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Relational Topology

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Abstract

In this text several concepts of topology, such as neighborhoods, transition to the open kernel etc., are integrated under one common relational roof. Transitions between them are made possible, not least via TITUREL programs.

Furthermore, a study of several approaches to spatial reasoning on discreteness, proximity, nearness, apartness is presented, which are frequently performed by logicians.

Also some ideas about how to work relationally on simplicial complexes are demonstrated at least in examples.

Keywords relational mathematics, relation algebra, Kronecker-, fork-, and join-operator, direct product, existential and inverse image, neighborhood, open set, continuity, nearness, apartness, proximity, simplicial complex

Contents

1 Introduction

There exist lots of logical concepts around topology: open sets, neighborhoods, proximity, nearness, betweenness, apartness, different concepts of contact and the like. Of course, they are heavily interrelated which one cannot immediately recognize, because they are usually discussed in quite different settings. We are going to identify the core concepts of those ideas and to show how they may be mutually deduced from one another.

It is a requirement for such research to be acquainted with the relation-algebraic methods as presented in [Sch11] and [SW14] as well as with the treatment of the Kronecker product, the strict fork and join operations.

This work is organized as follows: In Chapt. 2, we collect what has to be mentioned from known relational methods to make the article self-contained. Also several new findings of this kind are elaborated.

Known concepts of topology and continuity are recalled in Chapt. 3. They are then lifted to a point-free relational form, thus opening them to being handled relationally, using the existential image and the inverse image e.g., as it has partly been tried already in [Sch14].

Chapt. 4 mentions the less known Aumann contact relation — that later resurrected as betweenness — and its connections with topology.

Several concepts of the border zone between topology and logics are recalled in Chapt. 5, in the highly diverse forms in which they frequently appear. They are then brought to relational style and many of their interrelationships are exhibited and proved formally.

We study concepts of homology such as orientation, boundary operators, etc., in Chapt. 6. Again, these are brought to a point-free relational form and then applied to simplicial complexes.

This work is completely based on relational methods. Its examples are all discrete. This might persuade persons to believe that it is restricted to finiteness. This is not so: The relational formulae are just "abbreviations" of the predicate logic formulae with which topology is traditionally defined.

A remark by Dieudonné from [Die74] seems interesting in this context. He reconsiders what René Thom said concerning superiority of "continuous" considerations as compared with "discrete" ones. He rightly criticizes Kronecker for his one-sided view on mathematics as fully based on the concept of a number. ... But then Thom himself, says that the *continuum needs to be discretized*, and that since Poincaré the only way to understand topology somehow is the ever increasing application of algebra leading to topological invariants as objects of study.

We show many computer-generated examples of finite discrete topologies, generated with the language TituRel `http://mucob.dyndns.org:30531/~gs/TituRel/indexTituRel.html` to interpret relational terms and formulae. The well-known RelView system would considerably scale up the size of problems that may be tackled; see `http://www.informatik.uni-kiel.de/~progsys/relview/`.

Quite often, one will detect similarities between approaches that come from absolutely different sides. Sometimes, there is an attitude that one should study all these minor differences in ever new papers. Our approach is definitely different: Can we — led by the ideas of these differing approaches — find some relational 'ginder' carrying all the intertwingled theories that shows us a basis with several sound anchoring supports that are relationally related in a simple way and that may provide a firm starting point for research. Such a ginder should serve as a reference for further study; it should also be the measure against which any strengthening or weakening of the axioms should be discussed.

There was an additional motivation for this type of research: We further develop the algebra of the strict Kronecker, fork-, and join-operator from a rigorous relational axiomatization. It seems that this has so far never been systematically collected although it is important with regard to the difficult model questions around.

In total, a main aim was also to further grind, sharpen, and edge our relational tools. Thus, it is a further extension of [Sch11, SW14].

2 Prerequisites

Relational Methods still need a detailed introduction. We recall some basics, not least from [SS89, SS93, Sch11, SW14]. Some results, however, are new.

2.1 Preliminaries

The prerequisites presented routinely for relational work are by now fairly well-known: Boolean operations and predicates $\cup, \cap, \overline{}, \subseteq$, together with the least $\mathbb{⊥}$ and the greatest element $\mathbb{T}$; then the monoid operation of relational composition $;$ together with the identities $\mathbb{I}$, and finally transposition or conversion $.^{\mathsf{T}}$. The most immediate interpretation is that of Boolean matrices, i.e., **0**, **1**-matrices; therefore we explain effects often via rows, columns, and diagonals.

When a non-commutative composition is available, one usually looks for the left and the right residual, defined via

$$A;B \subseteq C \iff A \subseteq \overline{\overline{C};B^{\mathsf{T}}} =: C/B \qquad \text{and} \qquad A;B \subseteq C \iff B \subseteq \overline{A^{\mathsf{T}};\overline{C}} =: A\backslash C.$$

The relation $A\backslash C$ describes which columns of A are contained in which columns of C. Intersecting such residuals in $\mathtt{syq}(R,S) := \overline{R^{\mathsf{T}};\overline{S}} \cap \overline{\overline{R}^{\mathsf{T}};S}$, the *symmetric quotient* $\mathtt{syq}(R,S) : W \longrightarrow Z$ of two relations $R : V \longrightarrow W$ and $S : V \longrightarrow Z$ is defined. Symmetric quotients serve the purpose of 'column comparison'

$$\big[\,\mathtt{syq}(R,S)\big]_{wz} = \forall v \in V : R_{vw} \longleftrightarrow S_{vz}.$$

The symmetric quotient is used to introduce *membership relations* $\varepsilon : V \longrightarrow \mathcal{P}(V)$ between a set V and its powerset $\mathcal{P}(V)$ or $\mathbf{2}^V$. These can be characterized algebraically up to isomorphism demanding $\mathtt{syq}(\varepsilon,\varepsilon) \subseteq \mathbb{I}$ and surjectivity of $\mathtt{syq}(\varepsilon,R)$ for all R. With a membership ε, the powerset ordering is easily described as $\Omega = \overline{\varepsilon^{\mathsf{T}};\overline{\varepsilon}}$. The equivalent version $\Omega = \varepsilon\backslash\varepsilon$ makes indeed clear that columns of ε are investigated as to whether they are contained in columns of ε.

The most well-known properties of a relation Q are being *univalent*, i.e., a (possibly partial) *function*, ($Q^{\mathsf{T}};Q \subseteq \mathbb{I}$), being *injective* (when Q^{T} is univalent), being *total* ($\mathbb{I} \subseteq Q;Q^{\mathsf{T}}$ or equivalently $Q;\mathbb{T} = \mathbb{T}$), being *surjective* (when Q^{T} is total), and finally being a *mapping* (when univalent as well as total). The latter word is reserved for a total function.

There are three frequently applied rules that we recall here for convenience: When f is a mapping, always

$$A;f \subseteq B \qquad \Longleftrightarrow \qquad A \subseteq B;f^{\mathsf{T}},$$

a transition we refer to as *shunting*. When we call a transition *destroy and append*, we mean

$$(A;Q^{\mathsf{T}} \cap B);Q = A \cap B;Q$$

which holds for every univalent Q. Yet another rule is *masking* with a row-constant relation

$$(A \cap B;\mathbb{T});C = A;C \cap B;\mathbb{T},$$

which says that one may annihilate rows according to $B;\mathbb{T}$ before or after composition with C.

We present a novel and useful rule for composition of a univalent relation Q with a symmetric quotient which for $Q;\mathbb{T} = \mathbb{T}$ is an obvious generalization of Prop. 8.16.ii of [Sch11]:

2.1 Proposition. A univalent relation Q satisfies

$$Q;\mathtt{syq}(A,B) = Q;\mathbb{T} \cap \mathtt{syq}(A;Q^{\mathsf{T}},B).$$

Proof: $Q;\mathtt{syq}(A,B) = Q;\left[\,\overline{\overline{A}^{\mathsf{T}};B} \cap \overline{A^{\mathsf{T}};\overline{B}}\,\right]$ by definition

$= Q;\overline{\overline{A}^{\mathsf{T}};B} \cap Q;\overline{A^{\mathsf{T}};\overline{B}}$ since Q is univalent

$= \left[Q;\mathbb{T} \cap \overline{Q;\overline{A}^{\mathsf{T}};B}\right] \cap \left[Q;\mathbb{T} \cap \overline{Q;A^{\mathsf{T}};\overline{B}}\,\right]$ according to Prop. 5.6 of [Sch11]

$= Q;\mathbb{T} \cap \overline{Q;\overline{A^{\mathsf{T}}};B} \cap \overline{Q;A^{\mathsf{T}};\overline{B}}$

$= Q;\mathbb{T} \cap \overline{[Q;\mathbb{T} \cap \overline{Q;A^{\mathsf{T}}}];B} \cap \overline{Q;A^{\mathsf{T}};\overline{B}}$ again Prop. 5.6 of [Sch11]

$= Q;\mathbb{T} \cap \overline{Q;\mathbb{T} \cap \overline{Q;A^{\mathsf{T}}};B} \cap \overline{Q;A^{\mathsf{T}};\overline{B}}$ masking; see [SS93, Prop. 2.4.2] or [Sch11, Prop. 8.5]

$= Q;\mathbb{T} \cap \left[\overline{Q;\mathbb{T}} \cup \overline{\overline{Q;A^{\mathsf{T}}};B}\right] \cap \overline{Q;A^{\mathsf{T}};\overline{B}}$ De Morgan rule

$= \left[Q;\mathbb{T} \cap \overline{Q;\mathbb{T}} \cap \overline{Q;A^{\mathsf{T}};\overline{B}}\right] \cup \left[Q;\mathbb{T} \cap \overline{\overline{Q;A^{\mathsf{T}}};B} \cap \overline{Q;A^{\mathsf{T}};\overline{B}}\right]$ distributive

$= \mathbb{\perp\!\!\!\perp} \cup \left[Q;\mathbb{T} \cap \mathtt{syq}(A;Q^{\mathsf{T}},B)\right]$ $\square$

2.2 Power operations recalled

There exists an interesting interrelationship between relations and their counterparts between the corresponding powersets. It offers the possibility to work algebraically at situations where this has so far not been the classical approach.

2.2 Definition. Let any relation $R : X \longrightarrow Y$ be given together with membership relations $\varepsilon : X \longrightarrow \mathbf{2}^X, \varepsilon' : Y \longrightarrow \mathbf{2}^Y$. Then the corresponding **existential image mapping** is defined as $\vartheta_R := \mathtt{syq}(R^{\mathsf{T}};\varepsilon, \varepsilon')$. One may correspondingly study the **inverse image mapping** defined as $\vartheta_{R^{\mathsf{T}}} = \mathtt{syq}(R;\varepsilon', \varepsilon)$. $\square$

We further recall an interesting fact concerning the existential image; see [Sch11]. In total, we have for an existential image the equality

$$\varepsilon^{\mathsf{T}};R = \vartheta_R;\varepsilon'^{\mathsf{T}}.$$

Correspondingly, an appliction of this simulation rule to R^{T} instead of R reads

$$\varepsilon'^{\mathsf{T}};R^{\mathsf{T}} = \vartheta_{R^{\mathsf{T}}};\varepsilon^{\mathsf{T}}, \qquad \text{or else} \qquad R;\varepsilon' = \varepsilon;\vartheta_{R^{\mathsf{T}}}^{\mathsf{T}}.$$

The following rule is not unimportant when later continuity is studied.

2.3 Proposition. In the following, $N_X := \mathtt{syq}(\varepsilon_X, \overline{\varepsilon_X}) : \mathbf{2}^X \longrightarrow \mathbf{2}^X$ is powerset negation.

i) $\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}};\vartheta_{f^{\mathsf{T}}};\vartheta_f = \vartheta^{\mathsf{T}}_{f^{\mathsf{T}}};\vartheta^{\mathsf{T}}_f;\vartheta_f$ for an arbitrary mapping $f : X \longrightarrow Y$.

ii) $\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}};\mathbb{T} \cap \vartheta_f = \vartheta^{\mathsf{T}}_{f^{\mathsf{T}}} \cap \mathbb{T};\vartheta_f$ for an arbitrary mapping $f : X \longrightarrow Y$.

iii) $f : X \longrightarrow Y$ surjective mapping $\implies$ $\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}} \subseteq \vartheta_f$ or $\mathtt{syq}(\varepsilon_X, f;\varepsilon_Y) \subseteq \mathtt{syq}(f^{\mathsf{T}};\varepsilon_X, \varepsilon_Y)$.

iv) $N_X;\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}} = \vartheta^{\mathsf{T}}_{f^{\mathsf{T}}};N_Y$

Proof: of (iv); for the others see [SW14].

$$N_X;\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}} = N_X;\mathtt{syq}(\varepsilon_X, f;\varepsilon_Y) = \mathtt{syq}(\varepsilon_X;N_X, f;\varepsilon_Y) = \mathtt{syq}(\overline{\varepsilon_X}, f;\varepsilon_Y)$$
$$= \mathtt{syq}(\varepsilon_X, \overline{f;\varepsilon_Y}) = \mathtt{syq}(\varepsilon_X, f;\overline{\varepsilon_Y}) = \mathtt{syq}(\varepsilon_X, f;\varepsilon_Y;N_Y) = \mathtt{syq}(\varepsilon_X, f;\varepsilon_Y);N_Y = \vartheta^{\mathsf{T}}_{f^{\mathsf{T}}};N_Y \qquad \square$$

$$f = \begin{array}{r|cccc} & a & b & c & d \\ \hline 1 & 0&1&0&0 \\ 2 & 1&0&0&0 \\ 3 & 0&0&1&0 \\ 4 & 0&0&0&1 \\ 5 & 0&0&1&0 \end{array}$$

$$\vartheta_f = \begin{array}{r|cccccccccccccccc}
 & \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} & \{d\} & \{a,d\} & \{b,d\} & \{a,b,d\} & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\} \\ \hline
\{\} & 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{1\} & 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{2\} & 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{1,2\} & 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{3\} & 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\
\{1,3\} & 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0 \\
\{2,3\} & 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0 \\
\{1,2,3\} & 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\
\{4\} & 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\
\{1,4\} & 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0 \\
\{2,4\} & 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\
\{1,2,4\} & 0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0 \\
\{3,4\} & 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0 \\
\{1,3,4\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\
\{2,3,4\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0 \\
\{1,2,3,4\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\
\{5\} & 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\
\{1,5\} & 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0 \\
\{2,5\} & 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0 \\
\{1,2,5\} & 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\
\{3,5\} & 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\
\{1,3,5\} & 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0 \\
\{2,3,5\} & 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0 \\
\{1,2,3,5\} & 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\
\{4,5\} & 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0 \\
\{1,4,5\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\
\{2,4,5\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0 \\
\{1,2,4,5\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\
\{3,4,5\} & 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0 \\
\{1,3,4,5\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\
\{2,3,4,5\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0 \\
\{1,2,3,4,5\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{array}$$

$$\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}} = \begin{array}{r|cccccccccccccccc}
 & \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} & \{d\} & \{a,d\} & \{b,d\} & \{a,b,d\} & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\} \\ \hline
\{\} & 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{1\} & 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{2\} & 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{1,2\} & 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{3\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{1,3\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{2,3\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{1,2,3\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{4\} & 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\
\{1,4\} & 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0 \\
\{2,4\} & 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\
\{1,2,4\} & 0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0 \\
\{3,4\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{1,3,4\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{2,3,4\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{1,2,3,4\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{5\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{1,5\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{2,5\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{1,2,5\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{3,5\} & 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\
\{1,3,5\} & 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0 \\
\{2,3,5\} & 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0 \\
\{1,2,3,5\} & 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\
\{4,5\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{1,4,5\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{2,4,5\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{1,2,4,5\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{3,4,5\} & 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0 \\
\{1,3,4,5\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\
\{2,3,4,5\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0 \\
\{1,2,3,4,5\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{array}$$

Fig. 2.1 Existential and inverse image for a surjective mapping

These results imply not least that $\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}}$ is univalent, or a partial function, when f is surjective. With (ii), we have then also $\vartheta_f \cap \vartheta^{\mathsf{T}}_{f^{\mathsf{T}}}; \mathbb{T} = \vartheta^{\mathsf{T}}_{f^{\mathsf{T}}}$. The illustrations below stem from [SW14].

Property (iv) may be visualized by the right relation of Fig. 2.1: Multiplying N_X from the left means turning upside down, while N_Y composed from the right side flips left/right. This applies also to the non-surjective case as may be seen in Fig. 2.2. This means not least that $\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}}$ is univalent, or a partial function, when f is surjective.

$$
\begin{array}{c}
 \\
1\\2\\3
\end{array}
\begin{array}{c}
\begin{array}{cccc} a & b & c & d \end{array}\\
\begin{pmatrix}
0&0&1&0\\
0&1&0&0\\
1&0&0&0
\end{pmatrix}
\end{array}
$$

f

$$
\begin{array}{c|cccccccccccccccc}
 & \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} & \{d\} & \{a,d\} & \{b,d\} & \{a,b,d\} & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\}\\
\hline
\{\} & 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\{1\} & 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
\{2\} & 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\{1,2\} & 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
\{3\} & 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\{1,3\} & 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
\{2,3\} & 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
\{1,2,3\} & 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0
\end{array}
$$

ϑ_f

$$
\begin{array}{c|cccccccccccccccc}
 & \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} & \{d\} & \{a,d\} & \{b,d\} & \{a,b,d\} & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\}\\
\hline
\{\} & 1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
\{1\} & 0&0&0&0&1&0&0&0&0&0&0&0&1&0&0&0\\
\{2\} & 0&0&1&0&0&0&0&0&0&0&1&0&0&0&0&0\\
\{1,2\} & 0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&0\\
\{3\} & 0&1&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
\{1,3\} & 0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0\\
\{2,3\} & 0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&0\\
\{1,2,3\} & 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1
\end{array}
$$

$\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}}$

Fig. 2.2 Existential and inverse image for a non-surjective mapping

2.3 Some categorical considerations

We simply recall here relation-algebraic concepts we will use afterwards. Everything is fully based on the generic constructions of a direct sum, or product, etc. If any two heterogeneous relations π, ρ with common source are given, they are said to form a **direct product** if

$$\pi^{\mathsf{T}}; \pi = \mathbb{I}, \quad \rho^{\mathsf{T}}; \rho = \mathbb{I}, \quad \pi; \pi^{\mathsf{T}} \cap \rho; \rho^{\mathsf{T}} = \mathbb{I}, \quad \pi^{\mathsf{T}}; \rho = \mathbb{T}.$$

Thus, the relations π, ρ are mappings, usually called **projections**. In a similar way, any two heterogeneous relations ι, κ with common target are said to form the left, respectively right, **injection** of a **direct sum** if

$$\iota; \iota^{\mathsf{T}} = \mathbb{I}, \quad \kappa; \kappa^{\mathsf{T}} = \mathbb{I}, \quad \iota^{\mathsf{T}}; \iota \cup \kappa^{\mathsf{T}}; \kappa = \mathbb{I}, \quad \iota; \kappa^{\mathsf{T}} = \perp\!\!\!\perp.$$

2.4 Definition. Given any direct products by projections

$$\pi : X \times Y \longrightarrow X, \quad \rho : X \times Y \longrightarrow Y, \qquad \pi' : U \times V \longrightarrow U, \quad \rho' : U \times V \longrightarrow V,$$

we define the Kronecker product, the fork-, and the join-operator:

i) $(A \otimes B) := \pi; A; \pi'^{\mathsf{T}} \cap \rho; B; \rho'^{\mathsf{T}}$

ii) $(C \varolessthan D) := C; \pi'^{\mathsf{T}} \cap D; \rho'^{\mathsf{T}}$

iii) $(E \varogreaterthan F) := \pi; E \cap \rho; F$ □

It is sometimes helpful, not least for space reasons, to extrude a non-empty subset v out of its surrounding set X and, thus, to give the copy Θ_v of the subset v an own identity. For such an operation we provide notation as follows:

$$\theta_v : \Theta_v \longrightarrow X, \qquad \theta_v(x\!\rightarrowtail) = x \qquad \text{for all } x \in X$$

The injection of the copy Θ_v is denoted by θ_v; it satisfies $\quad \theta_v^\mathsf{T};\theta_v \subseteq \mathbb{I}_X, \quad \theta_v;\theta_v^\mathsf{T} = \mathbb{I}_{\Theta_v}$. One will find out that the constructs $\theta_v, \Theta_v, x\!\!\rightarrowtail$ are uniquely determined up to isomorphism. Should someone come and present, say, $f, F, \overline{x}$ instead with rules corresponding to those above, we are in a position to schematically define the transition as $\Phi := \theta_v;f^\mathsf{T} : \Theta_v \longrightarrow F$ with $\overline{x} = \Phi^\mathsf{T};x\!\!\rightarrowtail$.

$$\theta_v = \begin{array}{r} \\ \text{Jan}\rightarrow \\ \text{Jun}\rightarrow \\ \text{Jul}\rightarrow \end{array} \!\!\begin{array}{c} \begin{array}{cccccccccccc} \text{Jan} & \text{Feb} & \text{Mar} & \text{Apr} & \text{May} & \text{Jun} & \text{Jul} & \text{Aug} & \text{Sep} & \text{Oct} & \text{Nov} & \text{Dec} \end{array} \\ \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{array} : \Theta_v \longrightarrow X \qquad\qquad v = \begin{array}{r} \text{Jan} \\ \text{Feb} \\ \text{Mar} \\ \text{Apr} \\ \text{May} \\ \text{Jun} \\ \text{Jul} \\ \text{Aug} \\ \text{Sep} \\ \text{Oct} \\ \text{Nov} \\ \text{Dec} \end{array} \!\begin{pmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} \subseteq X$$

Fig. 2.3 Illustrating notation of subset extrusion

We recall the statements concerning the strict fork with respect to membership and powerset ordering from [SW14] and then extend them.

2.5 Proposition. $\varepsilon;(\Omega \mathbin{\text{\textcircled{$<$}}} \Omega) = (\varepsilon \mathbin{\text{\textcircled{$<$}}} \varepsilon) \quad$ and $\quad (\varepsilon;\Omega \mathbin{\text{\textcircled{$<$}}} \varepsilon;\Omega) = (\varepsilon \mathbin{\text{\textcircled{$<$}}} \varepsilon);(\Omega \otimes \Omega)$

Proof: We use $\varepsilon;\Omega = \varepsilon$ in:

$(\varepsilon \mathbin{\text{\textcircled{$<$}}} \varepsilon) = \varepsilon;\mathfrak{M}^\mathsf{T}$ Prop. 9.1.iv of [SW14]
$= \varepsilon;\Omega;\mathfrak{M}^\mathsf{T}$
$= \varepsilon;(\Omega \mathbin{\text{\textcircled{$<$}}} \Omega)$ Prop. 9.2.iv of [SW14]

The second claim is simply Prop. 9.2.vi of [SW14]. □

Extending the preceding, we prove the following proposition. Direction "$\supseteq$" of it is trivial and for other first factors than ε one can in general not say more due to unsharpness. The crucial point in proving "$\subseteq$" is a technically rather difficult application of the singleton injection $\sigma = \mathsf{syq}(\mathbb{I}, \varepsilon) : X \longrightarrow \mathbf{2}^X$ corresponding to ε. It rests on a result obtained by Jules Desharnais, [Des99]. However, this kinship is hard to realize.

2.6 Proposition. Given any relations $R, S : X \longrightarrow \mathbf{2}^X$ typed like membership relation and singleton injection $\varepsilon, \sigma : X \longrightarrow \mathbf{2}^X$, together with two relations $A : \mathbf{2}^X \longrightarrow Y$ and $B : \mathbf{2}^X \longrightarrow Z$, one may factorize

$$(R;A \mathbin{\text{\textcircled{$<$}}} S;B) = (R \mathbin{\text{\textcircled{$<$}}} S);(A \otimes B).$$

Proof: $(R;A \mathbin{\text{\textcircled{$<$}}} S;B) = ((R \cap \sigma;\mathbb{T});A \mathbin{\text{\textcircled{$<$}}} S;B)$ $\quad \sigma$ is a mapping, i.e. not least total
$= ((R \cap \sigma;\rho^\mathsf{T};\pi);A \mathbin{\text{\textcircled{$<$}}} S;B)$ $\quad$ property of the direct product π, ρ
$= ((R;\pi^\mathsf{T} \cap \sigma;\rho^\mathsf{T});\pi;A \mathbin{\text{\textcircled{$<$}}} S;B)$ $\quad$ destroy and append rule
$= (R;\pi^\mathsf{T} \cap \sigma;\rho^\mathsf{T});\pi;A;\pi'^\mathsf{T} \cap S;B;\rho'^\mathsf{T}$ $\quad$ definition of fork operator
$\subseteq \left[(R;\pi^\mathsf{T} \cap \sigma;\rho^\mathsf{T}) \cap \ldots\right];\left[\pi;A;\pi'^\mathsf{T} \cap (R;\pi^\mathsf{T} \cap \sigma;\rho^\mathsf{T})^\mathsf{T};S;B;\rho'^\mathsf{T}\right]$ $\quad$ Dedekind rule
$\subseteq (R;\pi^\mathsf{T} \cap \sigma;\rho^\mathsf{T});\left[\rho;\sigma^\mathsf{T};S;B;\rho'^\mathsf{T} \cap \pi;A;\pi'^\mathsf{T}\right]$ $\quad$ transposed and shuffled

$$
\begin{aligned}
&= (R \otimes\!\!\!\!\bigcirc\, \sigma) ; \big[(\mathbb{I} \otimes \sigma^{\mathsf{T}} ; S) ; \rho ; B ; \rho'^{\mathsf{T}} \cap \pi ; A ; \pi'^{\mathsf{T}}\big] && \text{since } \rho ; Q = (\mathbb{I} \otimes Q) ; \rho, \text{ Prop. 7.2.i of [SW14]} \\
&\subseteq (R \otimes\!\!\!\!\bigcirc\, \sigma) ; \big[(\mathbb{I} \otimes \sigma^{\mathsf{T}} ; S) \cap \ldots\big] ; \big[\rho ; B ; \rho'^{\mathsf{T}} \cap (\mathbb{I} \otimes S^{\mathsf{T}} ; \sigma) ; \pi ; A ; \pi'^{\mathsf{T}}\big] && \text{Dedekind rule again} \\
&\subseteq (R \otimes\!\!\!\!\bigcirc\, \sigma) ; (\mathbb{I} \otimes \sigma^{\mathsf{T}} ; S) ; \big[\rho ; B ; \rho'^{\mathsf{T}} \cap \pi ; A ; \pi'^{\mathsf{T}}\big] && \text{since } (\mathbb{I} \otimes \ldots) ; \pi \subseteq \pi \\
&\subseteq (R \otimes\!\!\!\!\bigcirc\, \sigma ; \sigma^{\mathsf{T}} ; S) ; (A \otimes B) && \text{Prop. 7.3.i of [SW14], definition of the Kronecker operator} \\
&= (R \otimes\!\!\!\!\bigcirc\, S) ; (A \otimes B) && \text{since } \sigma \text{ is an injective mapping} \qquad \square
\end{aligned}
$$

2.4 Lifting a Boolean algebra

For being self-contained, we need the following, which is more or less repeated verbally from [SW14]. A note seems necessary concerning Boolean algebras; here supported with visualization in a concrete example. The peculiar recursive and fractal symmetries of these examples often give additional insight — and have already triggered secretaries to stitch such patterns for a pot cloth.

Most people work with subsets $U \subseteq X$, while we distinguish between a subset in this standard form and the corresponding element e, considered as a point in the powerset. The two are related via the membership relation ε as shown in Fig. 2.4 together with the powerset ordering $\Omega = \overline{\varepsilon^{\mathsf{T}} ; \overline{\varepsilon}}$.

$$U = \varepsilon ; e \qquad e = \mathtt{syq}(\varepsilon, U)$$

$$
\begin{array}{c}
\begin{array}{c|cccccccccccccccc|}
 & \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} & \{d\} & \{a,d\} & \{b,d\} & \{a,b,d\} & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\} \\
a & 0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1 \\
b & 0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\
c & 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\
d & 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1
\end{array}
\quad
\begin{pmatrix} 0\\1\\0\\1 \end{pmatrix}
\\[1em]
(0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0) = e^{\mathsf{T}}
\end{array}
$$

$$
\begin{array}{c|cccccccccccccccc|}
 & \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} & \{d\} & \{a,d\} & \{b,d\} & \{a,b,d\} & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\} \\
\{\} & 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\
\{a\} & 0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1 \\
\{b\} & 0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\
\{a,b\} & 0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1 \\
\{c\} & 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\
\{a,c\} & 0&0&0&0&0&1&0&1&0&0&0&0&0&1&0&1 \\
\{b,c\} & 0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1 \\
\{a,b,c\} & 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1 \\
\{d\} & 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \\
\{a,d\} & 0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1 \\
\{b,d\} & 0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&1 \\
\{a,b,d\} & 0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1 \\
\{c,d\} & 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \\
\{a,c,d\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1 \\
\{b,c,d\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1 \\
\{a,b,c,d\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{array}
\quad
\begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1\\0\\0\\0\\0\\0 \end{pmatrix}
$$

Fig. 2.4 Subset U and corresponding point e in the powerset via ε, Ω

Theoreticians frequently consider Boolean algebras "with signature $\langle X, \cdot, +, -, \mathbf{0}, \mathbf{1} \rangle$". Following their idea, we find on X the operations $\cap, \cup, \overline{}, \perp\!\!\!\perp, \mathbb{T}$ on subsets of X.

