\mathsf{T}} \oslash \pi ; \pi^{\mathsf{T}}) \big) \big] ; (C \otimes C) ; \big((\pi \otimes \pi) \oslash (\rho \otimes \rho) \big) ; \big(\mathbb{M} \otimes \mathbb{M} \big) ; \eta\end{aligned}$$

By the way:

$\big((\pi ; \pi^{\mathsf{T}} \oslash \rho ; \rho^{\mathsf{T}}) \oslash (\rho ; \rho^{\mathsf{T}} \oslash \pi ; \pi^{\mathsf{T}}) \big) = \big((\pi ; \pi^{\mathsf{T}} \oslash \rho ; \rho^{\mathsf{T}}) \oslash (\rho ; \rho^{\mathsf{T}} \oslash \pi ; \pi^{\mathsf{T}}) \big)$

which is probably useful.

The "funny term" $[\mathbb{I} \cup \ldots]$ in the middle — instead of just $\mathbb{I}$, resp. simply nothing — is intended to capture this phenomenon: Consider numbers limited by 7 and the task

$\frac{3}{4} * \frac{7}{3}$

resulting in $\frac{21}{12} = \frac{7}{4}$ inside limitation, however with intermediate results outside. Switching to $\frac{3}{3} * \frac{7}{4}$ with "automatic" canceling C lets these stay inside — which needs to be proved.

	[(0,0)]	[(1,1)]	[(1,2)]	[(1,3)]	[(1,4)]	[(2,1)]	[(2,3)]	[(3,1)]	[(3,2)]	[(3,4)]	[(4,1)]	[(4,3)]
[(0,0)]	1	1	1	1	1	1	1	1	1	1	1	1
[(1,1)]	1	2	3	4	5	6	7	8	9	10	11	12
[(1,2)]	1	3	5			2		9	10		6	
[(1,3)]	1	4				7		2			12	
[(1,4)]	1	5				3		10			2	
[(2,1)]	1	6	2	7	3	11	12					
[(2,3)]	1	7				12						
[(3,1)]	1	8	9	2	10							
[(3,2)]	1	9	10									
[(3,4)]	1	10										
[(4,1)]	1	11	6	12	2							
[(4,3)]	1	12										

	[(0,0)]	[(1,1)]	[(1,2)]	[(1,3)]	[(1,4)]	[(2,1)]	[(2,3)]	[(3,1)]	[(3,2)]	[(3,4)]	[(4,1)]	[(4,3)]
[(0,0)]	1	1	1	1	1	1	1	1	1	1	1	1
[(1,1)]	1	2	3	4	5	6	7	8	9	10	11	12
[(1,2)]	1	3	5			2	4	9	10		6	7
[(1,3)]	1	4				7		2	3	5	12	
[(1,4)]	1	5				3		10			2	4
[(2,1)]	1	6	2	7	3	11	12		8	9		
[(2,3)]	1	7	4			12		6	2	3		
[(3,1)]	1	8	9	2	10		6					11
[(3,2)]	1	9	10	3		8	2					6
[(3,4)]	1	10		5		9	3				8	2
[(4,1)]	1	11	6	12	2					8		
[(4,3)]	1	12	7		4			11	6	2		

Fig. 11.1.6 Multiplication without and with the "funny term" — numbering as in second column

It seems that no other possible products stay undefined. This observation is not a proof, but together with the next Fig. 11.1.6 it gives enough evidence to justify that we proceed in exploring first.

We may even check whether the inverse of multiplication behaves correct. To this end, we have included Figs. 11.1.7 and 11.1.9.

$\mathfrak{Q}_{\mathbb{Q}} =$

	[(0,0)]	[(1,1)]	[(1,2)]	[(1,3)]	[(1,4)]	[(2,1)]	[(2,3)]	[(3,1)]	[(3,2)]	[(3,4)]	[(4,1)]	[(4,3)]
[(0,0)]	•	1	1	1	1	1	1	1	1	1	1	1
[(1,1)]		2	6	8	11	3	9	4	7	12	5	10
[(1,2)]		3	2	9	6	5	10		4	7		
[(1,3)]		4	7	2	12		3					5
[(1,4)]		5	3	10	2					4		
[(2,1)]		6	11			2	8	7	12		3	9
[(2,3)]		7	12	6		4	2					3
[(3,1)]		8				9		2	6	11	10	
[(3,2)]		9	8			10		3	2	6		
[(3,4)]		10	9		8			5	3	2		
[(4,1)]		11				6		12			2	8
[(4,3)]		12		11		7	6				4	2

Fig. 11.1.7 Division on $\mathbb{Q}$, limited to 4 — numbering as in second column

$$\mathfrak{Q}_{\mathbb{N}\times\mathbb{N}} := \big(\,(\pi \otimes \rho)_{;}\,\mathbb{M} \oslash (\rho \otimes \pi)_{;}\,\mathbb{M}\,\big) \quad \approx \quad \tfrac{a}{b} : \tfrac{c}{d} = \tfrac{ad}{bc}$$

