Relations and Partialities

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Abstract

We define the concept of partiality to cope with partial availability of arguments and results of program steps. To this end, relation algebras are investigated for which, in addition to the identity $\mathbb{I}$, a specific type of an ordering $E \supseteq \mathbb{I}$ is given in order to model increasing degrees of availability. It turns out that functions regulating non-strict transfer of partialities in processes are *lattice-continuous* with respect to such orderings.

One may also consider partialities with regard to their "atomic" constituents. We exhibit how *relations* between the atomic constituents before and after a process step are represented by such continuous partiality transfer functions. Our result is that they are images of a multiplicative embedding into a larger relation algebra.

1 Introduction

Parallel processes confront us with both, strict and non-strict situations. As long as no cooperation between processes is supposed to take place, one may consider them separately and need not ask for progress of the other processes. If, however, a composite result is to be delivered, it is important in which way it is built.

When dealing with *possibly partial* availability of information on elements of a set, it is a well-known technique to put an additional bottom element $\perp$ below all the others to obtain a flat ordering. More difficult situations are studied with cpo's and there exists a highly developed theory of orderings on semantic domains. A treatment of possibly partial availability of information may also be seen in descriptions of eager/data-driven evaluation as opposed to lazy/demand-driven evaluation.

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We feel that partiality is not yet handled satisfactorily, and that a unified theoretical basis is still missing. In the approach presented here, we try to lay a basis of algebraic rules around partiality.

From the beginning we assume given with every element the Boolean lattice describing the degrees to which this element may be partially available. Availability of an element is thus no longer conceived as an atomic qualification *available/non-available* or $\bullet, \bot$ symbolically. It may now be qualified in greater detail. Consider, e.g., the pair of objects (x, y) and assume x and y to be in some sense atomic or non-composite. First, none of the components may be available, denoted as $(\bot, \bot)$. Availability on the pair (x, y) could, however, increase to $(\bot, \bullet)$ indicating that the second component is already available but the first is not, or $(\bullet, \bullet)$ indicating that it is fully available. There is a natural way of dealing with such a situation, namely speaking of the *partiality* or the *degree of being partially available* and introducing an ordering $\sqsubseteq$ to express an increase in partiality, such that for instance $(\bot, \bot) \sqsubseteq (\bullet, \bot) \sqsubseteq (\bullet, \bullet)$.

Studied this way, every element is endowed its own partiality lattice. These partiality lattices may indeed vary over the elements of a domain: Consider the direct sum $X + (X \times X)$ of a set X and the set of pairs formed over X. Assume the items of X to be atomic. Then items of X have the partiality lattice $\mathbb{P}$ and the pairs have $\mathbb{P}^2$, where $\mathbb{P} = \{\bot, \bullet\}$.

An investigation on relations in the presence of an ordering on the domain and on the range side will therefore be given. It takes into account the possibly increasing degree of availability concerning the argument on the domain side and hence on the results of a relation on the range side. Increased information on an argument should result in at least as much information on the result as before.

2 Preliminaries

The reader is assumed to be familiar with relation algebra. We start with orderings as well as isotone and continuous mappings.

2.1 Ordering and Continuity

An element E of a relation algebra satisfying $\mathbb{I} \subseteq E$ *(reflexivity)*, $E^2 \subseteq E$ *(transitivity)* and $E \cap E^\intercal \subseteq \mathbb{I}$ *(antisymmetry)* is called an **ordering relation**. The adequate structure-preserving transition is the isotone mapping. Given two ordering relations E and E' we call a mapping φ **isotone** if $E ; \varphi \subseteq \varphi ; E'$.

An ordering relation gives rise to looking for the existence of upper resp. lower bounds (majorants resp. minorants), and least upper resp. greatest lower bounds. The following definitions may be found in greater detail in [SS85,SS89,SS93].