There is, however, a second "lifted" form, for which the *elements* are taken from $\mathbf{2}^X$ with corresponding operations consisting of

$$\mathfrak{M} = \mathtt{syq}(\,(\varepsilon \otimes\!\!\!\!\bigcirc\, \varepsilon)\,, \varepsilon), \quad \mathfrak{J} = \mathtt{syq}(\,(\overline{\varepsilon} \otimes\!\!\!\!\bigcirc\, \overline{\varepsilon})\,, \overline{\varepsilon}), \quad N,$$
$$(\overline{\varepsilon^{\mathsf{T}} ; \mathbb{T}} =)\ \mathtt{syq}(\varepsilon, \perp\!\!\!\perp), \quad (\overline{\overline{\varepsilon}^{\mathsf{T}} ; \mathbb{T}} =)\ \mathtt{syq}(\varepsilon, \mathbb{T}),$$

as defined below. Easiest to observe are the 0-ary operators or elements $\overline{\varepsilon^{\mathsf{T}} ; \mathbb{T}} \approx \mathbf{0}\,, \overline{\overline{\varepsilon}^{\mathsf{T}} ; \mathbb{T}} \approx \mathbf{1}$ for which obviously, looking at Fig. 2.4,

$$\perp\!\!\!\perp = \varepsilon ; \overline{\varepsilon^{\mathsf{T}} ; \mathbb{T}} = \varepsilon ; \mathtt{syq}(\varepsilon, \perp\!\!\!\perp), \qquad \mathbb{T} = \varepsilon ; \overline{\overline{\varepsilon}^{\mathsf{T}} ; \mathbb{T}} = \varepsilon ; \mathtt{syq}(\varepsilon, \mathbb{T}).$$

Next we study the unary operator

$$N := \mathtt{syq}(\overline{\varepsilon}, \varepsilon) \qquad\qquad N : \mathbf{2}^X \longrightarrow \mathbf{2}^X,$$

visualized in Fig. 2.5, for which we show in advance

$$\overline{\varepsilon}{;}N = \overline{\varepsilon}{;}\mathtt{syq}(\overline{\varepsilon}, \varepsilon) = \varepsilon \qquad\qquad \varepsilon{;}N = \varepsilon{;}\mathtt{syq}(\overline{\varepsilon}, \varepsilon) = \varepsilon{;}\mathtt{syq}(\varepsilon, \overline{\varepsilon}) = \overline{\varepsilon}$$

$$\mathbb{I} \subseteq \Omega = \overline{\varepsilon^{\mathsf{T}}{;}\overline{\varepsilon}} = \overline{\varepsilon^{\mathsf{T}}{;}\varepsilon{;}N} \quad \Longrightarrow \quad N \subseteq \overline{\varepsilon^{\mathsf{T}}{;}\varepsilon}.$$

Multiplying a relation with N from the left flips this relation upside/down, while multiplying from the right side flips it left/right. Sometimes, we have to apply N to both sides of a pair, for which purpose we also introduce

$$\mathcal{N} := (N \otimes N) = \pi{;}N{;}\pi^{\mathsf{T}} \cap \rho{;}N{;}\rho^{\mathsf{T}} : \mathbf{2}^X \times \mathbf{2}^X \longrightarrow \mathbf{2}^X \times \mathbf{2}^X.$$

We identify here disjointness $\overline{\varepsilon^{\mathsf{T}}{;}\varepsilon}$ which is shown in Fig. 2.5. It looks as if the powerset ordering Ω of Fig. 2.4 were rotated by an angle of -90 degrees, which may more mathematically be expressed as $\Omega{;}N = \overline{\varepsilon^{\mathsf{T}}{;}\varepsilon}$; this time flipping left/right.

$$\begin{array}{rc}
 & \begin{array}{cccccccccccccccc} \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} & \{d\} & \{a,d\} & \{b,d\} & \{a,b,d\} & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\} \end{array} \\
\begin{array}{r} \{\} \\ \{a\} \\ \{b\} \\ \{a,b\} \\ \{c\} \\ \{a,c\} \\ \{b,c\} \\ \{a,b,c\} \\ \{d\} \\ \{a,d\} \\ \{b,d\} \\ \{a,b,d\} \\ \{c,d\} \\ \{a,c,d\} \\ \{b,c,d\} \\ \{a,b,c,d\} \end{array} &
\left(\begin{array}{cccccccccccccccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}\right)
\end{array}
\qquad
\begin{array}{rc}
 & \begin{array}{cccccccccccccccc} \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} & \{d\} & \{a,d\} & \{b,d\} & \{a,b,d\} & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\} \end{array} \\
\begin{array}{r} \{\} \\ \{a\} \\ \{b\} \\ \{a,b\} \\ \{c\} \\ \{a,c\} \\ \{b,c\} \\ \{a,b,c\} \\ \{d\} \\ \{a,d\} \\ \{b,d\} \\ \{a,b,d\} \\ \{c,d\} \\ \{a,c,d\} \\ \{b,c,d\} \\ \{a,b,c,d\} \end{array} &
\left(\begin{array}{cccccccccccccccc}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0\\
1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0\\
1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0\\
1&1&1&1&0&0&0&0&1&1&1&1&0&0&0&0\\
1&0&1&0&0&0&0&0&1&0&1&0&0&0&0&0\\
1&1&0&0&0&0&0&0&1&1&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0\\
1&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0\\
1&1&0&0&1&1&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}\right)
\end{array}$$

Fig. 2.5 Negation N and disjointness $\overline{\varepsilon^{\mathsf{T}}{;}\varepsilon} = \Omega{;}N$ in the powerset

3 Applying Relations in Topology

Since its first appearence[1] in the book *Vorstudien zur Topologie* by Johann Benedict Listing of 1847, topology (then and for a long period termed ANALYSIS SITUS) has been given many facets; among the main ones are considerations of neighborhoods, open sets, and closed sets. We start here, giving the corresponding definitions lifted to point-free versions, showing how they are interrelated, thus exhibiting their cryptomorphism and offering the possibility to transform one version into the other, not least visualizing them via TITUREL programs.

[1]Citation: Es mag erlaubt sein, für diese Art Untersuchungen räumlicher Gebilde den Namen „Topologie“ zu gebrauchen statt der von Leibniz vorgeschlagenen Benennung „geometria situs“, welche an den Begriff des Maßes, der hier ganz untergeordnet ist, erinnert, und mit dem bereits für eine andere Art geometrischer Betrachtungen gebräuchlich gewordenen Namen „géométrie de position“ collidiert.

It has also been reported that Karl von Staudt in Erlangen with his *Geometrie der Lage* of 1848 has made one of the greatest achievements of Geometry over thousands of years; see [Fab59]. von Staudt does no longer talk on the length of a line, nor on the degree of an angle. Instead, he talks on points on lines, and incidence.

Early in the twentieth century, topology has split into 'general topology' or 'point set theory', mainly invented by Georg Cantor and later developed further by Felix Hausdorff, and what we today call 'algebraic topology'.

3.1 General properties of kernel forming

Partly recalled from [Sch11]: We consider some set X and its powerset $\mathbf{2}^X$, so that one automatically has also the membership relation $\varepsilon : X \longrightarrow \mathbf{2}^X$, the powerset ordering $\Omega : \mathbf{2}^X \longrightarrow \mathbf{2}^X$, the powerset negation $N : \mathbf{2}^X \longrightarrow \mathbf{2}^X$, and the powerset join and meet $\mathfrak{J}, \mathfrak{M} : \mathbf{2}^X \times \mathbf{2}^X \longrightarrow \mathbf{2}^X$.

We recall here for convenience the definitions of closure as well as kernel forming $\rho, \mathcal{K}$ with regard to some ordering Ω in general:

$$\begin{array}{lll} \rho \subseteq \Omega & \Omega \mathbin{;} \rho \subseteq \rho \mathbin{;} \Omega & \rho \mathbin{;} \rho \subseteq \rho \\ \mathcal{K} \subseteq \Omega^\mathsf{T} & \Omega \mathbin{;} \mathcal{K} \subseteq \mathcal{K} \mathbin{;} \Omega & \mathcal{K} \mathbin{;} \mathcal{K} \subseteq \mathcal{K} \end{array}$$

This expresses that these mappings shall be expanding, resp. contracting, isotonic, and idempotent. Kernel-forming — up to a trivial additional totality requirement and the distributivity in Prop. 3.2.iv below — will soon be recognized as being crytomorphic with a neighborhood topology.

We first investigate in which way a monotone mapping f and forming binary meets with $\mathfrak{M}$ are related. The interpretation of the following proposition is that when going from a pair of subsets to the intersection of their f-images, one may also first obtain the intersection of the two sets and take its f-image and find oneself below — and having to follow Ω to catch up with the former.

3.1 Proposition. For every monotone mapping f we have with regard to meet forming $\mathfrak{M}$

$$(f \otimes f) \mathbin{;} \mathfrak{M} \subseteq \mathfrak{M} \mathbin{;} f \mathbin{;} \Omega.$$

Proof: $(f \otimes f) \mathbin{;} \mathfrak{M} = (f \otimes f) \mathbin{;} \mathtt{syq}((\varepsilon \olessthan \varepsilon), \varepsilon)$ Def. 9.i of $\mathfrak{M}$ in [SW14]

$= \mathtt{syq}((\varepsilon \olessthan \varepsilon) \mathbin{;} (f \otimes f)^\mathsf{T}, \varepsilon)$ since $(f \otimes f)$ is a map

$= \mathtt{syq}((\varepsilon \olessthan \varepsilon) \mathbin{;} (f^\mathsf{T} \otimes f^\mathsf{T}), \varepsilon)$ transposition distributed

$= \mathtt{syq}((\varepsilon \mathbin{;} f^\mathsf{T} \olessthan \varepsilon \mathbin{;} f^\mathsf{T}), \varepsilon)$ Prop. 7.iii in [SW14]

$= \overline{(\pi \mathbin{;} f \mathbin{;} \varepsilon^\mathsf{T} \cap \rho \mathbin{;} f \mathbin{;} \varepsilon^\mathsf{T}) \mathbin{;} \overline{\varepsilon}} \cap \ldots$ by definition of the symmetric quotient

Now we proceed with only the first constituent:

$$\begin{array}{ll} & \overline{(\pi \mathbin{;} f \mathbin{;} \varepsilon^\mathsf{T} \cap \rho \mathbin{;} f \mathbin{;} \varepsilon^\mathsf{T}) \mathbin{;} \overline{\varepsilon}} \subseteq \mathfrak{M} \mathbin{;} f \mathbin{;} \Omega = \overline{\mathfrak{M} \mathbin{;} f \mathbin{;} \varepsilon^\mathsf{T} \mathbin{;} \overline{\varepsilon}} \\ \Longleftrightarrow & \mathfrak{M} \mathbin{;} f \mathbin{;} \varepsilon^\mathsf{T} \mathbin{;} \overline{\varepsilon} \subseteq (\pi \mathbin{;} f \mathbin{;} \varepsilon^\mathsf{T} \cap \rho \mathbin{;} f \mathbin{;} \varepsilon^\mathsf{T}) \mathbin{;} \overline{\varepsilon} \quad \text{negated} \\ \Longleftarrow & \mathfrak{M} \mathbin{;} f \mathbin{;} \varepsilon^\mathsf{T} \subseteq \pi \mathbin{;} f \mathbin{;} \varepsilon^\mathsf{T} \cap \rho \mathbin{;} f \mathbin{;} \varepsilon^\mathsf{T} \\ \Longleftrightarrow & \mathfrak{M} \mathbin{;} f \mathbin{;} \varepsilon^\mathsf{T} \subseteq \pi \mathbin{;} f \mathbin{;} \varepsilon^\mathsf{T} \quad \text{and} \quad \mathfrak{M} \mathbin{;} f \mathbin{;} \varepsilon^\mathsf{T} \subseteq \rho \mathbin{;} f \mathbin{;} \varepsilon^\mathsf{T}, \quad \text{from now on, we show only the first} \\ \Longleftrightarrow & \Omega^\mathsf{T} \mathbin{;} f \mathbin{;} \varepsilon^\mathsf{T} = \pi^\mathsf{T} \mathbin{;} \mathfrak{M} \mathbin{;} f \mathbin{;} \varepsilon^\mathsf{T} \subseteq f \mathbin{;} \varepsilon^\mathsf{T} \quad \text{shunting and Prop. 9.2.ii of [SW14]} \\ \Longleftrightarrow & f^\mathsf{T} \mathbin{;} \Omega^\mathsf{T} \mathbin{;} f \mathbin{;} \varepsilon^\mathsf{T} \subseteq \Omega^\mathsf{T} \mathbin{;} f^\mathsf{T} \mathbin{;} f \mathbin{;} \varepsilon^\mathsf{T} \subseteq \Omega^\mathsf{T} \mathbin{;} \varepsilon^\mathsf{T} = \varepsilon^\mathsf{T} \quad \text{shunting, isotony, and univalence} \end{array}$$

□

$$
\mathcal{K} = \begin{array}{r}
\\ \{\} \\ \{a\} \\ \{b\} \\ \{a,b\} \\ \{c\} \\ \{a,c\} \\ \{b,c\} \\ \{a,b,c\} \\ \{d\} \\ \{a,d\} \\ \{b,d\} \\ \{a,b,d\} \\ \{c,d\} \\ \{a,c,d\} \\ \{b,c,d\} \\ \{a,b,c,d\}
\end{array}
\begin{array}{c}
\{\}\ \{a\}\ \{b\}\ \{a,b\}\ \{c\}\ \{a,c\}\ \{b,c\}\ \{a,b,c\}\ \{d\}\ \{a,d\}\ \{b,d\}\ \{a,b,d\}\ \{c,d\}\ \{a,c,d\}\ \{b,c,d\}\ \{a,b,c,d\} \\
\begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix}
\end{array}
$$

Fig. 3.1 $\cap$-sub-distributive kernel forming, i.e., $(\mathcal{K} \otimes \mathcal{K}) ; \mathfrak{M} \subseteq \mathfrak{M} ; \mathcal{K} ; \Omega$; viz. $\{a,d\} \cap \{b,d\}$

We may qualify this as being sub-distributive: When starting from a pair with kernel forming and intersecting afterwards, one will end above what one reaches when intersecting first and forming then a kernel.

This shall now be specialized to kernel forming $\mathcal{K}$.

3.2 Proposition. Meet forming, projections and the open kernel mapping are related as

i) $(\mathcal{K} \otimes \mathcal{K}) ; \mathfrak{M} \subseteq \mathfrak{M} ; \mathcal{K} ; \Omega^\mathsf{T} \quad \Longrightarrow \quad (\mathcal{K} \otimes \mathcal{K}) ; \mathfrak{M} = \mathfrak{M} ; \mathcal{K}$,

ii) $\Omega ; \mathcal{K} \cap \Omega^\mathsf{T} = \mathcal{K}$,

iii) $\pi ; \mathcal{K} \cap \rho ; \mathcal{K} \subseteq \mathfrak{M} ; \mathcal{K}, \qquad$ or $\qquad (\mathcal{K} \ogreaterthan \mathcal{K}) \subseteq \mathfrak{M} ; \mathcal{K}$,

iv) $\pi ; \mathcal{K}^\mathsf{T} \cap \rho ; \mathcal{K}^\mathsf{T} \subseteq \mathfrak{M} ; \mathcal{K}^\mathsf{T}$.

Proof: i) When in addition the assumption of Prop. 3.1 $(\mathcal{K} \otimes \mathcal{K}) ; \mathfrak{M} \subseteq \mathfrak{M} ; \mathcal{K} ; \Omega^\mathsf{T}$, we have in total

$$(\mathcal{K} \otimes \mathcal{K}) ; \mathfrak{M} \subseteq \mathfrak{M} ; \mathcal{K} ; (\Omega \cap \Omega^\mathsf{T}) = \mathfrak{M} ; \mathcal{K},$$

which means even equality, since both sides are mappings.

ii) The direction "$\supseteq$" is trivial because Ω is reflexive and $\mathcal{K} \subseteq \Omega^\mathsf{T}$. "$\subseteq$" is more challenging:

$\Omega ; \mathcal{K} \cap \Omega^\mathsf{T}$
$= \Omega ; \mathcal{K} ; (\mathcal{K} \cap \mathbb{I}) \cap \Omega^\mathsf{T}$ since for the map $\mathcal{K}$ with idempotency $\mathcal{K} ; (\mathcal{K} \cap \mathbb{I}) = \mathcal{K} ; \mathcal{K} \cap \mathcal{K} = \mathcal{K}$
$= \Omega ; \mathcal{K} \cap \Omega^\mathsf{T} ; (\mathcal{K} \cap \mathbb{I})$ because $Q ; J \cap R = Q \cap R ; J$ whenever $J \subseteq \mathbb{I}$ 'filters columns'
$\subseteq \Omega ; \mathcal{K} \cap \Omega^\mathsf{T} ; \mathcal{K}$
$\subseteq \mathcal{K} ; \Omega \cap \mathcal{K} ; \Omega^\mathsf{T}$ two times monotony of $\mathcal{K}$
$\subseteq \mathcal{K} ; (\Omega \cap \Omega^\mathsf{T})$ $\mathcal{K}$ is univalent
$\subseteq \mathcal{K} ; \mathbb{I} = \mathcal{K}$ antisymmetry

iii) We make use of (ii), namely $\Omega{:}\mathcal{K} \cap \Omega^{\mathsf{T}} = \mathcal{K}$ and apply Prop. 9.2.i,iii of [SW14]:

$$\pi{:}\mathcal{K} \cap \rho{:}\mathcal{K} \subseteq \mathfrak{M}{:}\mathfrak{M}^{\mathsf{T}}{:}(\pi{:}\mathcal{K} \cap \rho{:}\mathcal{K}) \subseteq \mathfrak{M}{:}(\mathfrak{M}^{\mathsf{T}}{:}\pi{:}\mathcal{K} \cap \mathfrak{M}^{\mathsf{T}}{:}\rho{:}\mathcal{K}) = \mathfrak{M}{:}(\Omega{:}\mathcal{K} \cap \Omega{:}\mathcal{K}) = \mathfrak{M}{:}\Omega{:}\mathcal{K}$$
$$\pi{:}\mathcal{K} \cap \rho{:}\mathcal{K} \subseteq \pi{:}\Omega^{\mathsf{T}} \cap \rho{:}\Omega^{\mathsf{T}} = \mathfrak{M}{:}\Omega^{\mathsf{T}} \quad \text{Prop. 9.2.iv of [SW14]}$$

So in total

$$\pi{:}\mathcal{K} \cap \rho{:}\mathcal{K} \subseteq \mathfrak{M}{:}\Omega{:}\mathcal{K} \cap \mathfrak{M}{:}\Omega^{\mathsf{T}} = \mathfrak{M}{:}(\Omega{:}\mathcal{K} \cap \Omega^{\mathsf{T}}) = \mathfrak{M}{:}\mathcal{K} \quad \text{using (ii)}$$

iv) follows from $\pi \cap \rho \subseteq \mathfrak{M}$ and univalency of $\mathcal{K}$ via shunting:

$$(\pi \cap \rho){:}\mathcal{K}^{\mathsf{T}}{:}\mathcal{K} \subseteq \pi \cap \rho \subseteq \mathfrak{M} \quad \Longleftrightarrow \quad (\pi \cap \rho){:}\mathcal{K}^{\mathsf{T}} \subseteq \mathfrak{M}{:}\mathcal{K}^{\mathsf{T}} \qquad \square$$

(ii) expresses when $\mathcal{K}, \mathfrak{M}$ commute; appropriately modified, however, to cope with a binary and a unary mapping.

A first observation is the following counterplay of two relations $\mathcal{U}, \mathcal{K}$, studied before entering into the topology discussion proper:

3.3 Proposition. Based on an arbitrary membership relation $\varepsilon : X \longrightarrow \mathbf{2}^X$, we consider a pair of transitions of the type

$$\mathcal{U} \quad \mapsto \quad \mathcal{K} := \mathtt{syq}(\mathcal{U}, \varepsilon) : \mathbf{2}^X \longrightarrow \mathbf{2}^X \qquad \text{and} \qquad \mathcal{K} \quad \mapsto \quad \mathcal{U} := \varepsilon{:}\mathcal{K}^{\mathsf{T}} : X \longrightarrow \mathbf{2}^X.$$

i) Such transitions are inverses of one another and $\mathcal{K}$ is necessarily a mapping.

ii) The following two equivalences hold:

$$\varepsilon{:}\mathcal{K}^{\mathsf{T}} \text{ total} \quad \Longleftrightarrow \quad \mathcal{U} \text{ total}$$
$$\mathcal{K}{:}\mathcal{K} = \mathcal{K} \quad \Longleftrightarrow \quad \mathcal{U} = \mathcal{U}{:}\mathtt{syq}(\varepsilon, \mathcal{U})$$

Proof: i) The $\mathcal{K}$ defined on the left is certainly a mapping, since $\mathcal{K}^{\mathsf{T}}{:}\mathcal{K} \subseteq \mathtt{syq}(\varepsilon, \varepsilon) = \mathbb{I}$, and, since forming the symmetric quotient with ε on the right side of $\mathtt{syq}$ always gives a total relation.

$$\varepsilon{:}[\mathtt{syq}(\mathcal{U}, \varepsilon)]^{\mathsf{T}} = \varepsilon{:}\mathtt{syq}(\varepsilon, \mathcal{U}) = \mathcal{U}, \quad \text{since } \mathtt{syq}(\varepsilon, X) \text{ is always surjective}$$
$$\mathtt{syq}(\varepsilon{:}\mathcal{K}^{\mathsf{T}}, \varepsilon) = \mathcal{K}{:}\mathtt{syq}(\varepsilon, \varepsilon) = \mathcal{K}{:}\mathbb{I} = \mathcal{K} \quad \text{since } \mathcal{K} \text{ is a mapping}$$

ii) The first statement is trivial in view of the definitions. For "$\Longrightarrow$" of the second statement, we show using the definition of $\mathcal{U}$ and idempotency $\mathcal{U}{:}\mathtt{syq}(\varepsilon, \mathcal{U}) = \varepsilon{:}\mathcal{K}^{\mathsf{T}}{:}\mathcal{K}^{\mathsf{T}} = \varepsilon{:}\mathcal{K}^{\mathsf{T}} = \mathcal{U}$.

"$\Longleftarrow$": $\mathcal{K}{:}\mathcal{K} = \mathcal{K}{:}\mathtt{syq}(\mathcal{U}, \varepsilon) = \mathtt{syq}(\mathcal{U}{:}\mathcal{K}^{\mathsf{T}}, \varepsilon) = \mathtt{syq}(\mathcal{U}{:}\mathtt{syq}(\varepsilon, \mathcal{U}), \varepsilon) = \mathtt{syq}(\mathcal{U}, \varepsilon) = \mathcal{K}$ $\square$

3.2 Topology via neighborhoods and kernel forming

We recall the definition of a topology via a neighborhood system from [Fra60] mentioning that in the classical definition a set X endowed with a system $\mathcal{U}(p)$ of subsets for every $p \in X$ — called neighborhoods — is a **topological structure**, provided

- $p \in U$ for every neighborhood $U \in \mathcal{U}(p)$,
- if $U \in \mathcal{U}(p)$ and $V \supseteq U$, then $V \in \mathcal{U}(p)$,
- if $U_1, U_2 \in \mathcal{U}(p)$, then $U_1 \cap U_2 \in \mathcal{U}(p)$ and $X \in \mathcal{U}(p)$,
- for every $U \in \mathcal{U}(p)$ there is a $V \in \mathcal{U}(p)$ so that $U \in \mathcal{U}(y)$ for all $y \in V$.

Thus prepared we give the relational definition of a topology in point-free form.

3.4 Definition. A relation $\mathcal{U} : X \longrightarrow \mathbf{2}^X$ will be called a **neighborhood topology** if the following properties are satisfied:

i) $\mathcal{U};\mathbb{T} = \mathbb{T}$ and $\mathcal{U} \subseteq \varepsilon$,

ii) $\mathcal{U};\Omega \subseteq \mathcal{U}$,

iii) $(\mathcal{U} \mathbin{\text{\textcircled{<}}} \mathcal{U}) ; \mathfrak{M} \subseteq \mathcal{U}$,

iv) $\mathcal{U} \subseteq \mathcal{U};\overline{\varepsilon^{\mathsf{T}};\overline{\mathcal{U}}}$. □

An example isf given in Fig. 3.2 where (since it is finite and discrete) only the tightest neighborhood of every element is shown; others may be obtained forming arbitrary unions.

$$\mathcal{U} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccccccccccccccc} \rotatebox{90}{\{\}} & \rotatebox{90}{\{a\}} & \rotatebox{90}{\{b\}} & \rotatebox{90}{\{a,b\}} & \rotatebox{90}{\{c\}} & \rotatebox{90}{\{a,c\}} & \rotatebox{90}{\{b,c\}} & \rotatebox{90}{\{a,b,c\}} & \rotatebox{90}{\{d\}} & \rotatebox{90}{\{a,d\}} & \rotatebox{90}{\{b,d\}} & \rotatebox{90}{\{a,b,d\}} & \rotatebox{90}{\{c,d\}} & \rotatebox{90}{\{a,c,d\}} & \rotatebox{90}{\{b,c,d\}} & \rotatebox{90}{\{a,b,c,d\}} \end{array} \\ \begin{pmatrix} 0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1 \\ 0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1 \\ 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix} \end{array}$$

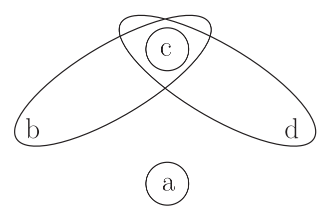


Fig. 3.2 A neighborhood topology and the basis of its open sets

Def. 3.4 obviously resembles being total and assigning only subsets as neighborhoods to an element it is indeed contained in (i), being up-closed (ii), admitting binary meets (iii), and providing open subsets. Property (iv) is not so easily recognized as providing an open kernel for every neighborhood; in [Sch14], a detailed hint is given.

3.5 Proposition. In every neighborhood topology according to the minimalistic properties of Def. 3.4, some stronger ones are satisfied.

i) $\mathcal{U};\Omega = \mathcal{U}$

ii) $(\mathcal{U} \otimes_{<} \mathcal{U});\mathfrak{M} = \mathcal{U}$

iii) $(\mathcal{U} \otimes_{<} \mathcal{U}) = \mathcal{U};\mathfrak{M}^\top$

iv) $\mathcal{U} = \mathcal{U};\overline{\varepsilon^\top;\overline{\mathcal{U}}}$

v) $\mathcal{U} = \mathcal{U};\overline{\mathcal{U}^\top;\overline{\varepsilon}}$

vi) $\mathcal{U} = \mathcal{U};\mathtt{syq}(\varepsilon,\mathcal{U}) = \mathcal{U};\mathcal{K}^\top$

vii) $(\mathcal{K} \otimes \mathcal{K});\mathfrak{M} = \mathtt{syq}((\mathcal{U} \otimes_{<} \mathcal{U}),\varepsilon)$

Proof: i) follows from Def. 3.4.ii since Ω is reflexive.

ii) In addition to Def. 3.4.iii:

$(\mathcal{U} \otimes_{<} \mathcal{U});\mathfrak{M} = [\mathcal{U};\pi^\top \cap \mathcal{U};\rho^\top];\mathfrak{M}$ expanded
$= [\mathcal{U};\Omega;\pi^\top \cap \mathcal{U};\Omega;\rho^\top];\mathfrak{M}$ (i)
$= [\mathcal{U};\mathfrak{M}^\top;\pi;\pi^\top \cap \mathcal{U};\mathfrak{M}^\top;\rho;\rho^\top];\mathfrak{M}$ Prop. 9.2.ii of [SW14]
$\supseteq [\mathcal{U};\mathfrak{M}^\top \cap \mathcal{U};\mathfrak{M}^\top];\mathfrak{M}$ projections are total
$= \mathcal{U};\mathfrak{M}^\top;\mathfrak{M} \supseteq \mathcal{U}$ since meet-forming $\mathfrak{M}$ is surjective

iii) Direction "$\subseteq$" is a trivial variant of Def. 3.4.iii obtained via shunting. The other direction $\mathcal{U};\mathfrak{M}^\top \subseteq \mathcal{U};\pi^\top \cap \mathcal{U};\rho^\top$ splits into two similar parts that are shown with Prop. 9.2.ii of [SW14] after having shunted:

$\mathcal{U};\mathfrak{M}^\top \subseteq \mathcal{U};\pi^\top \quad \Longleftarrow \quad \mathcal{U};\mathfrak{M}^\top;\pi = \mathcal{U};\Omega = \mathcal{U}$

iv) In addition to Def. 3.4.iv, we have $\mathcal{U};\overline{\varepsilon^\top;\overline{\mathcal{U}}} \subseteq \mathcal{U} \iff \mathcal{U}^\top;\overline{\mathcal{U}} \subseteq \varepsilon^\top;\overline{\mathcal{U}} \Longleftarrow \mathcal{U} \subseteq \varepsilon$

v) $\mathcal{U} = \mathcal{U};\Omega = \mathcal{U};\overline{\varepsilon^\top;\overline{\varepsilon}} \subseteq \mathcal{U};\overline{\mathcal{U}^\top;\overline{\varepsilon}}$, using (i), definition of Ω, and $\mathcal{U} \subseteq \varepsilon$. It remains to show the reverse direction.

