Partiality I: Embedding Relation Algebras

Gunther Schmidt

Institute for Software Technology, Department of Computing Science
Federal Armed Forces University Munich, 85577 Neubiberg
e-Mail: Schmidt@Informatik.UniBw-Muenchen.DE

Abstract

As long as no cooperation between processes is supposed to take place, one may consider them separately and need not ask for the progress of the respective other processes. If a composite result of processes is to be delivered, it is important in which way the result is built, only by non-strict/continuous "accumulation" (i.e., open for partial evaluation) or with additional intermittent strict/non-continuous "transactions". $A \Subset B$

We define the concept of partiality to cope with partial availability. To this end relations are handled under the aspect that orderings are defined in addition to the identities in every relation algebra. Only continuous functions with respect to these orderings are considered to regulate transfer of partialities. The partialities are afterwards recognized as the image of a multiplicative embedding of the relations on the atomic constituents into a bigger relation algebra. The latter will then give room for the necessary external arbiter who decides for strict transitions that all required components are available.

Once this is proved, it is handled rather easily. The proof, however, needs a considerable amount of formulae. Once these are known sufficiently well, one may start investigating matrices the coefficients of which are such partiality transfer functions in the forthcoming second part of the paper. Then universal characterizations of parallel products will be given and studied testing them with regard to correctness rules.

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1 Introduction

When dealing with *possibly partial* availability of information on elements of a set, it is a well-known technique to put an additional bottom-element $\bot$ below all the others to obtain a flat ordering. More difficult situations are studied with cpo's and there exists a highly developed theory of orderings on semantic domains. A treatment of possibly partial availability of information may also be seen in descriptions of eager/data-driven evaluation as opposed to lazy/demand-driven evaluation. Yet another example provide the highly developed lock/commit/rollback techniques of data base administration. The operating systems community strives to define ISO standards for transaction protocols brute-force testing them many steps ahead instead of proving them — again some sort of arranging properly with possibly partial availability of information.

We feel that partiality is not yet handled satisfactorily, and that a unified theoretical basis for all this is still missing. In the approach presented here, we try to lay a basis for later obtaining relation algebra rules around partiality.

First we exhibit the basic distinction between strict/non-continuous and non-strict/continuous work. Then we refer to rather recent results on Goguen categories [Win02a,Win02c,Win02b,Win03] expressing that strictness cannot be formulated *inside* a given environment and will need an *exterior* to talk and reason on being strict.

Accepting this basic setting, we try to develop the plexus of algebraic rules and identities to appropriately handle such situations with two levels, an interior (non-strict) and an exterior (strict) aspect. The *Munich Approach* (see [BHSV94]) provides strict products in as far as projection π in $(R;\pi^\mathsf{T} \cap S;\rho^\mathsf{T});\pi = R \cap S;\mathbb{T}$ requires also S to deliver a result. The *Rio Approach* was mainly interested in non-strict ones. As both were formulated relation-algebraically, there should be a chance to tackle both levels with relations.

From the beginning we assume given with every element the boolean lattice describing the degrees to which this element may be partially available. Availability of an element is thus no longer conceived as an atomic qualification *available/non-available* or $\bullet, \bot$ symbolically. It may now be qualified in greater detail. Consider, e.g., the pair of objects (x, y) and assume x and y to be in some sense atomic or non-composite. First, none of the components may be available, denoted as $(\bot, \bot)$. Availability on the pair (x, y) could, however, increase to $(\bot, \bullet)$ indicating that the second component is already available but the first is not, or $(\bullet, \bullet)$ indicating that it is fully available. There is a natural way of dealing with such a situation, namely speaking of the *partiality* or the *degree of being partially available* and introducing an ordering $\sqsubseteq$ to express an increase in partiality, such that for instance $(\bot, \bot) \sqsubseteq (\bullet, \bot) \sqsubseteq (\bullet, \bullet)$.

Studied this way, every element is endowed its own partiality lattice. These partiality lattices may indeed vary over the elements of a domain: Consider the direct sum $X + (X \times X)$ of a set X and the set of pairs formed over X. Assume the items of X to be atomic. Then items of X have the partiality lattice $\mathbb{P}$ and the pairs have $\mathbb{P}^2$, where $\mathbb{P} = \{\bot, \bullet\}$.

An investigation on relations in the presence of an ordering on the domain and on the range side will therefore be given. It takes into account the possibly increasing degree of availability concerning the argument on the domain side and hence on the results of a relation on the range side. Increased information on an argument should result in at least as much information on the result as before. Operations should therefore be isotonic with respect to the orderings. The restriction is, however, a bit more sophisticated as will be shown.

Accepting that we have now the possibility of getting better and better informed on items changed by a process, brings deep changes to some basic concepts. Astonishingly enough, we still stay in the realm of relation algebras. Therefore not everything has to be invented from the scratch. We can maintain a lot but have to adapt it or to reinterpret it from the changed point of view.

This article is organized as follows. After an introduction in the present Sect. 1, we collect known prerequisites symmetric quotients and direct powers in Sect. 2. This is followed by a brief motivation observing incrementation of availabilies in Sect. 3 as they occur in parallel processes. We then study in Sect. 4 an ordering relation E which in addition is an atomic complete boolean lattice.

Section 5 is devoted to the study of relations that are continuous mappings wrt. such lattices. It will turn out that the set of all these mappings constitutes a relation algebra again, if suitably modified operations are chosen. It is a relation algebra *but given together with the possibility of looking at its elements from outside* where also an ordering is available. So, a relational strictness operator may be formulated. It is important that composition and identity stay the same as before. In a short Outlook we announce in which way matrix algebras over these newly introduced relation algebras are useful for important applications. Not least, it is mentioned how one can construct categorical products of relations in the forthcoming second part of the article.

2 Preliminaries

The reader is assumed to be familiar with relation algebra; only some less well-known aspects are recalled as well as symmetric quotients and direct powers. Then we consider orderings as well as isotone and continuous mappings.

2.1 Elementary Additions

In order to avoid clumsy notation, we shall often say in a heterogeneous algebra "For every X ...", meaning "For every X for which the construct in question is defined ...". We are also a bit sloppy writing $\mathbb{T}$ when $\mathbb{T}_{A,B}$, e.g., would be more precise. We will use several times that

$$p \subseteq \mathbb{I} \quad \Longrightarrow \quad p;\overline{X} = p;\overline{p;X} \quad \text{for all } X. \tag{*}$$

The following formulae hold for arbitrary relations Q, R, S:

$$(Q \cap R;\mathbb{T});S = Q;S \cap R;\mathbb{T} \qquad (Q \cap (R;\mathbb{T})^\mathsf{T});S = Q;(S \cap R;\mathbb{T}) \qquad (\dagger)$$

We add one more fact on mappings showing that mappings (i.e., univalent and total relations) may slip under negations when multiplied from the left.

$$R \text{ is a mapping} \quad \Longrightarrow \quad R;\overline{S} = \overline{R;S} \text{ for all } S \qquad (\ddagger)$$

2.2 Symmetric Quotients

The concept of left residuals has been introduced, defining $R \backslash S := \overline{R^\mathsf{T};\overline{S}}$ which turns out to be the greatest relation X satisfying $R;X \subseteq S$ and often resulting in addition in $R;X = S$. Also right residuals have been studied. The following concept of a **symmetric quotient** has very successfully been used in various application fields, not least in [BSZ86,BSZ90,BGS94]. We recall its definition

$$\mathtt{syq}(A, B) := \overline{A^\mathsf{T};\overline{B}} \cap \overline{\overline{A}^\mathsf{T};B}$$

and the basic algebraic rules without proof:

2.2.1 Proposition.

i) $\mathtt{syq}(\overline{A}, \overline{B}) = \mathtt{syq}(A, B)$;

ii) $\mathtt{syq}(B, A) = [\mathtt{syq}(A, B)]^\mathsf{T}$;

iii) $A;\mathtt{syq}(A, A) = A$;

iv) $\mathbb{I} \subseteq \mathtt{syq}(A, A)$;

v) $\mathtt{syq}(A, B) \subseteq \mathtt{syq}(C;A, C;B)$ for every C;

vi) $F;\mathtt{syq}(A, B) = \mathtt{syq}(A;F^\mathsf{T}, B)$ for every mapping F;

vii) $\mathtt{syq}(A, B);F^\mathsf{T} = \mathtt{syq}(A, B;F^\mathsf{T})$ for every mapping F. □

Always $A;\mathtt{syq}(A, B) \subseteq B$ holds. $A;\mathtt{syq}(A, B)$ can differ from B only in a fairly regular fashion: a "column" of $A;\mathtt{syq}(A, B)$ is either equal to the corresponding column of B or it is a zero column.

2.2.2 Proposition. If $\mathtt{syq}(A, B)$ is total, or if $\mathtt{syq}(B, C)$ is surjective, then

$$\mathtt{syq}(A, B);\mathtt{syq}(B, C) = \mathtt{syq}(A, C).$$

Proof: Using the Schröder equivalences and Prop. 2.2.1.i,ii we get "$\subseteq$":

$(A^\mathsf{T};\overline{C} \cup \overline{A}^\mathsf{T};C);[\mathtt{syq}(B, C)]^\mathsf{T} = A^\mathsf{T};\overline{C};\mathtt{syq}(\overline{C}, \overline{B}) \cup \overline{A}^\mathsf{T};C;\mathtt{syq}(C, B) \subseteq A^\mathsf{T};\overline{B} \cup \overline{A}^\mathsf{T};B$

"$\supseteq$" may be obtained with the Dedekind rule and the partial result just proved:

$$\begin{aligned}&\mathtt{syq}(A, B);\mathbb{T} \cap \mathtt{syq}(A, C)\\ &\quad\subseteq \big(\mathtt{syq}(A, B) \cap \mathtt{syq}(A, C);\mathbb{T}^\mathsf{T}\big);\big(\mathbb{T} \cap [\mathtt{syq}(A, B)]^\mathsf{T};\mathtt{syq}(A, C)\big)\\ &\quad\subseteq \mathtt{syq}(A, B);\mathtt{syq}(B, A);\mathtt{syq}(A, C) \subseteq \mathtt{syq}(A, B);\mathtt{syq}(B, C).\end{aligned}$$ □

2.3 Direct Powers

Product and sum definitions have long been investigated by relation algebraists and computer scientists. We don't recall them and immediately proceed to the direct power. It is designed as a relational analog to the situation between a set

A and its power set $\mathcal{P}(A)$. The "is element of" relation between A and $\mathcal{P}(A)$ is specified, which has already been published in [BSZ86,BSZ90,BGS94], as follows.

2.3.1 Definition. A relation ε is called a **direct power** if it satisfies

i) $\mathtt{syq}(\varepsilon, \varepsilon) \subseteq \mathbb{I}$,
ii) $\mathtt{syq}(\varepsilon, X)$ is surjective for every relation X.

For a given direct power ε, we define the **power ordering** as $\Omega := \overline{\varepsilon^{\mathsf{T}} ; \overline{\varepsilon}}$ and the **complement transition** as $N := \mathtt{syq}(\varepsilon, \overline{\varepsilon})$. □

Instead of (ii), one may also say that $\mathtt{syq}(X, \varepsilon)$ shall be a mapping, i.e. univalent and total. One easily derives, that $\varepsilon ; \mathtt{syq}(\varepsilon, X) = X$ for all X.