$\mathbb{M}_{\mathbb{Q}} =$

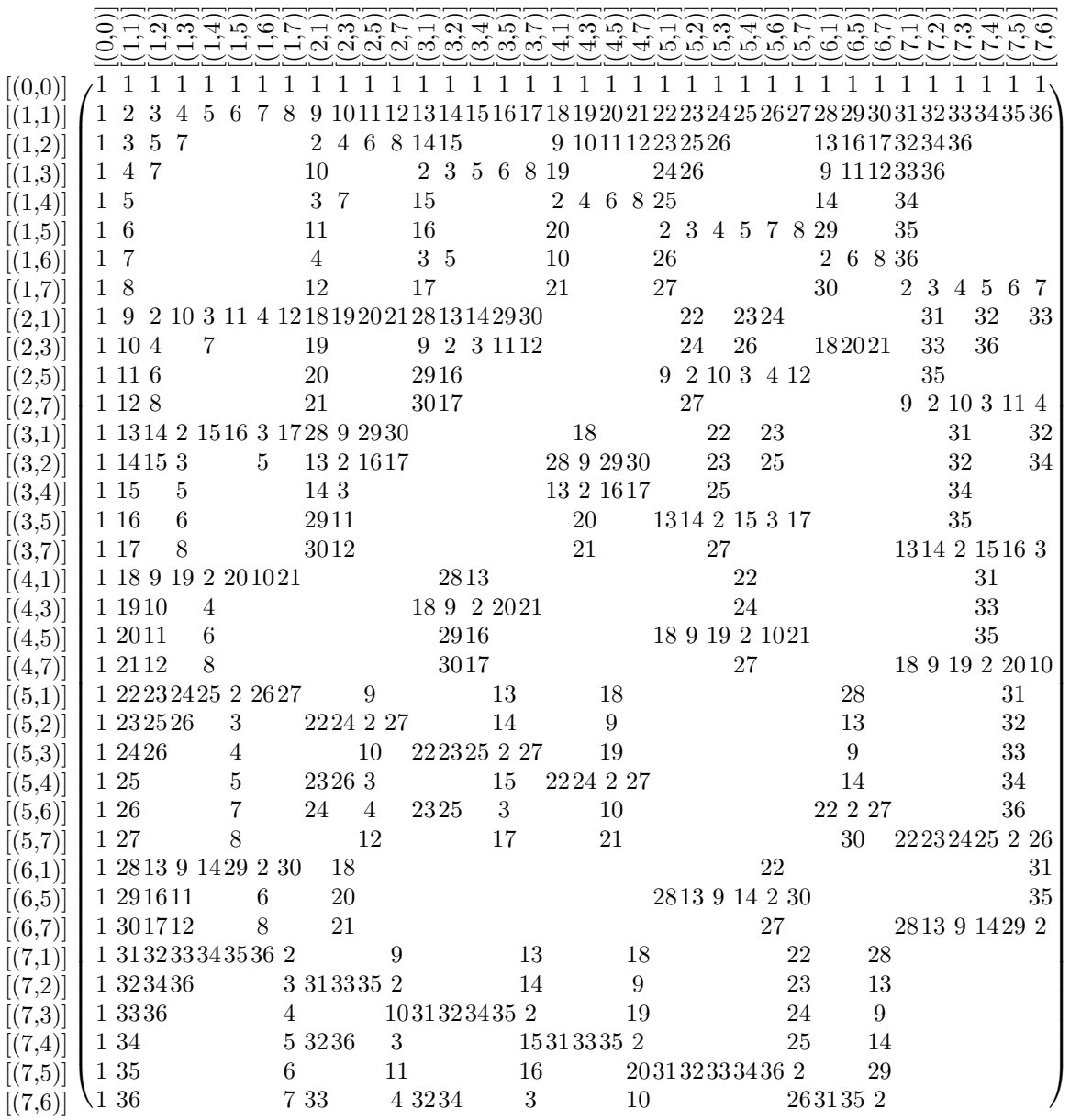


Fig. 11.1.8 Multiplication of