$\mathcal{U};\overline{\mathcal{U}^\top;\overline{\varepsilon}} \subseteq \mathcal{U};\overline{\varepsilon^\top;\overline{\mathcal{U}}};\overline{\mathcal{U}^\top;\overline{\varepsilon}}$ Def. 3.4.iv
$\subseteq \mathcal{U};\overline{\varepsilon^\top;\overline{\varepsilon}}$ see below
$= \mathcal{U};\Omega$ definition of Ω
$= \mathcal{U}$ (i)

The postponed part:

$\overline{\varepsilon}^\top;\mathcal{U} \subseteq \overline{\varepsilon}^\top;\mathcal{U}$ is certainly satisfied
$\iff \overline{\varepsilon};\overline{\overline{\varepsilon}^\top;\mathcal{U}} \subseteq \overline{\mathcal{U}}$
$\implies \varepsilon^\top;\overline{\varepsilon};\overline{\overline{\varepsilon}^\top;\mathcal{U}} \subseteq \varepsilon^\top;\overline{\mathcal{U}}$
$\iff \overline{\varepsilon^\top;\overline{\mathcal{U}}};\overline{\mathcal{U}^\top;\overline{\varepsilon}} \subseteq \overline{\varepsilon^\top;\overline{\varepsilon}}$

vi) We start from (iv) and get immediately

$$\mathcal{U} = \mathcal{U};\overline{\varepsilon^\top;\overline{\mathcal{U}}} = \mathcal{U};\overline{\varepsilon^\top;\overline{\overline{\varepsilon};\mathcal{K}^\top}} = \mathcal{U};\overline{\varepsilon^\top;\overline{\varepsilon}};\mathcal{K}^\top = \mathcal{U};\Omega;\mathcal{K}^\top = \mathcal{U};\mathcal{K}^\top = \mathcal{U};\mathtt{syq}(\varepsilon,\mathcal{U}).$$

vii) $(\mathcal{K} \otimes \mathcal{K});\mathfrak{M} = (\mathcal{K} \otimes \mathcal{K});\mathtt{syq}((\varepsilon \otimes_{<} \varepsilon),\varepsilon)$
$= \mathtt{syq}((\varepsilon \otimes_{<} \varepsilon);(\mathcal{K}^\top \otimes \mathcal{K}^\top),\varepsilon)$

$$
\begin{aligned}
&= \mathtt{syq}(\,(\varepsilon; \mathcal{K}^{\mathsf{T}} \oslash \varepsilon; \mathcal{K}^{\mathsf{T}})\,, \varepsilon) \\
&= \mathtt{syq}(\,(\mathcal{U} \oslash \mathcal{U})\,, \varepsilon)
\end{aligned}
$$

□

We will now study in which way the idea of Def. 3.4 may also be expressed in terms of conditions to be imposed on $\mathcal{K}$ instead of $\mathcal{U}$.

$$
\begin{array}{r}
\\
\{\} \\ \{a\} \\ \{b\} \\ \{a,b\} \\ \{c\} \\ \{a,c\} \\ \{b,c\} \\ \{a,b,c\} \\ \{d\} \\ \{a,d\} \\ \{b,d\} \\ \{a,b,d\} \\ \{c,d\} \\ \{a,c,d\} \\ \{b,c,d\} \\ \{a,b,c,d\}
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccccccc}
\{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} & \{d\} & \{a,d\} & \{b,d\} & \{a,b,d\} & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\}
\end{array} \\
\begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix}
\end{array}
\quad
\begin{array}{r}
\{\} \\ \{a\} \\ \{b\} \\ \{a,b\} \\ \{c\} \\ \{a,c\} \\ \{b,c\} \\ \{a,b,c\} \\ \{d\} \\ \{a,d\} \\ \{b,d\} \\ \{a,b,d\} \\ \{c,d\} \\ \{a,c,d\} \\ \{b,c,d\} \\ \{a,b,c,d\}
\end{array}
\begin{pmatrix}
1\\1\\0\\0\\1\\1\\1\\1\\0\\0\\0\\0\\1\\1\\1\\1
\end{pmatrix}
\quad
\begin{array}{r}
\{\} \\ \{a\} \\ \{b\} \\ \{a,b\} \\ \{c\} \\ \{a,c\} \\ \{b,c\} \\ \{a,b,c\} \\ \{d\} \\ \{a,d\} \\ \{b,d\} \\ \{a,b,d\} \\ \{c,d\} \\ \{a,c,d\} \\ \{b,c,d\} \\ \{a,b,c,d\}
\end{array}
\begin{pmatrix}
0\\1\\0\\0\\1\\0\\1\\0\\0\\0\\0\\0\\1\\0\\0\\0
\end{pmatrix}
$$

$$
\begin{array}{r}
a \\ b \\ c \\ d
\end{array}
\begin{pmatrix}
0&1&0&0&0&1&0&1&0&0&0&0&0&1&0&1\\
0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1\\
0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1
\end{pmatrix}
= \varepsilon_{\mathcal{O}} = \varepsilon \cap \mathbb{T}; \mathcal{O}^{\mathsf{T}} \subseteq \mathcal{U}
$$

Fig. 3.3 Kernel forming, open sets, their basis, and membership in open sets for Fig. 3.2

3.6 Definition. We call a relation $\mathcal{K} : \mathbf{2}^X \longrightarrow \mathbf{2}^X$ a **kernel-mapping topology**, if

i) $\mathcal{K}$ is a kernel forming, i.e., $\quad \mathcal{K} \subseteq \Omega^{\mathsf{T}}, \quad \Omega; \mathcal{K} \subseteq \mathcal{K}; \Omega, \quad \mathcal{K}; \mathcal{K} \subseteq \mathcal{K}$,

ii) $\varepsilon; \mathcal{K}^{\mathsf{T}}$ is total,

iii) $(\mathcal{K} \otimes \mathcal{K}); \mathfrak{M} \;=\; \mathfrak{M}; \mathcal{K}$. □

From Prop. 3.2.ii, we know already that the sharpened version of property (iii) demands that kernel and meet forming commute.

The following lemma may be helpful. It is intuitively clear when interpreted in the topology context.

3.7 Lemma. Any kernel forming operation $\mathcal{K}$ satisfies

i) $\Omega; \mathcal{K} \cap \mathcal{K}^{\mathsf{T}} = \mathcal{K}^{\mathsf{T}}; \mathcal{K} \subseteq \mathbb{I}$,

ii) $\Omega; \mathcal{K}^{\mathsf{T}} \cap \Omega^{\mathsf{T}} = \mathcal{K}^{\mathsf{T}}; \mathcal{K}$.

Proof: i) The containment in $\mathbb{I}$ follows simply from univalency. The term $\mathcal{K}^\mathsf{T}{:}\mathcal{K}$ is contained in $\Omega{:}\mathcal{K}$ because $\mathcal{K} \subseteq \Omega^\mathsf{T}$. The term $\mathcal{K}^\mathsf{T}{:}\mathcal{K}$ is also contained in $\mathcal{K}^\mathsf{T}$ since shunting makes this statement equivalent with $\mathcal{K}^\mathsf{T}{:}\mathcal{K}{:}\mathcal{K} \subseteq \mathbb{I}$, where the latter holds for the idempotent and univalent $\mathcal{K}$.

It remains to prove the other direction with

$$\Omega{:}\mathcal{K} \cap \mathcal{K}^\mathsf{T} \subseteq (\Omega \cap \mathcal{K}^\mathsf{T}{:}\mathcal{K}^\mathsf{T}){:}(\mathcal{K} \cap \Omega^\mathsf{T}{:}\mathcal{K}^\mathsf{T}) = (\Omega \cap \mathcal{K}^\mathsf{T}){:}(\mathcal{K} \cap \Omega^\mathsf{T}{:}\mathcal{K}^\mathsf{T}) \subseteq \mathcal{K}^\mathsf{T}{:}\mathcal{K}.$$

ii) "$\supseteq$" is trivial: $\mathcal{K}^\mathsf{T}{:}\mathcal{K} \subseteq \mathbb{I} = \Omega \cap \Omega^\mathsf{T}$ and $\mathcal{K}^\mathsf{T}{:}\mathcal{K} \subseteq \Omega{:}\mathcal{K}^\mathsf{T}$ is via shunting $\mathcal{K}^\mathsf{T}{:}\mathcal{K}{:}\mathcal{K} = \mathcal{K}^\mathsf{T}{:}\mathcal{K} \subseteq \Omega$. On the other hand $\Omega{:}\mathcal{K}^\mathsf{T} \cap \Omega^\mathsf{T} \subseteq \Omega{:}\Omega \cap \Omega^\mathsf{T} = \Omega \cap \Omega^\mathsf{T} = \mathbb{I}$ and similarly

$$\Omega{:}\mathcal{K}^\mathsf{T} \cap \Omega^\mathsf{T} \subseteq (\Omega \cap \Omega^\mathsf{T}{:}\mathcal{K}){:}(\mathcal{K}^\mathsf{T} \cap \Omega^\mathsf{T}{:}\Omega^\mathsf{T}) \subseteq \mathbb{I}{:}\mathcal{K}^\mathsf{T} = \mathcal{K}^\mathsf{T}.$$

Together

$$\ldots \subseteq \mathcal{K}^\mathsf{T} \cap \mathbb{I} = \mathcal{K}^\mathsf{T}{:}\mathcal{K}^\mathsf{T} \cap \mathbb{I} \subseteq (\mathcal{K}^\mathsf{T} \cap \mathbb{I}{:}\mathcal{K}){:}(\mathcal{K}^\mathsf{T} \cap \mathcal{K}{:}\mathbb{I}) \subseteq \mathcal{K}^\mathsf{T}{:}\mathcal{K}. \qquad \square$$

It is mainly the counterplay of Prop. 3.3 with which we study how a neighborhood topology and a kernel-mapping topology are bijectively interrelated.

3.8 Proposition. The properties demanded for a neighborhood topology $\mathcal{U}$ may also be expressed for $\mathcal{K}$, and vice versa:

i) Given any neighborhood topology $\mathcal{U}$, the construct $\mathcal{K} := \mathtt{syq}(\mathcal{U}, \varepsilon)$ is a kernel-mapping topology.

ii) Given any kernel-mapping topology $\mathcal{K}$, the construct $\mathcal{U} := \varepsilon{:}\mathcal{K}^\mathsf{T}$ results in a neighborhood topology.

Proof: i) Given the proofs of Prop. 3.3.i,ii, it remains to prove that $\mathcal{K}$ is contracting, monotonic and idempotent: Firstly, $\mathcal{K}$ is contracting, $\mathcal{K} \subseteq \Omega^\mathsf{T}$, because the negated version $\overline{\varepsilon}^\mathsf{T}{:}\varepsilon \subseteq \overline{\mathcal{U}}^\mathsf{T}{:}\varepsilon \cup \mathcal{U}^\mathsf{T}{:}\overline{\varepsilon} = \overline{\mathtt{syq}(\mathcal{U}, \varepsilon)} = \overline{\mathcal{K}}$ follows from $\mathcal{U} \subseteq \varepsilon$.

$\mathcal{K}$ is monotonic wrt. Ω: With Def. 3.4.ii, we obviously have

$$\Omega^\mathsf{T}{:}\mathcal{U}^\mathsf{T}{:}\overline{\varepsilon} \subseteq \overline{\mathcal{U}}^\mathsf{T}{:}\varepsilon \cup \mathcal{U}^\mathsf{T}{:}\overline{\varepsilon} = \overline{\mathtt{syq}(\mathcal{U}, \varepsilon)} = \overline{\mathcal{K}}.$$

With the Schröder rule this is equivalent to $\Omega{:}\mathcal{K} \subseteq \overline{\mathcal{U}^\mathsf{T}{:}\overline{\varepsilon}}$. Now we use that $\mathcal{U} = \varepsilon{:}\mathcal{K}^\mathsf{T}$ according to Prop. 3.3.i, and that $\mathcal{K}$ is a mapping, ending in $\Omega{:}\mathcal{K} \subseteq \mathcal{K}{:}\Omega$.

That $\mathcal{K}$ is idempotent follows using Prop 3.3.ii and Prop. 3.5.vi.

The second condition that $\varepsilon{:}\mathcal{K}^\mathsf{T}$ is total follows from Prop. 3.3.ii.

The third condition:

$$\begin{aligned}&(\mathcal{K} \otimes \mathcal{K}){:}\mathfrak{M} = \mathtt{syq}((\mathcal{U} \oslash \mathcal{U}), \varepsilon) \quad \text{due to Prop. 3.5.vii}\\ &= \mathtt{syq}(\mathcal{U}{:}\mathfrak{M}^\mathsf{T}, \varepsilon) \quad \text{Prop. 3.5.iii}\\ &= \mathfrak{M}{:}\mathtt{syq}(\mathcal{U}, \varepsilon) = \mathfrak{M}{:}\mathcal{K}\end{aligned}$$

ii) $\mathcal{U}$ is total in view of Def. 3.6.ii. Contraction $\mathcal{K} \subseteq \Omega^\mathsf{T}$ is equivalent with $\overline{\varepsilon}^\mathsf{T}{:}\varepsilon \subseteq \overline{\mathcal{K}}$, further with $\overline{\varepsilon}{:}\mathcal{K} \subseteq \overline{\varepsilon}$, and finally with $\mathcal{U} = \varepsilon{:}\mathcal{K}^\mathsf{T} \subseteq \varepsilon$ as demanded. In order to prove $\mathcal{U}{:}\Omega \subseteq \mathcal{U}$, we start with monotony, univalency, and shunting applied in

$$\varepsilon;\mathcal{K}^\mathsf{T};\Omega;\mathcal{K} \subseteq \varepsilon;\mathcal{K}^\mathsf{T};\mathcal{K};\Omega \subseteq \varepsilon;\Omega = \varepsilon \qquad\Longleftrightarrow\qquad \varepsilon;\mathcal{K}^\mathsf{T};\Omega = \mathcal{U};\Omega \subseteq \mathcal{U} = \varepsilon;\mathcal{K}^\mathsf{T}.$$

$$\begin{aligned} &\mathcal{U};\mathfrak{M}^\mathsf{T} = \varepsilon;\mathcal{K}^\mathsf{T};\mathfrak{M}^\mathsf{T} = \varepsilon;\mathfrak{M}^\mathsf{T};(\mathcal{K}^\mathsf{T}\otimes\mathcal{K}^\mathsf{T})\\ &= (\varepsilon \oslash \varepsilon);(\mathcal{K}^\mathsf{T}\otimes\mathcal{K}^\mathsf{T}) \quad \text{Prop. 9.1.iv of [SW14]}\\ &= (\varepsilon;\mathcal{K}^\mathsf{T}\oslash\varepsilon;\mathcal{K}^\mathsf{T}) = (\mathcal{U}\oslash\mathcal{U}) \quad \text{Prop. 7.3.iii of [SW14]}\end{aligned}$$

For the last property, we use Prop. 3.3.ii. □

The $\mathcal{K}$ of Fig. 3.1 does not satisfy $(\mathcal{K}\otimes\mathcal{K});\mathfrak{M} = \mathfrak{M}\mathcal{K}$ and, thus, fails to satisfy the requirements for a neighborhood topology. Also the mapping $\mathcal{K}_0 := \overline{\mathbb{T};\varepsilon}$ which sends everything to the empty set would be contracting, isotonic, and idempotent without $\varepsilon;\mathcal{K}_0^\mathsf{T}$ total; however it would lead to $\mathcal{U}_0 = \mathbb{L}$ which cannot be a neighborhood system.

The open sets are often defined identifying a subset of all open sets as a so-called *basis* with the idea that all their finite intersections and arbitrary unions will then produce them all. It is often convenient to restrict such a basis to just the smallest ones, i.e., those that are not non-trivial unions. Observe that the empty set is also an open one and would be the minimal one when not explicitly excluded. For finite cases at least, it is possible to characterize a basis of open sets as follows. The topology of the real axis, for example, does not allow such atomic open sets since the basis mapping β below turns out to be the singleton injection.

3.9 Proposition. We consider the construct $\beta := \mathtt{syq}(\overline{\overline{\varepsilon};\mathcal{U}^\mathsf{T}}, \varepsilon) : X \longrightarrow \mathbf{2}^X$.

i) β is a mapping.

ii) $\beta = \left[\mathtt{glb}_\Omega(\mathcal{U}^\mathsf{T})\right]^\mathsf{T}$

iii) $\beta \subseteq \mathcal{U} \quad\Longrightarrow\quad \beta;\Omega = \mathcal{U}$

Proof: i) β is a mapping since β^T is surjective by definition of the membership relation ε and because

$$\beta^\mathsf{T};\beta = \mathtt{syq}(\varepsilon, \overline{\overline{\varepsilon};\mathcal{U}^\mathsf{T}});\mathtt{syq}(\overline{\overline{\varepsilon};\mathcal{U}^\mathsf{T}}, \varepsilon) \subseteq \mathtt{syq}(\varepsilon,\varepsilon) = \mathbb{I}.$$

ii) See Prop. 9.10 of [Sch11] relating greatest lower bounds with symmetric quotients.

iii) While it is trivial that $\beta \subseteq \mathcal{U}$ implies $\beta;\Omega \subseteq \mathcal{U};\Omega = \mathcal{U}$, we have to apply the general result

$$X \subseteq \mathtt{ubd}_E\left(\mathtt{glb}_E(X)\right),$$

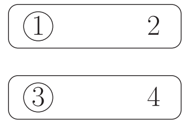
namely that X is contained in the upper bound set of its greatest lower bound with regard to some ordering E, to obtain the reverse inclusion. Applied to this case here, it simply says

$$\mathcal{U}^\mathsf{T} \subseteq \mathtt{ubd}_\Omega(\beta^\mathsf{T}) = \overline{\overline{\Omega}^\mathsf{T};\beta^\mathsf{T}} \iff \overline{\Omega}^\mathsf{T};\beta^\mathsf{T} \subseteq \overline{\mathcal{U}^\mathsf{T}} \iff \mathcal{U}^\mathsf{T};\beta \subseteq \Omega^\mathsf{T} \iff \mathcal{U}^\mathsf{T} \subseteq \Omega^\mathsf{T};\beta^\mathsf{T} \qquad □$$

We illustrate this result for a simple topology.

$$\begin{array}{c|c} & \{\}\ \{1\}\ \{2\}\ \{1,2\}\ \{3\}\ \{1,3\}\ \{2,3\}\ \{1,2,3\}\ \{4\}\ \{1,4\}\ \{2,4\}\ \{1,2,4\}\ \{3,4\}\ \{1,3,4\}\ \{2,3,4\}\ \{1,2,3,4\} \\ \begin{array}{c}1\\2\\3\\4\end{array} & \begin{pmatrix} 0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1\\ 0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1\\ 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix} \end{array}$$

$\mathcal{U}$



$$\begin{array}{c} \{\}\ \{1\}\ \{2\}\ \{1,2\}\ \{3\}\ \{1,3\}\ \{2,3\}\ \{1,2,3\}\ \{4\}\ \{1,4\}\ \{2,4\}\ \{1,2,4\}\ \{3,4\}\ \{1,3,4\}\ \{2,3,4\}\ \{1,2,3,4\} \\ \begin{pmatrix} 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0 \end{pmatrix} \end{array}$$

$\beta = \left[\mathtt{glb}_\Omega(\mathcal{U}^\mathsf{T})\right]^\mathsf{T}$

Fig. 3.4 A topology $\mathcal{U}$ with its basis mapping β

3.3 Qualifying a topology via its open sets

Qualifying a topology via a neighborhood system $\mathcal{U}$ or kernel mapping $\mathcal{K}$ has, thus, been shown to mean basically the same; $\mathcal{U}$ and $\mathcal{K}$ may be converted into one another. In what follows, we will use them interchangeably as required.

The next idea for topologies was to define them via their open sets. In a similar way as we could $\mathcal{U}, \mathcal{K}$ let more or less represent themselves mutually, we here have the versions $\mathcal{O}_V, \mathcal{O}_D$ positioned against the former two:

$$\begin{array}{ccc} \mathcal{U} & & \mathcal{O}_D \\ & \Longleftrightarrow & \\ \mathcal{K} & & \mathcal{O}_V \end{array}$$

The transitions up and down between $\mathcal{U}$ and $\mathcal{K}$ on the left of the diagram above have already been mentioned. Toggling between a vector $\mathcal{O}_V : \mathbf{2}^X \longrightarrow \mathbb{1}$ and the corresponding partial identity $\mathcal{O}_D : \mathbf{2}^X \longrightarrow \mathbf{2}^X$ is completely trivial and doesn't need any topological consideration:

$$\mathcal{O}_V = \mathcal{O}_D ; \mathbb{T} \qquad\qquad \mathcal{O}_D = \mathbb{I} \cap \mathcal{O}_V ; \mathbb{T}$$

Given any pair of a vector and a partial identity, we obtain the following results; we have, however, maintained the notations $\mathcal{O}_V, \mathcal{O}_D$ of a set of open sets.

3.10 Proposition. Given $\mathcal{O}_V$ resp. $\mathcal{O}_D$, two other relations

$$\varepsilon_\mathcal{O} := \varepsilon \cap \mathbb{T} ; \mathcal{O}_V^\mathsf{T} = \varepsilon ; \mathcal{O}_D \quad \text{ and } \quad \omega := \mathtt{syq}(\varepsilon_\mathcal{O}, \varepsilon)$$

are introduced for technical reasons. They satisfy the following properties:

i) ω is a mapping that satisfies $\omega^\mathsf{T} \subseteq \Omega$.

ii) $\varepsilon_\mathcal{O} ; \omega = \varepsilon \cap \mathbb{T} ; \omega$

iii) $\varepsilon ; \omega^\mathsf{T} = \varepsilon_\mathcal{O}$

iv) $\varepsilon_\mathcal{O} = \varepsilon \cap \mathbb{T} ; \varepsilon_\mathcal{O}$

Proof: i) The mapping property follows from the definition; furthermore

$$\omega^\mathsf{T} = \mathtt{syq}(\varepsilon, \varepsilon_\mathcal{O}) \subseteq \overline{\varepsilon^\mathsf{T} ; \overline{\varepsilon_\mathcal{O}}} \subseteq \overline{\varepsilon^\mathsf{T} ; \overline{\varepsilon}} = \Omega.$$

ii) $\varepsilon_\mathcal{O} ; \omega = \varepsilon_\mathcal{O} ; \mathtt{syq}(\varepsilon_\mathcal{O}, \varepsilon) = \varepsilon \cap \mathbb{T} ; \mathtt{syq}(\varepsilon_\mathcal{O}, \varepsilon)$ Prop. 8.12 of [Sch11]
$= \varepsilon \cap \mathbb{T} ; \omega$ by definition

iii) $\varepsilon;\omega^{\mathsf{T}} = \varepsilon;\mathtt{syq}(\varepsilon, \varepsilon_{\mathcal{O}}) = \varepsilon_{\mathcal{O}}$

iv) $\varepsilon_{\mathcal{O}} = \varepsilon \cap \mathbb{T};\mathcal{O}_V^{\mathsf{T}} = \varepsilon \cap (\mathbb{T};\varepsilon \cap \mathbb{T};\mathcal{O}_V^{\mathsf{T}}) = \varepsilon \cap \mathbb{T};(\varepsilon \cap \mathbb{T};\mathcal{O}_V^{\mathsf{T}}) = \varepsilon \cap \mathbb{T};\varepsilon_{\mathcal{O}}$ □

The global situation with several methods of characterizing a topology is best visualized with Fig. 3.5.

$$\begin{array}{cc}
 & \begin{array}{cccccccccccccccc} \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} & \{d\} & \{a,d\} & \{b,d\} & \{a,b,d\} & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\} \end{array} \\
\begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1 \\ 0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\ 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \end{pmatrix} \\
\begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1 \\ 0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1 \\ 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix} \\
\begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 0&1&0&0&0&1&0&1&0&0&0&0&0&1&0&1 \\ 0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1 \\ 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix}
\end{array}$$

$$\begin{array}{ccc}
 & \begin{array}{cccccccccccccccc} \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} & \{d\} & \{a,d\} & \{b,d\} & \{a,b,d\} & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\} \end{array} & \\
\begin{array}{r} \{\} \\ \{a\} \\ \{b\} \\ \{a,b\} \\ \{c\} \\ \{a,c\} \\ \{b,c\} \\ \{a,b,c\} \\ \{d\} \\ \{a,d\} \\ \{b,d\} \\ \{a,b,d\} \\ \{c,d\} \\ \{a,c,d\} \\ \{b,c,d\} \\ \{a,b,c,d\} \end{array} & \begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix} & \begin{pmatrix} 1\\1\\0\\0\\1\\1\\1\\1\\0\\0\\0\\0\\1\\1\\1\\1 \end{pmatrix}
\end{array}$$

ε $\quad \mathcal{U} \quad$ $\varepsilon_{\mathcal{O}} := \varepsilon \cap \mathbb{T};\mathcal{O}_V^{\mathsf{T}} = \varepsilon;\mathcal{K} \cap \varepsilon$ $\quad$ $\mathcal{K} := \mathtt{syq}(\mathcal{U}, \varepsilon)$ indicating $\mathcal{O}_D$ as diagonal $\quad \mathcal{O}_V$

Fig. 3.5 Open sets as membership and open kernel mapping

To relate the two, $\mathcal{O}_D, \mathcal{O}_V$, with $\varepsilon_{\mathcal{O}}$, the membership ε, but restricted to membership in sets qualified as being open, is a little complicated:

$$\varepsilon_{\mathcal{O}} = \varepsilon \cap \mathbb{T};\mathcal{O}_V^{\mathsf{T}} = \varepsilon;\mathcal{O}_D, \qquad \mathcal{O}_V = \varepsilon_{\mathcal{O}}^{\mathsf{T}};\mathbb{T} \cup \mathtt{syq}(\varepsilon, \perp\!\!\!\perp)$$

The disturbing term $\mathtt{syq}(\varepsilon, \perp\!\!\!\perp) = \overline{\varepsilon^{\mathsf{T}};\mathbb{T}}$ in the definition of $\mathcal{O}_V$ above owes its existence to the fact that also the empty set is by definition an open set, but does not contain any element. The marking of the empty set would not be shown in $\varepsilon_{\mathcal{O}}^{\mathsf{T}};\mathbb{T}$.

We may, thus expect some purely technical difficulties when always adding or deleting it. And this is the reason why we have decided for an alternative approach when relating the ‘membership-in-open-sets’ relation $\varepsilon_{\mathcal{O}}$ forth and back with the $\mathcal{U}, \mathcal{K}$ side. The mapping ω reproduces open environments. It does not, however, map environments to their open kernel, as $\mathcal{K}$, but to the empty set.

	{}	{a}	{b}	{a,b}	{c}	{a,c}	{b,c}	{a,b,c}	{d}	{a,d}	{b,d}	{a,b,d}	{c,d}	{a,c,d}	{b,c,d}	{a,b,c,d}
{}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{a}	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{b}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{a,b}	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{c}	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
{a,c}	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
{b,c}	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
{a,b,c}	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
{d}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{a,d}	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{b,d}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{a,b,d}	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{c,d}	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
{a,c,d}	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
{b,c,d}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
{a,b,c,d}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

	{}	{a}	{b}	{a,b}	{c}	{a,c}	{b,c}	{a,b,c}	{d}	{a,d}	{b,d}	{a,b,d}	{c,d}	{a,c,d}	{b,c,d}	{a,b,c,d}
{}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{a}	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{b}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{a,b}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{c}	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
{a,c}	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
{b,c}	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
{a,b,c}	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
{d}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{a,d}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{b,d}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{a,b,d}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{c,d}	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
{a,c,d}	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
{b,c,d}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
{a,b,c,d}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Fig. 3.6 $\mathcal{K}$ as opposed to ω

A topology may also be characterized when the open sets are given as a vector or its equivalent partial identity. One will recognize that in the following two definitions the first condition concerns declaring the empty as well as the full set to be open, the second that arbitrary unions and the third that finite intersections of open sets are open again.

3.11 Definition. A vector $\mathcal{O}_V$ along $\mathbf{2}^X$ will be called an **open set topology** provided

i) $\mathtt{syq}(\varepsilon, \perp\!\!\!\perp) \subseteq \mathcal{O}_V \qquad \mathtt{syq}(\varepsilon, \top\!\!\!\top) \subseteq \mathcal{O}_V$,

ii) $v \subseteq \mathcal{O}_V \quad \Longrightarrow \quad \mathtt{syq}(\varepsilon, \varepsilon; v) \subseteq \mathcal{O}_V$ for all vectors $v \subseteq \mathbf{2}^X$,

iii) $\mathfrak{M}^{\mathsf{T}}; (\mathcal{O}_V \oslash \mathcal{O}_V) \subseteq \mathcal{O}_V$. □

The following means largely the same. The only difference rests in the representation as a partial identity as opposed to a column vector which leads to minor technical changes.

3.12 Definition. A partial identity $\mathcal{O}_D$ on $\mathbf{2}^X$ is an **open diagonal topology** provided

i) $\mathtt{syq}(\varepsilon, \perp\!\!\!\perp) \subseteq \mathcal{O}_D; \top\!\!\!\top \qquad \mathtt{syq}(\varepsilon, \top\!\!\!\top) \subseteq \mathcal{O}_D; \top\!\!\!\top$,

ii) $v \subseteq \mathcal{O}_D; \top\!\!\!\top \quad \Longrightarrow \quad \mathtt{syq}(\varepsilon, \varepsilon; v) \subseteq \mathcal{O}_D; \top\!\!\!\top$ for all vectors $v \subseteq \mathbf{2}^X$,

iii) $(\mathcal{O}_D \otimes \mathcal{O}_D); \mathfrak{M} \subseteq \mathfrak{M}; \mathcal{O}_D$. □

It is relatively easy to see that these two versions mean the same, however, formulated with a vector $\mathcal{O}_V$ or a partial identity $\mathcal{O}_D$, respectively. Only the equivalence of the (iii)'s may need a bit of explanation:

$$\begin{aligned}
&\mathfrak{M}^{\mathsf{T}}; (\mathcal{O}_V \otimes \mathcal{O}_V) = \mathfrak{M}^{\mathsf{T}}; (\mathcal{O}_D;\mathbb{T} \otimes \mathcal{O}_D;\mathbb{T}) \quad \text{by definition}\\
&= \mathfrak{M}^{\mathsf{T}}; (\mathcal{O}_D \otimes \mathcal{O}_D);\mathbb{T} \quad \text{Prop. 7.3.iv of [SW14]}\\
&\subseteq \mathcal{O}_D; \mathfrak{M}^{\mathsf{T}} \subseteq \mathcal{O}_D;\mathbb{T} = \mathcal{O}_V \quad \text{Def. 3.12.iii}
\end{aligned}$$

$$\begin{aligned}
&(\mathcal{O}_D \otimes \mathcal{O}_D); \mathfrak{M} = (\mathbb{I} \cap \mathcal{O}_V;\mathbb{T} \otimes \mathbb{I} \cap \mathcal{O}_V;\mathbb{T}); \mathfrak{M} \quad \text{by definition}\\
&= \big[\, (\mathbb{I} \otimes \mathbb{I}) \cap (\mathcal{O}_V;\mathbb{T} \otimes \mathcal{O}_V;\mathbb{T}) \,\big]; \mathfrak{M} \quad \text{Prop. 7.3.vi of [SW14]}\\
&= \big[\mathbb{I} \cap (\mathcal{O}_V \otimes \mathcal{O}_V);\mathbb{T}\big]; \mathfrak{M}\\
&= \big[\mathbb{I} \cap \mathbb{T}; (\mathcal{O}_V^{\mathsf{T}} \oslash \mathcal{O}_V^{\mathsf{T}})\,\big]; \mathfrak{M} \quad \text{transposing a partial identity}\\
&\subseteq \mathfrak{M} \cap \mathbb{T}; (\mathcal{O}_V^{\mathsf{T}} \oslash \mathcal{O}_V^{\mathsf{T}}); \mathfrak{M}\\
&\subseteq \mathfrak{M};\mathbb{I} \cap \mathbb{T};\mathcal{O}_V^{\mathsf{T}} \quad \text{Def. 3.11.iii}\\
&\subseteq (\mathfrak{M} \cap \ldots);(\mathbb{I} \cap \mathfrak{M}^{\mathsf{T}};\mathbb{T};\mathcal{O}_V^{\mathsf{T}}) \quad \text{Dedekind rule}\\
&\subseteq \mathfrak{M};(\mathbb{I} \cap \mathbb{T};\mathcal{O}_V^{\mathsf{T}})\\
&= \mathfrak{M};\mathcal{O}_D
\end{aligned}$$

The following is a slight variant of the latter two definitions.