As a consequence of this definition, we have that $\varepsilon ; N = \varepsilon ; \mathtt{syq}(\varepsilon, \overline{\varepsilon}) = \overline{\varepsilon}$. It is easily proved that N is an involution using the rules of Prop. 2.2.1: $N^{\mathsf{T}} = \mathtt{syq}(\varepsilon, \overline{\varepsilon})^{\mathsf{T}} = \mathtt{syq}(\overline{\varepsilon}, \varepsilon) = \mathtt{syq}(\overline{\overline{\varepsilon}}, \overline{\varepsilon}) = \mathtt{syq}(\varepsilon, \overline{\varepsilon}) = N$. From $N ; N^{\mathsf{T}} = \mathtt{syq}(\varepsilon, \overline{\varepsilon}) ; \mathtt{syq}(\overline{\varepsilon}, \varepsilon) = \mathtt{syq}(\varepsilon, \varepsilon) = \mathbb{I}$ we observe that N is univalent and surjective, etc. In a — suitably arranged — representable setting, N corresponds to a counter-diagonal matrix in the same way as $\mathbb{I}$ corresponds to the diagonal matrix.

2.4 Ordering and Continuity

An element E of a relation algebra satisfying $\mathbb{I} \subseteq E$ *(reflexivity)*, $E^2 \subseteq E$ *(transitivity)* and $E \cap E^{\mathsf{T}} \subseteq \mathbb{I}$ *(antisymmetry)* is called an **ordering relation**. The adequate structure-preserving transition is the isotone mapping. Given two ordering relations E and E' we call a mapping φ **isotone** if $E ; \varphi \subseteq \varphi ; E'$.

An ordering relation gives rise to looking for the existence of upper resp. lower bounds (majorants resp. minorants), and least upper resp. greatest lower bounds. These constructs have already been transferred into relation-algebraic context. The following definitions may be found in greater detail in [SS85,SS89,SS93].

2.4.1 Definition. Given an ordering relation E, the **upper** and **lower bounds** of a relation X for which $E ; X$ exists can be formed. These as well as the **least upper** and **greatest lower bounds** may be defined as

$$\mathtt{ubd}_E(X) := \overline{\overline{E}^{\mathsf{T}} ; X} \qquad \mathtt{lbd}_E(X) := \overline{\overline{E} ; X}$$

$$\mathtt{lub}_E(X) := \mathtt{ubd}_E(X) \cap \mathtt{lbd}_E(\mathtt{ubd}_E(X)) := \overline{\overline{E}^{\mathsf{T}} ; X} \cap \overline{\overline{E} ; \overline{\overline{E}^{\mathsf{T}} ; X}}$$

$$\mathtt{glb}_E(X) := \mathtt{lbd}_E(X) \cap \mathtt{ubd}_E(\mathtt{lbd}_E(X)) := \overline{\overline{E} ; X} \cap \overline{\overline{E}^{\mathsf{T}} ; \overline{\overline{E} ; X}}$$ □

These functionals are always defined; the results may, however, be null relations. It is an easy task to prove that $\mathtt{lub}, \mathtt{glb}$ are always injective, resembling that such bounds are uniquely defined if they exist, see Ch. 3 of [SS85,SS89,SS93]. As an example we compute the least upper bound of the relation E itself, employing the well-known facts $\overline{\overline{E}^{\mathsf{T}};E} = E^{\mathsf{T}}$ and $\overline{\overline{E};E^{\mathsf{T}}} = E$ as well as antisymmetry of E:

$$\underset{E}{\texttt{lub}}(E) = \overline{\overline{E}^{\mathsf{T}};E} \cap \overline{\overline{E};\overline{\overline{E}^{\mathsf{T}};E}} = E^{\mathsf{T}} \cap \overline{\overline{E};E^{\mathsf{T}}} = E^{\mathsf{T}} \cap E = \mathbb{I}.$$

The adequate structure-preserving mappings for lattice orderings are continuous mappings, as defined below. They are sometimes also called additive.

2.4.2 Definition. A mapping f from an ordering E to an ordering E' is called **(upwards) continuous** wrt. E, E' if for every relation X with existing product $E;X$ we have that application of f commutes with forming the `lub`,

$$f^{\mathsf{T}};\texttt{lub}_E(X) = \texttt{lub}_{E'}(f^{\mathsf{T}};X). \qquad \square$$

Be aware, that this is a modified definition of continuity. Being continuous here requires, that least elements be mapped onto least elements. This is in slight contrast to cpo's. In a cpo, continuity is defined by the same formula, but restricted to *directed* sets X. A directed set is by definition nonempty, so the empty set with its `lub` equal to the least element is excluded from the definition in a cpo.

The advantage of having chosen our variant of defining continuity will become evident already by a simple counting argument: There exist 36 isotone mappings from the ordering of the powerset of a set consisting of 2 elements into itself, 25 satisfying the continuity condition for nonempty subsets — but there are 16 continuous ones according to our definition. Quite naively, at most the latter set may constitute relation algebra; see Theorem 5.2.3.

Just as an exercise, we prove in a component-free style that every upward continuous mapping φ is isotonic:

$$\varphi^{\mathsf{T}} = \varphi^{\mathsf{T}};\mathbb{I} = \varphi^{\mathsf{T}};\texttt{lub}_E(E) = \texttt{lub}_{E'}(\varphi^{\mathsf{T}};E) \subseteq \texttt{ubd}_{E'}(\varphi^{\mathsf{T}};E) = \overline{\overline{E'}^{\mathsf{T}};\varphi^{\mathsf{T}};E}$$

applying the continuity property to $X := E$. Now, we have $\overline{E'}^{\mathsf{T}};\varphi^{\mathsf{T}};E \subseteq \overline{\varphi}^{\mathsf{T}}$, from which we get $\varphi^{\mathsf{T}};E^{\mathsf{T}};\varphi \subseteq E'^{\mathsf{T}}$ or $\varphi^{\mathsf{T}};E\varphi \subseteq E'$, equivalent to the isotony condition using the Schröder rule.

Our aim is now to find out how some element E of a relation algebra may be qualified to constitute the ordering relation of a complete lattice. The qualification of E will follow the style of a *first-order* definition in the theory of relation algebra as quantification runs over all elements of that algebra. This is in contrast to the classical definition of a complete lattice where quantification runs over all subsets of elements making it a *second-order* construct.

2.4.3 Definition. The element E of a relation algebra is said to be a **complete lattice ordering relation**, if it is an ordering relation such that for all relations X with existing product $E;X$ the construct $\texttt{lub}_E(X)$ is surjective. $\square$

By mathematical folklore, with all `lub`'s "existing" also all the `glb`'s will "exist". After transfer into our relational setting, this means that with all `lub`'s surjective also all the `glb`'s will be surjective; see [SS89,SS93] 3.3.11.

3 Motivation by an Example

Before developing the algebraic apparatus aimed at, we present examples of the intended application.

3.1 Relations and Partialities

For a first approach, let some finite n-element set V be given together with the algebra of relations on this set. Then one has the well-known setting of a representable homogeneous relation algebra. Choosing an ordering on V, these relations may easily visualized as boolean $n \times n$-matrices.

From this setting we will now switch to another relation algebra — visualized as boolean $2^n \times 2^n$-matrices — with the given one embedded as the $n \times n$ rectangle of atoms, i.e., singleton sets.

In the strict setting, we have an n-tuple which may be available or not. Now we strive to handle all the possible degrees of having given the n-tuple, starting from a situation with none of the n constituents available and ending with all of them available; see the following ordering concerning availability of $(1, 2, 3)$.

$$E = \begin{array}{l} \{\} \\ \{1\} \\ \{2\} \\ \{1,2\} \\ \{3\} \\ \{1,3\} \\ \{2,3\} \\ \{1,2,3\} \end{array} \overset{\{\}\,\{1\}\{2\}\{1,2\}\{3\}\{1,3\}\{2,3\}\{1,2,3\}}{\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}}$$

Working strictly, one would consider transitions from an n-tuple to another one, and both are assumed to be fully given. Partial evaluation, e.g., requires that we may proceed in a process with not necessarily all constituents available. Think of forming $a \vee b$ for a pair (a, b) with a already known to be `True`, but b not yet delivered. Evaluating strictly, we get $\bot$; working nonstrictly, we get `True`. While this is trivial, it is not yet completely understood how to work algebraically with nets of communicating processes working non-strictly. The situation may improve once a suitable algebraic apparatus is established.

Assume that from the triple (a, b, c) of boolean values the triple $(a \vee b, c \wedge a, \neg a)$ shall be computed. If computing them strictly, the relation F on triples (without commas for reasons of space) expresses the transition. F is a mapping.

$$F = \begin{array}{c} (\mathbf{0\,0\,0}) \\ (\mathbf{0\,0\,1}) \\ (\mathbf{0\,1\,0}) \\ (\mathbf{0\,1\,1}) \\ (\mathbf{1\,0\,0}) \\ (\mathbf{1\,0\,1}) \\ (\mathbf{1\,1\,0}) \\ (\mathbf{1\,1\,1}) \end{array} \overset{\begin{array}{cccccccc} (\mathbf{0\,0\,0}) & (\mathbf{0\,0\,1}) & (\mathbf{0\,1\,0}) & (\mathbf{0\,1\,1}) & (\mathbf{1\,0\,0}) & (\mathbf{1\,0\,1}) & (\mathbf{1\,1\,0}) & (\mathbf{1\,1\,1}) \end{array}}{\begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \end{pmatrix}}$$

Let us now interpret this non-strictly, which requires a much more detailed representation. We don't need to know about b, c when computing the third component of the result, e.g. The **coefficient** for the transition from $(\mathbf{1}, \mathbf{1}, \mathbf{0})$ to $(\mathbf{1}, \mathbf{0}, \mathbf{0})$ should, therefore, look like

$$R = \begin{array}{c} a \\ b \\ c \end{array} \overset{\begin{array}{ccc} a & b & c \end{array}}{\begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \end{pmatrix}} \quad F_{(\mathbf{1},\mathbf{1},\mathbf{0}),(\mathbf{1},\mathbf{0},\mathbf{0})} = \begin{array}{c} \{\} \\ \{a\} \\ \{b\} \\ \{a,b\} \\ \{c\} \\ \{a,c\} \\ \{b,c\} \\ \{a,b,c\} \end{array} \overset{\begin{array}{cccccccc} \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} \end{array}}{\begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}}$$

R expresses, e.g., that *for this specific transition* the first component a determines the first one in the result completely, and — more trivially — the third. There is an intricate interrelationship between R and the blue field of the chosen coefficient of F. First, we observe that an available a fully determines the first and the third component of the result *at this position of the matrix* F. In the same way, if information on this b is available, we know the first component of the result. Finally, c here fully determines the second component as it is $\mathbf{0}$ and part of a conjunction. The blue zone does not coincide with R; however, it is very closely related to R. This can be seen at row a of R which contains two entries, and precisely the set of these two entries is assigned in F albeit outside the blue zone. The other entries of F may be computed following this idea. E.g., row $\{b, c\}$ has an entry at $\{a, b\}$ as looking at the atoms these two are determined.

We will later find out that F is necessarily a mapping which is continuous with respect to the subset-ordering E mentioned above; i.e., it satisfies $F^{\mathsf{T}};\mathtt{lub}_E(X) = \mathtt{lub}_E(F^{\mathsf{T}};X)$ for all relations X for which these constructs may be formed. This is the result of the "additive" construction just explained, giving us a strong algebraic property.