rationals limited by 7

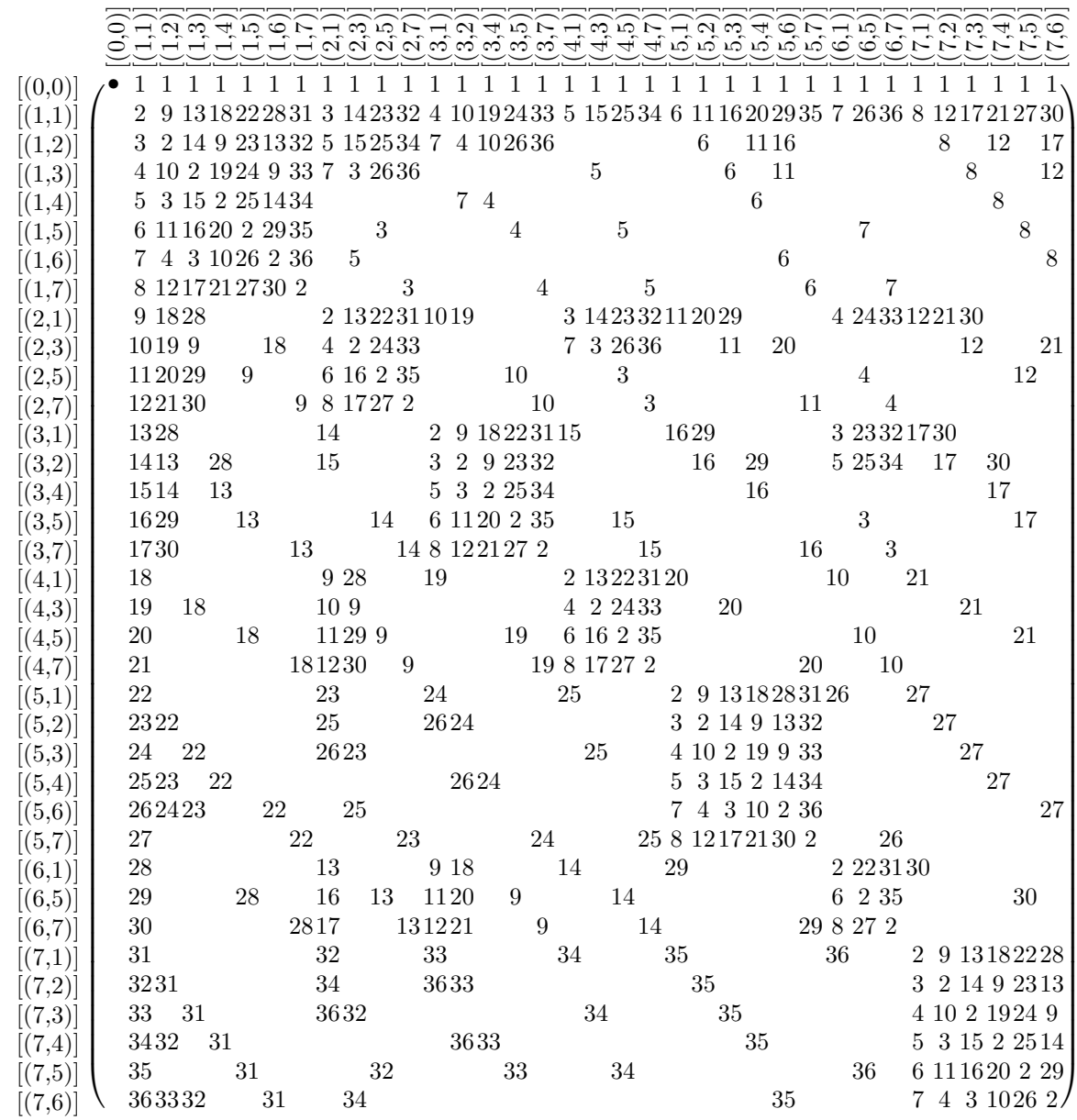
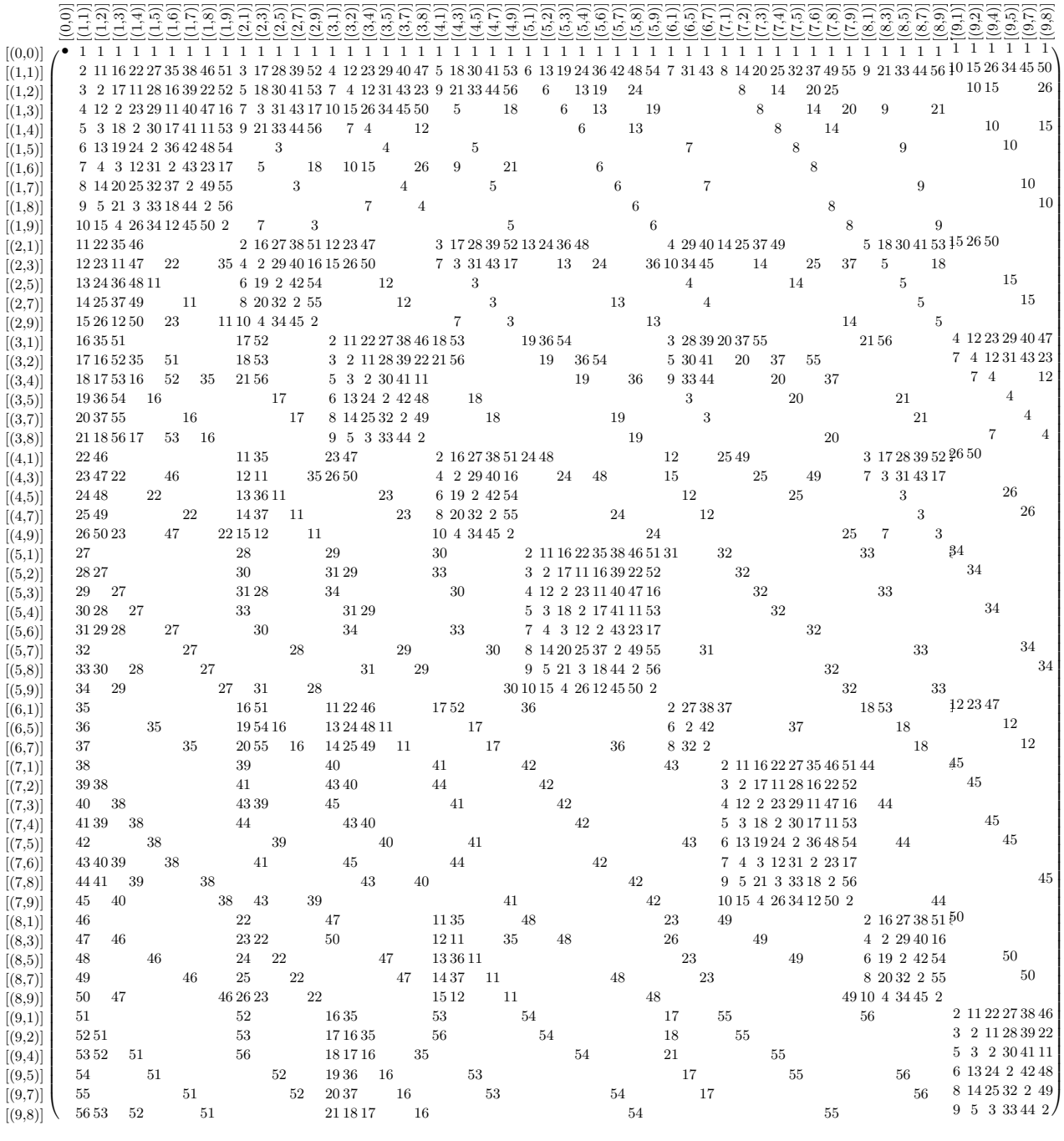