3.13 Definition. A relation $\varepsilon_{\mathcal{O}} : X \longrightarrow \mathbf{2}^X$ will be called a **membership-in-open-sets topology** provided

i) $\varepsilon_{\mathcal{O}};\mathbb{T} = \mathbb{T} \qquad \varepsilon_{\mathcal{O}} = \varepsilon \cap \mathbb{T};\varepsilon_{\mathcal{O}}$,

ii) $v \subseteq \varepsilon_{\mathcal{O}}^{\mathsf{T}};\mathbb{T} \quad \Longrightarrow \quad \mathtt{syq}(\varepsilon, \varepsilon;v) \subseteq \varepsilon_{\mathcal{O}}^{\mathsf{T}};\mathbb{T} \cup \overline{\varepsilon^{\mathsf{T}};\mathbb{T}}$ i.e. $= \omega^{\mathsf{T}};\mathbb{T}$ for all vectors $v \subseteq \mathbf{2}^X$,

iii) $(\varepsilon_{\mathcal{O}} \oslash \varepsilon_{\mathcal{O}}) \subseteq \varepsilon_{\mathcal{O}}; \mathfrak{M}^{\mathsf{T}}$. □

We will also indicate in which way this is cryptomorphic with, e.g., Def. 3.11.

$$\begin{aligned}
&\mathfrak{M}^{\mathsf{T}}; (\mathcal{O}_V \otimes \mathcal{O}_V) = \mathfrak{M}^{\mathsf{T}}; (\varepsilon_{\mathcal{O}}^{\mathsf{T}};\mathbb{T} \cup \overline{\varepsilon^{\mathsf{T}};\mathbb{T}} \otimes \varepsilon_{\mathcal{O}}^{\mathsf{T}};\mathbb{T} \cup \overline{\varepsilon^{\mathsf{T}};\mathbb{T}}) \quad \text{by definition}\\
&= \mathfrak{M}^{\mathsf{T}};\big[\pi;(\varepsilon_{\mathcal{O}}^{\mathsf{T}};\mathbb{T} \cup \overline{\varepsilon^{\mathsf{T}};\mathbb{T}}) \cap \rho;(\varepsilon_{\mathcal{O}}^{\mathsf{T}};\mathbb{T} \cup \overline{\varepsilon^{\mathsf{T}};\mathbb{T}})\big] \quad \text{expanding the join operator}\\
&= \mathfrak{M}^{\mathsf{T}};\big[(\pi;\varepsilon_{\mathcal{O}}^{\mathsf{T}};\mathbb{T} \cap \rho;\varepsilon_{\mathcal{O}}^{\mathsf{T}};\mathbb{T}) \cup (\ldots \cap \overline{\rho;\varepsilon^{\mathsf{T}};\mathbb{T}}) \cup (\overline{\pi;\varepsilon^{\mathsf{T}};\mathbb{T}} \cap \ldots) \cup (\overline{\pi;\varepsilon^{\mathsf{T}};\mathbb{T}} \cap \ldots)\big] \quad \text{distributivity}\\
&\subseteq \mathfrak{M}^{\mathsf{T}};(\pi;\varepsilon_{\mathcal{O}}^{\mathsf{T}};\mathbb{T} \cap \rho;\varepsilon_{\mathcal{O}}^{\mathsf{T}};\mathbb{T}) \cup \mathfrak{M}^{\mathsf{T}};\rho;\overline{\varepsilon^{\mathsf{T}};\mathbb{T}} \cup \mathfrak{M}^{\mathsf{T}};\pi;\overline{\varepsilon^{\mathsf{T}};\mathbb{T}} \cup \mathfrak{M}^{\mathsf{T}};\pi;\overline{\varepsilon^{\mathsf{T}};\mathbb{T}}\\
&\subseteq \mathfrak{M}^{\mathsf{T}}; (\varepsilon_{\mathcal{O}}^{\mathsf{T}};\mathbb{T} \otimes \varepsilon_{\mathcal{O}}^{\mathsf{T}};\mathbb{T}) \cup \overline{\varepsilon^{\mathsf{T}};\mathbb{T}} \quad \text{since } \mathfrak{M};\varepsilon^{\mathsf{T}} = \pi;\varepsilon^{\mathsf{T}} \cap \rho;\varepsilon^{\mathsf{T}} \cap \text{ in Prop. 9.1.iv of [SW14]}\\
&= \mathfrak{M}^{\mathsf{T}}; (\varepsilon_{\mathcal{O}}^{\mathsf{T}} \otimes \varepsilon_{\mathcal{O}}^{\mathsf{T}});\mathbb{T} \cup \overline{\varepsilon^{\mathsf{T}};\mathbb{T}}\\
&= \varepsilon_{\mathcal{O}}^{\mathsf{T}};\mathbb{T} \cup \overline{\varepsilon^{\mathsf{T}};\mathbb{T}} = \mathcal{O}_V \quad \text{Def. 3.13.iii}
\end{aligned}$$

$$\begin{aligned}
&(\varepsilon_{\mathcal{O}} \oslash \varepsilon_{\mathcal{O}}) = (\varepsilon \cap \mathbb{T};\mathcal{O}_V^{\mathsf{T}} \oslash \varepsilon \cap \mathbb{T};\mathcal{O}_V^{\mathsf{T}}) \quad \text{by definition}\\
&= (\varepsilon \oslash \varepsilon) \cap (\mathbb{T};\mathcal{O}_V^{\mathsf{T}} \oslash \mathbb{T};\mathcal{O}_V^{\mathsf{T}})\\
&= \varepsilon; \mathfrak{M}^{\mathsf{T}} \cap \mathbb{T};\mathcal{O}_V^{\mathsf{T}}; \mathfrak{M}^{\mathsf{T}} \quad \text{Prop. 9.1.iv of [SW14] and Def. 3.11.iii}\\
&= (\varepsilon \cap \mathbb{T};\mathcal{O}_V^{\mathsf{T}}); \mathfrak{M}^{\mathsf{T}} = \varepsilon_{\mathcal{O}}; \mathfrak{M}^{\mathsf{T}}
\end{aligned}$$

One may also find it difficult to see how to obtain $\mathtt{syq}(\varepsilon, \mathbb{T}) \subseteq \mathcal{O}_V$ of Def. 3.11.i, but

$$\mathbb{T} = \varepsilon_{\mathcal{O}};\mathbb{T} = (\varepsilon \cap \mathbb{T};\varepsilon_{\mathcal{O}});\mathbb{T} = \varepsilon;(\mathbb{T} \cap \varepsilon_{\mathcal{O}}^{\mathsf{T}};\mathbb{T}) = \varepsilon;\varepsilon_{\mathcal{O}}^{\mathsf{T}};\mathbb{T}$$

allows us to choose $v := \varepsilon_{\mathcal{O}}^{\mathsf{T}};\mathbb{T}$, so that $\mathcal{O}_V = \varepsilon_{\mathcal{O}}^{\mathsf{T}};\mathbb{T} \cup \overline{\varepsilon^{\mathsf{T}};\mathbb{T}} \supseteq \mathtt{syq}(\varepsilon, \varepsilon;\varepsilon_{\mathcal{O}}^{\mathsf{T}};\mathbb{T}) = \mathtt{syq}(\varepsilon, \mathbb{T})$. The cryptomorphy of the topology definitions Def. 3.4, Def. 3.6 and Def. 3.11 has, thus, slightly informally been established.

The transitions between $\mathcal{U}, \mathcal{K}$ and $\mathcal{O}_V$ shall now be investigated.

3.14 Proposition. Given an open set topology $\mathcal{O}_V$, the construct

$\mathcal{U} := \varepsilon;(\Omega \cap \mathcal{O}_V;\mathbb{T}) = (\varepsilon \cap \mathbb{T};\mathcal{O}_V^{\mathsf{T}});\Omega = \varepsilon;\mathcal{O}_D;\Omega = \varepsilon_{\mathcal{O}};\Omega$

constitutes a neighborhood topology.

Proof: The variants mentioned are obvious; in what follows we concentrate on the last. The numbering of the proof follows that of Def. 3.4.

i) We have $\mathbb{T} = \varepsilon;\mathtt{syq}(\varepsilon, \mathbb{T}) \subseteq \varepsilon;\mathcal{O}_V$ following Def. 3.11.i and may proceed with

$\mathcal{U};\mathbb{T} = \varepsilon;(\Omega \cap \mathcal{O}_V;\mathbb{T});\mathbb{T}$ by definition of $\mathcal{U}$
$= \varepsilon;(\Omega;\mathbb{T} \cap \mathcal{O}_V;\mathbb{T})$ masking
$= \varepsilon;\mathcal{O}_V;\mathbb{T}$ since $\Omega;\mathbb{T} = \mathbb{T}$
$= \mathbb{T};\mathbb{T} = \mathbb{T}$ see above
$\mathcal{U} = \varepsilon;(\Omega \cap \mathcal{O}_V;\mathbb{T}) \subseteq \varepsilon;\Omega = \varepsilon$

ii) $\mathcal{U};\Omega = \varepsilon;(\Omega \cap \mathcal{O}_V;\mathbb{T});\Omega = \varepsilon;(\Omega;\Omega \cap \mathcal{O}_V;\mathbb{T}) = \varepsilon;(\Omega \cap \mathcal{O}_V;\mathbb{T}) = \mathcal{U}$

iii) Here it appears more convenient to use the condition on $\mathcal{O}_D$.

$(\mathcal{U} \otimes \mathcal{U});\mathfrak{M} = (\varepsilon;\mathcal{O}_D;\Omega \otimes \varepsilon;\mathcal{O}_D;\Omega);\mathfrak{M}$ by definition
$= (\varepsilon \otimes \varepsilon);(\mathcal{O}_D;\Omega \otimes \mathcal{O}_D;\Omega);\mathfrak{M}$ Prop. 2.6
$= (\varepsilon \otimes \varepsilon);(\mathcal{O}_D \otimes \mathcal{O}_D);(\Omega \otimes \Omega);\mathfrak{M}$ since $(\mathcal{O}_D \otimes \mathcal{O}_D)$ is univalent
$= (\varepsilon \otimes \varepsilon);(\mathcal{O}_D \otimes \mathcal{O}_D);\mathfrak{M};\Omega$ Prop. 9.4.iv of [SW14]
$\subseteq (\varepsilon \otimes \varepsilon);\mathfrak{M};\mathcal{O}_D;\Omega$ Def. 3.12.iii
$= \varepsilon;\mathcal{O}_D;\Omega$ Prop. 9.2.v of [SW14]
$= \mathcal{U}$ by definition

iv) $\mathcal{U};\mathtt{syq}(\varepsilon, \mathcal{U}) = \varepsilon;(\Omega \cap \mathcal{O}_V;\mathbb{T});\mathtt{syq}(\varepsilon, \mathcal{U})$ by definition
$= \varepsilon;(\Omega;\mathtt{syq}(\varepsilon, \mathcal{U}) \cap \mathcal{O}_V;\mathbb{T})$ masking
$= \varepsilon;(\overline{\varepsilon^{\mathsf{T}};\overline{\varepsilon}};\mathtt{syq}(\varepsilon, \mathcal{U}) \cap \mathcal{O}_V;\mathbb{T})$ definition of Ω
$= \varepsilon;(\overline{\varepsilon^{\mathsf{T}};\overline{\varepsilon;\mathtt{syq}(\varepsilon, \mathcal{U})}} \cap \mathcal{O}_V;\mathbb{T})$ the $\mathtt{syq}$ is a transposed mapping
$= \varepsilon;(\overline{\varepsilon^{\mathsf{T}};\overline{\mathcal{U}}} \cap \mathcal{O}_V;\mathbb{T})$ property of the symmeric quotient
$= \varepsilon;(\overline{\varepsilon^{\mathsf{T}};\overline{\varepsilon;(\Omega \cap \mathcal{O}_V;\mathbb{T})}} \cap \mathcal{O}_V;\mathbb{T})$ expanded
$\supseteq \varepsilon;(\Omega \cap \mathcal{O}_V;\mathbb{T} \cap \mathcal{O}_V;\mathbb{T})$ trivial
$= \mathcal{U}$ by definition □

One may also go from $\mathcal{U}$ (always connected with its $\mathcal{K}$) to $\mathcal{O}_V$:

3.15 Proposition. Given any neighborhood topology $\mathcal{U}$, the construct $\mathcal{O}_V := \mathcal{K}^{\mathsf{T}};\mathbb{T}$ is an open set topology.

Proof: The numbering follows Def. 3.11.

i) We use Prop. 2.1 of [SW14] for the first inclusion:

$\mathcal{O}_V = \mathcal{K}^{\mathsf{T}};\mathbb{T} = \mathtt{syq}(\varepsilon, \mathcal{U});\mathbb{T} = (\overline{\overline{\varepsilon}^{\mathsf{T}};\mathcal{U}} \cap \overline{\varepsilon^{\mathsf{T}};\overline{\mathcal{U}}});\mathbb{T} \supseteq (\mathbb{I} \cap \overline{\varepsilon^{\mathsf{T}};\overline{\mathcal{U}}});\mathbb{T}$
$= \overline{(\mathbb{I} \cap \varepsilon^{\mathsf{T}};\overline{\mathcal{U}})};\mathbb{T} \supseteq \overline{\varepsilon^{\mathsf{T}};\overline{\mathcal{U}}};\mathbb{T} \supseteq \overline{\varepsilon^{\mathsf{T}}};\mathbb{T} = \mathtt{syq}(\varepsilon, \perp\!\!\!\perp)$

In order to prove the second inclusion, we define g as notation for the point $g := \mathtt{syq}(\varepsilon, \mathbb{T})$ and start showing $\Omega; g = \overline{\varepsilon^{\mathsf{T}}; \overline{\varepsilon; g}} = \overline{\varepsilon^{\mathsf{T}}; \overline{\varepsilon; \mathtt{syq}(\varepsilon, \mathbb{T})}} = \overline{\varepsilon^{\mathsf{T}}; \overline{\overline{\mathbb{T}}}} = \overline{\varepsilon^{\mathsf{T}}; \mathbb{⊥}} = \mathbb{T}$. Now we get $g \subseteq \mathcal{O}_V$:

$$g = \mathtt{syq}(\varepsilon, \mathbb{T}) = \mathtt{syq}(\varepsilon, \mathcal{U}; \mathbb{T}) = \mathtt{syq}(\varepsilon, \mathcal{U}; \Omega; g) = \mathtt{syq}(\varepsilon, \mathcal{U}; g) = \mathtt{syq}(\varepsilon, \mathcal{U}); g = \mathcal{K}^{\mathsf{T}}; g \subseteq \mathcal{K}^{\mathsf{T}}; \mathbb{T}$$

ii) We prove in advance that $v \subseteq \mathcal{O}_V = \mathcal{K}^{\mathsf{T}}; \mathbb{T}$ implies $\mathcal{K}^{\mathsf{T}}; v = v$:

$$v = \mathcal{K}^{\mathsf{T}}; \mathbb{T} \cap v \subseteq (\mathcal{K}^{\mathsf{T}} \cap v; \mathbb{T}); (\mathbb{T} \cap \mathcal{K}; v) = (\mathcal{K}^{\mathsf{T}} \cap v; \mathbb{T}); \mathcal{K}; v \subseteq \mathcal{K}^{\mathsf{T}}; \mathcal{K}; v \subseteq v,$$

since $\mathcal{K}$ is univalent, i.e. an equality. Therefore with idempotency

$$\mathcal{K}^{\mathsf{T}}; v = \mathcal{K}^{\mathsf{T}}; \mathcal{K}^{\mathsf{T}}; \mathcal{K}; v = \mathcal{K}^{\mathsf{T}}; \mathcal{K}; v = v.$$

Now follows $\mathcal{U}; v = \varepsilon; \mathcal{K}^{\mathsf{T}}; v = \varepsilon; v$, so that $\mathtt{syq}(\varepsilon, \mathcal{U}; v) = \mathtt{syq}(\varepsilon, \varepsilon; v) =: p$, which is necessarily a point; it represents the union in the powerset. For p, we prove

$$\varepsilon; p = \varepsilon; \mathtt{syq}(\varepsilon, \varepsilon; v) = \varepsilon; v = \mathcal{U}; v \subseteq \mathcal{U}; \Omega; p = \mathcal{U}; p \subseteq \varepsilon; p$$

using

$$\varepsilon; v \subseteq \varepsilon; p \quad \iff \quad \varepsilon^{\mathsf{T}}; \overline{\varepsilon; p} \subseteq \overline{v} \quad \iff \quad v \subseteq \Omega; p = \overline{\varepsilon^{\mathsf{T}}; \overline{\varepsilon}}; p = \overline{\varepsilon^{\mathsf{T}}; \overline{\varepsilon; p}}.$$

This allows us to reason

$$p = \mathtt{syq}(\varepsilon, \varepsilon; v) = \mathtt{syq}(\varepsilon, \varepsilon; \mathcal{K}^{\mathsf{T}}; v) = \mathtt{syq}(\varepsilon, \mathcal{U}; v) = \mathtt{syq}(\varepsilon, \mathcal{U}; p) = \mathtt{syq}(\varepsilon, \mathcal{U}); p = \mathcal{K}^{\mathsf{T}}; p$$

In total, we have shown that $v \subseteq \mathcal{O}_V$ implies $\mathtt{syq}(\varepsilon, \varepsilon; v) \subseteq \mathcal{O}_V$.

iii) $\mathfrak{M}^{\mathsf{T}}; (\mathcal{O}_V \oslash \mathcal{O}_V) = \mathfrak{M}^{\mathsf{T}}; (\mathcal{K}^{\mathsf{T}}; \mathbb{T} \oslash \mathcal{K}^{\mathsf{T}}; \mathbb{T})$
$= \mathfrak{M}^{\mathsf{T}}; (\mathcal{K}^{\mathsf{T}} \otimes \mathcal{K}^{\mathsf{T}}); \mathbb{T}$ Prop. 7.3.iv of [SW14]
$= \mathcal{K}^{\mathsf{T}}; \mathfrak{M}^{\mathsf{T}}; \mathbb{T}$ Def. 3.6.iii
$\subseteq \mathcal{K}^{\mathsf{T}}; \mathbb{T} = \mathcal{O}_V$ □

Having established the interrelationship, we proceed proving some additional formulae that quite intuitively characterize the different aspects of a topology.

3.16 Proposition.

i) $\mathcal{O}_D = \mathcal{K}^{\mathsf{T}}; \mathcal{K}$ $\qquad \varepsilon_{\mathcal{O}} = \varepsilon; \mathcal{K}^{\mathsf{T}}; \mathcal{K} = \mathcal{U}; \mathcal{K}$

ii) $\mathcal{K}; \omega = \mathcal{K}$ $\qquad \omega; \mathcal{K} = \omega$ $\qquad \mathbb{T}; \omega = \mathbb{T}; \mathcal{K}$

iii) $\varepsilon_{\mathcal{O}} = \mathcal{U}; \omega^{\mathsf{T}}$

iv) $\varepsilon_{\mathcal{O}}; \omega = \varepsilon_{\mathcal{O}} = \varepsilon_{\mathcal{O}}; \omega^{\mathsf{T}}$

v) $\omega; \omega = \omega$

vi) $\omega^{\mathsf{T}}; \omega = \mathcal{K}^{\mathsf{T}}; \mathcal{K} = \omega \cap \omega^{\mathsf{T}}; \mathbb{T}$

vii) $\varepsilon_{\mathcal{O}}; \mathcal{K}^{\mathsf{T}} = \mathcal{U}$

viii) $(\omega \otimes \omega); \mathfrak{M} = \mathtt{syq}((\varepsilon_{\mathcal{O}} \oslash \varepsilon_{\mathcal{O}}), \varepsilon)$

Proof: i) The first follows from the definition $\mathcal{O}_V := \mathcal{K}^\mathsf{T} ; \mathbb{T}$. The second:

$$\varepsilon_\mathcal{O} = \varepsilon ; \mathcal{O}_D = \varepsilon ; \mathcal{K}^\mathsf{T} ; \mathcal{K}$$

ii) We easily observe $\mathcal{K}^\mathsf{T} ; \mathcal{K} ; \mathcal{K}^\mathsf{T} = \mathcal{K}^\mathsf{T}$, so that

$$\mathcal{K} ; \omega = \mathcal{K} ; \mathtt{syq}(\varepsilon_\mathcal{O}, \varepsilon) = \mathtt{syq}(\varepsilon_\mathcal{O} ; \mathcal{K}^\mathsf{T}, \varepsilon) = \mathtt{syq}(\varepsilon ; \mathcal{K}^\mathsf{T} ; \mathcal{K} ; \mathcal{K}^\mathsf{T}, \varepsilon) = \mathtt{syq}(\varepsilon ; \mathcal{K}^\mathsf{T}, \varepsilon) = \mathtt{syq}(\mathcal{U}, \varepsilon) = \mathcal{K}.$$

For the second statement, we prove just $\omega ; \mathcal{K} \subseteq \omega$ from which equality follows since $\omega ; \mathcal{K}$ as well as ω are mappings. Via shunting this is equivalent with

$$\mathtt{syq}(\varepsilon_\mathcal{O}, \varepsilon) = \omega \subseteq \omega ; \mathcal{K}^\mathsf{T} = \mathtt{syq}(\varepsilon_\mathcal{O}, \varepsilon) ; \mathcal{K}^\mathsf{T} = \mathtt{syq}(\varepsilon_\mathcal{O}, \varepsilon ; \mathcal{K}^\mathsf{T}) = \mathtt{syq}(\varepsilon_\mathcal{O}, \mathcal{U}).$$

Expanding the symmetric quotients, we use $\mathcal{U} \subseteq \varepsilon$ to find out that it suffices to prove

$$\varepsilon_\mathcal{O}^\mathsf{T} ; \overline{\mathcal{U}} \subseteq \varepsilon_\mathcal{O}^\mathsf{T} ; \overline{\varepsilon}$$

which follows from

$$\varepsilon_\mathcal{O} ; \overline{\varepsilon_\mathcal{O}^\mathsf{T} ; \overline{\varepsilon}} = \varepsilon ; \mathcal{K}^\mathsf{T} ; \mathcal{K} ; \overline{\mathcal{K}^\mathsf{T} ; \mathcal{K} ; \varepsilon^\mathsf{T} ; \overline{\varepsilon}} = \varepsilon ; \mathcal{K}^\mathsf{T} ; \overline{\mathcal{K} ; \mathcal{K}^\mathsf{T} ; \mathcal{K} ; \varepsilon^\mathsf{T} ; \overline{\varepsilon}} = \varepsilon ; \mathcal{K}^\mathsf{T} ; \overline{\mathcal{K} ; \varepsilon^\mathsf{T} ; \overline{\varepsilon}} = \varepsilon ; \mathcal{K}^\mathsf{T} ; \mathcal{K} ; \overline{\overline{\varepsilon^\mathsf{T} ; \overline{\varepsilon}}}$$
$$= \varepsilon ; \mathcal{K}^\mathsf{T} ; \mathcal{K} ; \Omega = \varepsilon_\mathcal{O} ; \Omega \subseteq \mathcal{U} ; \Omega = \mathcal{U}.$$

$$\mathbb{T} ; \omega = \mathbb{T} ; \omega ; \mathcal{K} \subseteq \mathbb{T} ; \mathcal{K} \qquad\qquad \mathbb{T} ; \mathcal{K} = \mathbb{T} ; \mathcal{K} ; \omega \subseteq \mathbb{T} ; \omega$$

iii) $\mathcal{U} ; \omega^\mathsf{T} = \varepsilon ; \mathcal{K}^\mathsf{T} ; \omega^\mathsf{T} = \varepsilon ; \omega^\mathsf{T}$ using the second of (ii)

iv) $\varepsilon_\mathcal{O} ; \omega = \mathcal{U} ; \mathcal{K} ; \omega = \mathcal{U} ; \mathcal{K} = \varepsilon_\mathcal{O}$ employing (i,ii)

$\varepsilon_\mathcal{O} \subseteq \varepsilon_\mathcal{O} ; \omega ; \omega^\mathsf{T}$ ω is a mapping
$= \varepsilon_\mathcal{O} ; \omega^\mathsf{T}$ preceeding result
$= \varepsilon ; \mathcal{K}^\mathsf{T} ; \mathcal{K} ; \omega^\mathsf{T}$
$\subseteq \varepsilon ; \omega^\mathsf{T} = \varepsilon_\mathcal{O}$ (i)

v) $\omega ; \omega = \omega ; \mathtt{syq}(\varepsilon_\mathcal{O}, \varepsilon) = \mathtt{syq}(\varepsilon_\mathcal{O} ; \omega^\mathsf{T}, \varepsilon)$
$= \mathtt{syq}(\varepsilon_\mathcal{O}, \varepsilon)$ (iv)
$= \omega$

vi) $\mathcal{K}^\mathsf{T} ; \mathcal{K} = \omega^\mathsf{T} ; \mathcal{K}^\mathsf{T} ; \mathcal{K} ; \omega \subseteq \omega^\mathsf{T} ; \omega = \mathcal{K}^\mathsf{T} ; \omega^\mathsf{T} ; \omega ; \mathcal{K} \subseteq \mathcal{K}^\mathsf{T} ; \mathcal{K}$ using (ii) twice

The second equality follows using Prop. 2.1 for the univalent $\omega^\mathsf{T} ; \omega$:

$$\omega^\mathsf{T} ; \omega ; \mathtt{syq}(\varepsilon, \varepsilon) = \omega^\mathsf{T} ; \omega ; \mathbb{T} \cap \mathtt{syq}(\varepsilon ; \omega^\mathsf{T} ; \omega, \varepsilon).$$

Now, ω is total and $\mathtt{syq}(\varepsilon, \varepsilon) = \mathbb{I}$, so that this means — observing $\varepsilon_\mathcal{O} = \varepsilon ; \omega^\mathsf{T} ; \omega$ — in fact

$$\omega^\mathsf{T} ; \omega = \omega^\mathsf{T} ; \mathbb{T} \cap \omega.$$

vii) $\varepsilon_\mathcal{O} ; \mathcal{K}^\mathsf{T} = \varepsilon ; \mathcal{K}^\mathsf{T} ; \mathcal{K} ; \mathcal{K}^\mathsf{T} = \varepsilon ; \mathcal{K}^\mathsf{T} = \mathcal{U}$

viii) $(\omega \otimes \omega) ; \mathfrak{M} = (\omega \otimes \omega) ; \mathtt{syq}((\varepsilon \otimes\!\!\!< \varepsilon), \varepsilon) = \mathtt{syq}((\varepsilon ; \omega^\mathsf{T} \otimes\!\!\!< \varepsilon ; \omega^\mathsf{T}), \varepsilon)$
$= \mathtt{syq}((\varepsilon_\mathcal{O} \otimes\!\!\!< \varepsilon_\mathcal{O}), \varepsilon)$ □

Now follow some other transitions that might also be composed from preceding ones, but require other techniques.

3.17 Proposition. Given the membership-in-open-sets topology $\varepsilon_\mathcal{O}$ according to Def. 3.13, one will obtain via $\mathcal{U} := \varepsilon_\mathcal{O} ; \Omega$ a neighborhood topology.