3.2 Composition of Relations and Partialities

Our next observation concerns composition with the transition producing the result triple as $(u, v, w) \mapsto (v \wedge w, u \wedge \neg w, u \vee v \vee w)$. One will find the corresponding relation G for strict interpretation.

$$G = \begin{array}{c} \\ (\mathbf{0\,0\,0}) \\ (\mathbf{0\,0\,1}) \\ (\mathbf{0\,1\,0}) \\ (\mathbf{0\,1\,1}) \\ (\mathbf{1\,0\,0}) \\ (\mathbf{1\,0\,1}) \\ (\mathbf{1\,1\,0}) \\ (\mathbf{1\,1\,1}) \end{array} \begin{array}{cccccccc} (\mathbf{0\,0\,0}) & (\mathbf{0\,0\,1}) & (\mathbf{0\,1\,0}) & (\mathbf{0\,1\,1}) & (\mathbf{1\,0\,0}) & (\mathbf{1\,0\,1}) & (\mathbf{1\,1\,0}) & (\mathbf{1\,1\,1}) \\ 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&1&0&0 \end{array}$$

Interpreting non-strictly, we get the more detailed coefficient S for, e.g., the transition from position $(\mathbf{1}, \mathbf{0}, \mathbf{0})$ to $(\mathbf{0}, \mathbf{1}, \mathbf{1})$.

$$S = \begin{array}{c} \\ u \\ v \\ w \end{array} \begin{array}{c} u\,v\,w \\ \begin{pmatrix} \mathbf{0\,1\,1} \\ \mathbf{1\,0\,0} \\ \mathbf{1\,1\,0} \end{pmatrix} \end{array} G_{(\mathbf{1},\mathbf{0},\mathbf{0}),(\mathbf{0},\mathbf{1},\mathbf{1})} = \begin{array}{c} \\ \{\} \\ \{u\} \\ \{v\} \\ \{u,v\} \\ \{w\} \\ \{u,w\} \\ \{v,w\} \\ \{u,v,w\} \end{array} \begin{array}{cccccccc} \{\} & \{u\} & \{v\} & \{u,v\} & \{w\} & \{u,w\} & \{v,w\} & \{u,v,w\} \\ 1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&1 \end{array}$$

Now we compose the two computations resulting in

$$(a, b, c) \mapsto ((c \wedge a) \wedge \neg a, (a \vee b) \wedge \neg(\neg a), (a \vee b) \vee (c \wedge a) \vee \neg a).$$

It is clear what it means to compose two coefficients; it is relational composition $R \fatsemi S$, which in the case of F and G reduces to function composition. Matrix composition also requires some operation like addition to sum up the coefficient products. Considering availability, it will become clear that this will mean union of the sets. Or else: Union in the boolean matrix sense followed by forming the least upper bound with respect to the subset ordering E.

We now form the composition of $R \fatsemi S$ as well as of $F_{(\mathbf{1},\mathbf{1},\mathbf{0}),(\mathbf{1},\mathbf{0},\mathbf{0})} \fatsemi G_{(\mathbf{1},\mathbf{0},\mathbf{0}),(\mathbf{0},\mathbf{1},\mathbf{1})}$ resulting in the following matrices. One will find out the they indeed correspond to one another as before.

$$\begin{array}{c} \\ u \\ v \\ w \end{array} \begin{array}{c} u\;\;v\;\;w \\ \begin{pmatrix} \mathbf{1}&\mathbf{1}&\mathbf{1} \\ \mathbf{0}&\mathbf{1}&\mathbf{1} \\ \mathbf{1}&\mathbf{0}&\mathbf{0} \end{pmatrix} \end{array}, \qquad \text{resp.} \qquad \begin{array}{c} \\ \{\} \\ \{u\} \\ \{v\} \\ \{u,v\} \\ \{w\} \\ \{u,w\} \\ \{v,w\} \\ \{u,v,w\} \end{array} \begin{array}{cccccccc} \{\} & \{u\} & \{v\} & \{u,v\} & \{w\} & \{u,w\} & \{v,w\} & \{u,v,w\} \\ 1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1 \end{array}$$

Pushing tuples through a net of such boolean operations on tuples will quite frequently offer the opportunity to evaluate partially. Then one will no longer insist that this net operate synchronously as it would mean to wait always for the slowest — and possibly unimportant — computation.

3.3 The Heterogeneous Case

This basic idea carries over to the heterogeneous case. Assume that the triples considered so far originate from a singleton and a pair, i.e. we consider them as $(a, (b, c))$ instead of (a, b, c) so far. Let us investigate the projection operation on the singletons with π and on the pair with ρ, e.g.

$$\pi = \begin{array}{c} \\ (\mathbf{0}\,(\mathbf{0}\;\mathbf{0})) \\ (\mathbf{0}\,(\mathbf{0}\;\mathbf{1})) \\ (\mathbf{0}\,(\mathbf{1}\;\mathbf{0})) \\ (\mathbf{0}\,(\mathbf{1}\;\mathbf{1})) \\ (\mathbf{1}\,(\mathbf{0}\;\mathbf{0})) \\ (\mathbf{1}\,(\mathbf{0}\;\mathbf{1})) \\ (\mathbf{1}\,(\mathbf{1}\;\mathbf{0})) \\ (\mathbf{1}\,(\mathbf{1}\;\mathbf{1})) \end{array} \begin{array}{c} (\mathbf{0})(\mathbf{1}) \\ \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \end{array} \qquad \rho = \begin{array}{c} \\ (\mathbf{0}\,(\mathbf{0}\;\mathbf{0})) \\ (\mathbf{0}\,(\mathbf{0}\;\mathbf{1})) \\ (\mathbf{0}\,(\mathbf{1}\;\mathbf{0})) \\ (\mathbf{0}\,(\mathbf{1}\;\mathbf{1})) \\ (\mathbf{1}\,(\mathbf{0}\;\mathbf{0})) \\ (\mathbf{1}\,(\mathbf{0}\;\mathbf{1})) \\ (\mathbf{1}\,(\mathbf{1}\;\mathbf{0})) \\ (\mathbf{1}\,(\mathbf{1}\;\mathbf{1})) \end{array} \begin{array}{c} (\mathbf{0}\;\mathbf{0})(\mathbf{0}\;\mathbf{1})(\mathbf{1}\;\mathbf{0})(\mathbf{1}\;\mathbf{1}) \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Also matrices like π, ρ allow to introduce partialities. Coefficients — which again are continuous functions — will be represented by 8×2-matrices for π and 8×4-matrices in the case of ρ. Bear in mind, however, that our exposition here is not intended to describe an algorithm, nor shall it replace formal arguments. Rather we tried give a bit of intuition for the necessity as well as the way the methods to be presented will work.

4 Boolean Lattice Orderings

Our investigations at the borderline of non-strict/continuous and strict/non-continuous processes require that we be able to deal with two different aspects. First, we need a relational modeling of program transitions. This is by now well-known and has been handled in a diversity of approaches, e.g., [Sch81a,Sch81b].

Secondly, we need a modeling of the progress in availability of objects. Studies in connection with Goguen categories [Win02a,Win02c,Win02b,Win03] have shown that this cannot be formulated *inside* the relation algebra used for the first aspect. Our approach is, therefore, to embed the algebra for transitions somehow in a bigger relation algebra. Transactions that require certain availabilities will then be handled in the bigger algebra. After every transaction, transitions are again observed in the embedded algebra.

The switching between the embedded algebra and the embedding one needs algebraic properties of boolean lattice orderings formulated in componentfree form as these properties are afterwards used in a sensitive algebraic mechanism.

4.1 Selfsimilarity of Boolean Lattices

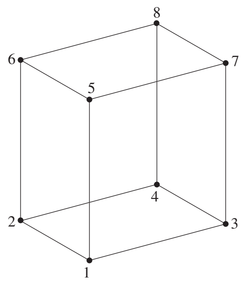
An ordering shall further be characterized to be a (possibly atomic and complete) boolean lattice. This is rather difficult a task as it requires not least to algebraically characterize the atoms. In addition to the algebraic considerations we will provide for a more intuitive view on the fractal style self-similarity of boolean lattices. The formal proofs later are, however, independent from this visualization.

We start our task by looking at pairs of elements having a common upper bound as given with the relation $E;E^{\mathsf{T}}$. Correspondingly, $E^{\mathsf{T}};E$ describes the relation between two elements of having a common lower bound. For an element and its negative in a boolean lattice it is characteristic that they don't have a common upper bound except for the greatest element and don't have a common lower bound except for the least element of the lattice. The approach is, therefore, to define the relations

$$D := E \cap \overline{E};\mathbb{T} \qquad\qquad F := E \cap \mathbb{T};\overline{E}.$$

Here, the vector $\overline{\overline{E};\mathbb{T}} = \texttt{lbd}(\mathbb{T})$ characterizes the least element, as only this is less or equal than all the others. Analogously, the vector $\overline{\mathbb{T};\overline{E}}^{\mathsf{T}} = \texttt{ubd}(\mathbb{T})$ characterizes the greatest element. Hence D is that part of the ordering relation E, which is restricted to rows representing elements strictly above the least element. Relation F restricts E to columns other than that corresponding to the greatest element.

For an example consider as boolean lattice the 3-dimensional cube, the ordering relation of which is given as the fractal style matrix E. Also D and F are shown.



$$E = \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1&2&3&4&5&6&7&8 \end{array} \\ \begin{pmatrix} 1&1&1&1&1&1&1&1\\ 0&1&0&1&0&1&0&1\\ 0&0&1&1&0&0&1&1\\ 0&0&0&1&0&0&0&1\\ 0&0&0&0&1&1&1&1\\ 0&0&0&0&0&1&0&1\\ 0&0&0&0&0&0&1&1\\ 0&0&0&0&0&0&0&1 \end{pmatrix} \end{array}$$

$$D = \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&1&0&1&0&1&0&1\\ 0&0&1&1&0&0&1&1\\ 0&0&0&1&0&0&0&1\\ 0&0&0&0&1&1&1&1\\ 0&0&0&0&0&1&0&1\\ 0&0&0&0&0&0&1&1\\ 0&0&0&0&0&0&0&1 \end{pmatrix} \qquad F = \begin{pmatrix} 1&1&1&1&1&1&1&0\\ 0&1&0&1&0&1&0&0\\ 0&0&1&1&0&0&1&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&1&1&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}$$

Using F, it is easy to present the relation

$$\overline{F;F^{\mathsf{T}}} = \overline{(E \cap \mathbb{T};\overline{E});(E \cap \mathbb{T};\overline{E})^{\mathsf{T}}} = \overline{E;(E \cap \mathbb{T};\overline{E})^{\mathsf{T}}} = \overline{(E \cap \mathbb{T};\overline{E});E^{\mathsf{T}}}$$

of "not admitting common upper bounds different from the greatest element". Analogously with D, one finds the relation

$$\overline{D^\mathsf{T};D} = \overline{(E \cap \overline{\overline{E};\mathbb{T}})^\mathsf{T};(E \cap \overline{\overline{E};\mathbb{T}})} = \overline{E^\mathsf{T};(E \cap \overline{\overline{E};\mathbb{T}})} = \overline{(E \cap \overline{\overline{E};\mathbb{T}})^\mathsf{T};E}$$

of "not admitting common lower bounds different from the least element". Again we observe the fractal construction.

$$\overline{D^\mathsf{T};D} = \begin{pmatrix} 1&1&1&1&1&1&1&1\\ 1&0&1&0&1&0&1&0\\ 1&1&0&0&1&1&0&0\\ 1&0&0&0&1&0&0&0\\ 1&1&1&1&0&0&0&0\\ 1&0&1&0&0&0&0&0\\ 1&1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0 \end{pmatrix} \qquad \overline{F;F^\mathsf{T}} = \begin{pmatrix} 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&1\\ 0&0&0&0&0&1&0&1\\ 0&0&0&0&1&1&1&1\\ 0&0&0&1&0&0&0&1\\ 0&0&1&1&0&0&1&1\\ 0&1&0&1&0&1&0&1\\ 1&1&1&1&1&1&1&1 \end{pmatrix}$$

We are not allowed, however, to make use of it in a naive form. Naive would mean to simply mirror the matrix $\overline{D^\mathsf{T};D}$ along the diagonal upper/right to lower/left in order to obtain $\overline{F;F^\mathsf{T}}$, e.g. Another naive operation would be to rotate E. The pictorial information may, however, be guidance to us, in order to obtain the desired algebraic versions of such transformations.

