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Contact, closure, topology, and the linking of row and column types of relations

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ABSTRACT

Forming closures of subsets of a set X is a technique that plays an important role in many scientific disciplines and there are many cryptomorphic mathematical structures that describe closures and their construction. One of them was introduced by Aumann in the year 1970 under the name *contact relation*. Using relation algebra, we generalize Aumann's notion of a contact relation between X and its powerset 2^X and that of a closure operation on 2^X from powersets to general membership relations and their induced partial orders. We also investigate the relationship between contacts and closures in this general setting and present some applications. In particular, we investigate the connections between contacts, closures and topologies and use contacts to establish a one-to-one correspondence between the column intersections space and the row intersections space of arbitrary relations.

1. Introduction

In various disciplines, not least in mathematics and computer science, closures of subsets of a set X are formed. Typically this task is combined with some predicate that holds for X and is $\cap$-hereditary, like "being a reflexive relation", "being a transitive relation", "being an axis-parallel rectangle", "being a subgroup" or "being a convex set". Such predicates lead to closure systems, i.e., subsets $\mathfrak{C}$ of the powerset 2^X of X that contain X and any intersection of subsets collected in $\mathfrak{C}$. It is well known that there is a one-to-one correspondence between the set of closure systems of 2^X and the set of extensive, monotone, and idempotent functions on 2^X (the closure operations on 2^X). Typical examples of closure operations are operations that yield the transitive closure of a relation, the convex hull of a set of points in real vector spaces or the subgroup generated by a set of elements of a given group.

The practical importance of closures is also documented by the fact that equivalent or at least very similar concepts have frequently been reinvented. Well-known examples of such equivalent concepts (in [5] called closure objects) are full implicational systems, dependency relations, entailment relations, Sperner villages, and join congruence relations. Also notational variants have been introduced and are still in use. For example, what in [12] is called a full implicational system is called a full family of functional dependencies in the theory of relational databases (see, e.g., [13]) or a closed family of implications in formal concept analysis (cf. [18]). Likewise a dependency relation in the sense of [12] is the same as a contact relation in the sense of [1,2] and one needs only to transpose such a relation and additionally to restrict its range to $2^X \setminus \{\emptyset\}$ in order to obtain an entailment relation in the sense of [14].

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Already in 1970, Aumann discovered that the concept of a closure on sets can also be described by a specific relation, namely — as he called it — a contact relation. When presenting this notion the first time in [1], one intention was to formalize the essential properties of a contact between objects and sets of objects, mainly to obtain a more suggestive access to topology for beginners than "traditional" axiom systems provide. In the introduction of his paper [1], Aumann also mentions sociological applications as motivation, but all examples of [1,2] are in fact from mathematics. A main result of [1] is that closure operations and contact relations are cryptomorphic mathematical structures in the sense of [12] and, hence, contact relations and the other concepts describing closures we have mentioned above are also cryptomorphic.

In the present paper, we generalize Aumann's concept of a contact relation between sets X and their powersets 2^X to contact relations given by an (almost) arbitrary relation M between two sets X and G, that may be interpreted as "an individual $x \in X$ is a member of a group $g \in G$ of individuals". Such an approach allows to treat also examples from sociology, political science, social choice theory and so forth. As we will show, every group membership relation M induces a partial order Ω_M on the groups of individuals. With respect to the relation Ω_M, we consider a notion of closure operation that directly arises out of the original one by replacing set inclusion by Ω_M. In this very general setting, we investigate contacts, their properties, and a construction similar to the lower/upper-derivative construction of formal concept analysis. The latter leads to a fixed point description of the set of contacts. Guided by Aumann's main result, we also study the relationship between general M-contacts and Ω_M-closures. We also will investigate the connections between contacts, closures and topologies. As an application, we, finally, use contacts to establish a one-to-one correspondence between the column intersections space and the row intersections space of a relation (or a Boolean matrix).

To carry out our investigations, we use heterogeneous abstract relation algebra in the sense of [21,22] which evolved from the homogeneous ones developed by A. Tarski and his collaborators (see [23,24]). This allows very concise and precise specifications and algebraic proofs that drastically reduce the danger of making mistakes. To give an example, when constructing closures from contacts, a subtle definedness condition plays a decisive role that can easily be overlooked when using the customary approach with closures being functions. Relation-algebraic specifications also allow to use tool support. For obtaining the results of this paper, the use of the relation-algebraic visualization and manipulation tool RelView (see [3,6]) for computing contacts and closures, testing properties, experimenting with concepts, drawing pictures etc. was very helpful.

2. Relation-algebraic preliminaries

In this section we present the facts on relation algebra that are needed in the remainder of the paper. We will work with heterogeneous relations. For more details on them and the corresponding relation algebra, see e.g., [11,21,22].

We denote the set (or type) of relations with source X and target Y, that is, the powerset $2^{X\times Y}$ of the direct product $X \times Y$, by $[X \leftrightarrow Y]$ and write $R : X \leftrightarrow Y$ instead of $R \in [X \leftrightarrow Y]$. If the sets X and Y are finite and of cardinality m and n, respectively, we may consider R as a Boolean $m \times n$ matrix. Since this interpretation is well suited for many purposes and also used by RelView as one of the means to depict relations, we will often use matrix notation and matrix terminology in this paper. In particular, we talk about rows, columns and entries of relations, and write frequently $R_{x,y}$ instead of $\langle x, y\rangle \in R$ or $x\,R\,y$.

Relational mathematics has gradually become a science and a craft. It is a science starting with the work of De Morgan and Peirce — who by the way is reported to have been so flexible as to immediately adopt the newly introduced matrices as a well-come tool. Since the second half of the last century, computers and computer science reached their present level and relational mathematics became also a craft with regard to all the practical tasks that may nowadays be solved using relations in reasoning and in computing.

Still there are certain differences between these two natures; while the former adheres to conceiving relations as being homogeneous, the latter often works with heterogeneous relations. Being very precise, a heterogeneous relation is a tripel consisting of a set of pairs, a source, being the set where the first components come from, and a target, being the set where the second components come from. Some sort of typing is necessary in either case. The heterogeneous community relies heavily on commonly understood techniques and well-established and broadly tested routines of computer science such as unification, data types and so fourth. It mistrusts, thus, to a certain extent any free-hand mathematics of type-inference along the diagonal across the infinite unrestricted domain evaluated separately along several rules that have not yet enjoyed the discussion by all the programmers world-wide. Heterogeneous relations with finite carrier sets, a correct typing of relations, and type checking during evaluations of relation-algebraic expressions is in our opinion indispensable if relations are (efficiently) manipulated via a computer system like RelView. The unsharpness problem seems to have for a long time escaped the attention of the homogeneous community.

Another point is whether one accepts or dislikes negation. People working homogeneously in an unrestricted universe are right in trying to avoid negation; often they use residuation instead. When something shall be computed, this does no longer seem justified; negation for heterogeneous relations is not problematical theoretically. In addition, it is an operation on bit values the compiler will take care of and that, thus, costs hardly anything. If binary decision diagrams (BDDs) are used to implement relations, as in the case of RelView, then almost all BDD-packages allow to realize the complement operation extremely efficient.

We assume the reader to be familiar with the basic operations on relations, viz. R^T (transposition, conversion), $\overline{R}$ (complement, negation), $R \cup S$ (union, join), $R \cap S$ (intersection, meet), $R;S$ (composition, product), the predicate indicating $R \subseteq S$

(inclusion), and the special relations O (empty or null relation), L (universal relation) and I (identity relation) the typing of which is usually suppressed. Each type $[X \leftrightarrow Y]$ with the operations $\overline{}$, $\cup$, $\cap$, the ordering $\subseteq$ and the constants O and L forms a complete Boolean lattice. Further well-known rules are, for instance,

$$R^{T^T} = R \qquad \overline{R^T} = \overline{R}^T$$
$$(R \cup S)^T = R^T \cup S^T \qquad (R \cap S)^T = R^T \cap S^T$$
$$Q;\,(R \cup S) = Q;\,R \cup Q;\,S \qquad Q;\,(R \cap S) \subseteq Q;\,R \cap Q;\,S$$

and that $R \subseteq S$ implies $R^T \subseteq S^T$ as well as $R;\,Q_1 \subseteq S;\,Q_1$ and $Q_2;\,R \subseteq Q_2;\,S$. The theoretical framework for these rules and many others to hold is that of an (axiomatic) relation algebra. As constants and operations of this algebraic structure we have those of concrete (i.e., set-theoretic) relations. The axioms of a relation algebra are those of a complete Boolean lattice for the Boolean part, the associativity and neutrality of identity relations for composition, the equivalences

$$Q;R \subseteq S \iff Q^T;\,\overline{S} \subseteq \overline{R} \iff \overline{S};R^T \subseteq \overline{Q} \tag{1}$$

(called the *Schröder rule*) or — equivalently to (1) — the inclusion

$$Q;R \cap S \subseteq (Q \cap S;R^T);\,(R \cap Q^T;S), \tag{2}$$

(called the *Dedekind rule*), and the implication

$$R \neq \mathsf{O} \implies \mathsf{L};R;\mathsf{L} = \mathsf{L} \tag{3}$$

(called the *Tarski rule*). In the proofs presented in this paper we will mention only these axioms and their "non-obvious" consequences. Well-known rules like those presented at the beginning of this section remain unmentioned.

A possibility to model sets in relation algebra are *vectors*, which are relations v which satisfy the equation $v = v;\,\mathsf{L}$. For a vector the target is irrelevant and we therefore consider mostly vectors $v : X \leftrightarrow \mathbf{1}$ with a specific singleton set $\mathbf{1} = \{\perp\}$ as target and omit in such cases the second subscript, i.e., write v_x instead of $v_{x,\perp}$. Such a vector can be considered as a Boolean matrix with exactly one column, i.e., as a Boolean column vector, and *represents* the subset $\{x \in X \mid v_x\}$ of its source X. For a relation $R : X \leftrightarrow Y$ and an element $y \in Y$, we define the y-column of R as the vector $R^{(y)} : X \leftrightarrow \mathbf{1}$ such that for all $x \in X$ it holds that $R^{(y)}_x$ if and only if $R_{x,y}$. Rows of relations can be defined as transposed columns of transposed relations.

The basic operations and constants just mentioned can be used for defining specific classes of relations in an algebraic way. We assume the reader to be familiar with relation-algebraic specifications of the most fundamental properties of relations, like univalence $R^T;\,R \subseteq \mathsf{I}$, totality $R;\,\mathsf{L} = \mathsf{L}$, reflexivity $\mathsf{I} \subseteq R$, antisymmetry $R \cap R^T \subseteq \mathsf{I}$, transitivity $R;\,R \subseteq R$ and so on. We also suppose that he is familiar with the symmetric quotient construction

$$\mathrm{syq}(R, S) := \overline{R^T;\,\overline{S}} \cap \overline{\overline{R}^T;\,S} \;:\; Y \leftrightarrow Z \tag{4}$$

of two relations $R : X \leftrightarrow Y$ and $S : X \leftrightarrow Z$ with the same source together with its main properties like the following ones.

$$\mathrm{syq}(R, S) = \mathrm{syq}(\overline{R}, \overline{S}) \qquad [\mathrm{syq}(R, S)]^T = \mathrm{syq}(S, R) \tag{5}$$

$$R;\,\mathrm{syq}(R, R) = R \qquad \mathrm{syq}(Q, R);\,\mathrm{syq}(R, S) \subseteq \mathrm{syq}(Q, S) \tag{6}$$

Otherwise, he may consult, for instance, [22], Sections 3.1, 4.2, and 4.4 or [21], Sections 4.4 and 8.5. In Boolean matrix terminology the symmetric quotient $\mathrm{syq}(R, S)$ of the relations $R : X \leftrightarrow Y$ and $S : X \leftrightarrow Z$ relates the elements $y \in Y$ and $z \in Z$ if and only if the y-column $R^{(y)} : X \leftrightarrow \mathbf{1}$ of R equals the z-column $S^{(z)} : X \leftrightarrow \mathbf{1}$ of S.

The set-theoretic symbol $\in$ gives rise to powerset relations $\varepsilon : X \leftrightarrow 2^X$ that relate $x \in X$ and $Y \in 2^X$ if and only if $x \in Y$. In [7,8] it is shown that for ε the formulae of (7) hold and these even characterize the powerset relation ε up to isomorphism.

$$\mathrm{syq}(\varepsilon, \varepsilon) = \mathsf{I} \qquad \mathsf{L};\,\mathrm{syq}(\varepsilon, R) = \mathsf{L} \quad \text{for all } R \text{ with row type } X \tag{7}$$

Based on (7), a lot of further set-theoretic constructions can be formalized in terms of relation algebra. In this paper, we need the following:

$$\iota := \mathrm{syq}(\mathsf{I}, \varepsilon) : X \leftrightarrow 2^X \qquad \Omega := \overline{\varepsilon^T;\,\overline{\varepsilon}} \;:\; 2^X \leftrightarrow 2^X \tag{8}$$

The relation ι is called singleton-set former, since it associates $x \in X$ with $Y \in 2^X$ if and only if $Y = \{x\}$. The relation Ω specifies the inclusion order on sets. Based on (7) and (8), the following properties are shown in [7], Theorem 4.3.3 (i), (ii), and (iv) or in [21] on page 145 .

Lemma 2.1. *If $\varepsilon : X \leftrightarrow 2^X$ is a powerset relation, then $\iota : X \leftrightarrow 2^X$ is an injective mapping (in the relational sense of Definition 4.2.1 of [22], or [21] Sections 5.1 and 5.2), $\Omega : 2^X \leftrightarrow 2^X$ is a partial order, and it holds that $\iota;\,\Omega = \varepsilon = \varepsilon;\,\Omega$.*

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The construction used in the relation-algebraic definition of the inclusion order via ε in (8) can be generalized to arbitrary relations. Given a relation $R : X \leftrightarrow Y$, we define

$$\Omega_R := \overline{R^T; \overline{R}} \; : Y \leftrightarrow Y. \tag{9}$$

Then the relation Ω_R is reflexive and transitive due to the Schröder rule. In Boolean matrix terminology it shows the "column-is-contained-preorder". That is, Ω_R relates the elements $y, z \in Y$ if and only if the y-column $R^{(y)}$ of R is contained in the z-column $R^{(z)}$ of R. When $\text{syq}(R, R) = \mathbb{I}$, i.e., if R does not possess duplicated columns, it is even antisymmetric due to

$$\Omega_R \cap \Omega_R^T = \overline{R^T; \overline{R}} \cap (\overline{R^T; \overline{R}}\,)^T = \overline{R^T; \overline{R}} \cap \overline{\overline{R}^T; R} = \text{syq}(R, R) = \mathbb{I}$$

and, thus, a partial order. Besides these partial order properties, we will apply the following fact.