2.1.1 Definition. Given an ordering relation E, the **upper** and **lower bounds** of a relation X for which $E;X$ exists can be formed. These as well as the **least upper** and **greatest lower bounds** may be defined as

$$\mathtt{ubd}_E(X) := \overline{\overline{E}^\mathsf{T};X} \qquad \mathtt{lbd}_E(X) := \overline{\overline{E};X}$$
$$\mathtt{lub}_E(X) := \mathtt{ubd}_E(X) \cap \mathtt{lbd}_E(\mathtt{ubd}_E(X)) := \overline{\overline{E}^\mathsf{T};X} \cap \overline{\overline{E};\overline{\overline{E}^\mathsf{T};X}}$$
$$\mathtt{glb}_E(X) := \mathtt{lbd}_E(X) \cap \mathtt{ubd}_E(\mathtt{lbd}_E(X)) := \overline{\overline{E};X} \cap \overline{\overline{E}^\mathsf{T};\overline{\overline{E};X}}$$ □

These functionals are always defined; the results may, however, be null relations. It is an easy task to prove that `lub`, `glb` are always injective, resembling that such bounds are uniquely defined if they exist, see [SS85] and Ch. 3 of [SS89,SS93]. As an example we compute the least upper bound of the relation E itself, employing the well-known facts $\overline{\overline{E}^\mathsf{T};E} = E^\mathsf{T}$ and $\overline{\overline{E};E^\mathsf{T}} = E$ as well as antisymmetry of E:

$$\mathtt{lub}_E(E) = \overline{\overline{E}^\mathsf{T};E} \cap \overline{\overline{E};\overline{\overline{E}^\mathsf{T};E}} = E^\mathsf{T} \cap \overline{\overline{E};E^\mathsf{T}} = E^\mathsf{T} \cap E = \mathbb{I}.$$

The adequate structure-preserving mappings for lattice orderings are continuous mappings, as defined below. They are sometimes also called additive.

2.1.2 Definition. A mapping f from an ordering E to an ordering E' is called **(upwards) continuous** wrt. E, E' if for every relation X with existing product $E;X$ we have that application of f commutes with forming the `lub`,

$$f^\mathsf{T};\mathtt{lub}_E(X) = \mathtt{lub}_{E'}(f^\mathsf{T};X).$$ □

2.2 Lattices

Be aware, that this is a modified definition of continuity. Being continuous here requires, that least elements be mapped onto least elements. In a cpo, continuity is defined by the same formula, but restricted to *directed* sets X. A directed set is by definition nonempty.

Our aim is now to find out how some element E of a relation algebra may be qualified to constitute the ordering relation of a complete lattice.

2.2.1 Definition. The element E of a relation algebra is said to be a **complete lattice ordering relation**, if it is an ordering relation such that for all relations X with existing product $E;X$ the construct $\mathtt{lub}_E(X)$ is surjective. □

By mathematical folklore, with all `lub`'s "existing" also all the `glb`'s will "exist". After transfer into our relational setting, this means that with all `lub`'s surjective also all the `glb`'s will be surjective; see [SS89,SS93] 3.3.11.

3 Boolean Lattice Orderings

Studies in connection with Goguen categories [Win02a,Win02c,Win02b,Win03] have shown that strictness cannot be formulated *inside* a given relation algebra. Our approach is, therefore, to embed the algebra somehow in a larger relation algebra. Transactions that strictly require certain availabilities will then be handled in the larger algebra.

The switching between the embedded algebra and the embedding one needs algebraic properties of Boolean lattice orderings formulated in componentfree form as these properties are afterwards used in a sensitive algebraic mechanism.

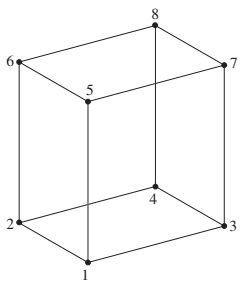
3.1 Selfsimilarity of Boolean Lattices

We start by looking at pairs of elements with a common upper bound as given by the relation $E;E^\intercal$. Correspondingly, $E^\intercal;E$ describes the relation between two elements of having a common lower bound. For an element and its negative in a Boolean lattice it is characteristic that they do not have a common upper bound except for the greatest element and do not have a common lower bound except for the least element of the lattice. We define the relations

$$D := E \cap \overline{E;\mathbb{T}} \qquad F := E \cap \mathbb{T};\overline{E}.$$

Here, the vector $\overline{E;\mathbb{T}} = \mathtt{lbd}_E(\mathbb{T})$ characterizes the least element, as only this is less or equal than all the others. Analogously, the vector $\overline{\mathbb{T};\overline{E}}^\intercal = \mathtt{ubd}_E(\mathbb{T})$ characterizes the greatest element.