Proof: We immediately have the counterplay of Prop. 3.3.i,ii between $\mathcal{U}$ and $\mathcal{K}$. The present result is shown for $\mathcal{U}$ only.

i) $\mathcal{U};\mathbb{T} = \varepsilon_{\mathcal{O}};\Omega;\mathbb{T} = \varepsilon_{\mathcal{O}};\mathbb{T} = \mathbb{T} \qquad \mathcal{U} = \varepsilon_{\mathcal{O}};\Omega \subseteq \varepsilon;\Omega = \varepsilon$

ii) $\mathcal{U};\Omega = \varepsilon_{\mathcal{O}};\Omega;\Omega = \varepsilon_{\mathcal{O}};\Omega = \mathcal{U}$ is completely trivial.

iii) $(\mathcal{U} \olessthan \mathcal{U}) = (\varepsilon_{\mathcal{O}};\Omega \olessthan \varepsilon_{\mathcal{O}};\Omega)$
$= (\varepsilon_{\mathcal{O}} \olessthan \varepsilon_{\mathcal{O}});(\Omega \otimes \Omega)$ due to Prop. 2.6
$\subseteq \varepsilon_{\mathcal{O}};\mathfrak{M}^\mathsf{T};(\Omega \otimes \Omega)$ Def. 3.13
$\subseteq \varepsilon_{\mathcal{O}};\mathfrak{M}^\mathsf{T};\mathfrak{M};\Omega;\mathfrak{M}^\mathsf{T}$ consequence of Prop. 9.4.iv from [SW14]
$\subseteq \varepsilon_{\mathcal{O}};\Omega;\mathfrak{M}^\mathsf{T} = \mathcal{U};\mathfrak{M}^\mathsf{T}$

iv) $\mathcal{U} = \varepsilon_{\mathcal{O}};\Omega = (\varepsilon \cap \mathbb{T};\varepsilon_{\mathcal{O}});\Omega = \varepsilon;(\Omega \cap \varepsilon_{\mathcal{O}}^\mathsf{T};\mathbb{T})$ using Def. 3.13.i
$= \varepsilon;(\Omega \cap \varepsilon_{\mathcal{O}}^\mathsf{T};\mathbb{T} \cap \varepsilon_{\mathcal{O}}^\mathsf{T};\mathbb{T})$
$\subseteq \varepsilon;(\overline{\varepsilon^\mathsf{T};\overline{\varepsilon;(\Omega \cap \varepsilon_{\mathcal{O}}^\mathsf{T};\mathbb{T})}} \cap \varepsilon_{\mathcal{O}}^\mathsf{T};\mathbb{T})$ trivial consequence of $\varepsilon;(\Omega \cap \varepsilon_{\mathcal{O}}^\mathsf{T};\mathbb{T}) \subseteq \varepsilon;(\Omega \cap \varepsilon_{\mathcal{O}}^\mathsf{T};\mathbb{T})$
$= \varepsilon;(\overline{\varepsilon^\mathsf{T};\overline{\mathcal{U}}} \cap \varepsilon_{\mathcal{O}}^\mathsf{T};\mathbb{T})$ see first line of this proof
$= (\varepsilon \cap \mathbb{T};\varepsilon_{\mathcal{O}});\overline{\varepsilon^\mathsf{T};\overline{\mathcal{U}}}$ mask shifting
$= \varepsilon_{\mathcal{O}};\overline{\varepsilon^\mathsf{T};\overline{\mathcal{U}}}$ Def. 3.13.i again
$\subseteq \varepsilon_{\mathcal{O}};\Omega;\overline{\varepsilon^\mathsf{T};\overline{\mathcal{U}}}$
$= \mathcal{U};\overline{\varepsilon^\mathsf{T};\overline{\mathcal{U}}}$ by definition $\square$

As a further proof of equivalence, we consider that from $\mathcal{O}_D$ to $\mathcal{U}$.

3.18 Proposition. Given an open diagonal topology $\mathcal{O}_D$, the construct
$\mathcal{U} := \varepsilon;\mathcal{O}_D;\Omega$
constitutes a neighborhood topology.

Proof: We follow the numbering of Def. 3.4.

i) $\mathcal{U};\mathbb{T} = \varepsilon;\mathcal{O}_D;\Omega;\mathbb{T} = \varepsilon;\mathcal{O}_D;\mathbb{T} \supseteq \varepsilon;\mathtt{syq}(\varepsilon,\mathbb{T}) = \mathbb{T}$ using Def. 3.12.i
$\mathcal{U} = \varepsilon;\mathcal{O}_D;\Omega \subseteq \varepsilon;\Omega = \varepsilon$ since $\mathcal{O}_D$ is a partial identity

ii) $\mathcal{U};\Omega = \varepsilon;\mathcal{O}_D;\Omega;\Omega = \varepsilon;\mathcal{O}_D;\Omega = \mathcal{U}$ is trivial

iii) $(\mathcal{U} \olessthan \mathcal{U});\mathfrak{M}$
$= (\varepsilon;\mathcal{O}_D;\Omega \olessthan \varepsilon;\mathcal{O}_D;\Omega);\mathfrak{M}$
$= (\varepsilon \olessthan \varepsilon);(\mathcal{O}_D \otimes \mathcal{O}_D);(\Omega \otimes \Omega);\mathfrak{M}$
$= (\varepsilon \olessthan \varepsilon);(\mathcal{O}_D \otimes \mathcal{O}_D);\mathfrak{M};\Omega$ Prop. 9.4.iv of [SW14]
$\subseteq (\varepsilon \olessthan \varepsilon);\mathfrak{M};\mathcal{O}_D;\Omega$ Prop. 3.12.iii
$= \varepsilon;\mathcal{O}_D;\Omega = \mathcal{U}$ Prop. 9.2.vi of [SW14]
$= \mathcal{U}$

iv) We start with the trivial fact

$$\varepsilon;\mathcal{O}_D;\Omega \subseteq \varepsilon;\mathcal{O}_D;\Omega \iff \varepsilon^\mathsf{T};\overline{\varepsilon;\mathcal{O}_D;\Omega} \subseteq \overline{\mathcal{O}_D;\Omega} \iff \mathcal{O}_D;\Omega \subseteq \overline{\varepsilon^\mathsf{T};\overline{\varepsilon;\mathcal{O}_D;\Omega}}.$$

This allows to estimate as follows:

$$\mathcal{U} = \varepsilon;\mathcal{O}_D;\Omega = \varepsilon;\mathcal{O}_D;\mathcal{O}_D;\Omega \subseteq \varepsilon;\mathcal{O}_D;\overline{\varepsilon^\mathsf{T};\overline{\varepsilon;\mathcal{O}_D;\Omega}} \subseteq \varepsilon;\mathcal{O}_D;\Omega;\overline{\varepsilon^\mathsf{T};\overline{\varepsilon;\mathcal{O}_D;\Omega}} = \mathcal{U};\overline{\varepsilon^\mathsf{T};\overline{\mathcal{U}}} \qquad \square$$

Now we investigate the reverse direction.

3.19 Proposition. Given any topology via $\mathcal{U}$ or $\mathcal{K}$, we obtain an open diagonal topology with the construct $\mathcal{O}_D := \mathcal{K}^\mathsf{T};\mathcal{K}$.

Proof: i) For the least element $\mathtt{syq}(\varepsilon, \perp\!\!\!\perp) =: n$ in the powerset, we have

$$n = \mathtt{syq}(\varepsilon, \perp\!\!\!\perp) = \overline{\varepsilon^\mathsf{T};\mathbb{T}} \subseteq \overline{\mathcal{U}^\mathsf{T};\mathbb{T}} = \overline{\mathcal{K};\varepsilon^\mathsf{T};\mathbb{T}} = \mathcal{K};\overline{\varepsilon^\mathsf{T};\mathbb{T}} = \mathcal{K};n.$$

This implies $n;n^\mathsf{T} \subseteq \mathcal{K}$ when shunting the point n. Transposing gives $n;n^\mathsf{T} \subseteq \mathcal{K}^\mathsf{T}$, shunting again $n \subseteq \mathcal{K}^\mathsf{T};n$, so that $n \subseteq \mathcal{K}^\mathsf{T};\mathbb{T} = \mathcal{K}^\mathsf{T};\mathcal{K};\mathbb{T} = \mathcal{O}_D;\mathbb{T}$.

For the greatest element $g := \mathtt{syq}(\varepsilon, \mathbb{T}) = \overline{\overline{\varepsilon}^\mathsf{T};\mathbb{T}} : \mathbf{2}^X \longrightarrow \mathbb{1}$, we reason as follows:

$$\mathcal{U};g = \mathcal{U};\Omega;g = \mathcal{U};\overline{\varepsilon^\mathsf{T};\overline{\varepsilon}};g = \mathcal{U};\overline{\varepsilon^\mathsf{T};\overline{\overline{\varepsilon};g}} = \mathcal{U};\overline{\varepsilon^\mathsf{T};\overline{\varepsilon;\mathtt{syq}(\varepsilon, \mathbb{T})}} = \mathcal{U};\overline{\varepsilon^\mathsf{T};\overline{\overline{\mathbb{T}}}} = \mathcal{U};\overline{\varepsilon^\mathsf{T};\perp\!\!\!\perp} = \mathcal{U};\overline{\perp\!\!\!\perp} = \mathcal{U};\mathbb{T} = \mathbb{T}$$
$$g = \mathtt{syq}(\varepsilon, \mathbb{T}) = \mathtt{syq}(\varepsilon, \mathcal{U};g) = \mathtt{syq}(\varepsilon, \mathcal{U});g = \mathcal{K}^\mathsf{T};g \subseteq \mathcal{K}^\mathsf{T};\mathbb{T} = \mathcal{K}^\mathsf{T};\mathcal{K};\mathbb{T} = \mathcal{O}_D;\mathbb{T}$$

ii) Assuming $v \subseteq \mathcal{K}^\mathsf{T};\mathcal{K};\mathbb{T}$, we get the equality $\mathcal{K}^\mathsf{T};\mathcal{K};v = v$, since

$$v \subseteq \mathcal{K}^\mathsf{T};\mathcal{K};\mathbb{T} \cap v = \mathcal{K}^\mathsf{T};\mathbb{T} \cap v \subseteq (\mathcal{K}^\mathsf{T} \cap v;\mathbb{T});(\mathbb{T} \cap \mathcal{K};v) \subseteq \mathcal{K}^\mathsf{T};\mathcal{K};v \subseteq v.$$

According to its definition, $e := \mathtt{syq}(\varepsilon, \varepsilon;v)$ is a point. Therefore, $\varepsilon;v = \varepsilon;e$ so that $v;e^\mathsf{T} \subseteq \Omega$ and finally

$$\mathcal{U};v;e^\mathsf{T} \subseteq \mathcal{U};\Omega = \mathcal{U},$$
$$\mathcal{U};v \subseteq \mathcal{U};e \subseteq \varepsilon;e = \varepsilon;\mathtt{syq}(\varepsilon, \varepsilon;v) = \varepsilon;v = \varepsilon;\mathcal{K}^\mathsf{T};\mathcal{K};v = \varepsilon;\mathcal{K}^\mathsf{T};\mathcal{K}^\mathsf{T};\mathcal{K};v = \mathcal{U};v,$$
$$e = \mathtt{syq}(\varepsilon, \varepsilon;v) = \mathtt{syq}(\varepsilon, \mathcal{U};e) = \mathtt{syq}(\varepsilon, \mathcal{U});e = \mathcal{K}^\mathsf{T};e \subseteq \mathcal{K}^\mathsf{T};\mathbb{T} = \mathcal{K}^\mathsf{T};\mathcal{K};\mathbb{T} = \mathcal{O}_D;\mathbb{T}.$$

iii) We find out that $\mathcal{K} = \mathcal{K};\mathcal{K} = \mathcal{K};\mathcal{K}^\mathsf{T};\mathcal{K};\mathcal{K}$, since $\mathcal{K}$ is univalent, and since "$\subseteq$" of the second equation is via shunting equivalent with $\mathcal{K}^\mathsf{T};\mathcal{K};\mathcal{K} \subseteq \mathcal{K}^\mathsf{T};\mathcal{K};\mathcal{K}$.

Now we may reason as follows:

$$\begin{aligned}
&(\mathcal{O}_D \otimes \mathcal{O}_D);\mathfrak{M} = (\mathcal{K}^\mathsf{T};\mathcal{K} \otimes \mathcal{K}^\mathsf{T};\mathcal{K});\mathfrak{M} = (\mathcal{K}^\mathsf{T};\mathcal{K};\mathcal{K} \otimes \mathcal{K}^\mathsf{T};\mathcal{K};\mathcal{K});\mathfrak{M} \quad \text{since } \mathcal{K} \text{ is idempotent}\\
&= (\mathcal{K}^\mathsf{T};\mathcal{K} \otimes \mathcal{K}^\mathsf{T};\mathcal{K});(\mathcal{K} \otimes \mathcal{K});\mathfrak{M} \quad \text{because } (\mathcal{K}^\mathsf{T};\mathcal{K} \otimes \mathcal{K}^\mathsf{T};\mathcal{K}) \text{ is univalent}\\
&= (\mathcal{K}^\mathsf{T};\mathcal{K} \otimes \mathcal{K}^\mathsf{T};\mathcal{K});\mathfrak{M};\mathcal{K} \quad \text{due to Def. 3.6.iii}\\
&= (\mathcal{O}_D \otimes \mathcal{O}_D);\mathfrak{M};\mathcal{K} \quad \text{by original definition}\\
&= (\mathcal{O}_D \otimes \mathcal{O}_D);\mathfrak{M};\mathcal{K};\mathcal{K}^\mathsf{T};\mathcal{K} \quad \text{since } \mathcal{K} \text{ is a mapping}\\
&= (\mathcal{O}_D \otimes \mathcal{O}_D);\mathfrak{M};\mathcal{K}^\mathsf{T};\mathcal{K} \quad \text{reversing the first four lines}\\
&\subseteq \mathfrak{M};\mathcal{K}^\mathsf{T};\mathcal{K} \quad \text{since } (\mathcal{O}_D \otimes \mathcal{O}_D) \text{ is a partial identity}\\
&= \mathfrak{M};\mathcal{O}_D \qquad \square
\end{aligned}$$

Of course, there are also all the widely symmetric concepts, namely

— the closed hull map $\mathcal{H} := N;\mathtt{syq}(\overline{\mathcal{U}}, \varepsilon)$,
— the closed sets diagonal $\mathcal{C}_D := \mathcal{H}^\mathsf{T};\mathcal{H} = \mathcal{H} \cap \mathbb{I}$,
— the closed sets vector $\mathcal{C}_V := \mathcal{C}_D;\mathbb{T}$,
— the membership restricted to closed sets $\varepsilon_\mathcal{C} := \varepsilon \cap \mathbb{T};\mathcal{C}_V^\mathsf{T}$.

3.4 Quotient topology

Before adding a remark on quotient topologies, we refer back to Prop. 4.3 of [SW14]. The denotation developed there is shown in Fig. 3.7.

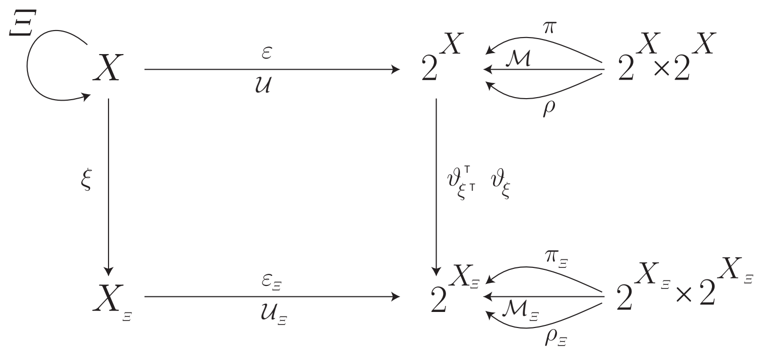


Fig. 3.7 Quotient of a topology

In a quotient topology, a subset shall be open precisely when its inverse image is. For quite some time, the authors were mislead to try the seemingly obvious version

$$\mathcal{U}_\Xi := \xi^\mathsf{T};\mathcal{U};\vartheta^\mathsf{T}_{\xi^\mathsf{T}} : X_\Xi \longrightarrow \mathbf{2}^{X_\Xi}.$$

Fig. 3.8 illustrates that this does not work; one has to concentrate on open sets first, with $\varepsilon_\mathcal{O}$, and include greater neighborhoods only later, i.e.

$$\varepsilon_{\mathcal{O}_\Xi} := \xi^\mathsf{T};\varepsilon_\mathcal{O};\vartheta^\mathsf{T}_{\xi^\mathsf{T}} : X_\Xi \longrightarrow \mathbf{2}^{X_\Xi}.$$

The proof is easiest to be executed for the open sets vector version $\mathcal{O}_V$, where we use that

$$\begin{aligned}
&\varepsilon_{\mathcal{O}_\Xi} = \varepsilon_\Xi \cap \mathbb{T};\mathcal{O}^\mathsf{T}_{V_\Xi} \\
&= \xi^\mathsf{T};\varepsilon;\vartheta^\mathsf{T}_{\xi^\mathsf{T}} \cap \mathbb{T};\mathcal{O}^\mathsf{T}_V;\vartheta^\mathsf{T}_{\xi^\mathsf{T}} \quad \text{see definitions in Prop. 3.20} \\
&= \xi^\mathsf{T};(\varepsilon;\vartheta^\mathsf{T}_{\xi^\mathsf{T}} \cap \mathbb{T};\mathcal{O}^\mathsf{T}_V;\vartheta^\mathsf{T}_{\xi^\mathsf{T}}) \quad \text{masking} \\
&= \xi^\mathsf{T};(\varepsilon \cap \mathbb{T};\mathcal{O}^\mathsf{T}_V);\vartheta^\mathsf{T}_{\xi^\mathsf{T}} \quad \text{since } \vartheta_{\xi^\mathsf{T}} \text{ is univalent} \\
&= \xi^\mathsf{T};\varepsilon_\mathcal{O};\vartheta^\mathsf{T}_{\xi^\mathsf{T}}
\end{aligned}$$

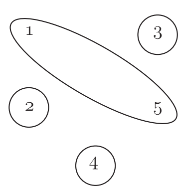
3.20 Proposition. Assume $\mathcal{O}_V \subseteq \mathbf{2}^X$ to be a topology on the set X and $\Xi : X \longrightarrow X$ an equivalence on that set. We consider its natural projection $\xi : X \longrightarrow X_\Xi$ as well as the membership $\varepsilon_\Xi : X_\Xi \longrightarrow \mathbf{2}^{X_\Xi}$ on the quotient. Furthermore, we introduce the existential image mapping $\vartheta_\xi := \mathtt{syq}(\xi^\mathsf{T};\varepsilon, \varepsilon_\Xi) : \mathbf{2}^X \longrightarrow \mathbf{2}^{X_\Xi}$ for ξ as well as the inverse image $\vartheta_{\xi^\mathsf{T}} := \mathtt{syq}(\xi;\varepsilon_\Xi, \varepsilon) : \mathbf{2}^{X_\Xi} \longrightarrow \mathbf{2}^X$. In this setting

$$\mathcal{O}_{V_\Xi} = \vartheta_{\xi^\mathsf{T}};\mathcal{O}_V \subseteq \mathbf{2}^{X_\Xi}$$

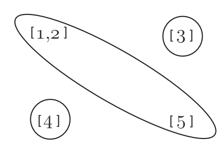
is again a topology.

$$\mathcal{U} = \begin{array}{c} \\ 1\\2\\3\\4\\5 \end{array} \begin{array}{c} \begin{array}{cccccccccccccccccccccccccccccccc} \{\} & \{1\} & \{2\} & \{1,2\} & \{3\} & \{1,3\} & \{2,3\} & \{1,2,3\} & \{4\} & \{1,4\} & \{2,4\} & \{1,2,4\} & \{3,4\} & \{1,3,4\} & \{2,3,4\} & \{1,2,3,4\} & \{5\} & \{1,5\} & \{2,5\} & \{1,2,5\} & \{3,5\} & \{1,3,5\} & \{2,3,5\} & \{1,2,3,5\} & \{4,5\} & \{1,4,5\} & \{2,4,5\} & \{1,2,4,5\} & \{3,4,5\} & \{1,3,4,5\} & \{2,3,4,5\} & \{1,2,3,4,5\} \end{array} \\ \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1 \\ 0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\ 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1 \end{pmatrix} \end{array}$$

$$\Xi = \begin{array}{c} \\ 1\\2\\3\\4\\5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1&2&3&4&5 \end{array} \\ \begin{pmatrix} 1&1&0&0&0 \\ 1&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix} \end{array} \qquad \xi^{\mathsf{T}} ; \mathcal{U} ; \vartheta_{\xi} = \xi^{\mathsf{T}} ; \mathcal{U} ; \vartheta^{\mathsf{T}}_{\xi^{\mathsf{T}}} = \begin{array}{c} \\ [1]\\ [3]\\ [4]\\ [5] \end{array} \begin{array}{c} \begin{array}{cccccccccccccccc} \{\} & \{[1]\} & \{[3]\} & \{[1],[3]\} & \{[4]\} & \{[1],[4]\} & \{[3],[4]\} & \{[1],[3],[4]\} & \{[5]\} & \{[1],[5]\} & \{[3],[5]\} & \{[1],[3],[5]\} & \{[4],[5]\} & \{[1],[4],[5]\} & \{[3],[4],[5]\} & \{[1],[3],[4],[5]\} \end{array} \\ \begin{pmatrix} 0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1 \\ 0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\ 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1 \end{pmatrix} \end{array}$$



$\mathcal{U}$



$\mathcal{U}_\Xi$

$$\xi^{\mathsf{T}} ; \varepsilon_{\mathcal{O}} ; \vartheta^{\mathsf{T}}_{\xi^{\mathsf{T}}} ; \Omega_\Xi = \begin{array}{c} \\ [1]\\ [3]\\ [4]\\ [5] \end{array} \begin{array}{c} \begin{array}{cccccccccccccccc} \{\} & \{[1]\} & \{[3]\} & \{[1],[3]\} & \{[4]\} & \{[1],[4]\} & \{[3],[4]\} & \{[1],[3],[4]\} & \{[5]\} & \{[1],[5]\} & \{[3],[5]\} & \{[1],[3],[5]\} & \{[4],[5]\} & \{[1],[4],[5]\} & \{[3],[4],[5]\} & \{[1],[3],[4],[5]\} \end{array} \\ \begin{pmatrix} 0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1 \\ 0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\ 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1 \end{pmatrix} \end{array}$$

Fig. 3.8 Quotient of a topology indicated via the open set bases

Proof: We convince ourselves in advance that the following hold

$\varepsilon_\Xi = \xi^{\mathsf{T}} ; \xi ; \varepsilon_\Xi$ since the natural projection ξ is a surjective mapping
$= \xi^{\mathsf{T}} ; \varepsilon ; \mathtt{syq}(\varepsilon, \xi ; \varepsilon_\Xi)$ property of a symmetric quotient with regard to ε
$\subseteq \xi^{\mathsf{T}} ; \varepsilon ; \mathtt{syq}(\xi^{\mathsf{T}} ; \varepsilon, \varepsilon_\Xi)$ shifting a surjective mapping, Prop. 2.3.iii
$\subseteq \varepsilon_\Xi$ cancellation,

resulting in equality and

$\varepsilon_\Xi = \xi^{\mathsf{T}} ; \varepsilon ; \vartheta^{\mathsf{T}}_{\xi^{\mathsf{T}}} = \xi^{\mathsf{T}} ; \varepsilon ; \vartheta_\xi$.

Furthermore

$\varepsilon ; \vartheta^{\mathsf{T}}_{\xi^{\mathsf{T}}} = \varepsilon ; \mathtt{syq}(\varepsilon, \xi ; \varepsilon_\Xi) = \xi ; \varepsilon_\Xi$,

besides the standard property $\varepsilon_\Xi ; \vartheta^{\mathsf{T}}_\xi = \xi^{\mathsf{T}} ; \varepsilon$ of an existential image.

Now follow the proofs of the topology properties numbered as in Def. 3.11:

i) We start from $\mathtt{syq}(\varepsilon, \perp\!\!\!\perp) \subseteq \mathcal{O}_V$, obtain $\vartheta_{\xi^{\mathsf{T}}} ; \mathtt{syq}(\varepsilon, \perp\!\!\!\perp) \subseteq \vartheta_{\xi^{\mathsf{T}}} ; \mathcal{O}_V = \mathcal{O}_{V_\Xi}$, where

$\vartheta_{\xi^{\mathsf{T}}} ; \mathtt{syq}(\varepsilon, \perp\!\!\!\perp) = \mathtt{syq}(\varepsilon ; \vartheta^{\mathsf{T}}_{\xi^{\mathsf{T}}}, \perp\!\!\!\perp) = \mathtt{syq}(\xi ; \varepsilon_\Xi, \perp\!\!\!\perp) = \mathtt{syq}(\xi ; \varepsilon_\Xi, \xi ; \perp\!\!\!\perp)$
$= \mathtt{syq}(\varepsilon_\Xi, \perp\!\!\!\perp)$ Prop. 8.16.i of [Sch11]

Analogously $\mathtt{syq}(\varepsilon_\Xi, \top\!\!\!\top) \subseteq \mathcal{O}_{V_\Xi}$.

ii) Assume $v \subseteq \mathcal{O}_{V_\Xi} = \vartheta_{\xi^{\mathsf{T}}} ; \mathcal{O}_V$, which gives via shunting $\vartheta^{\mathsf{T}}_{\xi^{\mathsf{T}}} ; v \subseteq \mathcal{O}_V$. Since $\mathcal{O}_V$ is an open-set-topology then

$\mathtt{syq}(\varepsilon, \varepsilon; \vartheta^\mathsf{T}_{\xi^\mathsf{T}}; v) \subseteq \mathcal{O}_V$.

This implies $\vartheta_{\xi^\mathsf{T}}; \mathtt{syq}(\varepsilon, \varepsilon; \vartheta^\mathsf{T}_{\xi^\mathsf{T}}; v) \subseteq \vartheta_{\xi^\mathsf{T}}; \mathcal{O}_V = \mathcal{O}_{V_\Xi}$, and consequently

$\mathtt{syq}(\varepsilon_\Xi, \varepsilon_\Xi; v) = \mathtt{syq}(\xi; \varepsilon_\Xi, \xi; \varepsilon_\Xi; v) = \mathtt{syq}(\varepsilon; \vartheta^\mathsf{T}_{\xi^\mathsf{T}}, \varepsilon; \vartheta^\mathsf{T}_{\xi^\mathsf{T}}; v) = \vartheta_{\xi^\mathsf{T}}; \mathtt{syq}(\varepsilon, \varepsilon; \vartheta^\mathsf{T}_{\xi^\mathsf{T}}; v) \subseteq \mathcal{O}_{V_\Xi}$.

iii) $(\vartheta_{\xi^\mathsf{T}} \otimes \vartheta_{\xi^\mathsf{T}}); \mathfrak{M}; \vartheta^\mathsf{T}_{\xi^\mathsf{T}} = (\vartheta_{\xi^\mathsf{T}} \otimes \vartheta_{\xi^\mathsf{T}}); \mathtt{syq}(\,(\varepsilon \otimes\!\!\!< \varepsilon)\,, \varepsilon); \vartheta^\mathsf{T}_{\xi^\mathsf{T}}$ by definition
$= \mathtt{syq}(\,(\varepsilon \otimes\!\!\!< \varepsilon); (\vartheta_{\xi^\mathsf{T}} \otimes \vartheta_{\xi^\mathsf{T}})^\mathsf{T}, \varepsilon; \vartheta^\mathsf{T}_{\xi^\mathsf{T}})$
$= \mathtt{syq}(\,(\varepsilon \otimes\!\!\!< \varepsilon); (\vartheta^\mathsf{T}_{\xi^\mathsf{T}} \otimes \vartheta^\mathsf{T}_{\xi^\mathsf{T}})\,, \varepsilon; \vartheta^\mathsf{T}_{\xi^\mathsf{T}})$
$= \mathtt{syq}(\,(\varepsilon; \vartheta^\mathsf{T}_{\xi^\mathsf{T}} \otimes\!\!\!< \varepsilon; \vartheta^\mathsf{T}_{\xi^\mathsf{T}})\,, \varepsilon; \vartheta^\mathsf{T}_{\xi^\mathsf{T}})$
$= \mathtt{syq}(\,(\xi; \varepsilon_\Xi \otimes\!\!\!< \xi; \varepsilon_\Xi)\,, \xi; \varepsilon_\Xi)$ see above
$= \mathtt{syq}(\xi; (\varepsilon_\Xi \otimes\!\!\!< \varepsilon_\Xi)\,, \xi; \varepsilon_\Xi)$ since ξ is univalent
$= \mathtt{syq}(\,(\varepsilon_\Xi \otimes\!\!\!< \varepsilon_\Xi)\,, \varepsilon_\Xi)$ Prop. 8.16.i of [Sch11] since ξ is a surjective map
$= \mathfrak{M}_\Xi$ by definition

$\mathfrak{M}^\mathsf{T}_\Xi; (\mathcal{O}_{V_\Xi} \otimes\!\!\!> \mathcal{O}_{V_\Xi}) = \vartheta_{\xi^\mathsf{T}}; \mathfrak{M}^\mathsf{T}; (\vartheta^\mathsf{T}_{\xi^\mathsf{T}} \otimes \vartheta^\mathsf{T}_{\xi^\mathsf{T}}); (\vartheta_{\xi^\mathsf{T}}; \mathcal{O}_V \otimes\!\!\!> \vartheta_{\xi^\mathsf{T}}; \mathcal{O}_V)$ see above
$= \vartheta_{\xi^\mathsf{T}}; \mathfrak{M}^\mathsf{T}; (\vartheta^\mathsf{T}_{\xi^\mathsf{T}} \otimes \vartheta^\mathsf{T}_{\xi^\mathsf{T}}); (\vartheta_{\xi^\mathsf{T}} \otimes \vartheta_{\xi^\mathsf{T}}); (\mathcal{O}_V \otimes\!\!\!> \mathcal{O}_V)$ since $(\vartheta_{\xi^\mathsf{T}} \otimes \vartheta_{\xi^\mathsf{T}})$ is univalent
$\subseteq \vartheta_{\xi^\mathsf{T}}; \mathfrak{M}^\mathsf{T}; (\vartheta^\mathsf{T}_{\xi^\mathsf{T}}; \vartheta_{\xi^\mathsf{T}} \otimes \vartheta^\mathsf{T}_{\xi^\mathsf{T}}; \vartheta_{\xi^\mathsf{T}}); (\mathcal{O}_V \otimes\!\!\!> \mathcal{O}_V)$
$= \vartheta_{\xi^\mathsf{T}}; \mathfrak{M}^\mathsf{T}; (\mathcal{O}_V \otimes\!\!\!> \mathcal{O}_V)$ because $\vartheta^\mathsf{T}_{\xi^\mathsf{T}}; \vartheta_{\xi^\mathsf{T}} = \mathbb{I}$
$\subseteq \vartheta_{\xi^\mathsf{T}}; \mathcal{O}_V$ since $\mathcal{O}_V$ was supposed to be an open-set-vector topology
$= \mathcal{O}_{V_\Xi}$ □

3.5 Inner points and tangent points

We have learned how to proceed from a powerset element to its open kernel element via $\mathcal{K}: a \mapsto a^\circ$. Now we investigate how this works when a subset is given as a vector $A \subseteq X$ and one asks for its open kernel $A^\circ \subseteq A$ or its closed hull $A \subseteq A^-$, respectively. Or else: we study when a point x is a tangent point of a subset A or an inner point of A.