Lemma 2.2. *For all relations $R : X \leftrightarrow Y$ we have $R; \Omega_R = R$.*

Proof. The inclusion $R \subseteq R; \Omega_R$ follows from the reflexivity of Ω_R, and with the help of the Schröder rule $R; \Omega_R \subseteq R$ is shown by

$$R^T; \overline{R} \subseteq R^T; \overline{R} \iff R; \overline{R^T; \overline{R}} \subseteq R \iff R; \Omega_R \subseteq R. \quad \square$$

As a last construction, we need the canonical epimorphism $\eta_E : X \leftrightarrow X/E$ induced by an equivalence relation $E : X \leftrightarrow X$. It relates each element $x \in X$ to the equivalence class $c \in X/E$ it belongs to. The following properties are immediate consequences of this point-wise specification; it can even be shown that they characterize canonical epimorphisms up to isomorphism.

$$\eta_E; \eta_E^T = E \qquad \eta_E^T; \eta_E = \mathbb{I} \tag{10}$$

Given $R : X \leftrightarrow Y$, in Sections 3 and 6 we will apply the canonical epimorphisms induced by the following two equivalence relations.

$$\Psi_R := \text{syq}(R, R) : Y \leftrightarrow Y \qquad \Phi_R := \text{syq}(R^T, R^T) : X \leftrightarrow X \tag{11}$$

In this context, the following additional property will be used.

Lemma 2.3. *For each relation $R : X \leftrightarrow Y$, the canonical epimorphism $\eta_{\Psi_R} : Y \leftrightarrow Y/\Psi_R$ induced by Ψ_R fulfils $\overline{R; \eta_{\Psi_R}} = \overline{R}; \eta_{\Psi_R}$.*

Proof. We abbreviate η_{Ψ_R} by η. Then, inclusion "$\subseteq$" follows from

$$\eta^T \text{ total} \implies \overline{\eta^T; R^T} \subseteq \eta^T; \overline{R^T} \iff \overline{R; \eta} \subseteq \overline{R}; \eta$$

using Proposition 4.2.4.i of [22] or Proposition 5.6 of [21], and inclusion "$\supseteq$" from

$$R \subseteq R \iff R; \text{syq}(R, R) \subseteq R \iff R; \eta; \eta^T \subseteq R \iff \overline{R}; \eta \subseteq \overline{R; \eta}$$

using the first rule of (6), the first axiom of (10), and the Schröder rule. $\square$

The relation Ψ_R of (11) is called the *column equivalence relation* since it relates the elements $x, y \in Y$ if and only if $R^{(x)} = R^{(y)}$, i.e., the corresponding columns coincide. Transposition yields that the *row equivalence relation* Φ_R of (11) relates $x, y \in X$ if and only if the corresponding rows are equal.

3. Contact relations

In the year 1970, Aumann introduced contacts between objects and sets of objects. If we formulate his original definition of a contact relation given in [1,2] in our notation, then a relation $A : X \leftrightarrow 2^X$ is an *(Aumann) contact relation* if the following conditions hold.

- (A_1) $\forall x : A_{x,\{x\}}$
- (A_2) $\forall x, Y, Z : A_{x,Y} \wedge Y \subseteq Z \rightarrow A_{x,Z}$
- (A_3) $\forall x, Y, Z : A_{x,Y} \wedge (\forall y : y \in Y \rightarrow A_{y,Z}) \rightarrow A_{x,Z}$

The simplest example of a contact relation is the powerset relation $\varepsilon : X \leftrightarrow 2^X$. Another simple mathematical example is given by graph reachability. If g is a directed graph with set X of vertices and we define a relation $K : X \leftrightarrow 2^X$ by $K_{x,Y}$ if there is a path from the vertex x to a vertex contained in Y, then K is a contact relation.

Our aim is to investigate contact relations by relation-algebraic means and supporting tools (like the manipulation system RelView [3,6]). Thereby Aumann's original approach will be generalized by replacing the powerset by a set G (of groups of individuals, political parties, alliances, organizations, …) and the set-theoretic membership relation $\varepsilon : X \leftrightarrow 2^X$ by a generalized membership relation $M : X \leftrightarrow G$ with regard to G. The latter point not only allows to treat mathematical examples for contact relationships as [1,2] does, but also examples from sociology, political science, social choice theory, game theory and so forth. In the following theorem, we present relation-algebraic versions of the above axioms. The proof of their correspondence consists of step-wise transformations of (A_1) to (A_3) into point-free versions using well-known correspondences between logical and relation-algebraic constructions. Doing so, (A_1) leads to a singleton-former ι and (A_2) to an inclusion order Ω as specified in (8).

Theorem 3.1. *A relation $A : X \leftrightarrow 2^X$ is an Aumann contact relation if and only if the following three inclusions hold:*

$$\iota \subseteq A \qquad A; \Omega \subseteq A \qquad A; \overline{\varepsilon^T; \overline{A}} \subseteq A$$

Proof. First, we show the equivalence of (A_1) and $\iota \subseteq A$.

$$\begin{aligned} \forall x : A_{x,\{x\}} &\iff \forall x, Y : Y = \{x\} \to A_{x,Y} \\ &\iff \forall x, Y : \iota_{x,Y} \to A_{x,Y} \\ &\iff \iota \subseteq A \end{aligned}$$

Next, we verify that (A_2) and $A; \Omega \subseteq A$ are equivalent.

$$\begin{aligned} \forall x, Y, Z : A_{x,Y} \wedge Y \subseteq Z \to A_{x,Z} &\iff \forall x, Z : (\exists Y : A_{x,Y} \wedge Y \subseteq Z) \to A_{x,Z} \\ &\iff \forall x, Z : (A; \Omega)_{x,Z} \to A_{x,Z} \\ &\iff A; \Omega \subseteq A \end{aligned}$$

The following calculation, treating (A_3) and $A; \overline{\varepsilon^T; \overline{A}} \subseteq A$, completes the proof.

$$\begin{aligned} &\forall x, Y, Z : A_{x,Y} \wedge (\forall y : y \in Y \to A_{y,Z}) \to A_{x,Z} \\ \iff &\forall x, Y, Z : A_{x,Y} \wedge \neg(\exists y : y \in Y \wedge \overline{A}_{y,Z}) \to A_{x,Z} \\ \iff &\forall x, Y, Z : A_{x,Y} \wedge \overline{\varepsilon^T; \overline{A}}_{Y,Z} \to A_{x,Z} \\ \iff &\forall x, Z : (\exists Y : A_{x,Y} \wedge \overline{\varepsilon^T; \overline{A}}_{Y,Z}) \to A_{x,Z} \\ \iff &\forall x, Z : (A; \overline{\varepsilon^T; \overline{A}})_{x,Z} \to A_{x,Z} \\ \iff &A; \overline{\varepsilon^T; \overline{A}} \subseteq A \qquad \square \end{aligned}$$

The relation-algebraic characterization of contacts just developed does not yet allow the generalization intended. We still have to remove the singleton-former, since such a construct need not exist in the general case of membership we want to deal with. The next theorem shows how this is possible.

Theorem 3.2. *A relation $A : X \leftrightarrow 2^X$ is an Aumann contact relation if and only if the following two inclusions hold:*

$$\varepsilon \subseteq A \qquad A^T; \overline{A} \subseteq \varepsilon^T; \overline{A}$$

Proof. Because of the Schröder rule, $A^T; \overline{A} \subseteq \varepsilon^T; \overline{A}$ is equivalent with $A; \overline{\varepsilon^T; \overline{A}} \subseteq A$. For the remaining parts, we start with "$\Longrightarrow$" and use Lemma 2.1 to show $\varepsilon \subseteq A$ by

$$\iota \subseteq A \implies \iota; \Omega \subseteq A; \Omega \iff \varepsilon \subseteq A; \Omega \implies \varepsilon \subseteq A.$$

For "$\Longleftarrow$", property $\iota \subseteq A$ follows from $\iota \subseteq \varepsilon$ and $\varepsilon \subseteq A$. Using the Schröder rule, we obtain $A; \Omega \subseteq A$ from

$$A^T; \overline{A} \subseteq \varepsilon^T; \overline{A} \subseteq \varepsilon^T; \overline{\varepsilon} = \overline{\Omega}. \qquad \square$$

Hence, we have that membership implies contact and for all $Y, Z \in 2^X$ from the existence of an element that is in contact with Y but not in contact with Z it follows that even a member of Y is not in contact with Z. In the literature such relations are also known as dependence or entailment relations and in particular considered in combination with so-called exchange properties. See [12,14] for example. And here is our generalization of Aumann's concept of a contact.

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Definition 3.1. A relation $K : X \leftrightarrow G$ is called an *(Aumann) contact relation with respect to the relation* $M : X \leftrightarrow G$ — in short: an *M-contact* — if the following properties hold.

$$(\mathrm{K}_1)\ \ M \subseteq K \qquad\qquad (\mathrm{K}_2)\ \ K^T;\ \overline{K} \subseteq M^T;\ \overline{K}$$

Axiom (K_2) is called the infectivity of a contact. We have chosen this form, since it proved to be particularly suitable for relation-algebraic reasoning. For concrete sociological or similar applications, frequently the equivalent version $K;\ \overline{M^T;\ \overline{K}} \subseteq K$ is more appropriate. E.g., in the case of persons and syndicates it says that if a person x is in contact to a syndicate Y_1 all of whose members are in contact to a syndicate Y_2, then also x is in contact to Y_2.

In real life, contacts between people and groups are frequently established by common interests. As an example, we consider a protesters scene of non-governmental organizations. There exist persons $x \in X$ willing to protest against several topics $t \in T$ (like new nuclear power plants, new coal-fired power plants or the introduction of the B.Sc./M.Sc. system of university education in Germany). Let this be formalized by the following relation.

$$J : X \leftrightarrow T \qquad\qquad J_{x,t} \ :\Longleftrightarrow\ x \text{ is in opposition to } t$$

Then typically a person $x \in X$ will get in touch with an activist group $g \in G$ if and only if x opposes to all topics all members of g oppose to. If we formalize the situation in predicate logic and afterwards translate this version into a relation-algebraic expression, we arrive at $\mathrm{mi}_J(\mathrm{ma}_J(M))_{x,g}$, where the relation $M : X \leftrightarrow G$ denotes activist group membership, the two functions $\mathrm{mi}_J : [T \leftrightarrow A] \to [X \leftrightarrow A]$ and $\mathrm{ma}_J : [X \leftrightarrow A] \to [T \leftrightarrow A]$ are defined as

$$\mathrm{mi}_J(R) = \overline{\overline{J}\,;R} \qquad\qquad \mathrm{ma}_J(S) = \overline{\overline{J}^{\,T};S} \tag{12}$$

and the complement of the above relation J now specifies the relationship "is not in opposition to". If J is a partial order, then mi_J and ma_J column-wise compute lower bounds and upper bounds, respectively; in the general case, they column-wise compute lower derivatives and upper derivatives, respectively, in the sense of formal concept analysis (see [18]). The next theorem shows that the above construction based on interest-relations J always leads to M-contacts.

Theorem 3.3. *Given arbitrary relations $M : X \leftrightarrow G$ and $J : X \leftrightarrow T$, we obtain an M-contact K if we define $K := \mathrm{mi}_J(\mathrm{ma}_J(M))$.*

Proof. Property (K_1) follows from

$$\begin{aligned}
\overline{J}^{\,T};M \subseteq \overline{J}^{\,T};M &\iff \overline{J};\ \overline{\overline{J}^{\,T};M} \subseteq \overline{M} && \text{Schröder rule}\\
&\iff M \subseteq \overline{\overline{J};\ \overline{\overline{J}^{\,T};M}}\\
&\iff M \subseteq \mathrm{mi}_J(\mathrm{ma}_J(M)) && \text{by (12)}\\
&\iff M \subseteq K,
\end{aligned}$$

and property (K_2) from

$$\begin{aligned}
&\overline{M^T;\ \overline{J}}\ ;\ \overline{J}^{\,T} \subseteq \overline{M^T;\ \overline{J}}\ ;\ \overline{J}^{\,T}\\
\iff\ &\overline{M^T;\ \overline{J}}\ ;\ \overline{J}^{\,T} \subseteq (\overline{J};\ \overline{\overline{J}^{\,T};M})^T\\
\iff\ &[\,\overline{\overline{J};\ \overline{\overline{J}^{\,T};M}}\,]^T;\ \overline{J} \subseteq M^T;\ \overline{J} && \text{Schröder rule}\\
\iff\ &[\mathrm{mi}_J(\mathrm{ma}_J(M))]^T;\ \overline{J} \subseteq M^T;\ \overline{J} && \text{by (12)}\\
\iff\ &K^T;\ \overline{J} \subseteq M^T;\ \overline{J}\\
\implies\ &K^T;\ \overline{J};\ \overline{\overline{J}^{\,T};M} \subseteq M^T;\ \overline{J};\ \overline{\overline{J}^{\,T};M}\\
\iff\ &K^T;\ \overline{\mathrm{mi}_J(\mathrm{ma}_J(M))} \subseteq M^T;\ \overline{\mathrm{mi}_J(\mathrm{ma}_J(M))} && \text{by (12)}\\
\iff\ &K^T;\ \overline{K} \subseteq M^T;\ \overline{K}. \qquad \square
\end{aligned}$$

The construction of Theorem 3.3 can immediately be transformed into RelView code. In the next example, we present a concrete application of the resulting program.

Example 3.1. We assume four persons, denoted by the natural numbers 1, 2, 3 and 4, three groups g_1, g_2 and g_3, and six topics A, B, C, D, E and F. If the group membership relation M is described by the left-most of the following three RelView matrices and the persons' interests relation J by the RelView matrix in the middle, then these relations lead to the contact K specified by the right-most RelView matrix.