An obvious idea is to look for the counter-diagonal, which resembles negation and may be determined as $N := \overline{D^\mathsf{T};D} \cap \overline{F;F^\mathsf{T}}$, using already the same notation as for complement transition in Def. 2.3.1. The counter diagonal normally has not an easy algebraic characterisation compared with the diagonal. Multiplying with N from the left turns upside down for a matrix. Multiplying with N from the right flips horizontally.

4.1.1 Definition. A complete lattice ordering relation E will be called a **complete boolean lattice ordering**, if the derived constructs

$$D := E \cap \overline{\overline{E};\mathbb{T}}, \qquad F := E \cap \overline{\mathbb{T};\overline{E}}, \qquad N := \overline{D^\mathsf{T};D} \cap \overline{F;F^\mathsf{T}}$$

satisfy the following conditions given in equational style

i) $\mathtt{ubd}(\overline{D^\mathsf{T};D}) = \overline{F;F^\mathsf{T}}$
ii) $\mathtt{lbd}(\overline{F;F^\mathsf{T}}) = \overline{D^\mathsf{T};D}$
iii) N is total, i.e. $N;\mathbb{T} = \mathbb{T}$ □

The interpretation of (i) is that for every element x the following holds: The cone of majorants of the set of "elements not admitting common lower bounds different from the least element" with x is equal to the set of "elements not admitting common upper bounds different from the greatest element" with x.

As we easily see that N is formed using a least upper bound, it is injective:

$$\begin{aligned} \texttt{lub}\,(\overline{D^{\mathsf{T}};D}) &= \texttt{ubd}\,(\overline{D^{\mathsf{T}};D}) \quad\cap\quad \texttt{lbd}\,(\texttt{ubd}\,(\overline{D^{\mathsf{T}};D})) \\ &= \overline{F;F^{\mathsf{T}}} \cap \texttt{lbd}\,(\overline{F;F^{\mathsf{T}}}) = \overline{F;F^{\mathsf{T}}} \cap \overline{D^{\mathsf{T}};D} = N \end{aligned}$$

Now, we can state that two elements are in relation N if at the same time they have no common upper bound except the greatest element and no common lower bound except for the least element. Univalence and totality follow from surjectivity and injectivity. Using symmetry of N, we get the involution property from surjectivity and univalence: $N;N = N^{\mathsf{T}};N = \mathbb{I}$.

We now investigate the following constructs a and € together with some of their surprising properties.

4.1.2 Definition. Given a boolean lattice ordering E together with the corresponding N, we define

$$a := (\overline{\overline{E};\overline{E}} \cap \overline{E};\mathbb{T} \cap \mathbb{T};\overline{E});N \qquad \text{€} := a;E. \qquad \square$$

These definitions need some visualization. The relation a will turn out to be the partial identity characterizing the atoms among the elements of the boolean lattice. In the matrix, this is certainly dependent in which order the elements are arranged. An indication that the order chosen may be a favourable one is given by €, which corresponds to the relation ε when omitting all rows full of **0** 's. We have chosen the Euro-symbol € for two reasons. It indicates nicely how several rows of E are taken according to a. In addition, it is not too different from the usual direct power symbol ε of Def. 2.3.1.

$$a = \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \qquad \text{€} = \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&1&0&1&0&1&0&1\\ 0&0&1&1&0&0&1&1\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&1&1&1&1\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}$$

We will explain this further starting from the observation that ordering relations representing boolean lattices may easily be generated recursively as follows:

$$E_0 := (1), \quad E_1 := \begin{pmatrix} E_0 & E_0 \\ \mathbb{⫫} & E_0 \end{pmatrix} = \begin{pmatrix} 1&1\\ 0&1 \end{pmatrix}, \quad E_2 := \begin{pmatrix} E_1 & E_1 \\ \mathbb{⫫} & E_1 \end{pmatrix} = \begin{pmatrix} 1&1&1&1\\ 0&1&0&1\\ 0&0&1&1\\ 0&0&0&1 \end{pmatrix}, \quad \ldots$$

$$E_{n+1} := \begin{pmatrix} E_n & E_n \\ \mathbb{⫫} & E_n \end{pmatrix}$$

Now consider the construct $\overline{\overline{E};\overline{E}}$. It is equal to $\texttt{lbd}_E(\overline{E})$. The column representing element x of relation $\overline{E}$ contains all the elements y that are strictly above x or are incomparable to x. Let us first consider the column corresponding to a greatest element x — if any. No element is in relation $\overline{E}$ to x, so that the lower bound set is the full set. Next, let us consider a column x corresponding to a negated atom of an atomic boolean lattice. The lower bound set of all elements related via $\overline{E}$ to x consists of the corresponding atom together with the zero element — if any. Finally, assuming an atomic lattice, every non-atomic non-zero element is incomparable or above of more than one atom; therefore the set of lower bounds of such a column contains just the least element.

Proofs of some key results may thus also be formulated by induction over this sequence. We demonstrate this with, e.g. showing how the construct $\overline{\overline{E};\overline{E}} \cap \overline{E};\mathbb{T} \cap \mathbb{T};\overline{E}$ keeps track of the atoms of the ordering[1]: It is certainly the case that we obtain $(\mathbf{0})$ and $\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{1} & \mathbf{0}\end{pmatrix}$ in the case of indices 0 and 1.

Let us now consider the induction step from n to $n+1$, where we omit the subscript n for reasons of space.

$$\begin{aligned}
&\overline{\overline{E_{n+1}};\overline{E_{n+1}}} \cap \overline{E_{n+1}};\mathbb{T}_{n+1} \cap \mathbb{T}_{n+1};\overline{E_{n+1}}\\
&\quad = \overline{\overline{\begin{pmatrix}E & E\\ \perp\!\!\!\perp & E\end{pmatrix}};\overline{\begin{pmatrix}E & E\\ \perp\!\!\!\perp & E\end{pmatrix}}} \cap \overline{\begin{pmatrix}E & E\\ \perp\!\!\!\perp & E\end{pmatrix}};\begin{pmatrix}\mathbb{T} & \mathbb{T}\\ \mathbb{T} & \mathbb{T}\end{pmatrix} \cap \begin{pmatrix}\mathbb{T} & \mathbb{T}\\ \mathbb{T} & \mathbb{T}\end{pmatrix};\overline{\begin{pmatrix}E & E\\ \perp\!\!\!\perp & E\end{pmatrix}}\\
&\quad = \overline{\begin{pmatrix}\overline{E} & \overline{E}\\ \mathbb{T} & \overline{E}\end{pmatrix};\begin{pmatrix}\overline{E} & \overline{E}\\ \mathbb{T} & \overline{E}\end{pmatrix}} \cap \begin{pmatrix}\overline{E} & \overline{E}\\ \mathbb{T} & \overline{E}\end{pmatrix};\begin{pmatrix}\mathbb{T} & \mathbb{T}\\ \mathbb{T} & \mathbb{T}\end{pmatrix} \cap \begin{pmatrix}\mathbb{T} & \mathbb{T}\\ \mathbb{T} & \mathbb{T}\end{pmatrix};\begin{pmatrix}\overline{E} & \overline{E}\\ \mathbb{T} & \overline{E}\end{pmatrix}\\
&\quad = \overline{\begin{pmatrix}\overline{E};\mathbb{T} & \overline{E};\overline{E}\\ \mathbb{T};\overline{E} \cup \overline{E};\mathbb{T} & \mathbb{T};\overline{E}\end{pmatrix}} \cap \begin{pmatrix}\overline{E};\mathbb{T} & \overline{E};\mathbb{T}\\ \mathbb{T} & \mathbb{T}\end{pmatrix} \cap \begin{pmatrix}\mathbb{T} & \mathbb{T};\overline{E}\\ \mathbb{T} & \mathbb{T};\overline{E}\end{pmatrix}\\
&\quad = \begin{pmatrix}\perp\!\!\!\perp & \overline{\overline{E};\overline{E}} \cap \overline{E};\mathbb{T} \cap \mathbb{T};\overline{E}\\ \overline{\mathbb{T};\overline{E}} \cap \overline{\overline{E};\mathbb{T}} & \perp\!\!\!\perp\end{pmatrix}
\end{aligned}$$

In the setting chosen here, the lower left matrix consists of a matrix with all $\mathbf{0}$-entries except for a $\mathbf{1}$ in the upper right corner, in this way giving one further atom in addition to those already present before. Atom number k appears in the counter-diagonal at position $(2^{k-1}+1, 2^n - 2^{k-1} - 1)$ with n the total number of atoms.

We have yet a further problem: There may be no atoms at all. One will verify that all the formulae stay valid for $E = N = (\mathbf{1})$ and $D = F = a = € = (\mathbf{0})$. While it is hard to imagine a "rowless" matrix ε with one column to obtain the ordering $E = (\mathbf{1}) = \overline{\varepsilon^{\mathsf{T}};\overline{\varepsilon}}$ on the powerset of the empty set, it is easy to work with a one-row $€ = (\mathbf{0})$.

As already announced by the example, a will turn out to be a partial identity characterizing the atoms of the ordering.

4.1.3 Proposition. $a \subseteq \mathbb{I}$

[1] We take this simpler construct at the cost of getting the atoms in the counter-diagonal instead of in the diagonal.

Proof: It suffices to prove $a;N \subseteq N$ as N is already known to be an involution, i.e. $N;N = \mathbb{I}$. The task is, therefore, to show

$$\overline{\overline{E};\overline{E}} \cap \overline{E};\mathbb{T} \cap \mathbb{T};\overline{E} \subseteq N.$$

Since $N = \overline{D^\mathsf{T};D} \cap \overline{F;F^\mathsf{T}}$, we prove two containments, beginning with

$$\overline{\overline{E};\overline{E}} \cap \overline{E};\mathbb{T} \cap \mathbb{T};\overline{E} \subseteq \overline{F;F^\mathsf{T}}$$

or equivalently with

$$F;F^\mathsf{T} \cap \overline{E};\mathbb{T} \cap \mathbb{T};\overline{E} \subseteq \overline{E};\overline{E}.$$

This, however, is indeed the case, as already

$$\begin{aligned} F;F^\mathsf{T} \cap \overline{E};\mathbb{T} &= (F \cap \overline{E};\mathbb{T});F^\mathsf{T} && \text{using Prop. } (\dagger)\\ &\subseteq (E \cap \overline{E};\mathbb{T});F^\mathsf{T} && \text{as } F \subseteq E \text{ by definition of } F\\ &= D;F^\mathsf{T} && \text{by definition of } D\\ &\subseteq \overline{E};\overline{E} && \text{using the forthcoming Lemma 4.1.6.ix.} \end{aligned}$$

The second part "$\subseteq \overline{D^\mathsf{T};D}$" is handled similarly. □

We will prove the algebraic property used together with a bulk of others.

4.1.4 Lemma. For D, E, F we always have $E;F = F$ and $D;E = D$.

Proof: $E;F = E;(E \cap \mathbb{T};\overline{E}) = E^2 \cap \mathbb{T};\overline{E} = E \cap \mathbb{T};\overline{E} = F$,
$D;E = (E \cap \overline{E};\mathbb{T});E = E;E \cap \overline{E};\mathbb{T} = E \cap \overline{E};\mathbb{T} = D$, using Prop. (†). □

It is well-known that $E^\mathsf{T};\overline{E;X} = \overline{E;X}$ for an ordering E and arbitrary X. Parts of this remain valid when switching from E to D or F.