For an example consider as Boolean lattice the 3-dimensional cube, the ordering relation of which is given as the fractal style matrix E. Also D and F are shown.



$$E = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix} \end{array}$$

$$D = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix} \qquad F = \begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

Using F, it is easy to present the relation

$$\overline{F;F^\intercal} = \overline{(E \cap \mathbb{T};\overline{E});(E \cap \mathbb{T};\overline{E})^\intercal} = \overline{E;(E \cap \mathbb{T};\overline{E})^\intercal} = \overline{(E \cap \mathbb{T};\overline{E});E^\intercal}$$

of "not admitting common upper bounds different from the greatest element". Analogously with D, one finds the relation

$\overline{D^\intercal ; D} = \overline{(E \cap \overline{E} ; \mathbb{T})^\intercal ; (E \cap \overline{E} ; \mathbb{T})} = \overline{E^\intercal ; (E \cap \overline{E} ; \mathbb{T})} = \overline{(E \cap \overline{E} ; \mathbb{T})^\intercal ; E}$

of "not admitting common lower bounds different from the least element". Again we observe the fractal construction.

$$\overline{D^\intercal ; D} = \begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \qquad \overline{F ; F^\intercal} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \end{pmatrix}$$

We are not allowed, however, to make use of it in a naive form. Naive would mean to simply mirror the matrix $\overline{D^\intercal ; D}$ along the diagonal upper/right to lower/left in order to obtain $\overline{F ; F^\intercal}$, e.g. Another naive operation would be to rotate E.

An obvious idea is to look for the counter-diagonal, which resembles negation and may be determined as $N := \overline{D^\intercal ; D} \cap \overline{F ; F^\intercal}$. The counter diagonal normally has not an easy algebraic characterisation compared with the diagonal. Multiplying with N from the left turns upside down for a matrix. Multiplying with N from the right flips horizontally.

3.1.1 Definition. A complete lattice ordering relation E will be called a **complete Boolean lattice ordering**, if the derived constructs

$D := E \cap \overline{E} ; \mathbb{T}, \qquad F := E \cap \mathbb{T} ; \overline{E}, \qquad N := \overline{D^\intercal ; D} \cap \overline{F ; F^\intercal}$

satisfy the following conditions given in equational style

i) $\mathtt{ubd}_E(\overline{D^\intercal ; D}) = \overline{F ; F^\intercal}$
ii) $\mathtt{lbd}_E(\overline{F ; F^\intercal}) = \overline{D^\intercal ; D}$
iii) N is total, , $N ; \mathbb{T} = \mathbb{T}$ □

The interpretation of (i) is that for every element x the following holds: The cone of upper bounds of the set of "elements not admitting common lower bounds different from the least element" with x is equal to the set of "elements not admitting common upper bounds different from the greatest element" with x.

As we easily see that N is formed using a least upper bound, it is injective:

$$\begin{aligned} \mathtt{lub}_E(\overline{D^\intercal ; D}) &= \mathtt{ubd}_E(\overline{D^\intercal ; D}) \cap \mathtt{lbd}_E(\mathtt{ubd}_E(\overline{D^\intercal ; D})) \\ &= \overline{F ; F^\intercal} \cap \mathtt{lbd}_E(\overline{F ; F^\intercal}) = \overline{F ; F^\intercal} \cap \overline{D^\intercal ; D} = N \end{aligned}$$

Now, we can state that two elements are in relation N if at the same time they have no common upper bound except the greatest element and no common lower bound except for the least element. Univalence and totality follow from surjectivity and injectivity. Using symmetry of N, we get the involution property from surjectivity and univalence: $N ; N = N^\intercal ; N = \mathbb{I}$.