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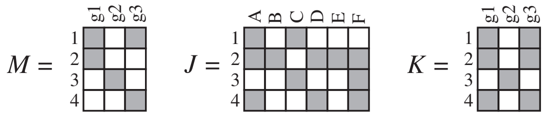

In these pictures of Boolean matrices, produced by the RelView tool, a black square means 1 as matrix entry and a white square means 0 as matrix entry so that, for instance, the first person is a member of the groups g_1 and g_3, it is in opposition to the topics A and C, and it is in contact with the two groups g_1 and g_3.

By definition, we have the inclusion $M \subseteq K$. In addition, we have the property $(4, g_1) \in K$, because wherever *all* persons of the group g_1 are jointly J-interested in a set of topics (here $\{1, 2\} \times \{A\}$), then also person 4 is J-interested in these topics. Also the relationship $(2, g_3) \in K$ is true; the rectangle $\{1, 4\} \times \{A\}$ indicates that all members of the group are jointly J-interested in the topic set $\{A\}$ and so is person 2.

We can even prove completeness of the construction of Theorem 3.3, i.e., that every M-contact K can be represented as an expression $\mathrm{mi}_J(\mathrm{ma}_J(M))$. As the next theorem shows, we may consider the groups as topics and K itself as interest relation J.

Theorem 3.4. *For all relations $M : X \leftrightarrow G$ and all M-contacts $K : X \leftrightarrow G$ we have the equation $K = \mathrm{mi}_K(\mathrm{ma}_K(M))$.*

Proof. "$\subseteq$": This inclusion is equivalent to property (K_2), since

$$
\begin{aligned}
K \subseteq \mathrm{mi}_K(\mathrm{ma}_K(M)) &\iff K \subseteq \overline{\overline{K}\,;\,\overline{\overline{K}^{\,T};M}} && \text{by (12)}\\
&\iff \overline{K}\,;\,\overline{\overline{K}^{\,T};M} \subseteq \overline{K} \\
&\iff \overline{K}^{\,T};K \subseteq \overline{K}^{\,T};M && \text{Schröder rule}\\
&\iff K^T;\,\overline{K} \subseteq M^T;\,\overline{K}\,.
\end{aligned}
$$

"$\supseteq$": Starting with the property (K_1), we get the result by

$$
\begin{aligned}
M \subseteq K &\iff \overline{K}\,;\mathbb{I} \subseteq \overline{M} \\
&\iff \overline{K}^{\,T};M \subseteq \overline{\mathbb{I}} && \text{Schröder rule}\\
&\iff \mathbb{I} \subseteq \overline{\overline{K}^{\,T};M} \\
&\implies \overline{K} \subseteq \overline{K}\,;\,\overline{\overline{K}^{\,T};M} \\
&\iff \overline{\overline{K}\,;\,\overline{\overline{K}^{\,T};M}} \subseteq K \\
&\iff \mathrm{mi}_K(\mathrm{ma}_K(M)) \subseteq K && \text{by (12).} \qquad \square
\end{aligned}
$$

From the two Theorems 3.3 and 3.4, we immediately obtain a fixed point characterization of the set of M-contacts.

Theorem 3.5. *Assume a group membership relation $M : X \leftrightarrow G$ to be given and consider all relations $R : X \leftrightarrow T$ for some set T. Then the function*

$$\tau_M : [X \leftrightarrow T] \to [X \leftrightarrow G] \qquad \tau_M(R) = \mathrm{mi}_R(\mathrm{ma}_R(M)),$$

will always produce an M-contact. The set $\mathfrak{K}_M$ of all M-contacts equals the set of fixed points of τ_M in case $T = G$.

Using relational fixed point enumeration techniques (cf. [4]), this property can be used to compute for small relations M all M-contacts by a tool like RelView. Note, however, that the function τ_M of Theorem 3.5 is not monotonic.

Since the underlying relation M is contained in each M-contact K, in its superrelation K a lot of columns will normally coincide. The column equivalence relation $\Psi_K = \mathrm{syq}(K, K)$ of (11) relates two groups if and only if the corresponding columns of the M-contact K are equal. Hence, we can remove duplicates of columns of K by multiplying it with the canonical epimorphism η_{Ψ_K} induced by the equivalence relation Ψ_K from the right. In the next theorem we prove that instead of K also $K;\,\eta_{\Psi_K}$ can be used.

Theorem 3.6. *For all relations $M : X \leftrightarrow G$ and all M-contacts $K : X \leftrightarrow G$ we have the equality $K = \mathrm{mi}_{K;\eta_{\Psi_K}}(\mathrm{ma}_{K;\eta_{\Psi_K}}(M))$.*

Proof. In the following calculation we abbreviate η_{Ψ_K} by η.

$$
\begin{aligned}
\mathrm{mi}_{K;\eta}(\mathrm{ma}_{K;\eta}(M)) &= \overline{\overline{K;\eta}\,;\,\overline{\overline{K;\eta}^{\,T};M}} && \text{by (12)}\\
&= \overline{\overline{K}\,;\eta;\,\overline{(\overline{K}\,;\eta)^T;M}} && \text{Lemma 2.3}
\end{aligned}
$$

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$$
\begin{aligned}
&= \overline{\overline{K}; \eta; \overline{\eta^T; \overline{K}^T; M}} && \\
&= \overline{\overline{K}; \overline{\eta; \eta^T; \overline{K}^T; M}} && \text{[22] Proposition 4.2.4.iii or [21], 5.13} \\
&= \overline{\overline{K}; \overline{\mathrm{syq}(K, K); \overline{K}^T; M}} && \text{by (10)} \\
&= \overline{\overline{K}; \overline{[\overline{K}; \mathrm{syq}(\overline{K}, \overline{K})]^T; M}} && \text{by (5)} \\
&= \overline{\overline{K}; \overline{\overline{K}^T; M}} && \text{by (6)} \\
&= K && \text{Theorem 3.4}
\end{aligned}
$$

Proposition 4.2.4.iii of [22] and [21] Proposition 5.13 state that R is a mapping if and only if $R; \overline{S} = \overline{R; S}$ for all relations S. □

4. Contacts and closures

Closure operations appear in many fields in computer science and mathematics. Usually, they are defined as extensive, monotone, and idempotent functions on powersets, i.e., functions $h : 2^X \to 2^X$ such that the following conditions hold.

$$
\begin{aligned}
&(\mathrm{H}_1) \quad \forall Y : Y \subseteq h(Y) \\
&(\mathrm{H}_2) \quad \forall Y, Z : Y \subseteq Z \to h(Y) \subseteq h(Z) \\
&(\mathrm{H}_3) \quad \forall Y : h(h(Y)) \subseteq h(Y)
\end{aligned}
$$

As in the case of Aumann contact relations, we start our investigations with a relation-algebraic characterization of closure operations. In the next theorem, the relation Ω denotes set inclusion on the powerset 2^X as specified in (8).

Theorem 4.1. *A mapping $H : 2^X \leftrightarrow 2^X$ is a closure operation if and only if the following three inclusions hold:*

$$
H \subseteq \Omega \qquad \Omega \subseteq H; \Omega; H^T \qquad H; H \subseteq H
$$

Proof. To enhance readability, in the calculations that follow we apply the common notation of function application also for H. The first calculation shows the equivalence of (H_1) and $H \subseteq \Omega$.

$$
\begin{aligned}
\forall Y : Y \subseteq H(Y) &\iff \forall Y, Z : H(Y) = Z \to Y \subseteq Z \\
&\iff \forall Y, Z : H_{Y,Z} \to \Omega_{Y,Z} \\
&\iff H \subseteq \Omega
\end{aligned}
$$

That (H_2) and $\Omega \subseteq H; \Omega; H^T$ are equivalent is shown by the second calculation.

$$
\begin{aligned}
& \forall Y, Z : Y \subseteq Z \to H(Y) \subseteq H(Z) \\
&\iff \forall Y, Z : Y \subseteq Z \to \exists U, V : H(Y) = U \wedge H(Z) = V \wedge U \subseteq V \\
&\iff \forall Y, Z : \Omega_{Y,Z} \to \exists U : H_{Y,U} \wedge \exists V : \Omega_{U,V} \wedge {H^T}_{V,Z} \\
&\iff \forall Y, Z : \Omega_{Y,Z} \to (H; \Omega; H^T)_{Y,Z} \\
&\iff \Omega \subseteq H; \Omega; H^T
\end{aligned}
$$

And here is the remaining proof that shows the equivalence of (H_3) and $H; H \subseteq H$. We start with

$$
\begin{aligned}
& \forall Y : H(H(Y)) \subseteq H(Y) \\
&\iff \forall Y, Z, U : H(Y) = U \wedge H(U) = Z \to (\exists W : H(Y) = W \wedge Z \subseteq W) \\
&\iff \forall Y, Z, U : H_{Y,U} \wedge H_{U,Z} \to (\exists W : H_{Y,W} \wedge \Omega_{Z,W}) \\
&\iff \forall Y, Z : (\exists U : H_{Y,U} \wedge H_{U,Z}) \to (\exists W : H_{Y,W} \wedge {\Omega^T}_{W,Z}) \\
&\iff \forall Y, Z : (H; H)_{Y,Z} \to (H; \Omega^T)_{Y,Z} \\
&\iff H; H \subseteq H; \Omega^T.
\end{aligned}
$$

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We already know that the property (H_1) equals $H \subseteq \Omega$, so that with the antisymmetry of the partial order Ω and the univalency of the mapping H we get

$$H;H \subseteq H;\Omega \cap H;\Omega^T = H;(\Omega \cap \Omega^T) \subseteq H;\mathbb{I} = H$$

from the above inclusion $H;H \subseteq H;\Omega^T$. That $H;H \subseteq H;\Omega^T$ follows from $H;H \subseteq H$ is a consequence of the reflexivity of Ω. □

The following calculation shows that the inclusion $H;H \subseteq H$ is in fact equivalent to the equality $H;H = H$ when H is a mapping; this corresponds to the well-known property that in axiom (H_3) even equality holds.

$$\begin{aligned}
H &= H;H;\mathsf{L} \cap H && \text{since } H \text{ is total}\\
&\subseteq (H;H \cap H;\mathsf{L}^T);(\mathsf{L} \cap (H;H)^T;H) && \text{Dedekind rule}\\
&\subseteq H;H;H^T;H && \text{because } H \text{ is transitive}\\
&\subseteq H;H && H \text{ is univalent}
\end{aligned}$$

Most textbooks combine (H_1) and (H_2) to prove $h(h(Y)) = h(Y)$ from $h(h(Y)) \subseteq h(Y)$. The last part of the proof of Theorem 4.1 shows that (H_1) is sufficient.

Because of Theorem 4.1, we are able to generalize the concept of a closure operation from powerset lattices to arbitrary partial order relations within the language of relation algebra as follows.

Definition 4.1. Given a partial order $P : X \leftrightarrow X$, a mapping $H : X \leftrightarrow X$ is called a *closure operation with respect to P* — in short: a *P-closure* — if the following conditions hold.

$$(\mathrm{C}_1)\ H \subseteq P \qquad (\mathrm{C}_2)\ P \subseteq H;P;H^T \qquad (\mathrm{C}_3)\ H;H \subseteq H$$

In [1] it is shown that there is a one-to-one correspondence between the set of all Aumann contact relations between X and 2^X and the set of all closure operations on 2^X. Without proof and reference to its origin, this correspondence is also mentioned in [12]. In the remainder of this section, we investigate the relationship between contact relations and closure operations in our general setting, i.e., in conjunction with M-contacts and Ω_M-closures, and using relation-algebraic means. As the only basic prerequisite on the relation $M : X \leftrightarrow G$ we assume $\mathrm{syq}(M, M) = \mathbb{I}$, i.e., pair-wise different columns, to ensure that Ω_M is a partial order; see Section 2. (Even this is not a really essential requirement.)

How to obtain M-contacts from Ω_M-closures is shown in the following theorem. In words, the theorem states that $x \in X$ is in contact with $g \in G$ if and only if x is a member of the closure $H(g)$ of the group g.

Theorem 4.2. *For all relations $M : X \leftrightarrow G$ satisfying $\mathrm{syq}(M, M) = \mathbb{I}$ and all Ω_M-closures $H : G \leftrightarrow G$, the relation $K := M;H^T : X \leftrightarrow G$ is an M-contact.*

Proof. For proving (K_1), we use (C_1) and Proposition 4.2.3 of [22] or [21] Proposition 5.9 in

$$M;\Omega_M \subseteq M \implies M;H \subseteq M \iff M \subseteq M;H^T \iff M \subseteq K.$$

Now, an application of Lemma 2.2 yields the desired result. The verification of property (K_2) is based on the following calculation.

$$\begin{aligned}
K;\overline{M^T;\overline{K}} &= M;H^T;\overline{M^T;\overline{M;H^T}} &&\\
&= M;H^T;\overline{M^T;\overline{M}};H^T && \text{[22] Proposition 4.2.4.iii or [21] Proposition 5.13}\\
&= M;H^T;\Omega_M;H^T &&\\
&\subseteq M;\Omega_M;H^T;H^T && \text{by } (\mathrm{C}_2) \text{ (cf. [22] p. 143 or [21] Proposition 5.45)}\\
&\subseteq M;\Omega_M;H^T && \text{by } (\mathrm{C}_3)\\
&= M;H^T && \text{Lemma 2.2}\\
&= K &&
\end{aligned}$$

An application of the Schröder rule to this inclusion completes the proof. □

It is also possible to obtain, in the reverse direction, a closure operation h_A from an Aumann contact relation A. In [1], the closure $h_A(Y)$ of a set Y is defined as the set of elements being in contact with Y — meaning in the highly specialized example of $A := \varepsilon$ that $h_\varepsilon(Y) = Y$. Relation-algebraically, this leads to the expression $\mathrm{syq}(A, \varepsilon)$ for the closure operation h_A — and to $\mathrm{syq}(\varepsilon, \varepsilon)$, i.e., the identity relation I, in this example.

In contrast to the transition from closure operations to contact relations, which also works in our general setting, the transition from M-contacts K to Ω_M-closures is problematic. The reason is that $\mathrm{syq}(K, M)$ may be non-total. However, if $\mathrm{syq}(K, M)$ is total, it is indeed an Ω_M-closure as the following theorem shows.