Fig. 11.1.9 Division limited by 7



Addition and subtraction are even more difficult to handle considering

$$\mathbb{A}_{\mathbb{Q}} := \eta_2^{\intercal} ; \big(\big((\pi \otimes \rho) ; \mathbb{M}_{\mathbb{N}} \otimes\ (\rho \otimes \pi) ; \mathbb{M}_{\mathbb{N}} \big) ; \mathbb{A}_{\mathbb{N}} \lessdot\ (\rho \otimes \rho) ; \mathbb{M}_{\mathbb{N}} \big) ; \eta \qquad \approx \qquad \tfrac{a}{b} + \tfrac{c}{d} = \tfrac{ad+bc}{bd}$$

$$\mathfrak{S}_{\mathbb{Q}} := \eta_2^{\intercal} ; \big(\big((\pi \otimes \rho) ; \mathbb{M}_{\mathbb{N}} \otimes\ (\rho \otimes \pi) ; \mathbb{M}_{\mathbb{N}} \big) ; \mathfrak{S}_{\mathbb{N}} \lessdot\ (\rho \otimes \rho) ; \mathbb{M}_{\mathbb{N}} \big) ; \eta \qquad \approx \qquad \tfrac{a}{b} - \tfrac{c}{d} = \tfrac{ad-bc}{bd}$$

together with the chance to exceed the limits given. So we start modestly assuming denominators to coincide as in

$$\mathbb{A}_{\mathbb{Q}}^{coinc} := \eta_2^{\intercal} ; \big((\pi \otimes \pi) ; \mathbb{A}_{\mathbb{N}} \lessdot\ (\rho \circledcirc \rho) \big) ; \eta \qquad \approx \qquad \tfrac{a}{b} + \tfrac{c}{b} = \tfrac{a+c}{b}$$

$$\mathfrak{S}_{\mathbb{Q}}^{coinc} := \eta_2^{\intercal} ; \big((\pi \otimes \pi) ; \mathfrak{S}_{\mathbb{N}} \lessdot\ (\rho \circledcirc \rho) \big) ; \eta \qquad \approx \qquad \tfrac{a}{b} - \tfrac{c}{b} = \tfrac{a-c}{b}$$

which is definitely insufficient, as Fig. 11.1.10 shows. However, $\eta_2^{\intercal}$ leads back to pairs of naturals, and thus possibly offers more than one as, e.g. $1 = \frac{1}{1} = \frac{2}{2} = \frac{3}{3} = \frac{4}{4}$, so that these relations contain

more entries than initially expected — but not all necessary ones. It produces immediately a first approach to subtraction as well as addition:

$$\mathbb{A}_{\mathbb{Q}}^{coinc} = \begin{array}{c} \\ [(0,0)] \\ [(1,1)] \\ [(1,2)] \\ [(1,3)] \\ [(1,4)] \\ [(2,1)] \\ [(2,3)] \\ [(3,1)] \\ [(3,2)] \\ [(3,4)] \\ [(4,1)] \\ [(4,3)] \end{array} \begin{array}{c} \begin{array}{cccccccccccc} [(0,0)] & [(1,1)] & [(1,2)] & [(1,3)] & [(1,4)] & [(2,1)] & [(2,3)] & [(3,1)] & [(3,2)] & [(3,4)] & [(4,1)] & [(4,3)] \end{array} \\ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 2 & 6 & 9 & 12 & & 8 & & 11 & & & & \\ 3 & 9 & 2 & & 10 & & & & 6 & & & \\ 4 & 12 & & 7 & & & 2 & & & & & \\ 5 & & 10 & & 3 & & & & & 2 & & \\ 6 & 8 & & & & 11 & & & & & & \\ 7 & & & 2 & & & 12 & & & & & \\ 8 & 11 & & & & & & & & & & \\ 9 & & 6 & & & & & & & & & \\ 10 & & & & 2 & & & & & & & \\ 11 & & & & & & & & & & & \\ 12 & & & & & & & & & & & \end{pmatrix} \end{array} \quad \begin{array}{c} \begin{array}{cccccccccccc} [(0,0)] & [(1,1)] & [(1,2)] & [(1,3)] & [(1,4)] & [(2,1)] & [(2,3)] & [(3,1)] & [(3,2)] & [(3,4)] & [(4,1)] & [(4,3)] \end{array} \\ \begin{pmatrix} 1 & & & & & & & & & & & \\ 2 & 1 & 3 & 7 & 10 & & 4 & & & 5 & & \\ 3 & & 1 & & 5 & & & & & & & \\ 4 & & & 1 & & & & & & & & \\ 5 & & & & 1 & & & & & & & \\ 6 & 2 & 9 & & & 1 & & & 3 & & & \\ 7 & & & 4 & & & 1 & & & & & \\ 8 & 6 & & & & 2 & & 1 & & & & \\ 9 & 3 & 2 & & & & & & 1 & & & \\ 10 & & 5 & & 3 & & & & & 1 & & \\ 11 & 8 & & & & 6 & & 2 & & & 1 & \\ 12 & 4 & & 2 & & & 7 & & & & & 1 \end{pmatrix} \end{array} = \mathfrak{S}_{\mathbb{Q}}^{coinc}$$

Fig. 11.1.10 Addition $\mathbb{A}_{\mathbb{Q}}^{coinc}$ and subtraction $\mathfrak{S}_{\mathbb{Q}}^{coinc}$ with denominators coinciding, limited by 4

A further problem shows up when addition of fractions would require to proceed to a common numerator surpassing the given limit; e.g. with numbers limited to 3 in

$\frac{3}{2} + \frac{1}{2} = \frac{4}{2} = 2.$

The 2 is inside limitation, 4 is not, but the final result 2 is. It is necessary to circumnavigate this problem using the associative rule and thus staying inside the limits:

$2 - \frac{1}{2} = (1 + 1) - \frac{1}{2} = 1 + (1 - \frac{1}{2}) = 1 + \frac{1}{2} = \frac{3}{2}.$

It seems advisable to introduce subtraction first, since subtraction in the numerator is less likely to transcede limitation.