We now investigate the following constructs a and $\in$ together with some of their surprising properties.

3.1.2 Definition. Given a Boolean lattice ordering E together with the corresponding N, we define

$$a := (\overline{\overline{E};\overline{E}} \cap \overline{E};\top \cap \top;\overline{E});N \qquad \in := a;E. \qquad \square$$

These definitions need some visualization. The relation a will turn out to be the partial identity characterizing the atoms among the elements of the Boolean lattice. In the matrix, this is certainly dependent in which order the elements are arranged. An indication that the order chosen may be a favourable one is given by $\in$, which corresponds to the relation ε when omitting all rows full of $\mathbf{0}$'s. We have chosen the Euro-symbol $\in$ for two reasons. It indicates nicely how several rows of E are taken according to a. In addition, it is not too different from the usual direct power symbol ε.

$$a = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \qquad \in = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

As already announced by the example, a will turn out to be a partial identity characterizing the atoms of the ordering, $a \subseteq \mathbb{I}$. There has been proved a bulk of algebraic properties around this, which cannot be presented here.

3.2 Power-Related Properties

From the following formulae, (ii) turns out to be a powerful one. It regulates two forms of negation, one on the normal relational level, the other with N.

3.2.1 Lemma. In the given setting of a Boolean lattice ordering, we have

i) $a;D^\mathsf{T} = a, \qquad a;E^\mathsf{T};a = a, \qquad F^\mathsf{T};a = a$

ii) $a;\overline{\in} = a;\overline{E} = \in;N$

iii) $a;\in = \in$

iv) $a;\in;\overline{\in^\mathsf{T};a;X} = a;\overline{X}$

v) $\in;\overline{\in^\mathsf{T};X} = a;\overline{X}$

vi) $a;\overline{\in};\overline{\in^\mathsf{T};a;X} = a;\overline{X}$ $\qquad \square$

The next observation concerns least upper bounds and symmetric quotients.

3.2.2 Proposition (*Connecting* `syq` *and* `lub`)**.** In an atomic complete Boolean lattice we have for all relations X with the product $\in ; X$ defined that

$\mathtt{lub}_E(X) = \mathtt{syq}(\in, \in ; X)$

$\in ; \mathtt{lub}_E(X) = \in ; \mathtt{syq}(\in, \in ; X) = \in ; X$

$\in ; \mathtt{syq}(\in, a ; X) = a ; X$ for all X. □

3.3 Atomicity of a Lattice Ordering

Finite Boolean lattices as we consider here are necessarily atomic. Nonetheless, we exhibit how existence of atoms may be expressed algebraically.

3.3.1 Definition. A complete Boolean lattice ordering relation will be called **atomic** provided it satisfies the condition $\overline{E} ; \mathbb{T} = E^\mathsf{T} ; a ; \mathbb{T}$. □

The additional property may be interpreted in the following way: Precisely all but the least element of the Boolean lattice ordering E offer the opportunity to reach an atom when going back against the ordering. The corresponding properties for anti-atoms reads as follows: $\mathbb{T} ; \overline{E} = \mathbb{T} ; a ; N ; E^\mathsf{T}$.

The partial identity a together with the variant $\in$ of the well-known direct power relation ε and atomicity just defined generate important new formulae exhibiting similarities between $\in$, E, and ε.

3.3.2 Lemma. In an atomic Boolean lattice ordering $E = \overline{\in^\mathsf{T} ; \overline{\in}} = \overline{E^\mathsf{T} ; \overline{E}}$. □

Useful results are now available:

$\mathtt{ubd}_E(\in) = \mathtt{ubd}_E(a ; E) = \overline{\overline{E}^\mathsf{T} ; a ; E} = \overline{\overline{E^\mathsf{T} ; a} ; a ; E} = \overline{\overline{\in^\mathsf{T} ; \in}} = E^\mathsf{T}$,