Theorem 4.3. *For all relations $M : X \leftrightarrow G$ satisfying $\mathrm{syq}(M, M) = \mathrm{I}$ and all M-contacts $K : X \leftrightarrow G$, the relation $H := \mathrm{syq}(K, M) : G \leftrightarrow G$ is an Ω_M-closure provided it is total.*

Proof. Since the totality of the relation H has been assumed as a prerequisite, we show by the subsequent calculation its univalence to establish H as a mapping.

$$
\begin{aligned}
H^T; H &= [\mathrm{syq}(K, M)]^T; \mathrm{syq}(K, M) \\
&= \mathrm{syq}(M, K); \mathrm{syq}(K, M) && \text{by (5)} \\
&\subseteq \mathrm{syq}(M, M) && \text{by (6)} \\
&= \mathrm{I}.
\end{aligned}
$$

Property (C_1) follows from (K_1), since

$$H = \mathrm{syq}(K, M) \subseteq \overline{K^T; \overline{M}} \subseteq \overline{M^T; \overline{M}} = \Omega_M.$$

In the following proof of (C_2) we use that totality of $\mathrm{syq}(K, M)$ implies surjectivity of $\mathrm{syq}(M, K)$ and that $\mathrm{syq}(M, K) = \mathrm{syq}(\overline{M}, \overline{K})$ (cf. (5) and Proposition 4.4.1.i and 4.4.1.ii of [22] or [21] Section 8.5). We start with

$$
\begin{aligned}
H; \overline{\Omega_M}; H^T &= [\mathrm{syq}(M, K)]^T; M^T; \overline{M}; \mathrm{syq}(M, K) && \text{by (5)} \\
&= [M; \mathrm{syq}(M, K)]^T; \overline{M}; \mathrm{syq}(\overline{M}, \overline{K}) && \text{by (5)} \\
&= K^T; \overline{K} && \text{[22] Proposition 4.4.2.ii or [21] 8.12.iii} \\
&\subseteq M^T; \overline{K} && \text{by } (\mathrm{K}_2) \\
&\subseteq M^T; \overline{M} && \text{by } (\mathrm{K}_1).
\end{aligned}
$$

Using that the total relation H is in fact a mapping, we get from this $\overline{H; \Omega_M; H^T} \subseteq \overline{\Omega_M}$, i.e., the desired inclusion $\Omega_M \subseteq H; \Omega_M; H^T$.

Also the first two calculations of the subsequent proof of property (C_3) use the surjectivity of $\mathrm{syq}(M, K) = \mathrm{syq}(\overline{M}, \overline{K})$. From (5) and Proposition 4.4.2.ii of [22] or [21] Proposition 8.12.iii and (K_1) we get the inclusion

$$K^T; \overline{M}; \mathrm{syq}(M, K) = K^T; \overline{M}; \mathrm{syq}(\overline{M}, \overline{K}) = K^T; \overline{K} \subseteq K^T; \overline{M}$$

and Proposition 4.4.2.ii of [22] or [21] Proposition 8.12.iii and (K_2) yield

$$\overline{K}^T; M; \mathrm{syq}(M, K) = \overline{K}^T; K = (K^T; \overline{K})^T \subseteq (M^T; \overline{K})^T = \overline{K}^T; M.$$

Putting these inclusions together, we obtain

$$(K^T; \overline{M} \cup \overline{K}^T; M); \mathrm{syq}(M, K) \subseteq K^T; \overline{M} \cup \overline{K}^T; M$$

that, due to the definition of $\mathrm{syq}(K, M)$ and (5), holds if and only if

$$\overline{\mathrm{syq}(K, M)}; [\mathrm{syq}(K, M)]^T \subseteq \overline{\mathrm{syq}(K, M)}$$

holds. An application of the Schröder rule to this result followed by the definition of H, finally, shows $H; H \subseteq H$. □

By the relation $\mathrm{syq}(K, M)$, groups $g_1, g_2 \in G$ are related if and only if g_2 consists of the individuals which are in contact with g_1. Theorem 4.3 says that if for all $g_1 \in G$ these individuals form one of the groups according to M, then $g_1 \mapsto g_2$ is a closure operation on G. Combining the last two theorems, we obtain for our general setting an injective embedding of the Ω_M-closures into the M-contacts.

Corollary 4.1. *Assume a relation $M : X \leftrightarrow G$ such that $\mathrm{syq}(M, M) = \mathsf{I}$ and let $\mathfrak{K}_M$ and $\mathfrak{H}_{\Omega_M}$ denote the set of M-contacts and Ω_M-closures, respectively. Then $\mathrm{con}_M : \mathfrak{H}_{\Omega_M} \to \mathfrak{K}_M$, defined by $\mathrm{con}_M(H) = M; H^T$, is an injective function.*

Proof. First, we show that $\mathrm{syq}(\mathrm{con}_M(H), M)$ is total for all $H \in \mathfrak{H}_{\Omega_M}$.

$$
\begin{aligned}
\mathrm{syq}(\mathrm{con}_M(H), M); \mathsf{L} &= \mathrm{syq}(M; H^T, M); \mathsf{L} && \text{definition of } \mathrm{con}_M(H) \\
&= H; \mathrm{syq}(M, M); \mathsf{L} && \text{[22] Proposition 4.4.1.vi or [21] 8.16.iii} \\
&= H; \mathsf{L} && \text{since } \mathrm{syq}(M, M) = \mathsf{I} \\
&= \mathsf{L} && H \text{ total}
\end{aligned}
$$

Hence, $\mathrm{syq}(\mathrm{con}_M(H), M)$ is an Ω_M-closure due to Theorems 4.2 and 4.3. The above calculation, furthermore, shows that the function

$$
\mathrm{clo}_M : \mathrm{con}_M(\mathfrak{H}_{\Omega_M}) \to \mathfrak{H}_{\Omega_M} \qquad \mathrm{clo}_M(K) = \mathrm{syq}(K, M)
$$

fulfils $\mathrm{clo}_M(\mathrm{con}_M(H)) = H$ for all $H \in \mathfrak{H}_{\Omega_M}$, and we are done. □

Specifying the well-known point-wise ordering of mappings relation-algebraically, we obtain for $H_1, H_2 \in \mathfrak{H}_{\Omega_M}$ that $H_1 \leq H_2$ if and only if $H_1 \subseteq H_2; \Omega_M^T$. In respect thereof, the following theorem shows that the function con_M is even an order embedding from the ordered set $(\mathfrak{H}_{\Omega_M}, \leq)$ into the ordered set $(\mathfrak{K}_M, \subseteq)$.

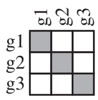
Theorem 4.4. *Under the assumptions of Corollary 4.1, we have the equivalence of $H_1 \subseteq H_2; \Omega_M^T$ and $M; H_1^T \subseteq M; H_2^T$.*

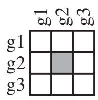
Proof. In the following calculation we combine the fact that H_1 and H_2 are mappings with Proposition 4.2.4.iii of [22] or [21] Proposition 5.13.

$$
\begin{aligned}
H_1 \subseteq H_2; \Omega_M^T &\iff H_1 \subseteq H_2; [\,\overline{M^T; \overline{M}}\,]^T \\
&\iff H_1 \subseteq H_2; \overline{\overline{M}^T; M} \\
&\iff H_1 \subseteq \overline{\overline{H_2; M^T}; M} && \text{Proposition 4.2.4.iii of [22] or [21] 5.13} \\
&\iff \overline{H_2; M^T}; M \subseteq \overline{H_1} \\
&\iff H_1; M^T \subseteq H_2; M^T && \text{Schröder rule} \qquad \square
\end{aligned}
$$

A little reflection shows that $(\mathfrak{K}_M, \subseteq)$ is a complete lattice. For the ordered set $(\mathfrak{H}_{\Omega_M}, \leq)$ this is not true in general. It is, however, true if the underlying set G on which the closure operations work is finite [19]. In general, we are not able to establish a one-to-one correspondence between contact relations and closure operations in our general setting without further assumptions on the underlying relation $M : X \leftrightarrow G$.

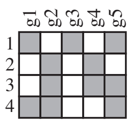
Examples 4.1. For the example of Section 3, e.g., RelView computed for the membership relation M and the M-contact K given there the following matrices for Ω_M and $\mathrm{syq}(K, M)$.

$\Omega_M =$ 

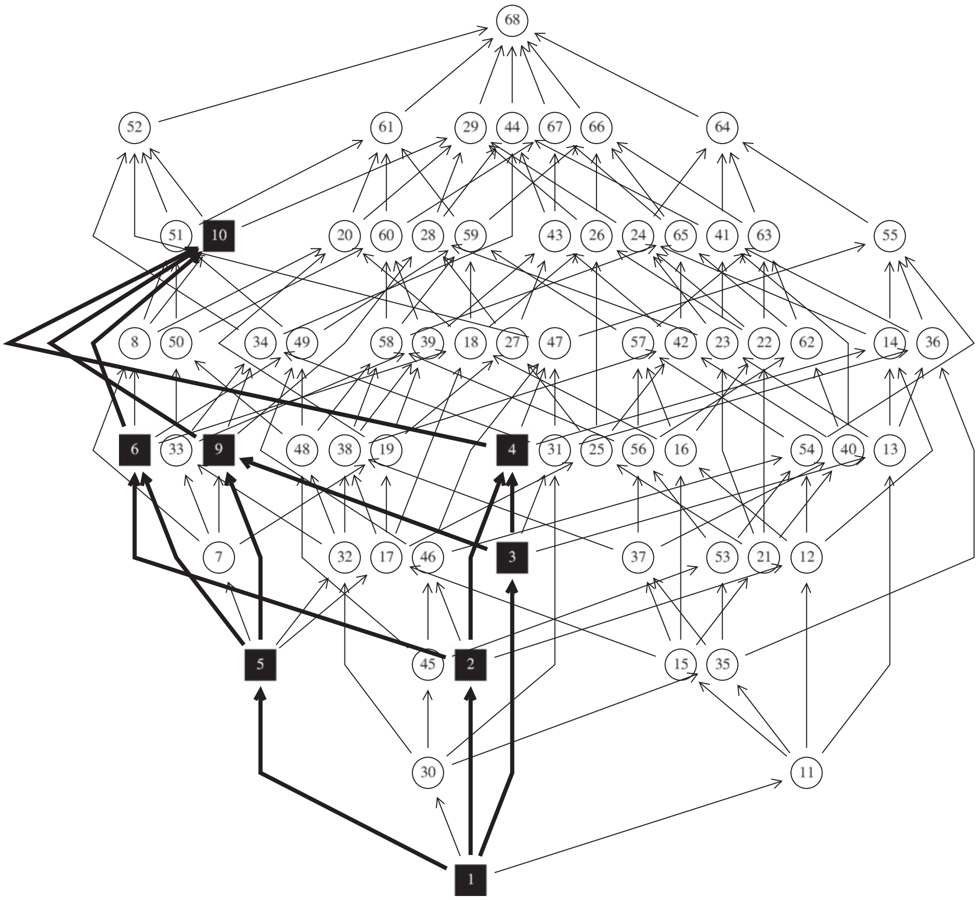
$\mathrm{syq}(K, M) =$ 

The relation Ω_M may be described as being the column-is-contained-preorder for M, while the relation $\mathrm{syq}(K, M)$ compares columns of K and M for being identical. Among the obviously 128 relations containing M the tool identified exactly 66 as M-contacts. Since Ω_M is the identity relation, however, there exists only one Ω_M-closure, viz. Ω_M.

To give an impression of the visualization potential of the RelView tool, we consider a second example. Let four individuals 1, 2, 3 and 4 and five groups g_1, g_2, g_3, g_4 and g_5 be given and consider the following relation of group membership.

$M =$ 

RelView computes for this example 68 M-contacts but only 8 Ω_M-closures. The following picture shows the Hasse diagram of the lattice $(\mathfrak{K}_M, \subseteq)$.



In this directed graph (drawn by RelView, too) the 8-element sublattice $\mathrm{con}_M(\mathfrak{H}_{\Omega_M})$ of $(\mathfrak{K}_M, \subseteq)$ is indicated by black squares and bold edges.

Using the RelView tool we also have discovered an oversight in [2] which was calculated by hand. On p. 98 a table with 10 functions together with the Hasse diagram of the point-wise ordering is represented and it is claimed that the functions are exactly all closure operations on the powerset of a two-element set. In fact, there exist only 7 such operations; the functions h_7 and h_8 of [2] are not closure operations. The number of different closure operations on 2^X is known only up to $|X| = 6$; these numbers are 2, 7, 61, 2480, 1385552 and 75973751474 (see [12] for example). Using RelView and the very simple programs developed in [5], we have been able to verify the cases $|X| \leq 5$. In Boolean matrix terminology, totality of $\mathrm{syq}(K, M)$ means that each column of $K : X \leftrightarrow G$ appears also as a column of M. Hence, this property should hold for G being a powerset 2^X and M being the powerset relation $\varepsilon : X \leftrightarrow 2^X$. And, in fact, totality of $\mathrm{syq}(K, \varepsilon)$ can be shown so that, together with the results already obtained, we are able to give not only a completely relation-algebraic proof of the above mentioned result of Aumann but also to show that the sets are isomorphic complete lattices.