3.21 Proposition. Given a neighborhood topology $\mathcal{U} : X \longrightarrow \mathbf{2}^X$ and any subset $A \subseteq X$, we obtain the open kernel and the closed hull of A as

$$A^\circ := \mathcal{U}; \overline{\varepsilon^\mathsf{T}; \overline{A}} \qquad A^- := \overline{\mathcal{U}; \overline{\varepsilon^\mathsf{T}; A}}.$$

Proof: To the element $a := \mathtt{syq}(\varepsilon, A)$ in the powerset that corresponds to A, we apply the mapping to its open kernel.

$$a^\circ = \mathcal{K}^\mathsf{T}; a = \mathtt{syq}(\varepsilon, \mathcal{U}); a = \mathtt{syq}(\varepsilon, \mathcal{U}; \Omega); a = \mathtt{syq}(\varepsilon, \mathcal{U}; \overline{\varepsilon^\mathsf{T}; \overline{\varepsilon}}); a = \mathtt{syq}(\varepsilon, \mathcal{U}; \overline{\varepsilon^\mathsf{T}; \overline{\varepsilon; a}})$$

and look for the corresponding vector

$$A^\circ = \varepsilon; a^\circ = \varepsilon; \mathtt{syq}(\varepsilon, \mathcal{U}; \overline{\varepsilon^\mathsf{T}; \overline{\varepsilon; a}}) = \mathcal{U}; \overline{\varepsilon^\mathsf{T}; \overline{\varepsilon; a}} = \mathcal{U}; \overline{\varepsilon^\mathsf{T}; \overline{A}}.$$

Correspondingly for the closed hull $\mathcal{H} = N; \mathtt{syq}(\overline{\mathcal{U}}, \varepsilon)$. □

Then, following [Sch11] Prop. 7.14, obviously

$$A^\circ = \mathcal{U}; \overline{\varepsilon^\mathsf{T}; \overline{A}} \subseteq \varepsilon; \overline{\varepsilon^\mathsf{T}; \overline{A}} = A,$$
$$A^- = \overline{\mathcal{U}; \overline{\varepsilon^\mathsf{T}; A}} \supseteq \overline{\varepsilon; \overline{\varepsilon^\mathsf{T}; A}} = A.$$

We may also consider the production of the open kernel or closed hull for all subsets simultaneously, i.e., obtaining

$$\varepsilon^{\circ} = \mathcal{U};\overline{\varepsilon^{\mathsf{T}};\overline{\varepsilon}} = \mathcal{U};\Omega = \mathcal{U}$$
$$\varepsilon^{-} = \overline{\mathcal{U};\overline{\varepsilon^{\mathsf{T}};\varepsilon}} = \overline{\mathcal{U};\overline{\varepsilon^{\mathsf{T}};\varepsilon;}}N;N = \overline{\mathcal{U};\overline{\varepsilon^{\mathsf{T}};\varepsilon;N}};N = \overline{\mathcal{U};\overline{\varepsilon^{\mathsf{T}};\overline{\varepsilon}}};N = \overline{\mathcal{U};\Omega};N = \overline{\mathcal{U}};N.$$

In addition, we convince ourselves that these definitions meet the traditional expectation combined with open kernel and closed hull:

$$A^{\circ}_x = [\mathcal{U};\overline{\varepsilon^{\mathsf{T}};\overline{A}}]_x = \exists u : \mathcal{U}_{xu} \wedge [\forall p : \varepsilon_{pu} \rightarrow A_p]$$
$$A^{-}_x = [\overline{\mathcal{U};\overline{\varepsilon^{\mathsf{T}};A}}]_x = \forall u : \mathcal{U}_{xu} \rightarrow [\exists p : \varepsilon_{pu} \wedge A_p]$$

3.6 Separation

A major question is to which extent points or subsets may be *distinguished* or even *separated* by environments or open sets. This gave rise to several definitions which we recall here first in their traditional form: Let a topology on X be given via neighborhoods, open sets, kernel mapping as required. It is then called a

- T_0-space (sometimes a Kolmogorov space) if for any two points in X an open set exists that contains one of them but not the other, i.e., points are topologically distinguishable.
- T_1-space when $\forall x, y : x \neq y \rightarrow \exists U, V \in \mathcal{O} : x \in U \wedge y \notin U \wedge y \in V \wedge x \notin V$,
- T_2-space, i.e., a topology satisfying the Hausdorff property, when
  $\forall x, y : x \neq y \rightarrow \exists U, V \in \mathcal{O} : x \in U \wedge y \in V \wedge \emptyset = U \cap V$.

Following our general guideline, we intend to lift these conditions to the relational level. Our first concern is as follows.

Concerning distinguishability, any given topology $\mathcal{U} : X \longrightarrow \mathbf{2}^X$ introduces the equivalence $\Xi := \texttt{syq}(\mathcal{U}^{\mathsf{T}},\mathcal{U}^{\mathsf{T}})$, the so-called *topological non-distinguishability* of points. We convince ourselves that always

$$\Xi := \texttt{syq}(\mathcal{U}^{\mathsf{T}},\mathcal{U}^{\mathsf{T}}) = \overline{\overline{\mathcal{U}};\mathcal{U}^{\mathsf{T}}} \cap \overline{\mathcal{U};\overline{\mathcal{U}}^{\mathsf{T}}} \quad \text{by definition of the symmetric quotient}$$
$$= \overline{\overline{\varepsilon_{\mathcal{O}};\mathcal{K}^{\mathsf{T}}};\mathcal{K};\varepsilon_{\mathcal{O}}^{\mathsf{T}}} \cap \overline{\varepsilon_{\mathcal{O}};\mathcal{K}^{\mathsf{T}};\overline{\mathcal{K};\varepsilon_{\mathcal{O}}^{\mathsf{T}}}} \quad \text{Prop. 3.16.vii}$$
$$= \overline{\overline{\varepsilon_{\mathcal{O}}};\mathcal{K}^{\mathsf{T}};\mathcal{K};\varepsilon_{\mathcal{O}}^{\mathsf{T}}} \cap \overline{\varepsilon_{\mathcal{O}};\mathcal{K}^{\mathsf{T}};\mathcal{K};\overline{\varepsilon_{\mathcal{O}}^{\mathsf{T}}}}$$
$$= \overline{\overline{\varepsilon_{\mathcal{O}}};\varepsilon_{\mathcal{O}}^{\mathsf{T}}} \cap \overline{\varepsilon_{\mathcal{O}};\overline{\varepsilon_{\mathcal{O}}^{\mathsf{T}}}} \quad \text{since } \varepsilon_{\mathcal{O}};\mathcal{K}^{\mathsf{T}};\mathcal{K} = \mathcal{U};\mathcal{K} = \varepsilon_{\mathcal{O}}$$
$$= \texttt{syq}(\varepsilon_{\mathcal{O}}^{\mathsf{T}}, \varepsilon_{\mathcal{O}}^{\mathsf{T}}).$$

When we divide it out, i.e. consider the quotient mapping ξ according to $\Xi = \xi;\xi^{\mathsf{T}}$, we obtain a topology that satisfies the T_0-property. We have, namely,

$$\texttt{syq}(\varepsilon_{\mathcal{O}_\Xi}^{\mathsf{T}}, \varepsilon_{\mathcal{O}_\Xi}^{\mathsf{T}}) = \texttt{syq}(\vartheta_{\xi^{\mathsf{T}}};\varepsilon_{\mathcal{O}}^{\mathsf{T}};\xi, \vartheta_{\xi^{\mathsf{T}}};\varepsilon_{\mathcal{O}}^{\mathsf{T}};\xi) = \texttt{syq}(\varepsilon_{\mathcal{O}}^{\mathsf{T}};\xi, \varepsilon_{\mathcal{O}}^{\mathsf{T}};\xi) \qquad \text{[Sch11] Prop. 8.16.i}$$
$$= \xi^{\mathsf{T}};\texttt{syq}(\varepsilon_{\mathcal{O}}^{\mathsf{T}}, \varepsilon_{\mathcal{O}}^{\mathsf{T}});\xi \qquad \text{[Sch11] Prop. 8.18.i}$$
$$= \xi^{\mathsf{T}};\Xi;\xi = \xi^{\mathsf{T}}\xi;\xi^{\mathsf{T}};\xi = \mathbb{I}.$$

For the following definition, we choose $\mathcal{U}$ as the most convenient ones among the diversity of topology definitions, but also $\mathcal{O}, \mathcal{K}, \varepsilon_{\mathcal{O}}$ might have been employed.

3.22 Definition. Let a topology $\mathcal{U}$ be given in relational form. It is called a

i) T_0-**space** or a **Kolmogorov** space if $\quad \overline{\mathbb{I}} \subseteq \overline{\mathtt{syq}(\mathcal{U}^\mathsf{T}, \mathcal{U}^\mathsf{T})}$,

ii) T_1-**space** if $\quad \overline{\mathbb{I}} \subseteq \mathcal{U} ; \overline{\mathcal{U}}^\mathsf{T}$,

iii) T_2-**space** or a **Hausdorff** space if $\quad \overline{\mathbb{I}} \subseteq \mathcal{U} ; N ; \mathcal{U}^\mathsf{T}$.

In all three cases this means in fact equality. □

Of course, we have the chain of implications

$$T_2\text{-space} \implies T_1\text{-space} \implies T_0\text{-space},$$

which can easily be proved observing

$$\mathcal{U} ; N ; \mathcal{U}^\mathsf{T} \subseteq \mathcal{U} ; N ; \varepsilon^\mathsf{T} = \mathcal{U} ; \overline{\varepsilon}^\mathsf{T} \subseteq \mathcal{U} ; \overline{\mathcal{U}}^\mathsf{T} \subseteq \overline{\mathtt{syq}(\mathcal{U}^\mathsf{T}, \mathcal{U}^\mathsf{T})}.$$

We establish equivalent versions using the membership-in-open-set topology definition.

3.23 Proposition. A topology given as $\mathcal{U}$, resp. $\varepsilon_\mathcal{O}$, is a

i) T_0-space $\iff$ $\overline{\mathbb{I}} \subseteq \mathtt{syq}(\varepsilon_\mathcal{O}^\mathsf{T}, \varepsilon_\mathcal{O}^\mathsf{T})$,

ii) T_1-space $\iff$ $\overline{\mathbb{I}} \subseteq \varepsilon_\mathcal{O} ; \overline{\varepsilon}^\mathsf{T}$ $\iff$ $\sigma \subseteq \varepsilon_\mathcal{C}$,

iii) T_2-space $\iff$ $\overline{\mathbb{I}} \subseteq \varepsilon_\mathcal{O} ; \overline{\varepsilon^\mathsf{T} ; \varepsilon} ; \varepsilon_\mathcal{O}^\mathsf{T}$.

Proof: i) The T_0 case follows from the initial remark on distinguishability.

ii) For the T_1 case we have $\varepsilon_\mathcal{O} ; \overline{\varepsilon}^\mathsf{T} = \mathcal{U} ; \mathcal{K} ; \overline{\varepsilon}^\mathsf{T} = \mathcal{U} ; \mathtt{syq}(\mathcal{U}, \varepsilon) ; \overline{\varepsilon}^\mathsf{T} = \mathcal{U} ; \mathtt{syq}(\overline{\mathcal{U}}, \overline{\varepsilon}) ; \overline{\varepsilon}^\mathsf{T} = \mathcal{U} ; \overline{\mathcal{U}}^\mathsf{T}$
and show in addition that this means that singleton sets are closed. If we write the claim voluntarily complicated, it offers itself for shunting:

$$\mathbb{I} ; \sigma \subseteq \overline{\mathcal{U}} ; \mathcal{K} ; N = \varepsilon_\mathcal{C}$$
$$\mathbb{I} \subseteq \overline{\mathcal{U}} ; \mathcal{K} ; N ; \sigma^\mathsf{T} = \overline{\mathcal{U}} ; \mathtt{syq}(\mathcal{U}, \varepsilon) ; N ; \sigma^\mathsf{T} = \overline{\mathcal{U}} ; \mathtt{syq}(\mathcal{U}, \overline{\varepsilon}) ; \sigma^\mathsf{T}$$
$$= \overline{\mathcal{U}} ; \mathtt{syq}(\mathcal{U}, \overline{\varepsilon} ; \sigma^\mathsf{T}) = \overline{\mathcal{U}} ; \mathtt{syq}(\mathcal{U}, \overline{\varepsilon ; \sigma^\mathsf{T}}) = \overline{\mathcal{U}} ; \mathtt{syq}(\mathcal{U}, \overline{\mathbb{I}}) = \overline{\mathcal{U}} ; \mathtt{syq}(\overline{\mathcal{U}}, \mathbb{I}) \subseteq \mathbb{I}$$

The T_2-case is easily shown using that $\mathcal{U} = \varepsilon_\mathcal{O} ; \Omega$ and, obviously, $\overline{\varepsilon^\mathsf{T} ; \varepsilon} ; \Omega^\mathsf{T} = \overline{\varepsilon^\mathsf{T} ; \varepsilon}$:

$$\varepsilon_\mathcal{O} ; \overline{\varepsilon^\mathsf{T} ; \varepsilon} ; \varepsilon_\mathcal{O}^\mathsf{T} = \varepsilon_\mathcal{O} ; \Omega ; \overline{\varepsilon^\mathsf{T} ; \varepsilon} ; \Omega^\mathsf{T} ; \varepsilon_\mathcal{O}^\mathsf{T} = \mathcal{U} ; \overline{\varepsilon^\mathsf{T} ; \overline{\varepsilon} ; N} ; \mathcal{U}^\mathsf{T} = \mathcal{U} ; \Omega ; N ; \mathcal{U}^\mathsf{T} = \mathcal{U} ; N ; \mathcal{U}^\mathsf{T}$$ □

3.7 Continuity

For a mathematical structure, one routinely defines its structure-preserving mappings. Traditionally, this is handled under the name of a homomorphism; it may be defined for relational structures as well as for algebraic ones in more or less the same standard way; it is available for a homogeneous as well as a heterogeneous structure. For topologies, however, the situation is different.

A neighborhood system requires a heterogeneous setting with two neighborhood topologies $\mathcal{U},\mathcal{U}'$ on sets X, X'. The continuity condition turns out to be a mixture of going forward and backwards as we will see.

$$\begin{tikzcd}
2^{X} & & 2^{X'} \arrow[ll, "\vartheta_{f^\mathsf{T}}"'] \\
X \arrow[u, "\mathcal{U}"] \arrow[rr, "f"'] & & X' \arrow[u, "\mathcal{U}'"']
\end{tikzcd}$$

Fig. 3.9 Typing around the continuity condition

The standard — i.e. not yet lifted — definition of topological continuity for a neighborhood topology runs as follows: Let any two neighborhood topologies $\mathcal{U},\mathcal{U}'$ be given on sets X, X', and consider a mapping $f : X \longrightarrow X'$. One says that f is continuous when

for every point $p \in X$ and every neighborhood $V \in \mathcal{U}'(f(p))$,
there exists a neighborhood $U \in \mathcal{U}(p)$ such that $f(U) \subseteq V$.

This definition has here only been recalled for convenience. Converting it gradually — but informally — to a point-free version is far from easy. Again, we must not quantify over subsets $U, V \subseteq X$ and move to quantifying over *points* $u, v \subseteq \mathbf{2}^X$ in the powerset.

For every $p \in X$ and every $V \in \mathcal{U}'(f(p))$, there exists a $U \in \mathcal{U}(p)$ such that $f(U) \subseteq V$.
$\forall p \in X : \forall V \in \mathcal{U}'(f(p)) : \exists U \in \mathcal{U}(p) : f(U) \subseteq V$
$\forall p \in X : \forall v \in \mathbf{2}^{X'} : \mathcal{U}'_{f(p),v} \longrightarrow \big(\exists u : \mathcal{U}_{pu} \wedge \big[\forall y : \varepsilon_{yu} \to \varepsilon'_{f(y),v}\big]\big)$
$\forall p : \forall v : (f;\mathcal{U}')_{pv} \longrightarrow \big(\exists u : \mathcal{U}_{pu} \wedge \big[\forall y : \varepsilon_{yu} \to (f;\varepsilon')_{yv}\big]\big)$
$\forall p : \forall v : (f;\mathcal{U}')_{pv} \longrightarrow \big(\exists u : \mathcal{U}_{pu} \wedge \overline{\exists y : \varepsilon_{yu} \wedge \overline{(f;\varepsilon')_{yv}}}\big)$
$\forall p : \forall v : (f;\mathcal{U}')_{pv} \longrightarrow \big(\exists u : \mathcal{U}_{pu} \wedge \overline{\varepsilon^\mathsf{T};\overline{f;\varepsilon'}}_{uv}\big)$
$\forall p : \forall v : (f;\mathcal{U}')_{pv} \longrightarrow \big(\mathcal{U};\overline{\varepsilon^\mathsf{T};\overline{f;\varepsilon'}}\big)_{pv}$
$f;\mathcal{U}' \subseteq \mathcal{U};\overline{\varepsilon^\mathsf{T};\overline{f;\varepsilon'}}$
$f;\mathcal{U}' \subseteq \mathcal{U};\vartheta^\mathsf{T}_{f^\mathsf{T}}$

The last step is proved as follows:

$\mathcal{U};\overline{\varepsilon^\mathsf{T};\overline{f;\varepsilon'}} \subseteq \mathcal{U};\overline{\varepsilon^\mathsf{T};\overline{f;\varepsilon';\vartheta_{f^\mathsf{T}};\vartheta^\mathsf{T}_{f^\mathsf{T}}}}$ because ϑ_{f^T} is total
$= \mathcal{U};\overline{\varepsilon^\mathsf{T};\overline{f;\varepsilon';\mathtt{syq}(f;\varepsilon',\varepsilon);\vartheta^\mathsf{T}_{f^\mathsf{T}}}}$ by definition of ϑ_{f^T}
$\subseteq \mathcal{U};\overline{\varepsilon^\mathsf{T};\overline{\varepsilon;\vartheta^\mathsf{T}_{f^\mathsf{T}}}}$ cancellation; always $A;\mathtt{syq}(A,B) \subseteq B$
$= \mathcal{U};\overline{\varepsilon^\mathsf{T};\overline{\varepsilon}};\vartheta^\mathsf{T}_{f^\mathsf{T}}$ since ϑ_{f^T} is a mapping
$= \mathcal{U};\Omega;\vartheta^\mathsf{T}_{f^\mathsf{T}}$
$= \mathcal{U};\vartheta^\mathsf{T}_{f^\mathsf{T}}$ Def. 3.4.ii

3.24 Definition. Consider two neighborhood topologies $\mathcal{U} : X \longrightarrow \mathbf{2}^X$ and $\mathcal{U}' : X' \longrightarrow \mathbf{2}^{X'}$ as well as a mapping $f : X \longrightarrow X'$. We call

f (neighborhood-)**continuous** $\quad :\Longleftrightarrow \quad f;\mathcal{U}' \subseteq \mathcal{U};\vartheta^\mathsf{T}_{f^\mathsf{T}}$.

The equivalent version $f;\mathcal{U}';\vartheta_{f^\mathsf{T}} \subseteq \mathcal{U}$ is obtained shunting the mapping ϑ_{f^T}. □

Observe that the mapping f cannot be shunted. This looks quite similar to a homomorphism condition, but it is definitely not a homomorphism. It allows, nevertheless, to be extended to iterated continuous mappings:

$$\begin{array}{l}\mathcal{U} : X \longrightarrow \mathbf{2}^X, \qquad \mathcal{U}' : X' \longrightarrow \mathbf{2}^{X'}, \qquad \mathcal{U}'' : X'' \longrightarrow \mathbf{2}^{X''}\\ \qquad\qquad f : X \longrightarrow X', \qquad g : X' \longrightarrow X''\\ f;\mathcal{U}' \subseteq \mathcal{U};\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}}, \quad g;\mathcal{U}'' \subseteq \mathcal{U}';\vartheta^{\mathsf{T}}_{g^{\mathsf{T}}} \quad \Longrightarrow \quad f;g;\mathcal{U}'' \subseteq f;\mathcal{U}';\vartheta^{\mathsf{T}}_{g^{\mathsf{T}}} \subseteq \mathcal{U};\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}};\vartheta^{\mathsf{T}}_{g^{\mathsf{T}}} = \mathcal{U};(\vartheta_{g^{\mathsf{T}}};\vartheta_{f^{\mathsf{T}}})^{\mathsf{T}}\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \mathcal{U};(\vartheta_{g^{\mathsf{T}};f^{\mathsf{T}}})^{\mathsf{T}} = \mathcal{U};(\vartheta_{(f;g)^{\mathsf{T}}})^{\mathsf{T}}\end{array}$$

What is not possible is "rolling the condition" to the same extent as for homomorphisms in [Sch11] Prop. 5.45 — except what has been shown above wrt. to rolling based on the mapping $\vartheta_{f^{\mathsf{T}}}$ alone. One has, thus, to apply the language of simulation as explained in [dRE98] and [Sch11] Prop. 19.17, calling $\mathcal{U}'^{\mathsf{T}}$ an $f^{\mathsf{T}}, \vartheta_{f^{\mathsf{T}}}$-$L^{\mathsf{T}}$-simulation of $\mathcal{U}^{\mathsf{T}}$ — or else an $\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}}, f^{\mathsf{T}}$-$U$-simulation of $\mathcal{U}^{\mathsf{T}}$.

We proceed defining continuity concepts for several topology versions, and prove afterwards that they all mean the same.

3.25 Definition. Given sets X, X' with topologies, we consider a mapping $f : X \longrightarrow X'$ together with its inverse image mapping $\vartheta_{f^{\mathsf{T}}} : \mathbf{2}^{X'} \longrightarrow \mathbf{2}^X$. Then we say that the pair $(f, \vartheta_{f^{\mathsf{T}}})$ is

i) (open-kernel-map-)**continuous** $:\Longleftrightarrow$ $\mathcal{K}^{\mathsf{T}}_2;\vartheta_{f^{\mathsf{T}}} \subseteq \overline{\varepsilon_2{}^{\mathsf{T}};f^{\mathsf{T}};\overline{\varepsilon_1}};\mathcal{K}^{\mathsf{T}}_1$

ii) (open-diagonal-)**continuous** $:\Longleftrightarrow$ $\mathcal{O}_{D2};\vartheta_{f^{\mathsf{T}}} \subseteq \vartheta_{f^{\mathsf{T}}};\mathcal{O}_{D1}$

iii) (open-set-)**continuous** $:\Longleftrightarrow$ $\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}};\mathcal{O}_{V_2} \subseteq \mathcal{O}_{V_1}$

iv) (membership-in-open-sets-)**continuous** $:\Longleftrightarrow$ $f;\varepsilon_{\mathcal{O}_2};\vartheta_{f^{\mathsf{T}}} \subseteq \varepsilon_{\mathcal{O}_1}$ □

The second and third definition obviously meet the classical form which says that inverse images of open sets shall be open again. In the first definition, one can recognize some sort of a homomorphism with respect to the converse of kernel-forming; however not with $\vartheta_{f^{\mathsf{T}}}$ on the right side, but with a residual slightly above.

One will observe that in the following proposition first a direct equivalence is proved and afterwards four statements cyclically.

3.26 Proposition. The diverse continuity conditions mean essentially the same:

i) (neighborhood-)**continuous** $\Longleftrightarrow$ (open-kernel-map-)**continuous**

ii) (neighborhood-)**continuous** $\Longrightarrow$ (open-diagonal-)**continuous**

iii) (open-diagonal-)**continuous** $\Longrightarrow$ (open-set-)**continuous**

iv) (open-set-)**continuous** $\Longrightarrow$ (membership-in-open-sets-)**continuous**

v) (membership-in-open-sets-)**continuous** $\Longrightarrow$ (neighborhood-)**continuous**

Proof: i) $f;\mathcal{U}_2;\vartheta_{f^\mathsf{T}} \subseteq \mathcal{U}_1 = \varepsilon_1;\mathcal{K}_1^\mathsf{T}$ assumption and expansion of $\mathcal{U}_1$

$\iff \quad f;\varepsilon_2;\mathcal{K}_2^\mathsf{T};\vartheta_{f^\mathsf{T}};\mathcal{K}_1 \subseteq \varepsilon_1$ expanding $\mathcal{U}_2$ and shunting

$\iff \quad {\varepsilon_2}^\mathsf{T};f^\mathsf{T};\overline{\varepsilon_1} \subseteq \overline{\mathcal{K}_2^\mathsf{T};\vartheta_{f^\mathsf{T}};\mathcal{K}_1}$ Schröder rule

$\iff \quad \mathcal{K}_2^\mathsf{T};\vartheta_{f^\mathsf{T}};\mathcal{K}_1 \subseteq \overline{{\varepsilon_2}^\mathsf{T};f^\mathsf{T};\overline{\varepsilon_1}}$ negated

$\iff \quad \mathcal{K}_2^\mathsf{T};\vartheta_{f^\mathsf{T}} \subseteq \overline{{\varepsilon_2}^\mathsf{T};f^\mathsf{T};\overline{\varepsilon_1}};\mathcal{K}_1^\mathsf{T}$ shunting again

ii) $\overline{\varepsilon_2^\mathsf{T};\overline{\mathcal{U}_2}} \subseteq \overline{\varepsilon_2^\mathsf{T};f^\mathsf{T};f;\overline{\mathcal{U}_2}} = \overline{\varepsilon_2^\mathsf{T};f^\mathsf{T};\overline{f;\mathcal{U}_2}} = \overline{\vartheta_{f^\mathsf{T}};\varepsilon_1^\mathsf{T};\overline{f;\mathcal{U}_2}} \subseteq \overline{\vartheta_{f^\mathsf{T}};\varepsilon_1^\mathsf{T};\overline{\mathcal{U}_1;\vartheta_{f^\mathsf{T}}^\mathsf{T}}} = \vartheta_{f^\mathsf{T}};\overline{\varepsilon_1^\mathsf{T};\overline{\mathcal{U}_1}};\vartheta_{f^\mathsf{T}}^\mathsf{T}$

$\implies \quad \mathcal{O}_{D2} = \mathbb{I} \cap \overline{\varepsilon_2^\mathsf{T};\overline{\mathcal{U}_2}} \subseteq \vartheta_{f^\mathsf{T}};\vartheta_{f^\mathsf{T}}^\mathsf{T} \cap \vartheta_{f^\mathsf{T}};\overline{\varepsilon_1^\mathsf{T};\overline{\mathcal{U}_1}};\vartheta_{f^\mathsf{T}}^\mathsf{T} = \vartheta_{f^\mathsf{T}}(\mathbb{I} \cap \overline{\varepsilon_1^\mathsf{T};\overline{\mathcal{U}_1}});\vartheta_{f^\mathsf{T}}^\mathsf{T} = \vartheta_{f^\mathsf{T}}\mathcal{O}_{D1};\vartheta_{f^\mathsf{T}}^\mathsf{T}$

iii) $\vartheta_{f^\mathsf{T}}^\mathsf{T};\mathcal{O}_{V_2} = \vartheta_{f^\mathsf{T}}^\mathsf{T};\mathcal{O}_{D2};\mathbb{T} = \vartheta_{f^\mathsf{T}}^\mathsf{T};\mathcal{O}_{D2}^\mathsf{T};\mathbb{T} \subseteq \mathcal{O}_{D1}^\mathsf{T};\vartheta_{f^\mathsf{T}}^\mathsf{T};\mathbb{T} = \mathcal{O}_{D1};\vartheta_{f^\mathsf{T}}^\mathsf{T};\mathbb{T} \subseteq \mathcal{O}_{D1};\mathbb{T} = \mathcal{O}_{V_1}$

iv) $f;\varepsilon_{\mathcal{O}_2};\vartheta_{f^\mathsf{T}} = f;(\varepsilon_2 \cap \mathbb{T};\mathcal{O}_{V_2}^\mathsf{T});\vartheta_{f^\mathsf{T}} = (f;\varepsilon_2 \cap f;\mathbb{T};\mathcal{O}_{V_2}^\mathsf{T});\vartheta_{f^\mathsf{T}}$

$= (\varepsilon_1;\vartheta_{f^\mathsf{T}}^\mathsf{T} \cap \mathbb{T};\mathcal{O}_{V_2}^\mathsf{T});\vartheta_{f^\mathsf{T}}$ following Def. 2.2.

$= \varepsilon_1 \cap \mathbb{T};\mathcal{O}_{V_2}^\mathsf{T};\vartheta_{f^\mathsf{T}}$ destroy and append

$\subseteq \varepsilon_1 \cap \mathbb{T};\mathcal{O}_{V_1}^\mathsf{T} = \varepsilon_{\mathcal{O}_1}$

v) $f;\mathcal{U}_2;\vartheta_{f^\mathsf{T}} = f;\varepsilon_{\mathcal{O}_2};\Omega_2;\vartheta_{f^\mathsf{T}}$

$\subseteq f;\varepsilon_{\mathcal{O}_2};\vartheta_{f^\mathsf{T}};\Omega_1$ Prop. 5.2.i of [SW14]

$\subseteq \varepsilon_{\mathcal{O}_1};\Omega_1$ assumption

$= \mathcal{U}_1$ □

This is a funny situation. Structure comparison mainly takes place in reverse direction, i.e. with $f^\mathsf{T}, \vartheta_{f^\mathsf{T}}$ and only the latter of the two is a mapping. "Rolling the homomorphism" may, thus, only be applied in a very restricted form.