$\mathtt{lub}_E(\in) = \mathtt{ubd}_E(\in) \cap \mathtt{lbd}_E(\mathtt{ubd}_E(\in)) = E^\mathsf{T} \cap \mathtt{lbd}_E(E^\mathsf{T}) = E^\mathsf{T} \cap E = \mathbb{I}$

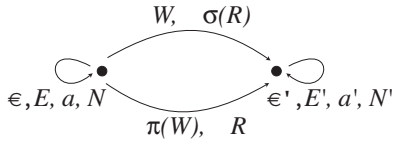
$\mathtt{syq}(\in, \in) = \mathtt{syq}(\in, \in ; \in) = \mathbb{I}$

4 Embedding Relation Algebras

As announced, a heterogeneous relation algebra will now be embedded into another one. The first step would be to embed relations between X and Y to relations on the singleton subsets $R = a ; R ; a' \subseteq \mathcal{P}(X) \times \mathcal{P}(Y)$. We will not denote this and start from the latter. When strictness is demanded, the surrounding relation algebra offers the opportunity to formulate an *external* arbiter to check for availability.

4.1 Embedding as a Galois Connection

As the mechanism of Galois connections is well-known, we need not give an introduction into this topic. Rather we formulate the effects directly addressing our particular case. Consider any two atomic complete Boolean lattice orderings E, E' together with their corresponding relations a, $\in, N$ and a', $\in', N'$. Then we have the following situation as our basic setting.



As a **general assumption** we assume two mappings relating relations with $R = a \fatsemi R \fatsemi a'$, i.e., subrelations of the "rectangle" $a \fatsemi \top \fatsemi a'$, to arbitrary relations W.

4.1.1 Theorem. The two constructs

$\sigma(R) := \overline{\in^{\intercal} \fatsemi a \fatsemi R \fatsemi a' \fatsemi \overline{\in'}}$ and $\pi(W) := a \fatsemi \overline{\in \fatsemi W \fatsemi \overline{\in'^{\intercal}}} \fatsemi a'$

form a Galois correspondence between the set of relations R with $R = a \fatsemi R \fatsemi a'$ and arbitrary relations W, i.e.,

$R \subseteq \pi(W) \quad \Longleftrightarrow \quad W \subseteq \sigma(R)$ □

As these two mappings form a Galois correspondence, the well-known consequences follow immediately without any additional assumptions on σ, π, not least that the composed mappings $\rho(R) := \pi(\sigma(R))$ and $\varphi(W) := \sigma(\pi(W))$ are expanding.

4.1.2 Proposition (*Injectivity of* σ)**.** In the special case of σ, π of our general assumption, i.e., considering the subset relations satisfying $R = a \fatsemi R \fatsemi a'$, the fixedpoint set $\mathcal{F}_\rho$ is always the full set of these relations $\subseteq a \fatsemi \top \fatsemi a'$, or equivalently

$R = \pi(\sigma(R))$ for all $R = a \fatsemi R \fatsemi a' \subseteq a \fatsemi \top \fatsemi a'$. □

Therefore, σ, π form what is usually called an adjoint pair. $\mathcal{F}_\varphi$ will not be the full set of all W. It is interesting, which relations may occur as images of σ. First we investigate their greatest lower bound.

4.1.3 Proposition. $\mathtt{syq}(R^{\intercal} \fatsemi \in, \in') = (\mathtt{glb}_{E'}(\sigma(R)^{\intercal}))^{\intercal}$ and

$f_R := \mathtt{syq}(R^{\intercal} \fatsemi \in, \in')$ is a continuous mapping. □

The relations f_R and R are capable of simulating one another via $\in$ and $\in'$:

$\in' \fatsemi f_R^{\intercal} = R^{\intercal} \fatsemi \in$.

The following proposition shows that one may call f_R "lower borderline" of $\sigma(R)$.