Corollary 4.2. *Given an arbitrary powerset relation $\varepsilon : X \leftrightarrow 2^X$, the two ordered sets $(\mathfrak{K}_\varepsilon, \subseteq)$ and $(\mathfrak{H}_\Omega, \leq)$ are isomorphic via the function $\mathrm{con}_\varepsilon : \mathfrak{H}_\Omega \rightarrow \mathfrak{K}_\varepsilon$ of Corollary 4.1 and its inverse function $\mathrm{clo}_\varepsilon : \mathfrak{K}_\varepsilon \rightarrow \mathfrak{H}_\Omega$, where $\mathrm{clo}_\varepsilon(K) = \mathrm{syq}(K, \varepsilon)$.*

Proof. For each $K \in \mathfrak{K}_\varepsilon$, (5) and the second axiom of (7) imply

$$\mathrm{syq}(K, \varepsilon); \mathsf{L} = (\mathsf{L}; \mathrm{syq}(K, \varepsilon)^T)^T = (\mathsf{L}; \mathrm{syq}(\varepsilon, K))^T = \mathsf{L}.$$

Because of Theorem 4.3, therefore, $\mathrm{clo}_\varepsilon(K)$ is defined for all $K \in \mathfrak{K}_\varepsilon$. From the proof of Corollary 4.1 we know already that

$$\mathrm{clo}_\varepsilon(\mathrm{con}_\varepsilon(H)) = H$$

holds for all $H \in \mathfrak{H}_\Omega$. Furthermore, we obtain for all $K \in \mathfrak{K}_\varepsilon$ the equation

$$\mathrm{con}_\varepsilon(\mathrm{clo}_\varepsilon(K)) = \varepsilon; \mathrm{syq}(K, \varepsilon)^T = \varepsilon; \mathrm{syq}(\varepsilon, K) = K$$

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using the second axiom of (7) in combination with Proposition 4.4.2.ii of [22] or [21] Section 8.5. These two properties show that the functions are bijective and inverses of one another. That the two mappings are order isomorphisms follows from Theorem 4.4. □

One might conjecture that in the case $\mathrm{syq}(M, M) = \mathsf{I}$ from an isomorphism between the sets $\mathfrak{K}_M$ and $\mathfrak{H}_{\Omega_M}$ also the second axiom of (7) follows, i.e., M is essentially a powerset relation. Unfortunately, this speculation is false, as the simple example with a single group, i.e., $G := \mathbf{1}$, and M as $\mathsf{L} : X \leftrightarrow \mathbf{1}$ shows.

5. Contacts and topology

Topology may be defined in many ways, for example via neighbourhood systems and topologically open (or closed) sets. In this section we show how these notions can be specified relation-algebraically and that they are heavily related with contacts studied so far. We also will demonstrate that a relation-algebraic description immediately leads to algorithms that allow to transform one description into another.

We need only a few additional relation-algebraic preliminaries. Let X and Y be sets. Then there are the two canonical projection functions which decompose a pair $u \in X \times Y$ into its first component u_1 and its second component u_2. In the following we always assume pairs u to be of the form $\langle u_1, u_2\rangle$, that is, u_1 denotes the first component and u_2 its second component.

For a relation-algebraic approach it is useful to consider instead of these functions the corresponding *projection relations* $\pi : X\times Y \leftrightarrow X$ and $\rho : X\times Y \leftrightarrow Y$ such that for all $u \in X \times Y$, $x \in X$ and $y \in Y$ we have $\pi_{u,x}$ if and only if $u_1 = x$ and $\rho_{u,y}$ if and only if $u_2 = y$. Projection relations enable us to specify the well-known pairing operation of functional programming relation-algebraically as follows: For relations $R : Z \leftrightarrow X$ and $S : Z \leftrightarrow Y$ we define their *pairing* (frequently also called *fork* or *tupeling*) by

$$[R, S] := R; \pi^T \cap S; \rho^T : Z \leftrightarrow X\times Y. \tag{13}$$

Then for all $z \in Z$ and $u \in X \times Y$ a simple reflection shows that $[R, S]_{z,u}$ if and only if R_{z,u_1} and S_{z,u_2}. By a combination of (13) with symmetric quotients a lot of the well-known operations on sets can be specified as relations. For dealing with topology, we only need the following two:

$$\mathsf{M} := \mathrm{syq}([\varepsilon, \varepsilon], \varepsilon) : 2^X\times 2^X \leftrightarrow 2^X \qquad \mathsf{C} := \mathrm{syq}(\varepsilon, \overline{\varepsilon}) : 2^X \leftrightarrow 2^X \tag{14}$$

In the specifications of (14), the relation $\varepsilon : X \leftrightarrow 2^X$ is again the powerset relation. The relation M specifies set intersection relation-algebraically, since for all pairs of sets $\langle S, T\rangle \in 2^X\times 2^X$ and all sets $U \in 2^X$ it holds that

$$\begin{aligned}
\mathsf{M}_{\langle S,T\rangle,U} &\iff \mathrm{syq}([\varepsilon, \varepsilon], \varepsilon)_{\langle S,T\rangle,U} \\
&\iff \forall x : [\varepsilon, \varepsilon]_{x,\langle S,T\rangle} \leftrightarrow \varepsilon_{x,U} \\
&\iff \forall x : (x \in S \wedge x \in T) \leftrightarrow x \in U \\
&\iff S \cap T = U.
\end{aligned}$$

In a similar way it can be shown that C realizes set complementation, i.e., for all $S, T \in 2^X$ it holds that $\mathsf{C}_{S,T}$ if and only if $\overline{S} = T$.

If we define topology via neighbourhoods, then a set X endowed with a system $\mathcal{N}(x)$ of nonempty subsets of 2^X for every element $x \in X$ is called a *topological structure* if we have $x \in U$ for all $x \in X$ and $U \in \mathcal{N}(x)$ and, hence, *neighbourhood systems* $\mathcal{N}(x)$ never contain the empty set, all neighbourhood systems $\mathcal{N}(x)$ are up-sets, all neighbourhood systems $\mathcal{N}(x)$ are closed under binary (and, thus, finite) intersections and for all $x \in X$ and $U \in \mathcal{N}(x)$ there exists $V \in \mathcal{N}(x)$ such that $U \in \mathcal{N}(y)$ for all $y \in V$. The neighbourhood systems function $\mathcal{N}$ that maps elements to their neighbourhood systems has source X and target 2^{2^X}. A function of such type can also be interpreted as relation of type $[X \leftrightarrow 2^X]$ that relates $x \in X$ and $U \in 2^X$ if and only if $U \in \mathcal{N}(x)$. Then the nonemptiness of the neighbourhood systems is equivalent to the totality of the relation since for all $x \in X$ there exists a neighbourhood $U \in \mathcal{N}(x)$, that is, a set $U \in 2^X$ such that $\mathcal{N}_{x,U}$ if we denote the relation with the same symbol as the neighbourhood systems function. Relation-algebraically, totality may be specified as $\mathcal{N}; \mathsf{L} = \mathsf{L}$. The following theorem shows how the remaining four requirements on neighbourhood systems look if translated into relation-algebraic formulae. Here $\varepsilon : X \leftrightarrow 2^X$ is a powerset relation and $\Omega : 2^X \leftrightarrow 2^X$ the induced inclusion order. The formulae of the theorem constitute a relation-algebraic specification of topology via neighbourhood systems.

Theorem 5.1. *Suppose a total relation $\mathcal{N} : X \leftrightarrow 2^X$. Then the four formulae*

$$\mathcal{N} \subseteq \varepsilon \qquad \mathcal{N}; \Omega \subseteq \mathcal{N} \qquad [\mathcal{N}, \mathcal{N}]; \mathsf{M} \subseteq \mathcal{N} \qquad \mathcal{N} \subseteq \mathcal{N}; \overline{\varepsilon^T; \overline{\mathcal{N}}}$$

hold if and only if the set X endowed with the system of subsets $\mathcal{N}(x) := \{U \in 2^X \mid \mathcal{N}_{x,U}\}$ for all $x \in X$ is a topological structure.

Proof. Obviously, the first formula $\mathcal{N} \subseteq \varepsilon$ corresponds to the fact that $x \in U$ for all $x \in X$ and $U \in \mathcal{N}(x)$, The first formula says not least that $\mathcal{N}$ does never relate with the empty set, that is, none of the neighbourhood systems $\mathcal{N}(x)$ contains $\emptyset$. That all sets $\mathcal{N}(x)$ are up-sets is relation-algebraically described by the second formula $\mathcal{N};\Omega \subseteq \mathcal{N}$ and that all neighbourhood systems $\mathcal{N}(x)$ are closed under binary intersections is relation-algebraically described by the third one. We demonstrate only the latter.

$$\begin{aligned}
[\mathcal{N},\mathcal{N}];\mathsf{M} \subseteq \mathcal{N} &\iff \forall x,U : ([\mathcal{N},\mathcal{N}];\mathsf{M})_{x,U} \to \mathcal{N}_{x,U} \\
&\iff \forall x,U : (\exists V_1,V_2 : [\mathcal{N},\mathcal{N}]_{x,\langle V_1,V_2\rangle} \wedge \mathsf{M}_{\langle V_1,V_2\rangle,U}) \to \mathcal{N}_{x,U} \\
&\iff \forall x,U : (\exists V_1,V_2 : \mathcal{N}_{x,V_1} \wedge \mathcal{N}_{x,V_2} \wedge (V_1 \cap V_2 = U)) \to \mathcal{N}_{x,U} \\
&\iff \forall x,U,V_1,V_2 : V_1 \in \mathcal{N}(x) \wedge V_2 \in \mathcal{N}(x) \wedge (V_1 \cap V_2 = U) \to U \in \mathcal{N}(x) \\
&\iff \forall x,V_1,V_2 : V_1 \in \mathcal{N}(x) \wedge V_2 \in \mathcal{N}(x) \to V_1 \cap V_2 \in \mathcal{N}(x)
\end{aligned}$$

From the calculation

$$\begin{aligned}
\mathcal{N} \subseteq \mathcal{N};\overline{\varepsilon^T;\overline{\mathcal{N}}} &\iff \forall x,U : \mathcal{N}_{x,U} \to (\mathcal{N};\overline{\varepsilon^T;\overline{\mathcal{N}}}\,)_{x,U} \\
&\iff \forall x,U : \mathcal{N}_{x,U} \to \exists V : \mathcal{N}_{x,V} \wedge \overline{\varepsilon^T;\overline{\mathcal{N}}}_{\,V,U} \\
&\iff \forall x,U : \mathcal{N}_{x,U} \to \exists V : \mathcal{N}_{x,V} \wedge \neg[\exists y : {\varepsilon^T}_{V,y} \wedge \overline{\mathcal{N}}_{y,U}] \\
&\iff \forall x,U : \mathcal{N}_{x,U} \to \exists V : \mathcal{N}_{x,V} \wedge [\forall y : \varepsilon_{y,V} \to \mathcal{N}_{y,U}] \\
&\iff \forall x,U : U \in \mathcal{N}(x) \to \exists V : V \in \mathcal{N}(x) \wedge [\forall y : y \in V \to U \in \mathcal{N}(y)]
\end{aligned}$$

we obtain, finally, that the last formula $\mathcal{N} \subseteq \mathcal{N};\overline{\varepsilon^T;\overline{\mathcal{N}}}$ corresponds to the last property of a topological structure. □

We call a relation of type $[X \leftrightarrow 2^X]$ that fulfils the properties of Theorem 5.1 a neighbourhood systems relation. For all $x \in X$ the x-column of the transpose of such a relation is a vector of type $[2^X \leftrightarrow \mathbf{1}]$ and represents a neighbourhood system of x. In total we have for neighbourhood systems relations that $\mathcal{N};\Omega = \mathcal{N} = \mathcal{N};\overline{\varepsilon^T;\overline{\mathcal{N}}}$, since Ω is reflexive and $\mathcal{N} \subseteq \varepsilon$. A lot of the basics of topology follow from the above description using relation-algebraic reasoning, not least the interconnection with the definition of a topological space via open sets.

If topology is described via open sets, then one considers a *topological space* $(X,\mathcal{O})$, where $\mathcal{O}$ is a subset of the powerset 2^X, called the *open set topology*. The four axioms for $\mathcal{O}$ are $\emptyset \in \mathcal{O}$, $X \in \mathcal{O}$, that arbitrary unions of sets of $\mathcal{O}$ belong to $\mathcal{O}$, and that binary (and, thus, also finite) intersections of sets of $\mathcal{O}$ belong to $\mathcal{O}$. It is easy to lift this definition to the relation-algebraic level if the set $\mathcal{O}$ is represented by a vector $\omega : 2^X \leftrightarrow \mathbf{1}$, i.e., if for all $S \in 2^X$ it holds that ω_S if and only if $S \in \mathcal{O}$.

Theorem 5.2. *Assume the subset $\mathcal{O}$ of 2^X to be represented by the vector $\omega : 2^X \leftrightarrow \mathbf{1}$. Then the set $\mathcal{O}$ satisfies the axioms of an open set topology if and only if the formulae*

$$\overline{\varepsilon^T;\mathsf{L}} \subseteq \omega \qquad \overline{\overline{\varepsilon}^T;\mathsf{L}} \subseteq \omega \qquad \mu \subseteq \omega \Rightarrow \mathsf{syq}(\varepsilon,\varepsilon;\mu) \subseteq \omega \qquad [\omega^T,\omega^T];\mathsf{M} \subseteq \omega^T$$

hold, where in the third formula the vector μ is assumed to be universally quantified.