4 Aumann Closure, Aumann Contact

Topology has been shown to be definable in cryptomorphically equivalent ways by neighborhood systems, open sets, and closed sets, e.g. It is less commonly known that also Aumann closures as originating from [Aum70, Aum74] give rise to topologies. These in turn often stem from contact relations.

4.1 Aumann contact related to topology

The following is some sort of a free re-interpretation of Aumanns concepts; see [Sch11].

4.1 Definition. We consider a set related to its powerset, with a membership relation $\varepsilon : X \longrightarrow \mathbf{2}^X$. Then a relation $C : X \longrightarrow \mathbf{2}^X$ is called an **Aumann**[2] **contact relation**, provided

[2]Georg Aumann (1906–1980) was a professor at TU München since 1960. Before that, he was at LMU München. Already in 1934/35 he visited the Institute for Advanced Studies in Princeton as a Rockefeller Fellow. Some have considered him as one of the more significant mathematicians of the first half of the 20th century, not least because of his book *Reelle Funktionen*. The first author knew him quite well, since in 1968 he has been with him among those who formally founded the by now famous Mathematics faculty of TUM that had as an offspin the equally famous Informatics faculty, — after its existence as an informal substructure of the old faculty of 'Allgemeine Wissenschaften'.

i) $\varepsilon \subseteq C \subseteq \mathbb{T};\varepsilon$

ii) $C;\overline{\varepsilon^{\mathsf{T}};\overline{C}} \subseteq C$, or equivalently, $C^{\mathsf{T}};\overline{C} \subseteq \varepsilon^{\mathsf{T}};\overline{C}$, (which means in fact "=").

We call C a **topological Aumann contact relation**, when in addition
$(C \circledless C) \subseteq C;\mathfrak{M}^{\mathsf{T}}$. □

One will easily observe that the relation C is up-closed, i.e.:

$$C;\Omega = C;\overline{\varepsilon^{\mathsf{T}};\overline{\varepsilon}} \subseteq C;\overline{\varepsilon^{\mathsf{T}};\overline{C}} \subseteq C.$$

This definition is slightly more restrictive than that of [Sch11, Def. 11.18] in as far as contact with the empty set is concerned; e.g., the first column of C in Fig. 4.1 is demanded to be a $\mathbf{0}$-column. Aumann contacts may always be generated from an arbitrary relation $R : X \longrightarrow Y$ using the membership relation $\varepsilon : X \longrightarrow \mathbf{2}^X$ as $C := \overline{\overline{R};\overline{\overline{R}^{\mathsf{T}};\varepsilon}} = R/(\varepsilon\backslash R)$. According to [Sch11] p. 281, we have in particular that every contact generates itself, i.e., $C = \overline{\overline{C};\overline{\overline{C}^{\mathsf{T}};\varepsilon}} = C/(\varepsilon\backslash C)$. This is easily shown remembering the upper and lower bound functionals:

$$\overline{\overline{C};\overline{\overline{C}^{\mathsf{T}};\varepsilon}} = \overline{\overline{C};\overline{\overline{C}^{\mathsf{T}};C}} \quad \text{due to (ii)}$$

$$= \overline{\overline{C};\overline{\overline{C}^{\mathsf{T}};\overline{\overline{C};\mathbb{I}}}} = \overline{\overline{C};\mathbb{I}} = C \quad \text{since } \mathtt{lbd}_C(\mathtt{ubd}_C(\mathtt{lbd}_C(\mathbb{I}))) = \mathtt{lbd}_C(\mathbb{I})$$

4.2 Propostion. Whenever C is an Aumann contact, then so is the possibly smaller relation $C' := C \cap \overline{\overline{\mathbb{I}};\sigma}$ with $\sigma := \mathtt{syq}(\mathbb{I}, \varepsilon)$ the singleton injection.

Proof: i) $\overline{\mathbb{I}};\sigma = \overline{\mathbb{I}};\mathtt{syq}(\mathbb{I}, \varepsilon) = \overline{\mathbb{I}};\mathtt{syq}(\overline{\mathbb{I}}, \overline{\varepsilon}) \subseteq \overline{\varepsilon}$, so that also $\varepsilon \subseteq \overline{\overline{\mathbb{I}};\sigma}$. Obviously $C' \subseteq C \subseteq \mathbb{T};\varepsilon$.

ii) We have to prove $(C \cap \overline{\overline{\mathbb{I}};\sigma})^{\mathsf{T}};(\overline{C} \cup \overline{\mathbb{I}};\sigma) \subseteq \varepsilon^{\mathsf{T}};(\overline{C} \cup \overline{\mathbb{I}};\sigma)$, from which the product with $\overline{C}$ is trivial since C is an Aumann contact by assumption. It suffices then to show that $(C \cap \overline{\overline{\mathbb{I}};\sigma})^{\mathsf{T}};\overline{\mathbb{I}};\sigma \subseteq \varepsilon^{\mathsf{T}};\overline{\mathbb{I}};\sigma$:

$$\begin{array}{ll}
\Longleftrightarrow & (C \cap \overline{\overline{\mathbb{I}};\sigma})^{\mathsf{T}};\overline{\mathbb{I}} \subseteq \varepsilon^{\mathsf{T}};\overline{\mathbb{I}};\sigma;\sigma^{\mathsf{T}} \qquad \text{shunting} \\
\Longleftarrow & (C \cap \overline{\overline{\mathbb{I}};\sigma})^{\mathsf{T}};\overline{\mathbb{I}} \subseteq \varepsilon^{\mathsf{T}};\overline{\mathbb{I}} \qquad \text{since } \sigma \text{ is total} \\
\Longleftarrow & (\mathbb{T};\varepsilon \cap \overline{\overline{\mathbb{I}};\sigma})^{\mathsf{T}};\overline{\mathbb{I}} \subseteq \varepsilon^{\mathsf{T}};\overline{\mathbb{I}} \qquad \text{because } C \subseteq \mathbb{T};\varepsilon \\
\Longleftrightarrow & \overline{\mathbb{I}};(\mathbb{T};\varepsilon \cap \overline{\overline{\mathbb{I}};\sigma}) \subseteq \overline{\mathbb{I}};\varepsilon \qquad \text{transposed} \\
\Longleftrightarrow & \overline{\mathbb{I}};(\varepsilon \cap \overline{\overline{\mathbb{I}};\sigma}) \subseteq \overline{\mathbb{I}};\varepsilon \quad \text{and} \quad \overline{\mathbb{I}};(\overline{\mathbb{I}};\varepsilon \cap \overline{\overline{\mathbb{I}};\sigma}) \subseteq \overline{\mathbb{I}};\varepsilon, \qquad \text{splitted } \mathbb{T} = \mathbb{I} \cup \overline{\mathbb{I}} \\
\Longleftarrow & \overline{\mathbb{I}};(\overline{\mathbb{I}};\varepsilon \cap \overline{\overline{\mathbb{I}};\sigma}) \subseteq \overline{\mathbb{I}};\varepsilon \quad \text{since the first one is trivial} \\
\Longleftarrow & \overline{\mathbb{I}};(\overline{\mathbb{I}};\varepsilon \cap \overline{\overline{\mathbb{I}};\sigma}) = \overline{\mathbb{I}};(\mathbb{T};\varepsilon \cap \overline{\mathbb{T};\sigma}) \qquad \text{see below} \\
 & \quad = \mathbb{T};\varepsilon \cap \overline{\mathbb{I}};\overline{\mathbb{T};\sigma} \quad \text{masking} \\
 & \quad \subseteq \mathbb{T};\varepsilon \cap \overline{\mathbb{T};\sigma} \\
 & \quad = \overline{\mathbb{I}};\varepsilon \cap \overline{\overline{\mathbb{I}};\sigma} \quad \text{again as shown below} \\
 & \quad \subseteq \overline{\mathbb{I}};\varepsilon
\end{array}$$

Here, we had been allowed to replace $\overline{\mathbb{I}}$ by $\mathbb{T}$; which is trivial — when interpreted in matrices:

$$\mathbb{T};\varepsilon \cap \overline{\mathbb{T};\sigma} = (\mathbb{I};\varepsilon \cup \overline{\mathbb{I}};\varepsilon) \cap \overline{\mathbb{I};\sigma \cup \overline{\mathbb{I}};\sigma} = (\mathbb{I};\varepsilon \cup \overline{\mathbb{I}};\varepsilon) \cap \overline{\sigma} \cap \overline{\overline{\mathbb{I}};\sigma} = (\mathbb{I};\varepsilon \cup \overline{\mathbb{I}};\varepsilon) \cap (\overline{\mathbb{I}};\varepsilon \cup \overline{\varepsilon}) \cap \overline{\overline{\mathbb{I}};\sigma}$$

$$= \left[\overline{\mathbb{I}};\varepsilon \cup (\varepsilon \cap \overline{\varepsilon})\right] \cap \overline{\overline{\mathbb{I}};\sigma} = \overline{\mathbb{I}};\varepsilon \cap \overline{\overline{\mathbb{I}};\sigma}$$ □

In C', it is no longer allowed that an element is in contact to a singleton set it is not contained in. Quite obviously, this is strongly related with the forth-coming Prop. 5.17.

$$
C = \begin{array}{c} \\ \mathrm{a} \\ \mathrm{b} \\ \mathrm{c} \\ \mathrm{d} \end{array}
\begin{array}{c}
\begin{array}{cccccccccccccccc} \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} & \{d\} & \{a,d\} & \{b,d\} & \{a,b,d\} & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\} \end{array} \\
\begin{pmatrix}
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
\end{array}
\qquad
\begin{array}{c} \\ \mathrm{a} \\ \mathrm{b} \\ \mathrm{c} \\ \mathrm{d} \end{array}
\begin{array}{c}
\begin{array}{cccccccccccccccc} \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} & \{d\} & \{a,d\} & \{b,d\} & \{a,b,d\} & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\} \end{array} \\
\begin{pmatrix}
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & \mathrm{O} & 1 & 1 & 1 & 1 & 1 & \mathrm{O} & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
\end{array}
= C'
$$

Fig. 4.1 Aumann contact with or without modification of Prop. 4.2

From such contact relation, we first get a closure operation $\rho' := \mathtt{syq}(C, \varepsilon)$ in a rather simple way, see Def. 17.13 of [Sch11], from which in turn a topology may be derived.

4.3 Proposition. Given a topological Aumann contact relation C, the construct

$$\mathcal{U} := C;\rho';\Omega = C;\overline{C^{\mathsf{T}};\overline{\varepsilon}}$$

is indeed a neighborhood topology as defined in Def. 3.4.

Proof: ii) is trivial. We follow the numbering scheme of Def. 3.4 and prove at the beginning

$$\rho';\Omega = \rho';\overline{\varepsilon^{\mathsf{T}};\overline{\varepsilon}} = \overline{\rho';\varepsilon^{\mathsf{T}};\overline{\varepsilon}} = \overline{\mathtt{syq}(C,\varepsilon);\varepsilon^{\mathsf{T}};\overline{\varepsilon}} = \overline{C^{\mathsf{T}};\overline{\varepsilon}}.$$

i) The relation $\mathcal{U}$ is total since $C;\rho';\Omega;\mathbb{T} = C;\rho';\mathbb{T} = C;\mathbb{T} \supseteq \varepsilon;\mathbb{T} = \mathbb{T}$.

$$C;\rho';\Omega \subseteq \varepsilon \quad\Longleftrightarrow\quad \overline{\varepsilon} = \overline{\varepsilon};\Omega^{\mathsf{T}} \subseteq \overline{C;\rho'} \quad\Longleftrightarrow\quad C;\rho' \subseteq \varepsilon \quad\Longleftrightarrow\quad C \subseteq \varepsilon;\rho'^{\mathsf{T}}.$$

iii) $\mathcal{U} = C;\rho';\Omega$ is typed like a membership relation, so that we may apply Prop. 2.6 and get

$$
\begin{aligned}
&(C;\rho';\Omega \olessthan C;\rho';\Omega) \;=\; (C;\rho' \olessthan C;\rho');(\Omega \otimes \Omega)\\
&= (C \olessthan C);(\rho' \otimes \rho');(\Omega \otimes \Omega)\\
&\subseteq C;\mathfrak{M}^{\mathsf{T}};(\rho' \otimes \rho');(\Omega \otimes \Omega)\\
&\subseteq C;\rho';\Omega;\mathfrak{M}^{\mathsf{T}};(\Omega \otimes \Omega) \qquad \text{Prop. 3.1}\\
&\subseteq C;\rho';\Omega;\Omega;\mathfrak{M}^{\mathsf{T}} \qquad \text{Prop. 9.4.iv of [SW14]}\\
&= C;\rho';\Omega;\mathfrak{M}^{\mathsf{T}}\\
&= \mathcal{U};\mathfrak{M}^{\mathsf{T}}
\end{aligned}
$$

iv) $C;\rho';\Omega = \overline{\overline{C;\rho';\Omega}} = \overline{\mathtt{lbd}_{\overline{C}}(\rho';\Omega)} = \overline{\mathtt{lbd}_{\overline{C}}(\mathtt{ubd}_{\overline{C}}(\mathtt{lbd}_{\overline{C}}(\rho';\Omega)))} = C;(\mathtt{ubd}_{\overline{C}}(\mathtt{lbd}_{\overline{C}}(\rho';\Omega)))$

$= C;\overline{C^{\mathsf{T}};\mathtt{lbd}_{\overline{C}}(\rho';\Omega)} = C;\overline{C^{\mathsf{T}};\overline{\overline{C;\rho';\Omega}}} = C;\overline{\rho';\varepsilon^{\mathsf{T}};\overline{\overline{C;\rho';\Omega}}} = C;\rho';\overline{\varepsilon^{\mathsf{T}};\overline{\overline{C;\rho';\Omega}}} \subseteq C;\rho';\Omega;\overline{\varepsilon^{\mathsf{T}};\overline{\overline{C;\rho';\Omega}}}$ $\square$

But also the other way round:

4.4 Proposition. Given a neighborhood topology $\mathcal{U}$, the construct $C := \overline{\mathcal{U}};N$ is always an Aumann contact.

Proof: i) $\varepsilon \subseteq \overline{\mathcal{U}};N \quad\Longleftrightarrow\quad \overline{\varepsilon} = \varepsilon;N \subseteq \overline{\mathcal{U}} \quad\Longleftrightarrow\quad \mathcal{U} \subseteq \varepsilon,$ which holds by definition.

We consider $g := \mathtt{syq}(\mathbb{T}, \varepsilon) = \overline{\mathbb{T};\overline{\varepsilon}}$, the mapping that sends every element to the powerset element corresponding to the greatest subset. Then obviously

$$\Omega;g^\mathsf{T} = \overline{\varepsilon^\mathsf{T};\overline{\varepsilon}};g^\mathsf{T} = \overline{\varepsilon^\mathsf{T};\overline{\varepsilon;g^\mathsf{T}}} = \overline{\varepsilon^\mathsf{T};\overline{\varepsilon;\mathtt{syq}(\varepsilon,\mathbb{T})}} = \overline{\varepsilon^\mathsf{T};\overline{\overline{\mathbb{T}}}} = \overline{\varepsilon^\mathsf{T};\perp\!\!\!\perp} = \mathbb{T}.$$

This allows us to proceed as follows

$$\mathbb{T} = \mathcal{U};\mathbb{T} = \mathcal{U};\Omega;g^\mathsf{T} = \mathcal{U};g^\mathsf{T} \quad\Longrightarrow\quad \mathbb{T};g \subseteq \mathcal{U} \quad \text{via shunting}$$

in order to finally arrive at

$$C = \overline{\mathcal{U}};N \subseteq \mathbb{T};\varepsilon \quad\Longleftrightarrow\quad \overline{\mathcal{U}} \subseteq \mathbb{T};\varepsilon;N = \mathbb{T};\overline{\varepsilon} = \overline{\mathbb{T};g}.$$

ii) $C;\overline{\varepsilon^\mathsf{T};\overline{C}} = \overline{\mathcal{U}};N;\overline{\varepsilon^\mathsf{T};\overline{\overline{\mathcal{U}};N}} \subseteq \overline{\mathcal{U}};N = C$

$$\Longleftarrow \quad \overline{\mathcal{U}};\overline{N;\varepsilon^\mathsf{T};\overline{\overline{\mathcal{U}}}} \subseteq \overline{\mathcal{U}} \quad\Longleftrightarrow\quad \overline{\mathcal{U}};\overline{\overline{\varepsilon}^\mathsf{T};\mathcal{U}} \subseteq \overline{\mathcal{U}} \quad\Longleftrightarrow\quad \mathcal{U};\overline{\mathcal{U}^\mathsf{T};\overline{\varepsilon}} \subseteq \mathcal{U} \quad\Longleftarrow\quad \text{Prop. 3.5.v} \qquad \square$$

The concepts underlying the Aumann contact and Aumann closure have attracted further attention. In the voluminous *Theory of convex structures*, [vdV93], the concept of betweenness is defined in predicate logic form which we lift to point-free style as follows.

4.5 Definition. A relation $B : X \longrightarrow \mathbf{2}^X$ will be called **betweenness** provided it satisfies in combination with the membership relation $\varepsilon : X \longrightarrow \mathbf{2}^X$ the following:

i) $B \subseteq \mathbb{T};\varepsilon$, i.e., no point is between the empty set

ii) $\varepsilon \subseteq B$

iii) $B^\mathsf{T};\overline{B} \subseteq \varepsilon^\mathsf{T};\overline{B}$ $\square$

It is evident that this concept coincides with the earlier one of an Aumann contact. A detailed study of certain aspects of betweenness may also be found in [AN98].

4.2 Overview and Examples

In total, we have the interrelationship of these topological concepts as shown in the following diagram. The result of Prop. 4.4 does not help in identifying the way back from $\mathcal{U}$ to C; it gives a different contact relation.

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
 & C := \bigwedge_R(\varepsilon) = \overline{\overline{R};\overline{\overline{R}^\mathsf{T};\varepsilon}} \arrow[r, shift right, "\rho = \mathtt{syq}(C,\varepsilon)"'] \arrow[d, "\mathcal{U} = C;\overline{C^\mathsf{T};\overline{\varepsilon}} = C;\rho;\Omega"] & \rho \arrow[l, shift right, "C = \varepsilon;\rho^\mathsf{T}"'] \\
 & \mathcal{U} \arrow[dl, shift right] \arrow[dr, shift left] & \\
\mathcal{K} \arrow[ur, shift right, "\begin{array}{l}\mathcal{K} = \mathtt{syq}(\mathcal{U},\varepsilon)\\ \mathcal{U} = \varepsilon;\mathcal{K}^\mathsf{T}\end{array}"] \arrow[rr, shift right, "\varepsilon_\mathcal{O} = \varepsilon;\mathcal{K} \cap \varepsilon"'] \arrow[d, shift left, "\begin{array}{l}\mathcal{K} = \mathtt{syq}(\varepsilon;\mathcal{O}_D;\Omega,\varepsilon)\\ \mathcal{O}_D = \mathcal{K}^\mathsf{T};\mathcal{K}\end{array}"] & & \varepsilon_\mathcal{O} \arrow[ul, shift left, "\begin{array}{l}\varepsilon_\mathcal{O} = \mathcal{U};\mathcal{K}\\ \mathcal{U} = \varepsilon_\mathcal{O};\Omega\end{array}"'] \arrow[ll, shift right, "\mathcal{K} = \mathtt{syq}(\varepsilon_\mathcal{O};\Omega,\varepsilon)"'] \arrow[d, shift left, "\begin{array}{r}\varepsilon_\mathcal{O} = \varepsilon \cap \mathbb{T};\mathcal{O}_V^\mathsf{T}\\ \mathcal{O}_V = \varepsilon_\mathcal{O}^\mathsf{T};\mathbb{T} \cup \overline{\varepsilon^\mathsf{T};\mathbb{T}}\end{array}"'] \\
\mathcal{O}_D \arrow[u, shift left] \arrow[rr, shift right, "\mathcal{O}_V = \mathcal{O}_D;\mathbb{T}"'] & & \mathcal{O}_V \arrow[u, shift left] \arrow[ll, shift right, "\mathcal{O}_D = \mathbb{I} \cap \mathcal{O}_V;\mathbb{T}"']
\end{tikzcd}
$$

To the lowest two we may go also directly from $\mathcal{U}$:

$$\mathcal{O}_D = \mathbb{I} \cap \overline{\varepsilon^\mathsf{T};\overline{\mathcal{U}}} \qquad \mathcal{O}_V = \overline{\varepsilon^\mathsf{T};\overline{\mathcal{U}}};\mathbb{T}.$$

This follows since

$$\mathcal{K}^\mathsf{T} = \mathtt{syq}(\varepsilon,\mathcal{U}) = \overline{\varepsilon^\mathsf{T};\overline{\mathcal{U}}} \cap \overline{\overline{\varepsilon}^\mathsf{T};\mathcal{U}} \supseteq \overline{\varepsilon^\mathsf{T};\overline{\mathcal{U}}} \cap \overline{\overline{\varepsilon}^\mathsf{T};\varepsilon} = \overline{\varepsilon^\mathsf{T};\overline{\mathcal{U}}} \cap \Omega^\mathsf{T}, \text{ but also } \overline{\varepsilon^\mathsf{T};\overline{\mathcal{U}}} \subseteq \overline{\varepsilon^\mathsf{T};\overline{\varepsilon}} = \Omega.$$

5 Proximity and Nearness

Proximity is introduced when trying to axiomatize the concept of being in some sense "near" that may hold from a set to another set. Far better known are point-to-set notions that characterize being element of a neighborhood or of an open set. Proximity was described in 1908 by Frigyes Riesz and then ignored. Others to be mentioned for having worked on such ideas include V. A. Efremovič in 1934 and A. N. Wallace in 1940. More recently, we found some work in [NW70, VDDB02, BD07].

5.1 Proximity

The conditions for a proximity relation δ are often formulated as being symmetric, only defined for nonempty sets, encompassing nonempty intersection, being join-distributive and satisfying a last not so easily describable law, which we will bring over several steps to a point-free version.

A proximity space (X, δ) is therefore a set X with a relation δ between subsets of X satisfying the following properties: For all subsets A, B and C of X

– $A\delta B \implies B\delta A$,

– $A\delta B \implies A \neq \emptyset$,

– $A \cap B \neq \emptyset \implies A\delta B$,

– $A\delta(B \cup C) \iff (A\delta B \text{ or } A\delta C)$,

– $\forall E, A\delta E \text{ or } B\delta(X\backslash E) \implies A\delta B$.

The first four items are required for a so-called *contact* in [BD07]; in [NW70] the last one is called the *strong axiom*. This indicates the importance of the last.

If $A\delta B$, one says that "A is δ-near B" or "A and B are δ-proximal". It is not too easy to rephrase the intention of the last property above in plain words: Two arbitrary sets A, B aren't δ-near when there exists a subset E and its complement so that A is not near to E and at the same time B not to its complement.

The main properties of such a set neighborhood relation obviously ask for an alternative axiomatic characterization lifted to point-free form; it is provided with Def. 5.1. We restrict to justifying the lifting process for the most complicated of these laws, the strong one, in some more detail:

$$\begin{array}{l}
\big[\forall E : A\delta E \vee B\delta(X\backslash E)\big] \longrightarrow A\delta B \\
\big[\neg\exists E : A\overline{\delta}E \wedge B(\overline{\delta};N)E\big] \longrightarrow A\delta B \\
A\overline{\overline{\delta};N;\overline{\delta}^{\mathsf{T}}}B \longrightarrow A\delta B \\
\overline{\overline{\delta};N;\overline{\delta}^{\mathsf{T}}} \subseteq \delta
\end{array}$$

This leads us to define in a completely point-free form as follows:

5.1 Definition. We speak of a **pre-proximity relation** on a set X if in addition to membership $\varepsilon : X \longrightarrow \mathbf{2}^X$ and join $\mathfrak{J} : \mathbf{2}^X \times \mathbf{2}^X \longrightarrow \mathbf{2}^X$ a relation $\Delta : \mathbf{2}^X \longrightarrow \mathbf{2}^X$ is given satisfying the following properties

i) $\Delta^{\mathsf{T}} \subseteq \Delta$,

ii) $\Delta;\mathbb{T} \subseteq \varepsilon^{\mathsf{T}};\mathbb{T}$,

iii) $\varepsilon^{\mathsf{T}};\varepsilon \subseteq \Delta$,

iv) $\mathfrak{J};\Delta = (\pi \cup \rho);\Delta$.

A **proximity relation** is a pre-proximity satisfying the strong $\overline{\Delta} \subseteq \overline{\Delta};N;\overline{\Delta}$ in addition. $\square$

The following remark indicates the coarsest proximity, so that there is no smaller one, and the biggest.

5.2 Remark. Given any membership relation $\varepsilon : X \longrightarrow \mathbf{2}^X$, the constructs $\Delta := \varepsilon^{\mathsf{T}}\mathbin{;}\mathbb{I}\mathbin{;}\varepsilon$ as well as $\Delta' := \varepsilon^{\mathsf{T}}\mathbin{;}\mathbb{T} \cap \mathbb{T}\mathbin{;}\varepsilon = \varepsilon^{\mathsf{T}}\mathbin{;}\mathbb{T}\mathbin{;}\varepsilon$ satisfy the requirements for a proximity.

Proof: i,ii,iii) are trivial in both cases.

iv) is shown simultaneously for both cases using Prop. 9.1.iii of [SW14]:

$$\varepsilon\mathbin{;}\mathfrak{J}^{\mathsf{T}} = \varepsilon\mathbin{;}\pi^{\mathsf{T}} \cup \varepsilon\mathbin{;}\rho^{\mathsf{T}} = \varepsilon\mathbin{;}(\pi^{\mathsf{T}} \cup \rho^{\mathsf{T}})$$

The additional "strong" property that makes these to proximities is also satisfied, which we show for the first variant with

$$\overline{\Delta} \subseteq \Omega\mathbin{;}\overline{\Delta} = \overline{\varepsilon^{\mathsf{T}}\mathbin{;}\overline{\varepsilon}}\mathbin{;}\overline{\Delta} = \overline{\varepsilon^{\mathsf{T}}\mathbin{;}\varepsilon}\mathbin{;}N\mathbin{;}\overline{\Delta} = \overline{\Delta}\mathbin{;}N\mathbin{;}\overline{\Delta}.$$

For the second, we recall that $l := \overline{\varepsilon^{\mathsf{T}}\mathbin{;}\mathbb{T}}$ is a point, namely the least element in $\mathbf{2}^X$, and that

$$l \subseteq l\mathbin{;}N\mathbin{;}l,$$

which is a consequence of shunting the point l

$$l \subseteq \overline{\varepsilon^{\mathsf{T}}\mathbin{;}\mathbb{T}}\mathbin{;}N\mathbin{;}l \qquad \Longleftrightarrow \qquad l\mathbin{;}l^{\mathsf{T}} \subseteq \overline{\varepsilon^{\mathsf{T}}\mathbin{;}\mathbb{T}}\mathbin{;}N = \overline{\varepsilon^{\mathsf{T}}\mathbin{;}\mathbb{T}\mathbin{;}N} = \overline{\varepsilon^{\mathsf{T}}\mathbin{;}\mathbb{T}}.$$

Therefore $\overline{\Delta'} = l \cup l^{\mathsf{T}} \subseteq (l \cup l^{\mathsf{T}})\mathbin{;}N\mathbin{;}(l \cup l^{\mathsf{T}}) = \overline{\Delta'}\mathbin{;}N\mathbin{;}\overline{\Delta'}$. □

Fig. 5.1 shows the proximity $\Delta = \varepsilon^{\mathsf{T}}\mathbin{;}\mathbb{I}\mathbin{;}\varepsilon$ mentioned above and the one obtained from the topology of Fig. 5.2 following Prop. 5.3. With Fig. 5.1, it is relatively easy to see that $\Delta' = \varepsilon^{\mathsf{T}}\mathbin{;}\mathbb{T}\mathbin{;}\varepsilon$ is the biggest conceivable proximity; biggest means: exactly first row and column with **0** s.

$$\begin{array}{r|cccccccccccccccc}
 & \{\} & \{1\} & \{2\} & \{1,2\} & \{3\} & \{1,3\} & \{2,3\} & \{1,2,3\} & \{4\} & \{1,4\} & \{2,4\} & \{1,2,4\} & \{3,4\} & \{1,3,4\} & \{2,3,4\} & \{1,2,3,4\} \\ \hline
\{\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{1\} & 0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1 \\
\{2\} & 0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\
\{1,2\} & 0&1&1&1&0&1&1&1&0&1&1&1&0&1&1&1 \\
\{3\} & 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\
\{1,3\} & 0&1&0&1&1&1&1&1&0&1&0&1&1&1&1&1 \\
\{2,3\} & 0&0&1&1&1&1&1&1&0&0&1&1&1&1&1&1 \\
\{1,2,3\} & 0&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1 \\
\{4\} & 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \\
\{1,4\} & 0&1&0&1&0&1&0&1&1&1&1&1&1&1&1&1 \\
\{2,4\} & 0&0&1&1&0&0&1&1&1&1&1&1&1&1&1&1 \\
\{1,2,4\} & 0&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1 \\
\{3,4\} & 0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1 \\
\{1,3,4\} & 0&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1 \\
\{2,3,4\} & 0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\
\{1,2,3,4\} & 0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1
\end{array}$$

$$\begin{array}{r|cccccccccccccccc}
 & \{\} & \{1\} & \{2\} & \{1,2\} & \{3\} & \{1,3\} & \{2,3\} & \{1,2,3\} & \{4\} & \{1,4\} & \{2,4\} & \{1,2,4\} & \{3,4\} & \{1,3,4\} & \{2,3,4\} & \{1,2,3,4\} \\ \hline
\{\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{1\} & 0&1&1&1&0&1&1&1&0&1&1&1&0&1&1&1 \\
\{2\} & 0&1&1&1&0&1&1&1&0&1&1&1&0&1&1&1 \\
\{1,2\} & 0&1&1&1&0&1&1&1&0&1&1&1&0&1&1&1 \\
\{3\} & 0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1 \\
\{1,3\} & 0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\
\{2,3\} & 0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\
\{1,2,3\} & 0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\
\{4\} & 0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1 \\
\{1,4\} & 0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\
\{2,4\} & 0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\
\{1,2,4\} & 0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\
\{3,4\} & 0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1 \\
\{1,3,4\} & 0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\
\{2,3,4\} & 0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\
\{1,2,3,4\} & 0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1
\end{array}$$

Fig. 5.1 Coarsest and a bigger proximity

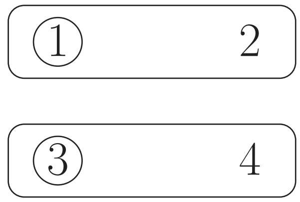


Fig. 5.2 The basis of open sets of the topology used for the right part of Fig. 5.1

Some interrelationships with topology seem obvious. The following proposition states that the relation between two points results in a pre-proximity when one takes their complements and finds a common point to which these are not neighborhoods.