4.1.5 Lemma. For D, E, F, the following equations hold:

i) $E^\mathsf{T};\overline{F;X} = \overline{F;X}$ for every X with existing $F;X$.
ii) $E;\overline{D^\mathsf{T};Y} = \overline{D^\mathsf{T};Y}$ for every Y with existing $D^\mathsf{T};Y$.

Proof: While $\supseteq$ is trivial, $\subseteq$ is obtained using Schröder's rule and 4.1.4. □

Starting herefrom, lots of formulae may be proved for D, E, F, N in a boolean lattice ordering.

4.1.6 Lemma. In a boolean lattice ordering the following hold

i) $N;E = \overline{F;F^\mathsf{T}} = E^\mathsf{T};N$
ii) $E;N = \overline{D^\mathsf{T};D} = N;E^\mathsf{T}$
iii) $\overline{D^\mathsf{T};D};N = E = N;\overline{F;F^\mathsf{T}} = N;E^\mathsf{T};N$
iv) $D;N = N;F^\mathsf{T}$
v) $N;D = F^\mathsf{T};N$

$$\begin{array}{rrcl}
\text{vi)} & \overline{E} & = & D^{\mathsf{T}};N;F^{\mathsf{T}} \\
\text{vii)} & D & \subseteq & D^{\mathsf{T}};D \\
\text{viii)} & F & \subseteq & F;F^{\mathsf{T}} \\
\text{ix)} & D;F^{\mathsf{T}} & \subseteq & \overline{E};\overline{E}
\end{array}$$

Proof: i) As N is an involution and the expression $\overline{F;F^{\mathsf{T}}}$ is obviously symmetric, only the first equation needs to be shown. For " $\subseteq$ ", we show $N;E \subseteq \overline{F;F^{\mathsf{T}}};E = \overline{F;F^{\mathsf{T}}}$ using (ii) of Lemma 4.1.5. From $\mathtt{lbd}(\overline{F;F^{\mathsf{T}}}) = \overline{D^{\mathsf{T}};D}$, or equivalently, from $\overline{E};\overline{F;F^{\mathsf{T}}} = D^{\mathsf{T}};D$, we easily deduce $N;\overline{F;F^{\mathsf{T}}} \subseteq \overline{D^{\mathsf{T}};D};\overline{F;F^{\mathsf{T}}} \subseteq E$. Now, the involution property of N together with isotone composition gives the desired result $\overline{F;F^{\mathsf{T}}} = N;N;\overline{F;F^{\mathsf{T}}} \subseteq N;E$. ii) is proved analogously.

iii) This follows easily from (i,ii) applying the involution property $N;N = \mathbb{I}$.

iv) $D;N = (E \cap \overline{E};\mathbb{T});N = E;N \cap \overline{E};\mathbb{T} = \overline{D^{\mathsf{T}};D} \cap \overline{E};\mathbb{T} = N;E^{\mathsf{T}} \cap \overline{N;E^{\mathsf{T}};N};\mathbb{T} = N;E^{\mathsf{T}} \cap N;\overline{E^{\mathsf{T}}};N;\mathbb{T} = N;E^{\mathsf{T}} \cap N;\overline{E^{\mathsf{T}}};\mathbb{T} = N;(E^{\mathsf{T}} \cap \overline{E^{\mathsf{T}}};\mathbb{T}) = N;F^{\mathsf{T}}$.

v) (iv) implies $D = D;N;N = N;F^{\mathsf{T}};N$, from which we get $N;D = N;N;F^{\mathsf{T}};N = F^{\mathsf{T}};N$.

vi) From (iii, v) we have $\overline{E} = \overline{N;\overline{F;F^{\mathsf{T}}}} = N;\overline{\overline{F;F^{\mathsf{T}}}} = N;F;F^{\mathsf{T}} = (F^{\mathsf{T}};N)^{\mathsf{T}};F^{\mathsf{T}} = (N;D)^{\mathsf{T}};F^{\mathsf{T}}$.

$$\begin{array}{rrll}
\text{vii)} & D & \subseteq E^{\mathsf{T}};D & E \text{ is reflexive} \\
& & = E^{\mathsf{T}};(E \cap \overline{E};\mathbb{T}) & \text{by definition of } D \\
& & = (E^{\mathsf{T}} \cap (\overline{E};\mathbb{T})^{\mathsf{T}});(E \cap \overline{E};\mathbb{T}) & \text{using Prop. } (\dagger) \\
& & = D^{\mathsf{T}};D & \text{by definition of } D
\end{array}$$

viii) is proved in an analogous way.

$$\begin{array}{rrll}
\text{ix)} & D;F^{\mathsf{T}} & \subseteq D^{\mathsf{T}};D;F;F^{\mathsf{T}} & \text{using (vii, viii)} \\
& & = \overline{E;N};\overline{N;E} & \text{using (i, ii)} \\
& & = \overline{E};N;N;\overline{E} & \text{using Prop. } (\ddagger)\text{.ii, since } N \text{ and } N^{\mathsf{T}} \text{ are mappings} \\
& & = \overline{E};\overline{E} & \text{as } N \text{ is an involution.}
\end{array}$$

□

One will quite easily observe that $\overline{D^{\mathsf{T}};D}$ is a lower cone, i.e. $E;\overline{D^{\mathsf{T}};D} = \overline{D^{\mathsf{T}};D}$. We give the proof of the nontrivial inclusion: By Schröder's rule we get $E^{\mathsf{T}};D^{\mathsf{T}};D \subseteq D^{\mathsf{T}};D$ which follows by monotony from $D;E = D$. Similarly, $\overline{F;F^{\mathsf{T}}}$ is an upper cone, i.e. $E^{\mathsf{T}};\overline{F;F^{\mathsf{T}}} = \overline{F;F^{\mathsf{T}}}$.

4.1.7 Lemma. N satisfies $\overline{E} \cup N;\overline{E} = \mathbb{T};\overline{E}$.

Proof: To prove the non-trivial direction " $\supseteq$ " of the formula, we start from

$$\begin{array}{rll}
E;(E \cap \mathbb{T};\overline{E})^{\mathsf{T}} & = (E \cap \mathbb{T};\overline{E});(E \cap \mathbb{T};\overline{E})^{\mathsf{T}} & \text{Prop. } (\dagger) \\
& = F;F^{\mathsf{T}} & \\
& \subseteq D^{\mathsf{T}};D \cup F;F^{\mathsf{T}} = \overline{N} & \text{see above}
\end{array}$$

Using Schröder's rule, we get $N;(E \cap \mathbb{T};\overline{E}) \subseteq \overline{E}$. As N is a mapping, it multiplies distributively from the left and may slip from the left under negations (‡) giving $\overline{N;\overline{E}} \cap \mathbb{T};\overline{E} \subseteq \overline{E}$, which is, up to boolean calculus, the non-trivial " $\supseteq$ ". □

Also this lemma may be proved by induction over the recursive generation of the E_n. It obviously holds for $E_1 := (\mathbf{1})$. The step from n to $n+1$ is as follows:

$$\begin{aligned}\overline{E_{n+1}} \cup N_{n+1}; \overline{E_{n+1}} &= \begin{pmatrix}\overline{E} & \overline{E} \\ \mathbb{T} & \overline{E}\end{pmatrix} \cup \begin{pmatrix}\perp\!\!\!\perp & N \\ N & \perp\!\!\!\perp\end{pmatrix}; \begin{pmatrix}\overline{E} & \overline{E} \\ \mathbb{T} & \overline{E}\end{pmatrix} \\ &= \begin{pmatrix}\mathbb{T} & \overline{E} \cup N; \overline{E} \\ \mathbb{T} & \overline{E} \cup N; \overline{E}\end{pmatrix} = \begin{pmatrix}\mathbb{T} & \mathbb{T} \\ \mathbb{T} & \mathbb{T}\end{pmatrix}; \begin{pmatrix}\overline{E} & \overline{E} \\ \mathbb{T} & \overline{E}\end{pmatrix} = \mathbb{T}; \overline{E_{n+1}}\end{aligned}$$

4.2 Power-related Properties

From the following formulae, (ii) turns out to be a powerful one. It regulates two forms of negation, one on the normal relational level, the other with N.

4.2.1 Lemma. In the given setting of a blo, we have

i) $a; D^\mathsf{T} = a, \qquad a; E^\mathsf{T}; a = a, \qquad F^\mathsf{T}; a = a$

ii) $a; \overline{\text{€}} = a; \overline{E} = \text{€}; N$

iii) $a; \text{€} = \text{€}$

Proof: i) By definition of a and D, the first equality requires to prove

$$(\overline{\overline{E}; \overline{E}} \cap \overline{E}; \mathbb{T} \cap \mathbb{T}; \overline{E}); N; D^\mathsf{T} = (\overline{\overline{E}; \overline{E}} \cap \overline{E}; \mathbb{T} \cap \mathbb{T}; \overline{E}); N.$$

With $N; D^\mathsf{T} = F; N$ from Lemma 4.1.6.iv and the involution property of N, this would follow from

$$(\overline{\overline{E}; \overline{E}} \cap \overline{E}; \mathbb{T} \cap \mathbb{T}; \overline{E}); F = \overline{\overline{E}; \overline{E}} \cap \overline{E}; \mathbb{T} \cap \mathbb{T}; \overline{E}.$$

Using $F = E \cap \mathbb{T}; \overline{E}$ and Prop. (†), the lefthand side evaluates to

$$(\overline{\overline{E}; \overline{E}} \cap \overline{E}; \mathbb{T} \cap \mathbb{T}; \overline{E}); E \cap \mathbb{T}; \overline{E},$$

so that we can immediately conclude that " $\supseteq$ ".

It remains to prove " $\subseteq$ ", for which

$$(\overline{\overline{E}; \overline{E}} \cap \overline{E}; \mathbb{T} \cap \mathbb{T}; \overline{E}); E \ \subseteq\ \overline{\overline{E}; \overline{E}} \cap \overline{E}; \mathbb{T}$$

is sufficient. This in turn is by Schröder's rule equivalent with

$$(\overline{E}; \overline{E} \cup \overline{\overline{E}; \mathbb{T}}); E^\mathsf{T} \ \subseteq\ \overline{E}; \overline{E} \cup \overline{\overline{E}; \mathbb{T}} \cup \overline{\mathbb{T}; \overline{E}}.$$

The two left terms are indeed contained in the corresponding terms on the righthand side, as $\overline{E}; E^\mathsf{T} = \overline{E}$.

$F^\mathsf{T}; a = a$ has, of course, a similar proof. An additional effort is necessary for proving $a; E^\mathsf{T}; a = a$. We observe that by definition of D we have $a; E^\mathsf{T}; a = a; (D \cup \overline{\overline{E}; \mathbb{T}})^\mathsf{T}; a$. The first part satisfies indeed $a; D^\mathsf{T}; a = a; a = a$. The second part vanishes as by definition $a \subseteq \overline{E};\mathbb{T};N \subseteq \overline{E};\mathbb{T}$. Therefore, $a;\mathbb{T} \subseteq \overline{E};\mathbb{T}$ and using Schröder's rule $a^\top;\overline{\overline{E};\mathbb{T}} \subseteq \perp\!\!\!\perp$.

$$
\begin{array}{lll}
\text{ii)}\ a;\overline{\text{€}} & = a;\overline{a;E} & \text{by definition of } \text{€} \\
 & = a;\overline{E} & (*) \\
 & = a;D^\top;N;F^\top & \text{Lemma 4.1.6.vi} \\
 & = a;N;F^\top & \text{(i)} \\
 & = a;D;N & \text{Lemma 4.1.6.iv} \\
 & = a;E;N & \text{using that } a;D = a;E\text{, see below} \\
 & = \text{€};N & \text{by definition of } \text{€}
\end{array}
$$

For $a;D = a;E$ it remains to show $a;E \subseteq a;D$, since $D \subseteq E$. By definition of the univalent a, we have $a;E = (\overline{\overline{E};\overline{E}} \cap \overline{E};\mathbb{T} \cap \mathbb{T};\overline{E});N;E \subseteq \overline{E};\mathbb{T}$, so that with $a;E = a^2;E$ it follows that $a;E = a;(E \cap a;E) \subseteq a;(E \cap \overline{E};\mathbb{T}) = a;D$.

iii) $a;\text{€} = a;a;E = a;E = \text{€}$ as $a \subseteq \mathbb{I}$ implies $a;a = a$. □

Now € offers itself in many respects as a substitute for a direct power ε.