This does not manage to handle properly $2 - \frac{2}{3} = \frac{4}{3}$ when restricted to 4, since this would require $\frac{6}{3} - \frac{2}{3} = \frac{4}{3}$. Other deficiencies are

$2 - \frac{2}{3} = \frac{4}{3} \qquad 3 - \frac{3}{2} = \frac{3}{2} \qquad \frac{4}{3} - \frac{2}{3} = \frac{2}{3}$

Employing $\mathbb{A}_{\mathbb{Q}}^{coinc}$ and subtraction $\mathfrak{S}_{\mathbb{Q}}^{coinc}$, we manage to arrive at

$$\mathfrak{S}_{\mathbb{Q}} := \pi_2^{\mathsf{T}_;} \left((\pi_2 ; \pi \oslash \rho_2) ; \mathfrak{S}_{\mathbb{Q}}^{coinc} \oslash (\rho_2 \oslash \pi_2 ; \rho) ; \mathfrak{S}_{\mathbb{Q}}^{coinc} \right) ; \mathbb{A}_{\mathbb{Q}}^{coinc} \cup$$
$$\pi_2^{\mathsf{T}_;} \left((\pi_2 ; \pi \oslash \rho_2) ; \mathfrak{S}_{\mathbb{Q}}^{coinc} \oslash (\pi_2 ; \rho \oslash \rho_2) ; \mathfrak{S}_{\mathbb{Q}}^{coinc} \right) ; \mathfrak{S}_{\mathbb{Q}}^{coinc}$$

Additions $\mathbb{A}_{\mathbb{Q}}^{coinc}$ and $\mathbb{A}_{\mathbb{Q}}$ are more easily obtained as right-inverses of subtraction.

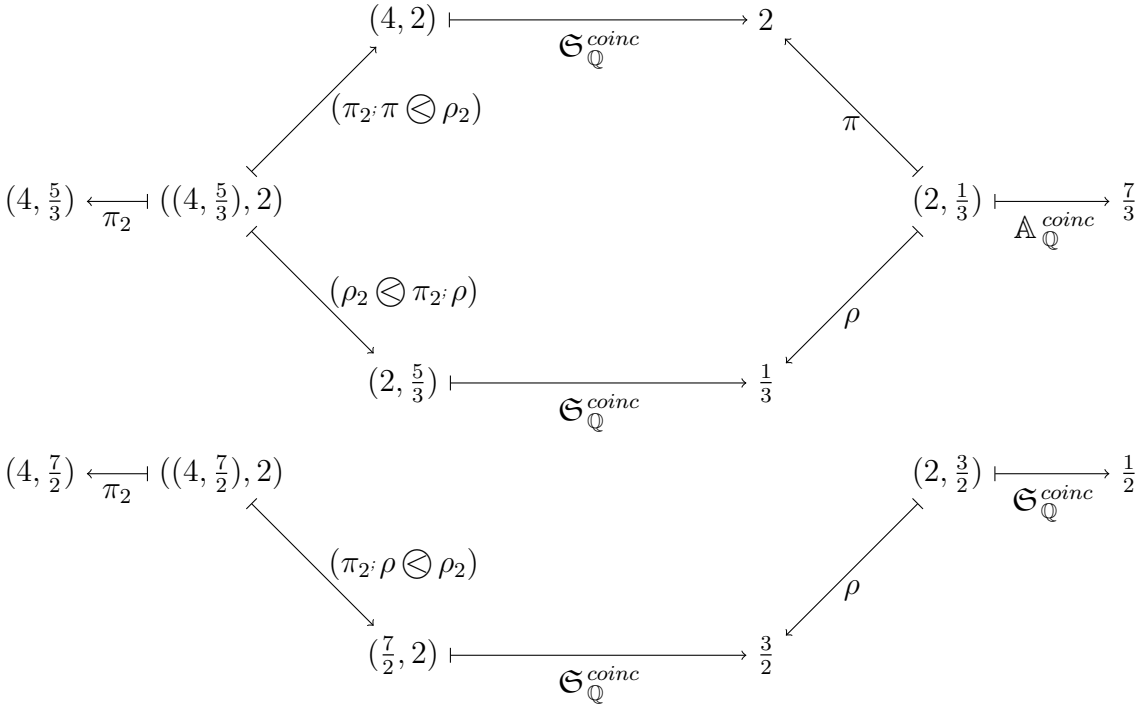


Fig. 11.1.11 Addition and subtraction for $\mathbb{Q}$, limited by 7, using an arbitrary natural, here e.g. 2

Resembling this, we provide several operations for pairs of pairs of naturals in order to finally arrive at addition.