Proof. We demonstrate the equivalence of the formulae and the above axioms for the first and third case only. Here is the translation from relation algebra into the first axiom.

$$\begin{aligned}
\overline{\varepsilon^T;\mathsf{L}} \subseteq \omega &\iff \forall S : \overline{\varepsilon^T;\mathsf{L}}_S \to \omega_S \\
&\iff \forall S : \neg[\exists x : {\varepsilon^T}_{S,x} \wedge \mathsf{L}_x] \to \omega_S \\
&\iff \forall S : \neg[\exists x : x \in S] \to S \in \mathcal{O} \\
&\iff \forall S : S = \emptyset \to S \in \mathcal{O} \\
&\iff \emptyset \in \mathcal{O}
\end{aligned}$$

Now, let an arbitrary subset $\mathcal{M}$ of 2^X be given and suppose that it is represented by the vector $\mu : 2^X \leftrightarrow \mathbf{1}$. Then we have $\mathsf{syq}(\varepsilon,\varepsilon;\mu) : 2^X \leftrightarrow \mathbf{1}$. Furthermore, for all $S \in 2^X$ we can calculate

$$\begin{aligned}
\mathrm{syq}(\varepsilon, \varepsilon; \mu)_S &\iff \forall x : \varepsilon_{x,S} \leftrightarrow (\varepsilon; \mu)_x \\
&\iff \forall x : \varepsilon_{x,S} \leftrightarrow \exists T : \varepsilon_{x,T} \wedge \mu_T \\
&\iff \forall x : x \in S \leftrightarrow \exists T : x \in T \wedge T \in \mathcal{M} \\
&\iff \forall x : x \in S \leftrightarrow x \in \bigcup \mathcal{M} \\
&\iff S = \bigcup \mathcal{M}
\end{aligned}$$

so that $\mathrm{syq}(\varepsilon, \varepsilon; \mu) : 2^X \leftrightarrow \mathbf{1}$ represents the union of the sets of $\mathcal{M}$. Hence, the implication expresses the fact that if $\mathcal{M}$ is a subset of $\mathcal{O}$ then $\bigcup \mathcal{M}$ belongs to $\mathcal{M}$, i.e., is equivalent to the third axiom. □

If a topological space $(X, \mathcal{O})$ is given relation-algebraically as vector $\omega : 2^X \leftrightarrow \mathbf{1}$ that represents the open set topology, then the vector $\mathsf{C}; \omega : 2^X \leftrightarrow \mathbf{1}$ represents the set of closed sets, i.e., the *closed set topology* $\mathcal{C}$, since for all $U \in 2^X$ it holds that

$$(\mathsf{C}; \omega)_U \iff \exists V : \mathsf{C}_{U,V} \wedge \omega_V \iff \exists V : \overline{U} = V \wedge V \in \mathcal{O} \iff \overline{U} \in \mathcal{O}.$$

It should be observed that in the last formula of Theorem 5.1, although residing on the larger instead of the smaller side, the same construction is used as in Theorem 3.1 when defining the decisive contact condition of an Aumann contact relation A:

$$\mathcal{N} \subseteq \mathcal{N}; \overline{\varepsilon^T; \overline{\mathcal{N}}} \quad \text{as opposed to} \quad A \supseteq A; \overline{\varepsilon^T; \overline{A}}$$

So, there may be an interesting relationship to Aumann contact relations that we are now going to exhibit.[1] Suppose $A : X \leftrightarrow 2^X$ to be an Aumann contact relation. From Section 4 we already know that in this case the relation $H := \mathrm{syq}(A, \varepsilon) : 2^X \leftrightarrow 2^X$ is a closure operation on 2^X. A little reflection shows that the vector $s := (H \cap \mathsf{I}); \mathsf{L} : 2^X \leftrightarrow \mathbf{1}$ represents the fixed points of H, that is, the closure system $\mathcal{S}$ induced by the closure operation H (see, e.g., [12]). It is a standard result of topology that one obtains from a closure system $\mathcal{S}$ a topological structure by defining the neighbourhood system of $x \in X$ as the sets of all sets that contain an $S \in \mathcal{S}$ with $x \in S$:

$$\mathcal{N}(x) := \{T \in 2^X \mid \exists S \in \mathcal{S} : x \in S \wedge S \subseteq T\}$$

Translated into the language of relation algebra, the construction of a neighbourhood systems relation from an Aumann contact relation via the induced closure looks as follows.

Theorem 5.3. *Let an Aumann contact relation $A : X \leftrightarrow 2^X$ be given together with the closure operation $H := \mathrm{syq}(A, \varepsilon)$ and the vector representation $s := (H \cap \mathsf{I}); \mathsf{L}$ of the closure system of H. Then $\mathcal{N} := (\varepsilon \cap \mathsf{L}; s^T); \Omega : X \leftrightarrow 2^X$ is a neighbourhood systems relation.*

From the neighbourhood systems $\mathcal{N}(x)$ in turn, again by well-known standard constructions of topology, the formation of the so-called *open kernel* of a subset S of X as well as the system $\mathcal{O}$ of the open sets in the topology thus defined may be obtained. The open kernel of S is the set of all elements $x \in X$ for which $S \in \mathcal{N}(x)$ and the set $\mathcal{O}$ consists of all sets which coincide with its open kernel (i.e., are fixed points of the kernel operation). Again translated into relation algebra, we have the following result:

Theorem 5.4. *Assume $\mathcal{N} : X \leftrightarrow 2^X$ to be the neighbourhood systems relation of a topological structure. If we define the open kernel relation $K : X \leftrightarrow 2^X$ as $K := \mathrm{syq}(\mathcal{N}, \varepsilon)$ and $\omega := (K \cap \mathsf{I}); \mathsf{L}$ as vector representing its fixed points, then the subset of 2^X represented by $\omega : 2^X \leftrightarrow \mathbf{1}$ is an open set topology.*

The way back from the open set vector ω to the neighbourhood systems relation $\mathcal{N}$ is also possible. It is known that an open set topology $\mathcal{O}$ yields the neighbourhood systems $\mathcal{N}(x)$ consisting of all sets which contain an open set S such that $x \in S$. In terms of relation algebra, the construction of the neighbourhood systems relation $\mathcal{N}$ is, thus, described as in Theorem 5.3, but with s taken as vector ω.

Theorem 5.5. *If the vector $\omega : 2^X \leftrightarrow \mathbf{1}$ represents the open set topology of a topological space $(X, \mathcal{O})$, then $\mathcal{N} := (\varepsilon \cap \mathsf{L}; \omega^T); \Omega : X \leftrightarrow 2^X$ defines a neighbourhood systems relation.*

[1] Recall that it was one of the intentions of Aumann to provide for beginners a more suggestive access to topology by means of contacts.

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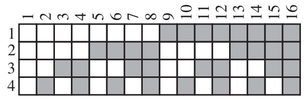
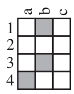
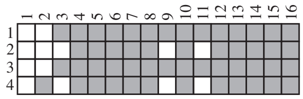
The construction of a neighbourhood systems relation from an Aumann contact relation and the exchange of the possibilities to describe a topology immediately can be transformed into the programming language of RelView. In the following example we want to visualize the last results by means of some RelView pictures.

Example 5.1. We start with the set $X := \{1, 2, 3, 4\}$, the powerset 2^X of X as set of groups and $\varepsilon : X \leftrightarrow 2^X$ as group membership relation. Here is the picture of ε.

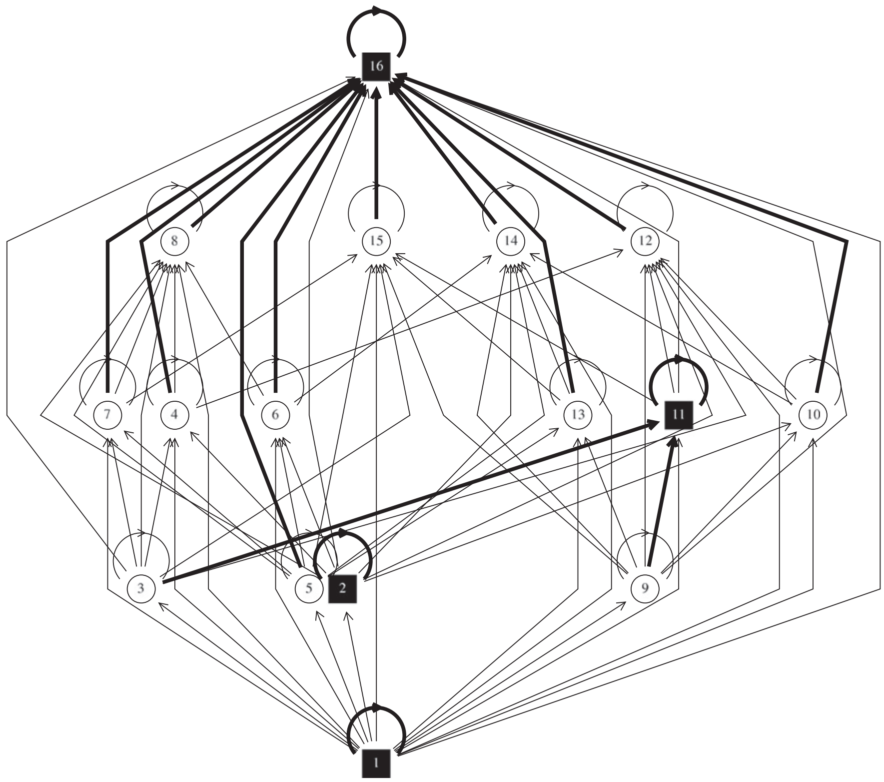
$\varepsilon =$ 

For reasons of space we do not use sets as column-labels — although RelView allows such labels. Instead we use the natural numbers from 1 to 16 to refer to the columns of ε. The correspondence between the numbers and the sets they stand for immediately follows from the black squares of the columns. For example, the label 4 stands for the set $\{3, 4\}$ and the label 8 stands for the set $\{2, 3, 4\}$.

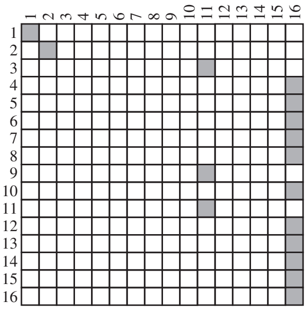
Now, we consider the following interest relation J with three topics a, b and c and the Aumann contact relation A obtained from ε and J by the construction of Theorem 3.3.

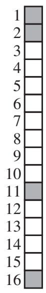
$J =$  $A =$ 

Based on the two relations ε and A, we next use Corollary 4.2 and construct the closure operation $H = \mathrm{syq}(A, \varepsilon) : 2^X \leftrightarrow 2^X$ and then the corresponding vector $s : 2^X \leftrightarrow \mathbf{1}$ that represents the fixed points of H (the induced closure system $\mathcal{S}$). The following picture shows the set inclusion relation Ω as directed graph, drawn by RelView, where the edges corresponding to the pairs of H are drawn in bold and the fixed points of H are emphasized as black squares.



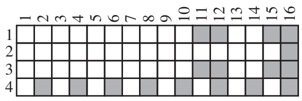
Two of the three properties of closure operations can immediately be verified by examining the picture. H is extensive, since each bold edge is an edge of Ω, and idempotent, since each H-path leads into a loop over at most one non-loop edge. Monotonicity of H can be recognized by pointwise comparisons. The picture also clearly visualizes that each element of 2^X is either a fixed point, i.e., contained in the closure system $\mathcal{S}$, or is mapped to the least element of $\mathcal{S}$ above it. And here are the RelView matrices for the relation H and the vector s.

$H =$ 

$s =$ 

If we compare the columns of ε with the black squares of the graph and the 1-entries of s, respectively, then we obtain that the sets $\emptyset$, $\{4\}$, $\{1, 3\}$ and $\{1, 2, 3, 4\}$ are the fixed points of H, i.e., form the closure system induced by H.

The next picture shows the neighbourhood systems relation $\mathcal{N} : X \leftrightarrow 2^X$ computed from s using the construction of Theorem 5.3.

$\mathcal{N} =$ 

From the rows of this relation and the columns of ε we obtain the following neighbourhood systems for the elements of X, i.e., the topological structure induced by A:

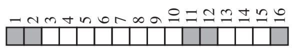
$$\mathcal{N}(1) = \{\, \{1, 3\}, \{1, 3, 4\}, \{1, 2, 3\}, \{1, 2, 3, 4\} \,\}$$

$$\mathcal{N}(2) = \{\, \{1, 2, 3, 4\} \,\}$$

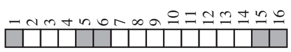
$$\mathcal{N}(3) = \{\, \{1, 3\}, \{1, 3, 4\}, \{1, 2, 3\}, \{1, 2, 3, 4\} \,\}$$

$$\mathcal{N}(4) = \{\, \{4\}, \{3, 4\}, \{2, 4\}, \{2, 3, 4\}, \{1, 4\}, \{1, 3, 4\}, \{1, 2, 4\}, \{1, 2, 3, 4\} \,\}$$

Finally, we pass from the above neighbourhood systems relation $\mathcal{N}$ to the open set topology $\mathcal{O}$, now using Theorem 5.4, and then to the closed sets $\mathcal{C}$, too. For reason of space the following picture shows the transpose of the vector $\omega : 2^X \leftrightarrow \mathbf{1}$ that represents the open set topology, that is, it shows the relation (row vector) $\omega^T : \mathbf{1} \leftrightarrow 2^X$.

$\omega^{\mathsf{T}} =$ 

A comparison of this picture and the columns 1, 2, 11, 12 and 16 of the relation ε yields the open sets $\emptyset$, $\{4\}$, $\{1, 3\}$, $\{1, 3, 4\}$ and $\{1, 2, 3, 4\}$, i.e., the topological space given by $\mathcal{N}$. The transpose of the vector representation $\gamma : 2^X \leftrightarrow \mathbf{1}$ of the closed set topology $\mathcal{C}$ is obtained as $\gamma^T = \omega^T; \mathsf{C}^T = \omega^T; \mathsf{C}$, since C is symmetric. This vector looks as follows:

$\gamma^{\mathsf{T}} =$ 

Hence, the closed set topology of the topology induced by the Aumann contact consists of the sets $\emptyset$, $\{2\}$, $\{2, 4\}$ $\{1, 2, 3\}$ and $\{1, 2, 3, 4\}$.

In Theorem 5.3 we have shown that Aumann contact relations lead to topologies. If the carrier set X is finite, then the converse direction is also possible. Recall that in this case a closure system w.r.t. the ordered set $(2^X, \subseteq)$ is a subset of 2^X that is closed under binary intersections and contains X. Hence, the pair $(X, \mathcal{O})$ is a topological space if and only if $\mathcal{O}$ is a closure system w.r.t. $(2^X, \subseteq)$ and w.r.t. $(2^X, \supseteq)$. As closure system w.r.t. $(2^X, \subseteq)$ it yields a closure operation and this, in turn, leads to an Aumann contact relation. Note, however, that there is no one-to-one correspondence between the topologies and the closures and contacts, respectively. The number of topologies is much smaller than the number of closures/contacts. We have already mentioned the number of different closure operations (contact relations) on 2^X up to $|X| = 6$; the corresponding numbers of topologies are 1, 4, 29, 355, 6942, 209527 (see [17]).