4.2.2 Lemma. $a;\text{€};\overline{\text{€}^\top;a;X} = a;\overline{X}$ $\qquad$ $\text{€};\overline{\text{€}^\top;X} = a;\overline{X}$

$$
\begin{array}{lll}
\textbf{Proof:}\ a;\text{€};\overline{\text{€}^\top;a;X} & = a;a;E;\overline{E^\top;a;a;X} & \text{by definition of } \text{€} \\
 & = a;E;\overline{E^\top;a;X} & \text{since } a \subseteq \mathbb{I} \\
 & = a;\overline{E^\top;a;X} & \text{as always } E;\overline{E^\top;Y} = \overline{E^\top;Y} \\
 & = a;\overline{a;E^\top;a;X} & (*) \\
 & = a;\overline{a;X} & \text{Lemma 4.2.1.i} \\
 & = a;\overline{X} & (*)
\end{array}
$$

The second version is a consequence as $\text{€} = a;\text{€}$ following Lemma 4.2.1.iii. □

4.2.3 Lemma. $a;\overline{\text{€}};\overline{\overline{\text{€}}^\top;a;X} = a;\overline{X}$

$$
\begin{array}{lll}
\textbf{Proof:}\ a;\overline{\text{€}};\overline{\overline{\text{€}}^\top;a;X} & = a;\overline{\text{€}};\overline{(a;\overline{\text{€}})^\top;a;X} & \text{since } a \subseteq \mathbb{I} \\
 & = \text{€};N;\overline{(\text{€};N)^\top;a;X} & \text{Lemma 4.2.1.ii} \\
 & = \text{€};N;N;\overline{\text{€}^\top;a;X} & N, N^\top \text{ are mappings} \\
 & = \text{€};\overline{\text{€}^\top;a;X} & N \text{ is an involution} \\
 & = a;\overline{X} & \text{Lemma 4.2.2}
\end{array}
$$
□

The first observation concerns least upper bounds and symmetric quotients.

4.2.4 Proposition (*Connecting* `syq` *and* `lub`)**.** In an atomic complete boolean lattice we have for all relations X with the product $\text{€};X$ defined that

$$\texttt{lub}_E(X) = \texttt{syq}(\text{€}, \text{€};X)$$

$$\text{and} \quad \in ; \mathtt{lub}_E(X) = \in ; \mathtt{syq}(\in, \in ; X) = \in ; X$$

Proof: We use the formula apparatus developed for $\in$.

$$
\begin{array}{rll}
\mathtt{syq}(\in, \in ; X) & = \overline{\overline{\in}^\top ; \in ; X} \cap \overline{\in^\top ; \overline{\in ; X}} & \text{definition of } \mathtt{syq} \\
& = \overline{\overline{\in}^\top ; \in ; X} \cap \overline{\in^\top ; a ; \overline{\in ; X}} & \text{Lemma 4.2.1.iii} \\
& = \overline{\overline{\in}^\top ; \in ; X} \cap \overline{\in^\top ; a ; \overline{\in} ; \overline{\overline{\in}^\top ; a ; \in ; X}} & \text{Lemma 4.2.3} \\
& = \overline{\overline{\in}^\top ; \in ; X} \cap \overline{\in^\top ; \overline{\in} ; \overline{\overline{\in}^\top ; \in ; X}} & \text{Lemma 4.2.1.iii} \\
& = \overline{\overline{E}^\top ; X} \cap \overline{\overline{E} ; \overline{\overline{E}^\top ; X}} & \text{Lemma 4.3.2} \\
& = \mathtt{lub}_E(X) & \text{Def. 2.4.1}
\end{array}
$$

The second line is an easy consequence of the first: For symmetric quotients always $A ; \mathtt{syq}(A, B) = B$ if $\mathtt{syq}(A, B)$ is surjective, and it has already been shown that $\mathtt{syq}(\in, \in ; X)$ is indeed surjective. □

Slightly generalized this produces the following lemma.

4.2.5 Lemma. $\in ; \mathtt{syq}(\in, a ; X) = a ; X$ for all X.

Proof: We start with Lemma 4.2.2 and have $a ; X = a ; \overline{\overline{X}} = \in ; \overline{\in^\top ; \overline{X}}$. Then

$$
\begin{array}{rll}
\in ; \mathtt{syq}(\in, a ; X) & = \in ; \mathtt{syq}(\in, \in ; \overline{\in^\top ; \overline{X}}) & \text{as before} \\
& = \in ; \overline{\in^\top ; \overline{X}} & \text{Prop. 4.2.4} \\
& = a ; X & \text{as before}
\end{array}
$$
□

4.3 Atomicity of a lattice ordering

Finite boolean lattices as we consider here are necessarily atomic. Nonetheless, we exhibit how existence of atoms may be expressed algebraically.

4.3.1 Definition. A complete boolean lattice ordering relation will be called **atomic** provided it satisfies the condition $\overline{E} ; \mathbb{T} = E^\top ; a ; \mathbb{T}$. □

The additional property may be interpreted in the following way: Precisely all but the least element of the boolean lattice ordering E offer the opportunity to reach an atom when going back against the ordering. The corresponding properties for anti-atoms reads as follows: $\mathbb{T} ; \overline{E} = \mathbb{T} ; a ; N ; E^\top$.

The partial identity a together with the variant $\in$ of the well-known direct power relation ε and atomicity just defined generate important new formulae exhibiting similarities between $\in$, E, and ε.

4.3.2 Lemma. In an atomic boolean lattice ordering $E = \overline{\varepsilon^{\top};\overline{\varepsilon}} = \overline{E^{\top};\overline{E}}$.

Proof: Observing that $E = \overline{E^{\top};\overline{E}}$ is trivial, we prove $\overline{E} = \varepsilon^{\top};\overline{\varepsilon}$:

$$
\begin{array}{rll}
\overline{E} & = D^{\top};N;F^{\top} & \text{Lemma 4.1.6.vi} \\
& = D^{\top};D;N & \text{Lemma 4.1.6.iv} \\
& = \varepsilon^{\top};\varepsilon;N & \text{see below} \\
& = \varepsilon^{\top};a;\overline{\varepsilon} & \text{Lemma 4.2.1.ii} \\
& = \varepsilon^{\top};\overline{\varepsilon} & \text{since by Lemma 4.2.1.iii } \varepsilon = a;\varepsilon
\end{array}
$$

It remains to prove $D^{\top};D = \varepsilon^{\top};\varepsilon$. From $\varepsilon = a;E = E \cap a;\mathbb{T} \subseteq D$ we have "$\supseteq$". For the other direction, we start showing

$$
\begin{array}{rll}
D & = \overline{E};\mathbb{T} \cap E & \text{by definition of } D \\
& = E^{\top};a;\mathbb{T} \cap E & \text{since } E \text{ is atomic, Def. 4.3.1} \\
& \subseteq (E^{\top} \cap E;(a;\mathbb{T})^{\top});(a;\mathbb{T} \cap E;E) & \text{Dedekind rule} \\
& = (E \cap a;\mathbb{T})^{\top};(E \cap a;\mathbb{T}) & a \subseteq \mathbb{I}, E^2 \subseteq E \\
& = \varepsilon^{\top};\varepsilon &
\end{array}
$$

This together with $\varepsilon;D = a;E;D \subseteq a;E;E = a;E = \varepsilon$ allow to prove the opposite direction: $D^{\top};D \subseteq D^{\top};\varepsilon^{\top};\varepsilon = (\varepsilon;D)^{\top};\varepsilon = \varepsilon^{\top};\varepsilon$. □

A useful result is now available:

$$\mathtt{ubd}(\varepsilon) = \mathtt{ubd}(a;E) = \overline{\overline{E}^{\top};a;E} = \overline{\overline{E^{\top};a};a;E} = \overline{\overline{\varepsilon^{\top}};\varepsilon} = E^{\top},$$

so that

$$\mathtt{lub}_E(\varepsilon) = \mathtt{ubd}_E(\varepsilon) \cap \mathtt{lbd}_E(\mathtt{ubd}_E(\varepsilon)) = E^{\top} \cap \mathtt{lbd}_E(E^{\top}) = E^{\top} \cap E = \mathbb{I}.$$

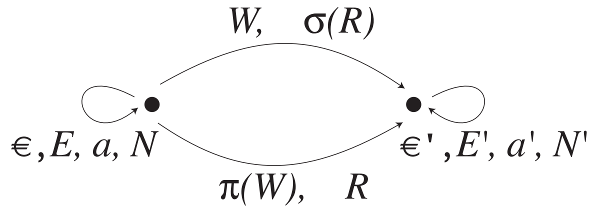
As a consequence $\mathtt{syq}(\varepsilon, \varepsilon) = \mathtt{syq}(\varepsilon, \varepsilon;\varepsilon) = \mathbb{I}$.

5 Embedding Relation Algebras

As announced, a heterogeneous relation algebra will now be embedded into another one. The first step would be to embed relations between X and Y to relations on the singleton subsets $R = a;R;a' \subseteq \mathcal{P}(X) \times \mathcal{P}(Y)$. We will not denote this and start from the latter. When strictness is demanded, the surrounding relation algebra offers the opportunity to formulate an *external* arbiter to check for availability.

5.1 Embedding as a Galois Connection

As the mechanism of Galois connections is well-known, we need not give an introduction into this topic. Rather we formulate the effects directly addressing our particular case. Consider any two atomic complete boolean lattice orderings E, E' together with their corresponding relations a, ε, N and a', ε', N'. Then we have the following situation as our basic setting.



Now, we assume as a **general assumption** two mappings relating relations with the property $R = a;R;a'$, i.e. subrelations of the "rectangle" $a;\mathbb{T};a'$, to arbitrary relations W.