	[(0,0)]	[(1,1)]	[(1,2)]	[(1,3)]	[(1,4)]	[(2,1)]	[(2,3)]	[(3,1)]	[(3,2)]	[(3,4)]	[(4,1)]	[(4,3)]
[(0,0)]	1	2	3	4	5	6	7	8	9	10	11	12
[(1,1)]	2	6	9	12		8		11				
[(1,2)]	3	9	2		10				6			
[(1,3)]	4	12		7			2					
[(1,4)]	5		10		3					2		
[(2,1)]	6	8				11						
[(2,3)]	7			2			12					6
[(3,1)]	8	11										
[(3,2)]	9		6						8			
[(3,4)]	10				2					9		
[(4,1)]	11											
[(4,3)]	12						6					

	[(0,0)]	[(1,1)]	[(1,2)]	[(1,3)]	[(1,4)]	[(2,1)]	[(2,3)]	[(3,1)]	[(3,2)]	[(3,4)]	[(4,1)]	[(4,3)]
[(0,0)]	1											
[(1,1)]	2	1	3	7	10		4			5		
[(1,2)]	3		1		5							
[(1,3)]	4			1								
[(1,4)]	5				1							
[(2,1)]	6	2	9			1	12		3			7
[(2,3)]	7			4			1					
[(3,1)]	8	6				2		1	9			
[(3,2)]	9	3	2						1	10		
[(3,4)]	10		5		3					1		
[(4,1)]	11	8				6		2			1	
[(4,3)]	12	4		2			7					1

Fig. 11.1.12 Addition and subtraction in general, limited by 4

That these concepts are sufficient, certainly needs a proof. As yet this has not been elaborated on in order to proceed with other ideas.