4.1.4 Proposition. The relation $f_R := \mathtt{syq}(R^\intercal \mathbin{;} \in, \in')$ satisfies the following properties for given R with $R = a \mathbin{;} R \mathbin{;} a'$:

i) $f_R \mathbin{;} E' = \sigma(R)$.
ii) $R = \pi(f_R)$. □

It should be pointed out that the operation $R \mapsto f_R$ is different from the frequently studied power transpose, since the latter lifts a relation $R \subseteq A \times B$ to a relation $R \subseteq A \times \mathcal{P}(B)$ what could be formulated as $\mathtt{syq}(R^\intercal, \in)$. It is also different from the construct of an existential image proposed by Oege de Moor and Richard Bird, since this is defined as lifting a relation $R \subseteq A \times B$ to a relation $\exists R \subseteq \mathcal{P}(A) \times \mathcal{P}(B)$ satisfying $(\exists R)(x) := \{b \mid \exists a \in x : (a, b) \in R\}$ which is not monotonic but also maps multiplicatively.

4.2 Multiplicative Embedding

The multiplicative structure stays the same when embedding.

4.2.1 Proposition. i) The embedding $R \mapsto f_R$ is multiplicative.
ii) σ is multiplicative, i.e., $\sigma(R) \mathbin{;} \sigma(S) = \sigma(R \mathbin{;} S)$.
iii) π is multiplicative when restricted to images of σ, i.e.,

$$\pi(\sigma(R)) \mathbin{;} \pi(\sigma(S)) = \pi(\sigma(R) \mathbin{;} \sigma(S)).$$

iv) π is multiplicative when restricted to images of f_R, i.e.,

$$\pi(f_R) \mathbin{;} \pi(f_S) = \pi(f_{R \mathbin{;} S}).$$

v) $f_a = \mathbb{I}$. □

One should, however, observe that transposition does not commute with the embedding, i.e., that in general

$$(\sigma(R))^\intercal \neq \sigma'(R^\intercal), \qquad \text{where} \quad \sigma'(R^\intercal) = \overline{\in'^\intercal \mathbin{;} R^\intercal \mathbin{;} \overline{\in}}.$$

Now we formulate our main result.

4.2.2 Theorem. Let a heterogeneous relation algebra $\mathcal{R}$ be given. Assume that for every object $A \in \mathcal{OBJ}_\mathcal{R}$ in the underlying category there is — in addition to the identity $\mathbb{I}_A$ — also given some relation E_A which is an atomic complete Boolean lattice ordering. In every morphism set $\mathcal{MOR}_{AB}$, we consider the subset of mappings lattice-continuous with respect to the orderings E_A, E_B. On the subsets $\mathcal{F}_{AB} \subseteq \mathcal{MOR}_{AB}$ defined in this way, we introduce the following operations. To avoid confusion, they are denoted differently but analogously.

0-ary operations or constants

$$\mathbb{L} := f_{a \mathbin{;} \mathbb{L} \mathbin{;} a'} \qquad \mathbb{I} = f_a, \qquad \mathbb{T} := f_{a \mathbin{;} \mathbb{T} \mathbin{;} a'}$$

1-ary operations:

$$f^\smile = f_{a \mathbin{;} \overline{\pi(f)} \mathbin{;} a'} \qquad f^\dagger = f_{\pi(f)^\intercal}$$

2-ary operations

$$f \sqcup f' := f_{\pi(f) \cup \pi(f')} \qquad f \sqcap f' := f_{\pi(f) \cap \pi(f')}$$

$$f ; f' = f_{\pi(f) ; \pi(f')}$$
$$f \sqsubseteq f' \ :\Longleftrightarrow \ \pi(f) \subseteq \pi(f')$$

The definitions above result in a heterogeneous relation algebra $\mathcal{F}$. □

While the relation algebra $\mathcal{R}$ initially given is a relation algebra of its own right, the relation algebra $\mathcal{F}$ is derived from $\mathcal{R}$ in connection with the family $(E_A)_{A \in \mathcal{OBJ}_R}$. As far as the operations in $\mathcal{R}$ are concerned, this is more or less immaterial. As far as one is interested in switching between strict and non-strict behaviour, it becomes important as it is now possible to define strictness which could not be done out of lattice-theoretic considerations alone.

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