5.1.1 Theorem. The two constructs

$$\sigma(R) := \overline{\varepsilon^{\mathsf{T}};a;R;a';\overline{\varepsilon'}} \quad \text{and} \quad \pi(W) := a;\overline{\varepsilon;W;\overline{\varepsilon'}^{\mathsf{T}}};a'$$

form a Galois correspondence between the set of relations R with $R = a;R;a'$ and arbitrary relations W, i.e.

$$R \subseteq \pi(W) \qquad \Longleftrightarrow \qquad W \subseteq \sigma(R)$$

Proof:		
	$W \subseteq \sigma(R)$	
$\Longleftrightarrow$	$W \subseteq \overline{\varepsilon^{\mathsf{T}};a;R;a';\overline{\varepsilon'}}$	by definition of σ
$\Longleftrightarrow$	$\varepsilon^{\mathsf{T}};a;R;a';\overline{\varepsilon'} \subseteq \overline{W}$	negated
$\Longleftrightarrow$	$\varepsilon;W \subseteq \overline{a;R;a';\overline{\varepsilon'}}$	Schröder
$\Longleftrightarrow$	$a;R;a';\overline{\varepsilon'} \subseteq \overline{\varepsilon;W}$	negated
$\Longleftrightarrow$	$\varepsilon;W;\overline{\varepsilon'}^{\mathsf{T}} \subseteq \overline{a;R;a'}$	Schröder
$\Longleftrightarrow$	$a;R;a' \subseteq \overline{\varepsilon;W;\overline{\varepsilon'}^{\mathsf{T}}}$	negated
$\Longleftrightarrow$	$a;R;a' \subseteq a;\overline{\varepsilon;W;\overline{\varepsilon'}^{\mathsf{T}}};a'$	a, a' idempotent, $a \subseteq \mathbb{I}$, $a' \subseteq \mathbb{I}$
$\Longleftrightarrow$	$R = a;R;a' \subseteq \pi(W)$	definition of π

It is precisely the last line of the proof where we employ the specific property $R = a;R;a'$ to obtain $R \subseteq \pi(W)$ instead of just $a;R;a' \subseteq \pi(W)$. □

As these two mappings form a Galois correspondence, the well-known consequences follow immediately without any additional assumptions on σ, π:

5.1.2 Theorem (*Standard Galois properties*)**.** The following holds if a Galois correspondence is given, i.e., a pair of mappings σ, π satisfying

$$R \subseteq \pi(W) \quad \Longleftrightarrow \quad W \subseteq \sigma(R).$$

i) The composed mappings $\rho(R) := \pi(\sigma(R))$ and $\varphi(W) := \sigma(\pi(W))$ are expanding, i.e., satisfy $R \subseteq \rho(R)$ and $W \subseteq \varphi(W)$.
ii) The mappings σ, π are antitonic.
iii) The composed mappings ρ, φ are idempotent. For all R and W, the images $\sigma(R)$ resp. $\pi(W)$ are fixedpoints under φ resp. ρ, i.e., satisfy

$$\sigma(R) = \sigma(\pi(\sigma(R))) \qquad\qquad \pi(W) = \pi(\sigma(\pi(W)))$$

iv) There is anti-continuity for σ, π, i.e.

$$\sigma(\mathtt{sup}_{R\in\mathcal{X}} R) = \mathtt{inf}_{R\in\mathcal{X}}(\sigma(R)) \qquad \pi(\mathtt{sup}_{W\in\mathcal{Y}} W) = \mathtt{inf}_{W\in\mathcal{Y}}(\pi(W)).$$

v) The mappings σ, π determine each other uniquely, i.e.

$$\pi(W) = \mathtt{sup}\{R \mid W \subseteq \sigma(R)\} \qquad \sigma(R) = \mathtt{sup}\{W \mid R \subseteq \pi(W)\}$$

vi) The fixedpoint sets of the mappings ρ, φ defined as

$$\mathcal{F}_\rho = \{R \mid R = \rho(R)\} \qquad \mathcal{F}_\varphi = \{W \mid W = \varphi(W)\}$$

are mapped bijectively onto one another by $\sigma|_{\mathcal{F}_\rho}$ and $\pi|_{\mathcal{F}_\varphi}$. □

One will have noticed that in this theorem indeed no reference has been made to the specific σ, π defined before. Now we look also for these specific properties.

5.1.3 Proposition (*Injectivity of* σ)**.** In the special case of σ, π introduced as a general assumption, i.e. considering the subset relations satisfying $R = aRa'$, the fixedpoint set $\mathcal{F}_\rho$ is always the full set of these relations $\subseteq a\mathbb{T}a'$, or equivalently

$$R = \pi(\sigma(R)) \quad \text{for all } R = a;R;a' \subseteq a;\mathbb{T};a'.$$

Therefore, σ, π form what is usually called an adjoint pair.

Proof: We prove this showing that σ is an injection.

$$\begin{aligned}
\pi(\sigma(R)) &= a;\overline{\varepsilon;\overline{\varepsilon^\top;a;R;a';\overline{\varepsilon'}};\overline{\varepsilon'}^\top};a' && \text{by the definition in Theorem 5.1.1}\\
&= a;\overline{a;\overline{a;R;a';\overline{\varepsilon'}};\overline{\varepsilon'}^\top};a' && \text{Lemma 4.2.2}\\
&= a;\overline{a;\overline{R;a';\overline{\varepsilon'}};\overline{\varepsilon'}^\top};a' && (*)
\end{aligned}$$

$$= a;\overline{a;\overline{R;a';\overline{\overline{\varepsilon'}}};\overline{\varepsilon'}^{\mathsf T}};a';a' \qquad (*)$$

$$= a;\overline{a;\overline{R};a'};a' \qquad \text{Lemma 4.2.3 in transposed form}$$

$$= a;\overline{\overline{R};a'};a' \qquad (*)$$

$$= a;\overline{\overline{R}};a' = a;R;a' = R \qquad (*) \qquad \square$$

$\mathcal{F}_\varphi$ will not be the full set of all W. It is interesting, which relations may occur as images of σ. First we investigate their greatest lower bound.

5.1.4 Proposition. $\mathtt{syq}(R^{\mathsf T};\varepsilon, \varepsilon') = \mathtt{glb}_{E'}(\sigma(R)^{\mathsf T})^{\mathsf T}$

Proof: We start evaluating $\mathtt{lbd}_{E'}(\sigma(R)^{\mathsf T})$ first.

$$
\begin{aligned}
\mathtt{lbd}_{E'}(\sigma(R)^{\mathsf T}) &= \overline{\overline{E'};\sigma(R)^{\mathsf T}} && \text{by definition of } \mathtt{lbd}\\
&= \overline{\varepsilon'^{\mathsf T};\overline{\varepsilon'};\overline{\overline{\varepsilon'^{\mathsf T};a';R^{\mathsf T};a;\varepsilon}}} && \text{4.3.2, definition of } \sigma\\
&= \overline{\varepsilon'^{\mathsf T};a';\overline{\varepsilon'};\overline{\overline{\varepsilon'^{\mathsf T};a';R^{\mathsf T};a;\varepsilon}}} && \varepsilon'^{\mathsf T} = \varepsilon'^{\mathsf T}a'\\
&= \overline{\varepsilon'^{\mathsf T};a';\overline{\overline{R^{\mathsf T};a;\varepsilon}}} && \text{Lemma 4.2.3}\\
&= \overline{\varepsilon'^{\mathsf T};\overline{\overline{R^{\mathsf T};a;\varepsilon}}} && \varepsilon'^{\mathsf T} = \varepsilon'^{\mathsf T}a'\\
\mathtt{glb}_{E'}(\sigma(R)^{\mathsf T}) &= \overline{\varepsilon'^{\mathsf T};\overline{\overline{R^{\mathsf T};a;\varepsilon}}} \cap \overline{\overline{\varepsilon'}^{\mathsf T};\varepsilon';\overline{\varepsilon'^{\mathsf T};\overline{\overline{R^{\mathsf T};a;\varepsilon}}}} && \text{definition of } \mathtt{glb}\text{, 4.3.2}\\
&= \overline{\varepsilon'^{\mathsf T};\overline{\overline{R^{\mathsf T};\varepsilon}}} \cap \overline{\overline{\varepsilon'}^{\mathsf T};a';R^{\mathsf T};\varepsilon} && \text{4.2.2, double negation}\\
&= \overline{\varepsilon'^{\mathsf T};\overline{\overline{R^{\mathsf T};\varepsilon}}} \cap \overline{\overline{\varepsilon'}^{\mathsf T};a';a';R^{\mathsf T};a;\varepsilon} && a;R;a' = R\\
&= \overline{\varepsilon'^{\mathsf T};\overline{\overline{R^{\mathsf T};\varepsilon}}} \cap \overline{\overline{\varepsilon'}^{\mathsf T};a';R^{\mathsf T};a;\varepsilon} && a' \text{ idempotent}\\
&= \overline{\varepsilon'^{\mathsf T};\overline{\overline{R^{\mathsf T};\varepsilon}}} \cap \overline{\overline{\varepsilon'}^{\mathsf T};R^{\mathsf T};\varepsilon} && a;R;a' = R\\
&= \mathtt{syq}(\varepsilon', R^{\mathsf T};\varepsilon) && \text{by definition of } \mathtt{syq}\\
&= (\mathtt{syq}(R^{\mathsf T};\varepsilon, \varepsilon'))^{\mathsf T} && \square
\end{aligned}
$$

Now this symmetric quotient is considered in more detail.

5.1.5 Proposition. $f_R := \mathtt{syq}(R^{\mathsf T};\varepsilon, \varepsilon')$ is a continuous mapping.

Proof: f_R is univalent since

$$
\begin{aligned}
f_R^{\mathsf T};f_R &= \mathtt{syq}(\varepsilon', R^{\mathsf T};\varepsilon);\mathtt{syq}(R^{\mathsf T};\varepsilon, \varepsilon') \subseteq \mathtt{syq}(\varepsilon', \varepsilon')\\
&= \overline{\overline{\varepsilon'}^{\mathsf T};\varepsilon'} \cap \overline{\varepsilon'^{\mathsf T};\overline{\varepsilon'}} = E'^{\mathsf T} \cap E' = \mathbb{I}
\end{aligned}
$$

It is in addition total as it has just been defined as a transposed greatest lower bound. This $\mathtt{glb}$ is indeed surjective as the ordering has been postulated to be an atomic complete boolean lattice. In this lattice `lub` will always be surjective and by mathematical folklore then also `glb`.

We prove the continuity condition using the abbreviation $g := \mathtt{lub}_E(X)^\mathsf{T}$:

$$\begin{array}{rll}
f_R^\mathsf{T};\mathtt{lub}_E(X) & = \mathtt{syq}(R^\mathsf{T};\varepsilon,\ \varepsilon')^\mathsf{T};\mathtt{lub}_E(X) & \text{by definition} \\
& = \mathtt{syq}(\varepsilon', R^\mathsf{T};\varepsilon);g^\mathsf{T} & \text{Prop. 2.2.1.ii, definition of } g \\
& = \mathtt{syq}(\varepsilon', R^\mathsf{T};\varepsilon;g^\mathsf{T}) & \text{Prop. 2.2.1.vii; } g \text{ is mapping} \\
& = \mathtt{syq}(\varepsilon', R^\mathsf{T};\varepsilon;\mathtt{lub}_E(X)) & \text{expanding } g \\
& = \mathtt{syq}(\varepsilon', R^\mathsf{T};\varepsilon;X) & \text{due to Prop. 4.2.4} \\
& = \mathtt{syq}(\varepsilon', \varepsilon';\mathtt{syq}(\varepsilon', R^\mathsf{T};\varepsilon);X) & \text{Lemma 4.2.5 as } R = a;R;a' \\
& = \mathtt{lub}_E(\mathtt{syq}(\varepsilon', R^\mathsf{T};\varepsilon);X) & \text{Prop. 4.2.4} \\
& = \mathtt{lub}_E(f_R^\mathsf{T};X) & \text{definition of } f_R \qquad \square
\end{array}$$

The relations f_R and R are capable of simulating one another via ε and ε'. There exist several concepts of simulation doing with relations what homomorphisms do with mappings.

5.1.6 Proposition. $\varepsilon';f_R^\mathsf{T} = R^\mathsf{T};\varepsilon$.

Proof:

$$\begin{array}{rll}
\varepsilon';f_R^\mathsf{T} & = \varepsilon';(\mathtt{syq}(R^\mathsf{T};\varepsilon,\ \varepsilon'))^\mathsf{T} & \text{by definition} \\
& = \varepsilon';\mathtt{syq}(\varepsilon', R^\mathsf{T};\varepsilon) & \text{transposing } \mathtt{syq} \\
& = R^\mathsf{T};\varepsilon & \text{Lemma 4.2.5 as } R = a;R;a' \qquad \square
\end{array}$$

The following proposition shows that f_R is indeed the lower borderline of $\sigma(R)$.