$\mathbb{A}_{\mathbb{Q}} =$

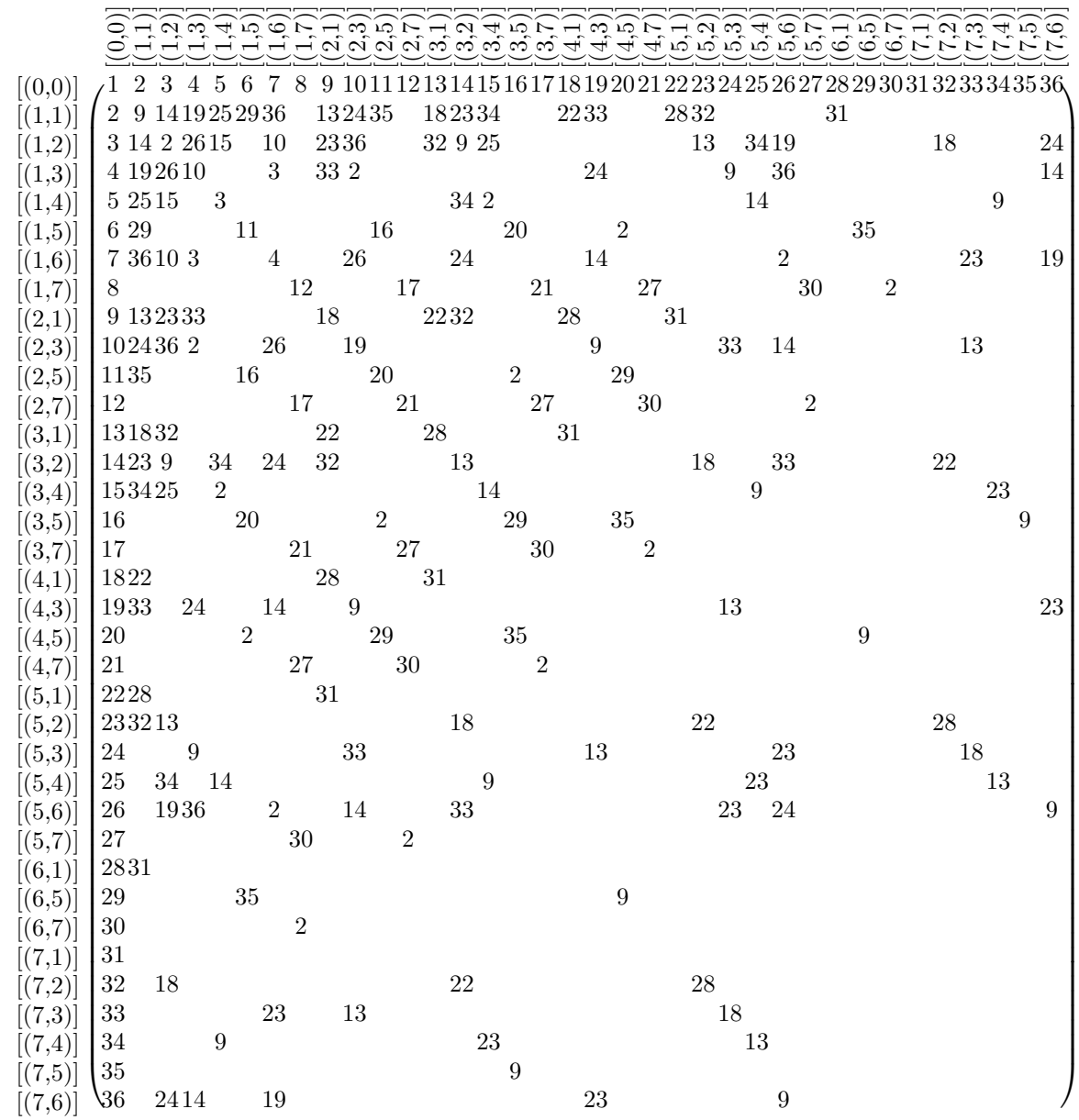


Fig. 11.1.13 Addition limited by 7 — and numbered via first column

$\mathfrak{S}_{\mathbb{Q}} =$

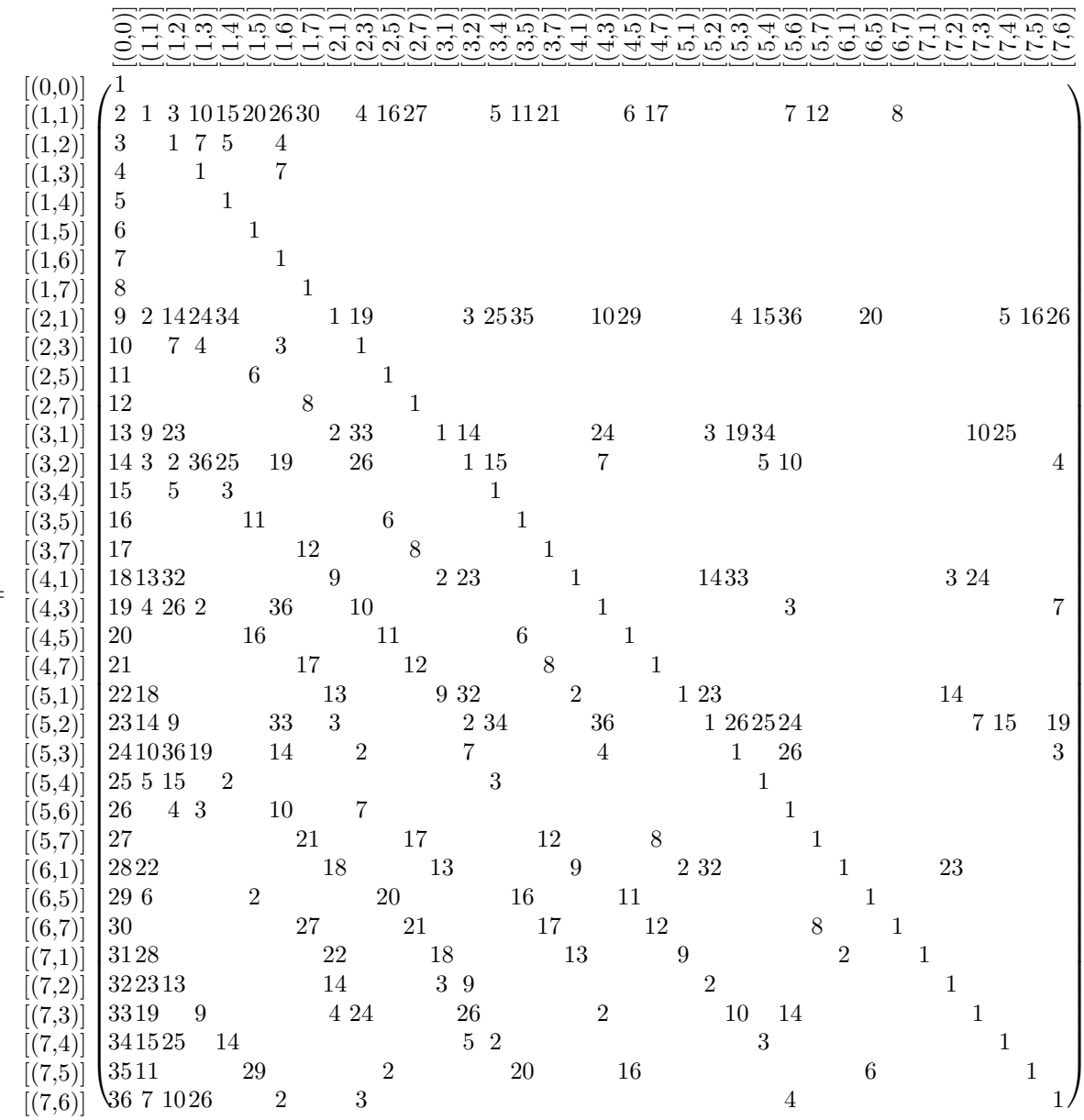


Fig. 11.1.14 Subtraction limited by 7 — and numbered via first column

Symbols

Sets

Union and intersection are denoted as $M \cup N$ and $M \cap N$ — in the same way as later for relations. The complement is $\overline{M}$, provided the ground set is tacitly given. For a one-element set we provide $\mathbb{1}$ as standard notation. The Cartesian product of sets is $M \times N$; the direct sum is $M + N$.

Logic

For metalanguage consequence, equivalence, and definition, "$\Longrightarrow$", "$\Longleftrightarrow$", and "$:\Longleftrightarrow$" are used. Definitional equality is denoted as ":=". The set of Boolean truth-values is $\mathbb{B} = \{\, \mathbf{0}\,,\, \mathbf{1}\,\}$. In the context of propositional logic, "$\wedge$", "$\vee$" are used for "and" respectively "or", together with "$\rightarrow$" for "if… then" and "$\leftrightarrow$" for "precisely when". In the context of predicate logic, "$\exists$" and "$\forall$" denote the existential quantifier and the universal quantifier.

Relations

$R : X \longrightarrow Y$	relation with source and target		7
$\mathcal{P}(X)$	powerset of X		7
$\mathbf{2}^X$	powerset of X, variant		7
$R \cup S$	union		7
$R \cap S$	intersection		7
$\mathbb{L}$	empty relation	provided the	8
$\mathbb{T}$	universal relation	ground sets	8
$\mathbb{I}$	identity	are tacitly given	8
$R^\intercal$	transposed relation, converse		8
$R \mathbin{;} S$	product, composition		8
R^+	transitive closure of R		8
R^*	reflexive-transitive closure of R		44
H_B	Hasse relation of strictorder B		39
$\pi : X \times Y \longrightarrow X$	projection of a product on to its first component X		7
$\rho : X \times Y \longrightarrow Y$	projection of a product on to its second component Y		7
$\iota : X \longrightarrow X + Y$	injection of the first variant X into the sum $X + Y$		7
$\kappa : Y \longrightarrow X + Y$	injection of the second variant Y into the sum $X + Y$		7
$\texttt{Inject}\ v : V \longrightarrow X$	natural injection for a non-empty subset v of X		8
$\texttt{Project}\ \Xi : X \longrightarrow X_\Xi$	natural projection for an equivalence Ξ on the set X		8
$\texttt{vec}(R)$	vectorization of R		14
$\texttt{rel}(v)$	relation belonging to vector v		14
D_v	the diagonal for the vector v		12
$\texttt{ets}(X, Y)$	set of matrix entries in $X \times Y$		19
$S \backslash R$	right residual		22
$S / R,$	left residual		22
$\texttt{syq}(R, S)$	symmetric quotient		8