6. Linking column and row types of a relation

We assume $M \in \mathbb{R}^{n \times n}$ to be a quadratic and real-valued matrix with columns $c_1, \ldots, c_n \in \mathbb{R}^n$ and transposed rows $r_1, \ldots, r_n \in \mathbb{R}^n$. If we define

$$M_c := \left\{ \sum_{i=1}^{n} x_i c_i \mid x_1, \ldots, x_n \in \mathbb{R} \right\} \qquad M_r := \left\{ \sum_{i=1}^{n} x_i r_i \mid x_1, \ldots, x_n \in \mathbb{R} \right\} \tag{15}$$

as the sets of all linear combinations of columns and of transposed rows of M, respectively, then the two subsets M_c and M_r of $\mathbb{R}^n$ form subspaces of the vector space $\mathbb{R}^n$ and $\dim M_c = \dim M_r$. Now, let $M \in \mathbb{B}^{n\times n}$ be a Boolean matrix with columns= $c_1, \ldots, c_n \in \mathbb{B}^n$ and transposed rows $r_1, \ldots, r_n \in \mathbb{B}^n$. Guided by the constructions of (15), we now consider Boolean coefficients $x_1, \ldots, x_n \in \mathbb{B}$ that select columns or rows, respectively. Let $I := \{\, i \mid x_i \,\}$ be the coefficient set thus selected, so that we concentrate on

$$M_c^{\cap} := \left\{ \bigcap_{i\in I} c_i \mid I \subseteq \{1, \ldots, n\} \right\} \qquad M_r^{\cap} := \left\{ \bigcap_{i\in I} r_i \mid I \subseteq \{1, \ldots, n\} \right\} \tag{16}$$

as the sets of all intersections (of possibly empty sets) of columns and of transposed rows of M, respectively. If we replace conjunction by disjunction, then the intersection of Boolean vectors becomes their union. Hence, besides the sets of (16) it seems also to be of interest to consider the constructions

$$M_c^{\cup} := \left\{ \bigcup_{i\in I} c_i \mid I \subseteq \{1, \ldots, n\} \right\} \qquad M_r^{\cup} := \left\{ \bigcup_{i\in I} r_i \mid I \subseteq \{1, \ldots, n\} \right\}, \tag{17}$$

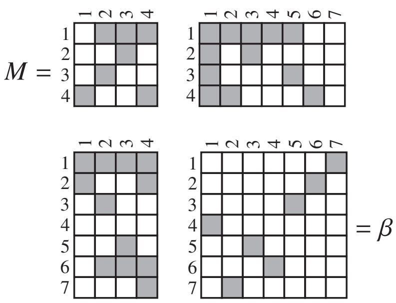
which introduce the sets of all possible unions of columns and of transposed rows of M, respectively. In a very natural way now the question arises whether, as in the case of the sets of (15), there is also a close connection between the sets of (16) and of (17), respectively. In this section we will give a positive answer, not only for quadratic Boolean matrices but even for arbitrary relations. Since unions of rows of a Boolean matrix M may, of course, also be considered as complements of intersections of complemented rows, for the following, we decide to treat mainly intersections. Although this looks more complicated introducing complements, it gives better guidance along residuation.

Recall from Section 2 that for a relation $M : X \leftrightarrow Y$ and an element $y \in Y$, the vector $M^{(y)} : X \leftrightarrow 1$ denotes the y-column of M. Then a generalization of the constructions of (16) to arbitrary relations reads as follows.

Definition 6.1. Given a relation $M : X \leftrightarrow Y$, by $M_c^{\cap} := \{\bigcap_{y\in I} M^{(y)} \mid I \in 2^Y\}$ the set of all possible intersections of columns of M is defined and by $M_r^{\cap} := (M^T)_c^{\cap}$ the set of all possible intersections of rows of M is defined.

Especially for obtaining the results of this section, experimenting and playing with the RelView system was very helpful. By the following example with four RelView pictures we want to describe the situation we are going to investigate.

Example 6.1. The following picture shows four relations, depicted as RelView matrices. We have a 4×4 Boolean matrix M, a 4×7 Boolean matrix right besides M, a 7×4 Boolean matrix below M and a 7×7 Boolean matrix β.



The 4×7 Boolean matrix on the right of M shows all possible intersections of columns of M, each of the seven results represented by a column of the matrix. For example, the intersection of the columns $M^{(2)}$ and $M^{(4)}$ is represented by column 4. Note that the universal vector as first column of the 4×7 Boolean matrix is obtained by intersecting the empty set of columns of M. In the same way the 7×4 Boolean matrix below M enumerates all intersections of sets of rows of M. Again we have seven different results, now represented by the matrix rows. Finally, the 7×7 matrix β bijectively links the column intersections and the row intersections of M.

The relation M of Example 6.1 is homogeneous in the sense that source and target coincide. The properties demonstrated in it remain valid if M is heterogeneous (source and target may be different). An extreme example is that M is the universal vector $\mathsf{L} : X \leftrightarrow 1$. In this case the only element of $\mathsf{L}_c^{\cap}$ is L itself, the only element of $\mathsf{L}_r^{\cap}$ is the universal relation of type $[1 \leftrightarrow 1]$, and the relation β that bijectively links these relations is the identity relation of type $[1 \leftrightarrow 1]$.

It is evident that several combinations of rows may produce the same union. When considering ε_X^T, multiplied from the left to M, where $\varepsilon_X : X \leftrightarrow 2^X$ is the powerset relation of X, one will probably obtain many identical unions of rows in the result $\varepsilon_X^T; M$. In order to eliminate multiply occurring unions, one may, of course, wish to identify them. A little reflection shows in an analogous way that all intersections of rows of M are given by the rows of $R := \overline{\varepsilon_X^T; \overline{M}} : 2^X \leftrightarrow X$. The elimination of multiple rows of R is obtained via $\eta_\Xi^T; R : 2^X/\Xi \leftrightarrow Y$, where $\eta_\Xi : 2^X \leftrightarrow 2^X/\Xi$ is the canonical epimorphism induced by the row equivalence relation $\Xi := \mathrm{syq}(R^T, R^T) : 2^X \leftrightarrow 2^X$. Equivalence classes of rows so obtained will be called row types.

We will use contacts for bijectively linking the row types of a relation with its column types. The corresponding reflection, namely, shows that all intersections of columns of M are given by the columns of the relation $C := \overline{\overline{M}; \varepsilon_Y} : X \leftrightarrow 2^Y$, where $\varepsilon_Y : Y \leftrightarrow 2^Y$ is the powerset relation of Y, so that we proceed with these relations.

Definition 6.2. Given the relations $M : X \leftrightarrow Y$, $C := \overline{\overline{M}; \varepsilon_Y}$, and $R := \overline{\varepsilon_X^T; \overline{M}}$, we define the *column intersection types relation* as $C; \eta_\Psi : X \leftrightarrow 2^Y/\Psi$ and the *row intersection types relation* as $\eta_\Xi^T; R : 2^X/\Xi \leftrightarrow Y$.

To visualize the constructions of Definition 6.2, we consider again the 4×4 Boolean matrix M of the above example.

Example 6.2. Let the relation / matrix $M : X \leftrightarrow Y$ be as in Example 6.1 and, hence, the source X as well as the target Y of M be equal to the set $\{1, 2, 3, 4\}$. Then the following RelView matrices represent the membership relation $\varepsilon_Y : Y \leftrightarrow 2^Y$ and the relation $C : Y \leftrightarrow 2^Y$, respectively.

$$\varepsilon_Y = \begin{array}{c|cccccccccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ \hline 1 & 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \\ 2 & 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 3 & 0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\ 4 & 0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1 \end{array} \qquad C = \begin{array}{c|cccccccccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ \hline 1 & 1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0 \\ 2 & 1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 3 & 1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\ 4 & 1&1&0&0&0&0&0&0&1&1&0&0&0&0&0&0 \end{array}$$

The relationship between the two matrices / relations ε_Y and C is as follows. If a subset I of the target Y of M is represented by a column of ε_Y in the sense of Section 2, then the vector $\bigcap_{y \in I} M^{(y)} : X \leftrightarrow \mathbf{1}$ equals the column $C^{(I)}$ of C. For example, for $I = \{3, 4\}$ (cf. the fourth column of ε_Y) we get that the intersection $M^{(3)} \cap M^{(4)}$ equals the fourth column of C and for $I = \{1, 3\}$ (cf. the column 11 of ε_Y) we get that $M^{(1)} \cap M^{(3)}$ is empty (cf. the column 11 of C).

If we transform C into the column intersection types relation $C; \eta_\Psi$ by the elimination of all duplicate columns, we exactly obtain the result already shown above, i.e., the 4×7 Boolean matrix right besides M.

It is a remarkable fact that there exists a close connection between the row and the column types relation. By the following bijection, one may feel reminded that for a real-valued matrix the row rank equals the column rank. Some ideas from the approach stem from real-valued matrices as presented e.g., in [20]. For the proof we need that symmetric quotients are difunctional in the sense that the following inclusion holds.

$$\mathrm{syq}(P, Q); [\mathrm{syq}(P, Q)]^T; \mathrm{syq}(P, Q) \subseteq \mathrm{syq}(P, Q), \tag{18}$$

This property immediately follows from Corollary 4.4.4 of [22] or [21] Section 8.5. The next theorem is the most decisive result of this section.

Theorem 6.1. *Given an arbitrary relation $M : X \leftrightarrow Y$ together with the derived relations*

$$C := \overline{\overline{M}; \varepsilon_Y} \qquad R := \overline{\varepsilon_X^T; \overline{M}} \qquad \Psi := \mathrm{syq}(C, C) \qquad \Xi := \mathrm{syq}(R^T, R^T),$$

there exists a bijective mapping (in the relational sense) of type $[2^X/\Xi \leftrightarrow 2^Y/\Psi]$.

Proof. The idea is to compare the contact relation $\mathrm{mi}_M(\mathrm{ma}_M(\varepsilon_X)) = \mathrm{mi}_M(R^T)$ and the lower derivative $\mathrm{mi}_M(\varepsilon_Y) = C$ via a symmetric quotient construction; so we define (equality of the two versions is easy to prove by expansion) an auxiliary relation A as follows.

$$A := \mathrm{syq}(\mathrm{mi}_M(R^T), C) = \mathrm{syq}(R^T, \mathrm{ma}_M(C)) : 2^X \leftrightarrow 2^Y \tag{19}$$

The relation A is total and surjective. For totality, we calculate

$$\begin{aligned} A &= \mathrm{syq}(\mathrm{mi}_M(R^T), C) && \text{by (19)} \\ &= \mathrm{syq}(\overline{\overline{M}; \overline{\overline{M}^T; \varepsilon_X}}, \overline{\overline{M}; \varepsilon_Y}) && \text{definition of } R \text{ and } C \\ &= \mathrm{syq}(\overline{M}; \overline{\overline{M}^T; \varepsilon_X}, \overline{M}; \varepsilon_Y) && \text{by (5)} \\ &\supseteq \mathrm{syq}(\overline{\overline{M}^T; \varepsilon_X}, \varepsilon_Y) && \text{[22] Proposition 4.4.1.v or [21] Section 8.5} \end{aligned}$$

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and apply then that syq($\overline{\overline{M}^T; \varepsilon_X}$, ε_Y) is total by (7) and (5) . To prove surjectivity, we reason in the same way, but use the other variant of A.

Next, we have a look at the row equivalence relation $\Xi' := \mathrm{syq}(A^T, A^T)$ and the column equivalence relation $\Psi' := \mathrm{syq}(A, A)$. It so happens that $\Xi = \Xi'$ and $\Psi = \Psi'$ via a general cancelling rule for symmetric quotients that follows from the laws of [22], Section 4.4 or [21] Proposition 8.20.iii. E.g., the second equality is shown by

$$
\begin{aligned}
\Psi' &= \mathrm{syq}(A, A) && \\
&= \mathrm{syq}(\mathrm{syq}(\mathrm{mi}_M(R^T), C), \mathrm{syq}(\mathrm{mi}_M(R^T), C)) && \text{by (19)} \\
&= \mathrm{syq}(C, C) && \text{cancelling} \\
&= \Psi. &&
\end{aligned}
$$

Based on $A : 2^X \leftrightarrow 2^Y$ and the canonical epimorphisms $\eta_\Xi : 2^X \leftrightarrow 2^X/\Xi$ and $\eta_\Psi : 2^Y \leftrightarrow 2^Y/\Psi$, we now define the following relation by simple composition.

$$
\beta := \eta_\Xi^T; A; \eta_\Psi : 2^X/\Xi \leftrightarrow 2^Y/\Psi \tag{20}
$$

This is a matching, defined as a relation that is at the same time univalent and injective. Using the Schröder rule, for the proof of univalency we start with

$$
A^T; A \subseteq \overline{\overline{A}^T; A} \iff A; \overline{A}^T; A \subseteq \overline{A} \iff A; A^T; \overline{A} \subseteq \overline{A} \iff A; A^T; A \subseteq A.
$$

This yields $A^T; A \subseteq \overline{\overline{A}^T; A}$ and, by transposition, also $A^T; A \subseteq \overline{A^T; \overline{A}}$, since symmetric quotients are difunctional due to (18). So, we have $A^T; A \subseteq \mathrm{syq}(A, A)$. If we combine this property with $\Xi = \Xi' = \mathrm{syq}(A^T, A^T)$ and Proposition 4.4.1.iii of [22] or [21] Section 8.5, we get

$$
A^T; \Xi; A = A^T; \mathrm{syq}(A^T, A^T); A = A^T; A \subseteq \mathrm{syq}(A, A) = \Psi' = \Psi.
$$

Now, the univalency of the relation β can be shown as follows.

$$
\begin{aligned}
\beta^T; \beta &= [\eta_\Xi^T; A; \eta_\Psi]^T; \eta_\Xi^T; A; \eta_\Psi && \text{by (20)} \\
&= \eta_\Psi^T; A^T; \eta_\Xi; \eta_\Xi^T; A; \eta_\Psi && \\
&= \eta_\Psi^T; A^T; \Xi; A; \eta_\Psi && \text{by (10)} \\
&\subseteq \eta_\Psi^T; \Psi; \eta_\Psi && \text{see above} \\
&= \eta_\Psi^T; \eta_\Psi; \eta_\Psi^T; \eta_\Psi && \text{by (10)} \\
&= \mathbb{I} && \text{by (10)}
\end{aligned}
$$

Transpositions of difunctional relations obviously are also difunctional relations. This implies $A; A^T \subseteq \mathrm{syq}(A^T, A^T)$ and from this fact we obtain, analogously to the above calculations, first the inclusion $A; \Psi; A^T \subseteq \Xi$ and then injectivity $\beta; \beta^T \subseteq \mathbb{I}$.