5.1.7 Proposition. The relation $f_R := \mathtt{syq}(R^\mathsf{T};\varepsilon,\ \varepsilon')$ satisfies the following properties for given R with $R = a;R;a'$:

i) $f_R;E' = \sigma(R)$.
ii) $R = \pi(f_R)$.

Proof: i) One direction is obtained from

$$\begin{array}{rll}
f_R;E' & = \mathtt{syq}(R^\mathsf{T};\varepsilon,\ \varepsilon');E' & \text{by definition} \\
& \subseteq \overline{\varepsilon^\mathsf{T};R;\overline{\varepsilon'}};\overline{\varepsilon'^\mathsf{T};\overline{\varepsilon'}} & \text{definition of } \mathtt{syq}\text{, Lemma 4.3.2} \\
& \subseteq \overline{\varepsilon^\mathsf{T};R;\overline{\varepsilon'}} = \sigma(R) & \text{as } R = a;R;a'\text{; see below}
\end{array}$$

Here, the step to the last line is justified using Schröder's rule, since

$$\varepsilon^\mathsf{T};R;\overline{\varepsilon'};\overline{\overline{\varepsilon'}^\mathsf{T};\varepsilon'} \subseteq \varepsilon^\mathsf{T};R;\overline{\varepsilon'}$$

is true by Lemma 4.2.3 using $R = aRa'$ and $\varepsilon' = a';\varepsilon'$. Now we prove the other direction in the form $\sigma(R)^\mathsf{T} \subseteq E'^\mathsf{T};f_R^\mathsf{T}$. This is equivalent with $\sigma(R)^\mathsf{T};f_R \subseteq E'^\mathsf{T}$.

This via Schröder's rule means $\overline{\in'}^{\mathsf{T}} ; \in' ; \mathtt{syq}(\in', R^{\mathsf{T}} ; \in) \subseteq \overline{\in'}^{\mathsf{T}} ; R^{\mathsf{T}} ; \in$ which is true as $R = a ; R ; a'$.

ii) $\pi(f_R) = a ; \overline{\in ; f_R ; \overline{\in'}^{\mathsf{T}}} ; a'$ by definition

$= a ; \overline{\in ; \overline{f_R ; \in'^{\mathsf{T}}}} ; a'$ Prop. (‡), since f_R is a mapping

$= a ; \overline{\in ; \overline{\in^{\mathsf{T}} ; R}} ; a'$ Lemma 5.1.6

$= a ; \overline{a ; \overline{R}} ; a'$ Lemma 4.2.2

$= a ; \overline{\overline{R}} ; a'$ (*)

$= a ; R ; a' = R$ □

It should be pointed out that the operation $R \mapsto f_R$ is different from the frequently studied power transpose, since the latter lifts a relation $R \subseteq A \times B$ to a relation $R \subseteq A \times \mathcal{P}(B)$ what could be formulated as $\mathtt{syq}(R^{\mathsf{T}}, \in)$. It is also different from the construct of an existential image proposed by Oege de Moor and Richard Bird, since this is defined as lifting a relation $R \subseteq A \times B$ to a relation $\exists R \subseteq \mathcal{P}(A) \times \mathcal{P}(B)$ satisfying $(\exists R)(x) := \{b \mid \exists a \in x : (a,b) \in R\}$ which is not monotonic but also maps multiplicatively.

5.2 Multiplicative Embedding

The multiplicative structure stays the same when embedding.

5.2.1 Proposition. i) The embedding $R \mapsto f_R$ is multiplicative.
ii) σ is multiplicative, i.e. $\sigma(R) ; \sigma(S) = \sigma(R ; S)$.
iii) π is multiplicative when restricted to images of σ, i.e.

$$\pi(\sigma(R)) ; \pi(\sigma(S)) = \pi(\sigma(R) ; \sigma(S)).$$

iv) π is multiplicative when restricted to images of f_R, i.e.

$$\pi(f_R) ; \pi(f_S) = \pi(f_{R;S}).$$

v) $f_a = \mathbb{I}$.

Proof: i) $f_R ; f_S = f_R ; \mathtt{syq}(S^{\mathsf{T}} ; \in', \in'')$ definition of f_S

$= \mathtt{syq}(S^{\mathsf{T}} ; \in' ; f_R^{\mathsf{T}}, \in'')$ Prop. 2.2.1.vi

$= \mathtt{syq}(S^{\mathsf{T}} ; R^{\mathsf{T}} ; \in, \in'')$ Prop. 5.1.6

$= \mathtt{syq}((R ; S)^{\mathsf{T}} ; \in, \in'')$

$= f_{R;S}$

ii) Using (i) and Prop. 5.1.7.i, we obtain one direction

$$\sigma(R) ; \sigma(S) = f_R ; E' ; f_S ; E'' \supseteq f_R ; f_S ; E'' = f_{R;S} ; E'' = \sigma(R ; S).$$

The other follows from $\overline{\overline{\in'}^{\mathsf{T}} ; R^{\mathsf{T}}} ; R \subseteq \in'^{\mathsf{T}}$, which holds due to Schröder, applying isotony of composition, $R = a ; R ; a'$, and Lemma 4.2.2 to obtain

$$\overline{\overline{\in'}^{\mathsf{T}} ; R^{\mathsf{T}} ; \in} ; \in^{\mathsf{T}} ; R ; S ; \overline{\in''} \subseteq \in'^{\mathsf{T}} ; S ; \overline{\in''}.$$

Using Schröder again, we have $\sigma(R){;}\sigma(S) \subseteq \sigma(R{;}S)$.
iii) By a simple argument using (ii), Prop. 5.1.3, and Prop. 5.1.7.ii, we see that π is multiplicative on images of σ:

$$\pi(\sigma(R){;}\sigma(S)) = \pi(\sigma(R{;}S)) = R{;}S = \pi(\sigma(R)){;}\pi(\sigma(S)).$$

iv) $\pi(f_R){;}\pi(f_S) = R{;}S$ due to Prop. 5.1.7.ii
$= \pi(f_{R;S})$ again due to Prop. 5.1.7.ii
$= \pi(f_R{;}f_S)$ (i)

v) $f_a = \mathtt{syq}(a^\mathsf{T}{;}\in, \in) = \mathtt{syq}(a{;}\in, \in) = \mathtt{syq}(\in, \in) = \mathbb{I}$ □

One should, however, observe that transposition does not commute with the embedding, i.e. that in general

$$(\sigma(R))^\mathsf{T} \neq \sigma'(R^\mathsf{T}), \qquad \text{where} \quad \sigma'(R^\mathsf{T}) = \overline{{\in'}^\mathsf{T}{;}R^\mathsf{T}{;}\overline{\in}}.$$

5.2.2 Proposition (*Function ordering*)**.** $f \sqsubseteq f' \iff f \subseteq f'{;}{E'}^\mathsf{T}$

Proof:

$$\begin{aligned}
f \subseteq f'{;}{E'}^\mathsf{T} &\iff {f'}^\mathsf{T}{;}f \subseteq {E'}^\mathsf{T} && \text{rolling mappings}\\
&\iff {f'}^\mathsf{T} \subseteq {E'}^\mathsf{T}{;}f^\mathsf{T} && \text{rolling mappings}\\
&\iff {f'}^\mathsf{T} \subseteq \overline{\overline{\in'}^\mathsf{T}{;}\in'}{;}f^\mathsf{T} && \text{Lemma 4.3.2}\\
&\iff f' \subseteq f{;}\overline{{\in'}^\mathsf{T}{;}\overline{\in'}} && \text{transposition}\\
&\iff f' \subseteq \overline{f{;}{\in'}^\mathsf{T}{;}\overline{\in'}} && f \text{ is a mapping}\\
&\iff f' \subseteq \overline{\in^\mathsf{T}{;}\pi(f){;}\overline{\in'}} && \text{using Prop. 5.1.6}\\
&\iff f' \subseteq \sigma(\pi(f)) && \text{definition of } \sigma \text{ and } \pi(f) = a{;}\pi(f){;}a'\\
&\iff \pi(f) \subseteq \pi(f') && \text{Galois property of } \sigma, \pi\\
&\iff f \sqsubseteq f' && \text{definition of } \sqsubseteq
\end{aligned}$$ □

Now we formulate our main result, the proof of which follows immediately as everything has been traced back to the embedded relation algebra.

5.2.3 Theorem. Let a heterogeneous relation algebra $\mathcal{R}$ be given. Assume that for every object $A \in \mathcal{OBJ}_\mathcal{R}$ in the underlying category there is — in addition to the identity $\mathbb{I}_A$ — also given some relation E_A which is an atomic complete boolean lattice ordering. In every morphism set $\mathcal{MOR}_{AB}$, we consider the subset of mappings lattice-continuous with respect to the orderings E_A, E_B. On the subsets $\mathcal{F}_{AB} \subseteq \mathcal{MOR}_{AB}$ defined in this way, we introduce the following operations. To avoid confusion, they are denoted differently but analogously.

0-ary operations or constants

$$⫫ := f_{a;⫫;a'} \qquad \mathbb{I} = f_a, \qquad ⫪ := f_{a;⫪;a'}$$

1-ary operations:

$$f^{\smile} = f_{a;\overline{\pi(f)};a'} \qquad f^{\dagger} = f_{\pi(f)^{\mathsf{T}}}$$

2-ary operations

$$f \sqcup f' := f_{\pi(f)\cup\pi(f')} \qquad f \sqcap f' := f_{\pi(f)\cap\pi(f')}$$

$$f ; f' = f_{\pi(f);\pi(f')}$$

$$f \sqsubseteq f' \;:\Longleftrightarrow\; \pi(f) \;\subseteq\; \pi(f')$$

The definitions above result in a heterogeneous relation algebra $\mathcal{F}$. □

While the relation algebra $\mathcal{R}$ initially given is a relation algebra of its own right, the relation algebra $\mathcal{F}$ is derived from $\mathcal{R}$ in connection with the family $(E_A)_{A\in\mathcal{OBJ}_R}$. As far as the operations in $\mathcal{R}$ are concerned, this is more or less immaterial. As far as one is interested in switching between strict and non-strict behaviour, it becomes important as it is now possible to define strictness which could not be done out of lattice-theoretic considerations alone.

6 Outlook

We have shown how a relation algebra may be embedded in a bigger one such that a relation $R \in X \times Y$ may afterwards also be conceived as a continuous mapping $f_R \in \mathcal{P}(X) \times \mathcal{P}(Y)$. As there are only rather few such continuous mappings, lots of other relations in $\mathcal{P}(X)\times\mathcal{P}(Y)$ will exist. Thus an embedding has taken place. In some sense, we have provided a model for the universal characterizations to be defined in Part II of this article.

Part II of this article will contain how these continuous functions describing transfer of partialities may be used as matrix coefficients. Putting it in another way, we will define partialities as a generalization of relations. The basic difference will be that in addition to the identities there will always also exist an ordering. For relations this ordering can well be neglected as it is the trivial ordering equal to the identity. For partialities it will be different from the identity in the way indicated by the examples before: For every element there will exist a boolean lattice describing possible incrementation.

Once this is defined, we have the possibility to decide whether an object is fully available, and a strict process may proceed.

Then a universal characterization will be given for the parallel or non-strict product. It will use the external arbiter `strict` to decide full availability. As we have seen here, information "disperses". When preparing common availability of (flight,hotel) for a holiday arrangement, several processes will try to find some. In case a transaction is made, there are investigated but not booked flights or hotels. The operation `strict` may be seen as cleaning this up for descriptional purposes.

This universal equational characterization will then be used, and tested, with regard to partial correctness, total correctness, and weakest preconditions.

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