$(R \oslash S)$	strict fork operator	8
$(R \oslash S)$	strict join operator	8
$(R \otimes S)$	Kronecker product	8
$\mathfrak{J}_2$	lifted join	8
$\mathfrak{M}_2$	lifted meet	8
ϑ_R	existential image of relation R	22
$\vartheta_{R^\intercal}$	inverse image of relation R	22
$\varepsilon : X \longrightarrow \mathbf{2}^X$	membership relation	7
$\sigma : X \longrightarrow \mathbf{2}^X$	singleton injection relation	7
$\Omega : \mathbf{2}^X \longrightarrow \mathbf{2}^X$	powerset ordering	7
$\mathcal{N} : \mathbf{2}^X \longrightarrow \mathbf{2}^X$	powerset negation	7
$E : X \longrightarrow X$	arbitrary ordering	8

Binary relational operations

$\mathfrak{P} : X \times Y \longrightarrow Y \times X$	pair exchange flip	21
$PO : X \times Y \longrightarrow \mathbb{B}$	pair ordering	27
$PSO : X \times Y \longrightarrow \mathbb{B}$	pair strictorder	27
$\mathfrak{T}$	associativity shuffling	23
$\mathfrak{D}$	distributivity shuffling	25
$\mathfrak{K}$	Kronecker-fork shuffling	26
$\hookrightarrow\!\!R$	left inverse of binary relational operation	29
$R\!\!\hookleftarrow$	right inverse of binary relational operation	29
$\bullet$	indicates non-univalence at just one entry	158
Φ	indicates a row with just one 0 and otherwise 1 in a table	206

Numbers

$\mathbb{N}$	possibly finite initial section of the naturals	45
$\mathbb{S} : \mathbb{N} \longrightarrow \mathbb{N}$	successor relation on possibly finite initial section of $\mathbb{N}$	47
$\mathbb{E}_6$	extruded subset of even numbers ≥ 6 of $\mathbb{N}$	285
$\mathtt{eve} : \mathbb{N} \longrightarrow \mathbb{N}$	partial identity of even numbers on $\mathbb{N}$	47
$\mathtt{odd} : \mathbb{N} \longrightarrow \mathbb{N}$	partial identity of odd numbers on $\mathbb{N}$	47
$\mathtt{FlipTD} : \mathbb{N} \longrightarrow \mathbb{N}$	flipping least to greatest on finite initial section of $\mathbb{N}$	49
$\mathtt{FlipPairsTD} : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N} \times \mathbb{N}$	flipping pairs in this way	49
$\mathbb{V}$	ordering on $\mathbb{N}$, i.e. right upper triangle relation	65
$\mathbb{V}$	left upper triangle relation on finite initial section of $\mathbb{N}$	69
$\mathbb{V}_n$	lower right border of left upper triangle	71
$\mathbb{V}_n$	left upper triangle	74
$n_{\mathbb{S}}$	greatest number on finite initial section of $\mathbb{N}$	49
$g_{\mathbb{S}}$	greatest point in finite initial section of $\mathbb{N}$	49
$\mathbb{A} : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$	addition relation on $\mathbb{N}$	82
$B : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$	addition relation on $\mathbb{N}$ variant	82
$\mathfrak{S} : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$	subtraction relation on $\mathbb{N}$	95
$\mathtt{S} : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$	subtraction relation on $\mathbb{N}$ variant	95
$\mathbb{X} : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$	absolute difference mapping on $\mathbb{N}$	96
$\mathbb{M} : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$	multiplication relation on $\mathbb{N}$	135
$\wp_1 : \mathbb{N} \longrightarrow \mathbb{N}$	partial identity with primes ≥ 1 on $\mathbb{N}$	14
$\mathbb{P}_1$	extruded set of primes starting at 1	235
$\wp_2 : \mathbb{N} \longrightarrow \mathbb{N}$	partial identity with primes ≥ 2 on $\mathbb{N}$	14

$\mathbb{P}_2$	extruded set of all primes	14
$\wp : \mathbb{N} \longrightarrow \mathbb{N}$	partial identity with primes ≥ 3 on $\mathbb{N}$	14
$\mathbb{P}$	extruded set of odd primes	14
$\mathfrak{Q} : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$	division relation on $\mathbb{N}$	147
$\mathcal{D} : \mathbb{N} \longrightarrow \mathbb{N}$	divides lattice	175
$\mathcal{D}_{\mathrm{s}} : \mathbb{N} \longrightarrow \mathbb{N}$	strict divides relation	175
$\mathtt{gcd} : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$	greatest common divisor relation on $\mathbb{N}$	182
$\mathtt{lcm} : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$	least common multiple relation on $\mathbb{N}$	182
$\mathtt{fact} : \mathbb{N} \longrightarrow \mathbb{N}$	factorial relation on $\mathbb{N}$	212
$\Gamma : \mathbb{N} \longrightarrow \mathbb{N}$	Gamma function on $\mathbb{N}$	212
$\mathtt{Euclid} : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N} \times \mathbb{N}$	Euclidean relation	167
`quot`	quotient forming in HASKELL	159
`rem`	remainder forming in HASKELL	159
`GD`	Goldbach decomposition function	284
`ND`	prime distance function	272
`NP`	next prime function	272
ϑ_p	first in prime pair to all lexicographic ones with p first	307
$[p^{\rightarrow}]$	the prime p, however, considered as $[p^{\rightarrow}] \in \mathbb{P}$	14

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