Since canonical epimorphisms and their transpositions are total and surjective and these properties pass on to compositions, by construction β is also total and surjective, i.e., the bijective mapping we have been looking for. □

In the next theorem we show how the bijective mapping β relates the intersections of columns and transposed rows by means of the bound functionals mi and ma.

Theorem 6.2. *Assume the five relations $M : X \leftrightarrow Y$, $C : X \leftrightarrow 2^Y$, $R : 2^X \leftrightarrow Y$, $\Xi : 2^X \leftrightarrow 2^X$ and $\Psi : 2^Y \leftrightarrow 2^Y$ to be as in Theorem 6.1 and the relation $\beta : 2^X/\Xi \leftrightarrow 2^Y/\Psi$ to be as defined in (20). Then the following two equations hold:*

$$
\mathrm{mi}_M(R^T; \eta_\Xi); \beta = C; \eta_\Psi \qquad \mathrm{ma}_M(C; \eta_\Psi); \beta^T = R^T; \eta_\Xi
$$

Proof. The first equation is shown by the following calculation.

$$
\begin{aligned}
\mathrm{mi}_M(R^T; \eta_\Xi); \beta &= \overline{\overline{M}; R^T; \eta_\Xi}; \eta_\Xi^T; A; \eta_\Psi && \text{by (20)} \\
&= \overline{\overline{M}; R^T; \eta_\Xi; \eta_\Xi^T}; A; \eta_\Psi && \text{[22] Prop. 4.2.4.iii or [21] Prop 5.13} \\
&= \overline{\overline{M}; R^T; \Xi}; A; \eta_\Psi && \text{by (10)} \\
&= \overline{\overline{M}; R^T; \mathrm{syq}(R^T, R^T)}; A; \eta_\Psi && \text{definition of } \Xi \\
&= \overline{\overline{M}; R^T}; A; \eta_\Psi && \text{[22] Prop. 4.4.1.iii or [21] Section 8.5}
\end{aligned}
$$

$$\begin{aligned}
&= \mathrm{mi}_M(R^T); A; \eta_\Psi \\
&= \mathrm{mi}_M(R^T); \mathrm{syq}(\mathrm{mi}_M(R^T), C); \eta_\Psi && \text{by (19)} \\
&= C; \eta_\Psi && \text{[22] Prop. 4.4.2.ii or [21] Section 8.5}
\end{aligned}$$

In a completely analogous way, $\mathrm{ma}_M(C; \eta_\Psi); \beta^T = R^T; \eta_\Xi$ can be verified. □

If we apply the common notation of function application for β, then this theorem says how the rows of $\eta_\Xi^T; R$ and the columns of $C; \eta_\Psi$ are linked together. Let $[I]$ denote the equivalence class of $I \in 2^X$ w.r.t. the equivalence relation Ξ and $[J]$ denote the equivalence class of $J \in 2^Y$ w.r.t. Ψ. Then we have:

(1) If $v^T : \mathbf{1} \leftrightarrow Y$ is the $[I]$-row of $\eta_\Xi^T; R$ and $w : X \leftrightarrow \mathbf{1}$ is the $\beta([I])$-column of $C; \eta_\Psi$, then $w = \mathrm{mi}_M(v)$.
(2) If $w : X \leftrightarrow \mathbf{1}$ is the $[J]$-column of $C; \eta_\Psi$ and $v^T : \mathbf{1} \leftrightarrow Y$ is the $[J]$-row of $\eta_\Xi^T; R$, then $v = \mathrm{ma}_M(w)$.

For instance, in Example 6.1 the transposed universal vector $\mathsf{L}^T : \mathbf{1} \leftrightarrow \{1, 2, 3, 4\}$ is the 1-row of the relation $\eta_\Xi^T; R$ (the matrix below M). Since each row of M contains a 0-entry, $\overline{M}$ is total. This yields $\mathrm{mi}_M(\mathsf{L}) = \overline{\overline{M}; \mathsf{L}} = \overline{\mathsf{L}} = \mathsf{O}$. And, indeed, the empty vector is the 7-column (that is, the $\beta(1)$-column) of $C; \eta_\Psi$ (the matrix right besides M). For the transposed empty vector $\mathsf{O}^T : \mathbf{1} \leftrightarrow \{1, 2, 3, 4\}$, the 4-row of $\eta_\Xi^T; R$, we get $\mathrm{mi}_M(\mathsf{O}) = \overline{\overline{M}; \mathsf{O}} = \overline{\mathsf{O}} = \mathsf{L}$. The latter vector coincides with the $\beta(4)$-column of $C; \eta_\Psi$.

The following result concerning the column intersection types relation also holds in an analogous manner for the row intersection types relation.

Theorem 6.3. *If $M : X \leftrightarrow Y$, $C : X \leftrightarrow 2^Y$ and $\Psi : 2^Y \leftrightarrow 2^Y$ are as in Theorem 6.1, then we have:*

(a) *$C; \eta_\Psi : X \leftrightarrow 2^Y/\Psi$ is a $C; \eta_\Psi$-contact.*
(b) *$\mathrm{syq}(C; \eta_\Psi, C; \eta_\Psi) : 2^Y/\Psi \leftrightarrow 2^Y/\Psi$ is a $\Omega_{C;\eta_\Psi}$-closure.*

Proof. (a) An immediate consequence of Theorem 6.2 is the following equation.

$$\mathrm{mi}_M(\mathrm{ma}_M(C; \eta_\Psi)) = \mathrm{mi}_M(\mathrm{ma}_M(C; \eta_\Psi); \beta^T; \beta) = \mathrm{mi}_M(R^T; \eta_\Xi; \beta) = \mathrm{mi}_M(R^T; \eta_\Xi); \beta = C; \eta_\Psi$$

Hence, Theorem 3.3 implies that $C; \eta_\Psi$ is a $C; \eta_\Psi$-contact.

(b) The relation $\mathrm{syq}(C; \eta_\Psi, C; \eta_\Psi)$ is reflexive and does not possess multiple columns. Relation-algebraically this is expressed by $\mathsf{I} = \mathrm{syq}(C; \eta_\Psi, C; \eta_\Psi)$. From this equation we obtain that $\mathrm{syq}(C; \eta_\Psi, C; \eta_\Psi)$ is total and the inclusion relation $\Omega_{C;\eta_\Psi}$ is a partial order on the equivalence classes. Now, Theorem 4.3 shows that $\mathrm{syq}(C; \eta_\Psi, C; \eta_\Psi)$ is a $\Omega_{C;\eta_\Psi}$-closure. □

Let us, after these digressions, return to the main topic of the section, the cardinality of the sets $M_c^\cap$ and $M_r^\cap$. Due to Theorem 6.1 the two sets $2^X/\Xi$ and $2^Y/\Psi$ have the same cardinality. To show that the cardinality of $M_c^\cap$ and $M_r^\cap$ are equal, it remains to prove that the same holds for $M_c^\cap$ and $2^Y/\Psi$ and for $M_r^\cap$ and $2^X/\Xi$, respectively. This is done by the following theorem, in which $[I]$ again denotes the equivalence class of $I \in 2^Y$ w.r.t. the equivalence relation Ψ.

Theorem 6.4. *Let the relations $M : X \leftrightarrow Y$, $C : X \leftrightarrow 2^Y$ and $\Psi : 2^Y \leftrightarrow 2^Y$ be as in Theorem 6.1 Then the function $f : 2^Y/\Psi \to M_c^\cap$, defined by $f([I]) = \bigcap_{y \in I} M^{(y)}$, is bijective.*

Proof. Let $\varepsilon : Y \leftrightarrow 2^Y$ denote the powerset relation on Y. If we assume an arbitrarily given set $I \in 2^Y$, then we have for all $x \in X$ the following equivalence.

$$\begin{aligned}
(\overline{\overline{M}; \varepsilon})_x^{(I)} &\iff (\overline{\overline{M}; \varepsilon})_{x,I} \\
&\iff \neg\exists y : \overline{M}_{x,y} \wedge \varepsilon_{y,I} \\
&\iff \neg\exists y : y \in I \wedge \overline{M}_{x,y} \\
&\iff \forall y : y \in I \to M_{x,y} \\
&\iff (\textstyle\bigcap_{y \in I} M^{(y)})_x
\end{aligned}$$

This shows for the vectors $(\overline{\overline{M}; \varepsilon})^{(I)} : X \leftrightarrow \mathbf{1}$ and $\bigcap_{y \in I} M^{(y)} : X \leftrightarrow \mathbf{1}$ that $(\overline{\overline{M}; \varepsilon})^{(I)} = \bigcap_{y \in I} M^{(y)}$. Using this property, we obtain for all $I, J \in 2^Y$ the following equivalence.

$$
\begin{array}{rlll}
[I] = [J] & \iff & \Psi_{I,J} & \text{classes w.r.t. } \Psi \\
& \iff & \mathrm{syq}(\,\overline{\overline{M}\,;\,\varepsilon}\,,\ \overline{\overline{M}\,;\,\varepsilon}\,)_{I,J} & \text{definition of } \Psi \text{ and } C \\
& \iff & (\,\overline{\overline{M}\,;\,\varepsilon}\,)^{(I)} = (\,\overline{\overline{M}\,;\,\varepsilon}\,)^{(J)} & \\
& \iff & \bigcap_{y\in I} M^{(y)} = \bigcap_{y\in J} M^{(y)} & \text{see above} \\
& \iff & f([I]) = f([J]) & \text{definition of } f
\end{array}
$$

Hence, f is well-defined and injective. Surjectivity of f is trivial. □

From this result we get that the sets $M_c^{\cap}$ and $2^Y/\Psi$ have the same cardinality. Transposing M yields that also the cardinalities of the sets $M_r^{\cap}$ and $2^X/\Xi$ coincide. Altogether, we get the desired result as already demonstrated by means of the introductory example of this section.

Corollary 6.1. *Let a relation $M : X \leftrightarrow Y$ be given. Then the sets $M_c^{\cap}$ and $M_r^{\cap}$ have the same cardinality.*

Note that all constructions of Theorem 6.1 and its proof and also the constructions of Theorem 6.2 are relation-algebraic expressions, that is, algorithmic. As a consequence, they can immediately be translated into RelView code,[2] such that the tool can be used to compute for a given relation its column intersection types relation as well as its row intersection types relation and the mapping that bijectively links the rows of the latter with the columns of the first one. The same holds for the computation of the column of the column intersection types relation that corresponds to a given row of the row intersection types relation and vice versa.

Similarly to Definition 6.2, also column union types relations and row union types relations can be introduced and then an analogon of Theorem 6.1 and its consequences, respectively, holds for these constructions. It shows that, with $M_c^{\cup} := \{\bigcup_{y\in I} M^{(y)} \mid I \in 2^Y\}$ as the set of all possible unions of columns of $M : X \leftrightarrow Y$ and $M_r^{\cup} := (M^T)_c^{\cup}$ as the set of all possible unions of rows, the cardinality of $M_c^{\cup}$ equals the cardinality of $M_r^{\cup}$.

7. Conclusion

Using relation algebra as conceptual and methodical tool, we have generalized Aumann's notion of a contact relation between a set X and its powerset 2^X and that of a closure operation on 2^X from powersets to general membership relations M and their induced partial orders Ω_M. We have provided a construction of M-contacts from interest relations J similar to the lower/upper-derivative construction of formal concept analysis that leads to a fixed point description of the set of M-contacts. We also have investigated the relationship between the lattice of M-contacts and the lattice of Ω_M-closures in this general setting. In general it is given as an order embedding of the Ω_M-closures into the M-contacts; in the specific case of powersets it becomes an order isomorphism. Moreover, we have shown how the connections between contacts, closures and topologies can relation-algebraically be described and used for practical computations using, e.g., the RelView system. As another application, we have used contacts to establish a one-to-one correspondence between the column intersections space and the row intersections space of arbitrary relations.

At the end of Section 4, we have remarked that a one-to-one correspondence between the M-contacts and the Ω_M-closures may also exist for the relation M not being (isomorphic to) a set-theoretic membership relation. Presently, we are looking for simple conditions on M which ensure that the sets $\mathfrak{K}_M$ and $\mathfrak{H}_{\Omega_M}$ are isomorphic. In this context, it is also interesting to study whether these conditions imply that Ω_M belongs to a specific class of partial orders. In respect thereof, a first result is that each relation M that, using matrix terminology, is obtained from a powerset relation ε by adding additional rows consisting of 1's only has as many M-contacts as Ω_M-closures and in this case Ω_M is isomorphic to Ω.

Besides Aumann contacts, another concept of contacts is discussed in the literature. These contacts arise in the context of qualitative geometry and are mainly used for reasoning about spatial regions. If we transform the axioms (1.4)–(1.6) of a contact relation on a set W of regions given in [15] into the language of relation algebra, then $C : W \leftrightarrow W$ is a contact relation if and only if C is reflexive, symmetric and $\mathrm{syq}(C^T, C^T) = \mathbb{I}$. In most cases (see e.g., [16]), it is additionally assumed that the underlying structure is a Boolean lattice, i.e., essentially a powerset ordered by set inclusion. This fact leads in a natural way to the task of detecting the interdependencies between the two concepts of a contact (if such are) and whether it is also possible and reasonable to generalize the latter one similar to our generalization of Aumann contacts to M-contacts, a work that is planned for the future.

Based on the results of the present paper and [5], another future work is the relation-algebraic treatment of important subclasses of contact relations and closure operations (like the topological ones), of other closure objects (like full implicational structures, Sperner villages and join-congruences) and of objects which are closely related to closure objects (like orders and pre-orders on finite sets; cf. [17]). This includes also the development of relational algorithms and RelView

[2] A relational program that computes the canonical epimorphism of an equivalence relation can be found in [9].

programs for their enumeration and counting, for recognizing them and for transforming them from one into another, if possible.

We end citing Garrett Birkhoff of 1967 from [10]: "…for the two centuries preceding the development of the computer, broadly speaking, the main progress in applied mathematics was concerned with continuum analysis. Whereas over the last 20 years I think that the most conspicuous feature of the revolution that has taken place, and is continuing to take place, is a transition back from continuum mathematics towards digital mathematics; and one of the big questions, of course, is how far will this trend go or can it go?" In particular when considering the current trend of mathematizing also social considerations and concepts in the humanities, such a transition of known concepts from the continuous area down to the discrete world seems extremely promising.

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