5.3 Proposition. From an arbitrary neighborhood topology determined equivalently by $\mathcal{U}, \mathcal{K}, \mathcal{H}$, one may obtain the pre-proximity relation $\Delta := N;\overline{\mathcal{U}}^{\mathsf{T}};\overline{\mathcal{U}};N = \mathcal{H};\overline{\Omega};\mathcal{K}^{\mathsf{T}};N$.

Proof: Prior to the proof, we show equivalence of the variants:

$$N;\overline{\mathcal{U}}^{\mathsf{T}};\overline{\mathcal{U}};N = N;\overline{\mathcal{K};\varepsilon^{\mathsf{T}}};\overline{\varepsilon;\mathcal{K}^{\mathsf{T}}};N = N;\mathcal{K};\overline{\varepsilon}^{\mathsf{T}};\overline{\varepsilon};\mathcal{K}^{\mathsf{T}};N = N;\mathcal{K};N;\varepsilon^{\mathsf{T}};\overline{\varepsilon};\mathcal{K}^{\mathsf{T}};N = \mathcal{H};\overline{\Omega};\mathcal{K}^{\mathsf{T}};N$$

i) Δ defined by the first variant is obviously symmetric by construction.

ii) We use that $\mathcal{U}$ is total, $\mathcal{U};\Omega = \mathcal{U}$ and $\overline{\Omega} = \varepsilon^{\mathsf{T}};\overline{\varepsilon}$ to show $\mathbb{T} = \mathcal{U};\mathbb{T} = \mathcal{U};(\Omega \cup \overline{\Omega}) \subseteq \mathcal{U} \cup \mathbb{T};\overline{\varepsilon}$ $\iff \overline{\mathcal{U}}^{\mathsf{T}} \subseteq \overline{\varepsilon}^{\mathsf{T}};\mathbb{T} \iff N;\overline{\mathcal{U}}^{\mathsf{T}} \subseteq \varepsilon^{\mathsf{T}};\mathbb{T}$. Now obviously $\Delta;\mathbb{T} \subseteq N;\overline{\mathcal{U}}^{\mathsf{T}};\mathbb{T} \subseteq \varepsilon^{\mathsf{T}};\mathbb{T}$.

iii) $\varepsilon^{\mathsf{T}};\varepsilon \subseteq N;\overline{\mathcal{U}}^{\mathsf{T}};\overline{\mathcal{U}};N \iff N;\varepsilon^{\mathsf{T}};\varepsilon;N = \overline{\varepsilon}^{\mathsf{T}};\overline{\varepsilon} \subseteq \overline{\mathcal{U}}^{\mathsf{T}};\overline{\mathcal{U}} \impliedby \mathcal{U} \subseteq \varepsilon$

iv) $\mathfrak{J};\Delta = \mathfrak{J};N;\mathcal{K};\overline{\Omega}^{\mathsf{T}};\mathcal{H}^{\mathsf{T}}$ using the above variant in transposed form
$= \mathcal{N};\mathfrak{M};\mathcal{K};\overline{\Omega}^{\mathsf{T}};\mathcal{H}^{\mathsf{T}}$ point-free De Morgan rule, Prop. 9.2.i of [SW14]
$= \mathcal{N};(\mathcal{K} \otimes \mathcal{K});\mathfrak{M};\overline{\Omega}^{\mathsf{T}};\mathcal{H}^{\mathsf{T}}$ Def. 3.6.iii
$= \mathcal{N};(\mathcal{K} \otimes \mathcal{K});\overline{\mathfrak{M};\Omega^{\mathsf{T}}};\mathcal{H}^{\mathsf{T}}$ since $\mathfrak{M}$ is a mapping
$= \mathcal{N};(\mathcal{K} \otimes \mathcal{K});\overline{\pi;\Omega^{\mathsf{T}} \cap \rho;\Omega^{\mathsf{T}}};\mathcal{H}^{\mathsf{T}}$ Prop. 9.2.iv of [SW14]
$= \mathcal{N};(\mathcal{K} \otimes \mathcal{K});(\pi;\overline{\Omega}^{\mathsf{T}} \cup \rho;\overline{\Omega}^{\mathsf{T}});\mathcal{H}^{\mathsf{T}}$
$= \mathcal{N};(\mathcal{K} \otimes \mathcal{K});\pi;\overline{\Omega}^{\mathsf{T}};\mathcal{H}^{\mathsf{T}} \cup \mathcal{N};(\mathcal{K} \otimes \mathcal{K});\rho;\overline{\Omega}^{\mathsf{T}};\mathcal{H}^{\mathsf{T}}$
$= \mathcal{N};\pi;\mathcal{K};\overline{\Omega}^{\mathsf{T}};\mathcal{H}^{\mathsf{T}} \cup \mathcal{N};\rho;\mathcal{K};\overline{\Omega}^{\mathsf{T}};\mathcal{H}^{\mathsf{T}}$ Prop. 7.2.ii of [SW14] with $\mathcal{K}$ total
$= \pi;N;\mathcal{K};\overline{\Omega}^{\mathsf{T}};\mathcal{H}^{\mathsf{T}} \cup \rho;N;\mathcal{K};\overline{\Omega}^{\mathsf{T}};\mathcal{H}^{\mathsf{T}}$
$= \pi;\Delta^{\mathsf{T}} \cup \rho;\Delta^{\mathsf{T}}$
$= (\pi \cup \rho);\Delta$ using (i) □

We do not give here the proof of the additional "strong" axiom to establish a proximity (and not just a pre-proximity). It would need to assume some separation such as by the Hausdorff-property. An example in Fig. 5.3 shows that a topology not necessarily results in a proximity via Prop. 5.3; again based on the topology of Fig. 3.2.

$$
\Delta_1 = \begin{array}{r}
\\ \{\} \\ \{a\} \\ \{b\} \\ \{a,b\} \\ \{c\} \\ \{a,c\} \\ \{b,c\} \\ \{a,b,c\} \\ \{d\} \\ \{a,d\} \\ \{b,d\} \\ \{a,b,d\} \\ \{c,d\} \\ \{a,c,d\} \\ \{b,c,d\} \\ \{a,b,c,d\}
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccccccc}
\rotatebox{90}{\{\}} & \rotatebox{90}{\{a\}} & \rotatebox{90}{\{b\}} & \rotatebox{90}{\{a,b\}} & \rotatebox{90}{\{c\}} & \rotatebox{90}{\{a,c\}} & \rotatebox{90}{\{b,c\}} & \rotatebox{90}{\{a,b,c\}} & \rotatebox{90}{\{d\}} & \rotatebox{90}{\{a,d\}} & \rotatebox{90}{\{b,d\}} & \rotatebox{90}{\{a,b,d\}} & \rotatebox{90}{\{c,d\}} & \rotatebox{90}{\{a,c,d\}} & \rotatebox{90}{\{b,c,d\}} & \rotatebox{90}{\{a,b,c,d\}}
\end{array} \\
\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1\\
0&0&1&1&1&1&1&1&0&0&1&1&1&1&1&1\\
0&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1\\
0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1\\
0&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1
\end{pmatrix}
\end{array}
$$

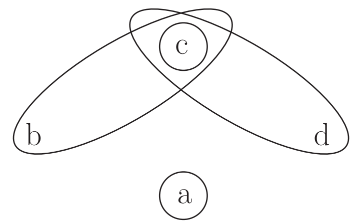


Fig. 5.3 Example of a pre-proximity which is not a proximity because of b, d; cf. Fig. 3.2

5.2 Another proximity concept

A similar concept is provided in the following definition; see also [DV06, DL12]. For purposes of distinct notation, we will call the concept a DV-pre-proximity; others have sometimes termed it *contact*, which would be misleading in the present context. Such a DV-pre-proximity $\mathcal{D}$ on a Boolean algebra B (with $0, \leq, -, \cup$) is given provided the following properties hold:

- $x\mathcal{D}y \implies y\mathcal{D}x$,
- $x\mathcal{D}y \implies x, y \neq 0$,
- $x \neq 0 \implies x\mathcal{D}x$,
- $x\mathcal{D}(y \cup z) \implies (x\mathcal{D}y \text{ or } x\mathcal{D}z)$,
- $x\mathcal{D}y$ and $y \leq z \implies x\mathcal{D}z$.

These are the basic rules. Also here additional properties are often demanded to hold, such as:

- $\mathcal{D}(x) = \mathcal{D}(y)$ implies $x = y$,
- If $(\forall z)(x\mathcal{D}z \text{ or } y\mathcal{D} - z)$ then $x\mathcal{D}y$.

The translation of this still partly predicate-logical version to a point-free form is immediate:

5.4 Definition. Given $\varepsilon, \Omega, \mathfrak{J}$ as usual, the relation $\mathcal{D}$ is called a **DV-pre-proximity relation**, provided

i) $\mathcal{D}^{\mathsf{T}} \subseteq \mathcal{D}$,

ii) $\mathcal{D} \subseteq \varepsilon^{\mathsf{T}}; \mathbb{T}; \varepsilon$,

iii) $\mathbb{I} \cap \varepsilon^{\mathsf{T}}; \mathbb{T} \subseteq \mathcal{D}$,

iv) $\mathcal{D}; \mathfrak{J}^{\mathsf{T}} \subseteq \mathcal{D}; (\pi \cup \rho)^{\mathsf{T}}$,

v) $\mathcal{D}; \Omega \subseteq \mathcal{D}$. □

Of course, the requirements of Def. 5.1 and of Def. 5.4 appear to be somehow similar. We prove that this is indeed the case.

5.5 Proposition. Every DV-pre-proximity is a pre-proximity and vice versa.

Proof: First we prove "Def. 5.1 $\Longrightarrow$ Def. 5.4". Properties (i,iv) are obvious.

ii) From Def. 5.1.ii, we get $\Delta \subseteq \Delta; \mathbb{T} \subseteq \varepsilon^{\mathsf{T}}; \mathbb{T}$ as well as by symmetry $\Delta \subseteq \mathbb{T}; \varepsilon$, so that with masking

$$\Delta \subseteq \varepsilon^{\mathsf{T}}; \mathbb{T} \cap \mathbb{T}; \varepsilon = (\mathbb{T} \cap \varepsilon^{\mathsf{T}}; \mathbb{T}); \varepsilon = \varepsilon^{\mathsf{T}}; \mathbb{T}; \varepsilon.$$

iii) From Def. 5.1.iii, we get with the Dedekind rule

$$\varepsilon^{\mathsf{T}}; \mathbb{T} \cap \mathbb{I} \subseteq (\varepsilon^{\mathsf{T}} \cap \mathbb{I}; \mathbb{T}); (\mathbb{T} \cap \varepsilon; \mathbb{I}) = \varepsilon^{\mathsf{T}}; \varepsilon \subseteq \Delta.$$

v) $\Delta; \Omega = \Delta; \pi^{\mathsf{T}}; \mathfrak{J}$ Prop. 9.2.iii of [SW14]
$\subseteq \Delta; (\pi^{\mathsf{T}} \cup \rho^{\mathsf{T}}); \mathfrak{J}$
$= \Delta; \mathfrak{J}^{\mathsf{T}}; \mathfrak{J}$ Def. 5.1.iv
$\subseteq \Delta$ since $\mathfrak{J}$ is univalent

Now we switch to proving "Def. 5.1 $\Longleftarrow$ Def. 5.4":

i) follows from Def. 5.4.i.

ii) $\mathcal{D}; \mathbb{T} \subseteq \varepsilon^{\mathsf{T}}; \mathbb{T}$ follows from Def. 5.4.ii because $\mathbb{T}; \varepsilon; \mathbb{T} \subseteq \mathbb{T}$.

iii) From Def. 5.4.i,v, we get $\Omega^{\mathsf{T}}; \mathcal{D}; \Omega \subseteq \mathcal{D}$. Applying this to (iii) and using Lemma 4.2.viii of [SW14] produces

$$\varepsilon^{\mathsf{T}}; \varepsilon = (\Omega^{\mathsf{T}} \cap \mathbb{T}; \varepsilon); (\Omega \cap \varepsilon^{\mathsf{T}}; \mathbb{T}) = \Omega^{\mathsf{T}}; (\mathbb{I} \cap \mathbb{T}; \varepsilon); (\mathbb{I} \cap \varepsilon^{\mathsf{T}}; \mathbb{T}); \Omega = \Omega^{\mathsf{T}}; (\mathbb{I} \cap \varepsilon^{\mathsf{T}}; \mathbb{T}); \Omega \subseteq \Omega^{\mathsf{T}}; \mathcal{D}; \Omega \subseteq \mathcal{D}.$$

iv) follows from Def. 5.4.iv as far as "$\subseteq$" is concerned. Regarding "$\supseteq$", we prove, e.g.,

$$\mathcal{D}; \pi^{\mathsf{T}} \subseteq \mathcal{D}; \mathfrak{J}^{\mathsf{T}} \iff \mathcal{D}; \pi^{\mathsf{T}}; \mathfrak{J} \subseteq \mathcal{D} \iff \mathcal{D}; \Omega \subseteq \mathcal{D}$$

shunting and using Prop. 9.2.ii of [SW14] as well as the present property 5.4.v. □

As this is now proved, we may use either of these definitions together with the strong axiom mentioned. With the following proposition, we see that a pre-proximity may arise from fairly trivial sources.

5.6 Proposition. Given any reflexive and symmetric relation $R : X \longrightarrow X$ together with the corresponding membership $\varepsilon : X \longrightarrow \mathbf{2}^X$, the construct $\mathcal{D} := \varepsilon^{\mathsf{T}}{;}R{;}\varepsilon$ turns out to be a DV-pre-proximity relation.

Proof: i) and (ii) are trivial for symmetric R. (v) follows from $\varepsilon{;}\Omega = \varepsilon$.

For (iii), we have obviously $\varepsilon{;}\mathbb{I} \subseteq R{;}\varepsilon$ when R is reflexive; therefore

$$\varepsilon^{\mathsf{T}}{;}\overline{R{;}\varepsilon} \subseteq \overline{\mathbb{I}}, \qquad \mathbb{I} \cap \varepsilon^{\mathsf{T}}{;}\overline{R{;}\varepsilon} \subseteq \perp\!\!\!\perp,$$

so that, splitting $\mathbb{T} = \left[\overline{R{;}\varepsilon} \cup R{;}\varepsilon\right]$,

$$\mathbb{I} \cap \varepsilon^{\mathsf{T}}{;}\mathbb{T} = \mathbb{I} \cap \varepsilon^{\mathsf{T}}{;}\left[\overline{R{;}\varepsilon} \cup R{;}\varepsilon\right] = \mathbb{I} \cap \left[\varepsilon^{\mathsf{T}}{;}\overline{R{;}\varepsilon} \cup \varepsilon^{\mathsf{T}}{;}R{;}\varepsilon\right] \subseteq \perp\!\!\!\perp \cup \varepsilon^{\mathsf{T}}{;}R{;}\varepsilon = \varepsilon^{\mathsf{T}}{;}R{;}\varepsilon = \mathcal{D}.$$

iv) We recall Prop. 9.1.iii of [SW14], namely $\varepsilon{;}\mathfrak{J}^{\mathsf{T}} = \varepsilon{;}(\pi^{\mathsf{T}} \cup \rho^{\mathsf{T}})$, implying

$$\mathcal{D}{;}\mathfrak{J}^{\mathsf{T}} = \varepsilon^{\mathsf{T}}{;}R{;}\varepsilon{;}\mathfrak{J}^{\mathsf{T}} = \varepsilon^{\mathsf{T}}{;}R{;}\varepsilon{;}(\pi^{\mathsf{T}} \cup \rho^{\mathsf{T}}) = \mathcal{D}{;}(\pi \cup \rho)^{\mathsf{T}}.$$ □

It shall now even be shown that there is a one-to-one correspondence between reflexive and symmetric relations R and DV-pre-proximity relations $\mathcal{D}$. In the following, we first recall the folklore properties of a Galois correspondence.

5.7 Proposition. Let be given any relations $A : X \longrightarrow Y$ and $B : U \longrightarrow V$.

i) Then there is a Galois correspondence between relations $R : X \longrightarrow U$ and $C : Y \longrightarrow V$

when defining $\quad \pi(C) := \overline{A{;}\overline{C}{;}B^{\mathsf{T}}}, \qquad \sigma(R) := A^{\mathsf{T}}{;}R{;}B, \qquad$ i.e.

$$R \subseteq \pi(C) \quad \Longleftrightarrow \quad C \supseteq \sigma(R).$$

ii) Specializing to $A := B := \varepsilon : X \longrightarrow \mathbf{2}^X, \qquad \sigma(R) := \varepsilon^{\mathsf{T}}{;}R{;}\varepsilon \qquad$ is an embedding.

Proof: i) We use the Schröder rule to obtain

$$R \subseteq \overline{A{;}\overline{C}{;}B^{\mathsf{T}}} \iff A{;}\overline{C}{;}B^{\mathsf{T}} \subseteq \overline{R} \iff R{;}B \subseteq \overline{A{;}\overline{C}} \iff A{;}\overline{C} \subseteq \overline{R{;}B} \iff A^{\mathsf{T}}{;}R{;}B \subseteq C.$$

ii) With the property of the membership relation ε, shown as [Sch11], Prop. 7.14, we get

$$\pi(\sigma(R)) = \overline{\varepsilon{;}\overline{\varepsilon^{\mathsf{T}}{;}R{;}\varepsilon}{;}\varepsilon^{\mathsf{T}}} = \overline{\overline{R{;}\varepsilon}{;}\varepsilon^{\mathsf{T}}} = \overline{\overline{R}} = R,$$

so that σ must be injective, i.e., an embedding making this an adjunction. □

We now concentrate on the special case of R being reflexive and symmetric and we see what it means in Prop. 5.6. It is obvious that symmetry propagates from R to $\mathcal{D}$ and vice versa. Starting from $\mathcal{D}$, we are in a position to prove that R is reflexive by

$$R = \overline{\varepsilon{;}\overline{\mathcal{D}}{;}\varepsilon^{\mathsf{T}}} \supseteq \overline{\varepsilon{;}\overline{\varepsilon^{\mathsf{T}}{;}\varepsilon}{;}\varepsilon^{\mathsf{T}}} = \overline{\overline{\varepsilon}{;}\varepsilon^{\mathsf{T}}} = \overline{\overline{\mathbb{I}}{;}\varepsilon{;}\varepsilon^{\mathsf{T}}} = \overline{\overline{\mathbb{I}}} = \mathbb{I}.$$

Nevertheless, it is possible as before to start from an arbitrary R and obtain the contact C with closure forming ρ.

$$
\begin{array}{c}
\\ \text{a} \\ \text{b} \\ \text{c} \\ \text{d}
\end{array}
\begin{array}{c}
\begin{array}{cccc} \text{a} & \text{b} & \text{c} & \text{d} \end{array} \\
\begin{pmatrix}
1 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0
\end{pmatrix}
\end{array}
\qquad
\begin{array}{c}
\begin{array}{cccccccccccccccc} \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} & \{d\} & \{a,d\} & \{b,d\} & \{a,b,d\} & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\} \end{array} \\
\begin{pmatrix}
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
\end{array}
$$

$$
\begin{array}{c|cccccccccccccccc}
 & \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} & \{d\} & \{a,d\} & \{b,d\} & \{a,b,d\} & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\} \\
\hline
\{\} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{a\} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{b\} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{a,b\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\{c\} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{a,c\} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{b,c\} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{a,b,c\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\{d\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\{a,d\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\{b,d\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\{a,b,d\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\{c,d\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\{a,c,d\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\{b,c,d\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\{a,b,c,d\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

Fig. 5.4 Non-symmetric, non-reflexive R with contact C and closure mapping $\rho := \mathtt{syq}(C, \varepsilon)$

We are by Prop. 5.5 entitled to use properties of Def. 5.1 and Def. 5.4 jointly when we show that proximities give rise to topologies.

5.8 Proposition. From an arbitrary proximity relation Δ, one may obtain the neighborhood topology $\mathcal{U} := \sigma; \overline{\Delta}; N$ (or equivalently $= \overline{\sigma; \Delta; N}$).

Proof: We recall in advance

$\varepsilon = \mathbb{I}; \varepsilon = \sigma; \varepsilon^{\mathsf{T}}; \varepsilon$ Lemma 4.2.i of [SW14]
$\subseteq \sigma; \Delta$ Def. 5.1.iii
$= \sigma; \Delta^{\mathsf{T}} \subseteq \sigma; \mathbb{T}; \Delta^{\mathsf{T}} \subseteq \sigma; \mathbb{T}; \varepsilon = \mathbb{T}; \varepsilon$ Def. 5.1.ii and σ is a mapping

i) $\mathcal{U} = \overline{\sigma; \Delta; N} \subseteq \overline{\varepsilon; N} = \overline{\overline{\varepsilon}} = \varepsilon,$ see above
$\mathcal{U} = \overline{\sigma; \Delta; N} \supseteq \overline{\mathbb{T}; \varepsilon; N} = \overline{\mathbb{T}; \overline{\varepsilon}} \supseteq \mathtt{syq}(\mathbb{T}, \varepsilon)$
Thus $\mathcal{U}$ is total since the definition of ε demands that every $\mathtt{syq}(\varepsilon, X)$ be surjective.

ii) We have rather obviously $N; \Omega^{\mathsf{T}} = \Omega; N$ and $\Delta; \Omega \subseteq \Delta$ due to Def. 5.4.v, so that
$\mathcal{U}; \Omega = \overline{\sigma; \Delta; N}; \Omega \subseteq \overline{\sigma; \Delta; N} = \mathcal{U} \iff \sigma; \Delta; N; \Omega^{\mathsf{T}} = \sigma; \Delta; \Omega; N \subseteq \sigma; \Delta; N.$

iii) $(\mathcal{U} \oslash \mathcal{U}) ; \mathfrak{M} = (\sigma; \overline{\Delta}; N \oslash \sigma; \overline{\Delta}; N) ; \mathfrak{M}$ by definition
$= \sigma; (\overline{\Delta} \oslash \overline{\Delta}); (N \otimes N); \mathfrak{M}$
$= \sigma; (\overline{\Delta} \oslash \overline{\Delta}); \mathcal{N} ; \mathfrak{M}$
$= \sigma; (\overline{\Delta} \oslash \overline{\Delta}); \mathfrak{J} ; N$ point-free De Morgan rule, Prop. 9.2.i of [SW14]
$= \sigma; (\overline{\Delta}; \pi^{\mathsf{T}} \cap \overline{\Delta}; \rho^{\mathsf{T}}); \mathfrak{J} ; N$
$= \sigma; (\overline{\Delta; \pi^{\mathsf{T}}} \cap \overline{\Delta; \rho^{\mathsf{T}}}); \mathfrak{J} ; N$
$= \sigma; \overline{\Delta; [\pi^{\mathsf{T}} \cup \rho^{\mathsf{T}}]}; \mathfrak{J} ; N$
$= \sigma; \overline{\Delta; \mathfrak{J}^{\mathsf{T}}}; \mathfrak{J} ; N = \sigma; \overline{\Delta}; \mathfrak{J}^{\mathsf{T}}; \mathfrak{J} ; N$ Prop. 5.1.iv
$\subseteq \sigma; \overline{\Delta}; N = \mathcal{U}$

iv) In order to show $\mathcal{U} \subseteq \mathcal{U};\overline{\varepsilon^{\mathsf{T}};\overline{\mathcal{U}}}$, we use that $\Delta;\Omega \subseteq \Delta$; see above. In addition, $\sigma;\varepsilon^{\mathsf{T}};\overline{\varepsilon} = \mathbb{I};\overline{\varepsilon} = \overline{\varepsilon}$, so that we may employ $\sigma^{\mathsf{T}};\varepsilon \subseteq \overline{\varepsilon^{\mathsf{T}};\overline{\varepsilon}} = \Omega$.

$\mathcal{U} = \sigma;\overline{\Delta};N \subseteq \sigma;\overline{\Delta};N;\overline{\Delta};N$ using the "strong" condition for a proximity
$\subseteq \sigma;\overline{\Delta};N;\overline{\Omega^{\mathsf{T}};\Delta;N}$ since N^{T} is a mapping and $\Delta;\Omega \subseteq \Delta$
$\subseteq \sigma;\overline{\Delta};N;\overline{\varepsilon^{\mathsf{T}};\sigma;\Delta;N}$ see above
$= \mathcal{U};\overline{\varepsilon^{\mathsf{T}};\overline{\mathcal{U}}}$ □

We may also go the other way round which has already been shown with Prop. 5.3, i.e., from $\mathcal{U}$ to Δ.

As for every mathematical structure, one has also defined a structure-preserving mapping f for proximity in [NW70]. This not yet lifted definition demands that

$(A, B) \in \Delta_1 \longrightarrow (f(A), f(B)) \in \Delta_2$,

from which we derive the following

5.9 Definition. Given proximities $\Delta_i : \mathbf{2}^{X_i} \longrightarrow \mathbf{2}^{X_i},\ i = 1, 2$, and a mapping $f : X_1 \longrightarrow X_2$ of the underlying sets, we call

f a **proximity mapping** $\quad :\Longleftrightarrow \quad \Delta_1;\vartheta_f \subseteq \vartheta_f;\Delta_2$. □

We have had problems to apply the traditional homomorphism scheme to continuity, when we define "traditional" to mean

structure $\times$ *mapping* $\quad\subseteq\quad$ *mapping* $\times$ *structure.*

For proximity mappings, we can say that their definition adheres more or less to the traditional form; there is only a slight deviation, because f is given, but the definition is based on its existential image ϑ_f.

In Prop. 5.8, we have identified a topology for every proximity. It is remarkable that proximity mappings lead to continuous mappings between such topologies.

5.10 Proposition. Any surjective proximity mapping is continuous with respect to the neighborhood topologies according to Prop. 5.8.

Proof: We have to prove $f;\mathcal{U}_2 \subseteq \mathcal{U}_1;\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}}$, which expands to

$f;\overline{\sigma_2;\Delta_2;N_2} \subseteq \overline{\sigma_1;\Delta_1;N_1};\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}} = \overline{\sigma_1;\Delta_1;N_1;\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}}}$.
$\Longleftrightarrow \quad f^{\mathsf{T}};\sigma_1;\Delta_1;N_1;\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}} = f^{\mathsf{T}};\sigma_1;\Delta_1;\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}};N_2 \subseteq \sigma_2;\Delta_2;N_2$ since $N_1;\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}} = \vartheta^{\mathsf{T}}_{f^{\mathsf{T}}};N_2$, Prop. 2.3.iv
$\Longleftrightarrow \quad f^{\mathsf{T}};\sigma_1;\Delta_1;\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}} \subseteq \sigma_2;\Delta_2$
$\Longleftrightarrow \quad \sigma_1;\Delta_1;\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}} \subseteq f;\sigma_2;\Delta_2$ shunting

This shall now be proved:

$\sigma_1;\Delta_1;\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}} \subseteq \sigma_1;\Delta_1;\vartheta_f$ Prop. 2.3.iii for surjective f
$\subseteq \sigma_1;\vartheta_f;\Delta_2$ Def. 5.9
$= f;\sigma_2;\Delta_2$ Prop. 5.3.ii of [SW14] □

It seems to be an interesting task to study how the additional "strong" properties sometimes demanded for pre-proximity as well as for DV-pre-proximity relations are related with one another.

5.3 Nearness

Closely related with "proximity" is the concept of "nearness". We have said "A and B are δ-proximal" if $A\delta B$. Now we proceed to saying that "B is in a δ-neighborhood of A", written $A \ll B$ when $A\delta(X\backslash B)$ is false. This changes the axioms slightly. The main properties of this set neighborhood relation are listed below. They also provide an alternative axiomatic characterization of proximity. For all subsets A, B, C, and D of the set X in question, one demands

- $X \ll X$
- $A \ll B \quad\Longrightarrow\quad A \subseteq B$
- $A \subseteq B \ll C \subseteq D \quad\Longrightarrow\quad A \ll D$
- $(A \ll B \text{ and } A \ll C) \quad\Longrightarrow\quad A \ll B \cap C$
- $A \ll B \quad\Longrightarrow\quad X\backslash B \ll X\backslash A$
- $A \ll B \quad\Longrightarrow\quad \exists E : A \ll E \ll B$

This is now lifted this to a point-free version.

5.11 Definition. We call the relation $R : \mathbf{2}^X \longrightarrow \mathbf{2}^X$ a **nearness**, provided

i) $\overline{\varepsilon^{\mathsf{T}};\mathbb{T}} \cup \overline{\mathbb{T};\overline{\varepsilon}} \subseteq R$ (or more intuitively $\mathtt{syq}(\varepsilon, \perp\!\!\!\perp) \subseteq R$ $\quad\mathtt{syq}(\varepsilon, \mathbb{T}) \subseteq R$)

ii) $R \subseteq \Omega$

iii) $\Omega;R;\Omega \subseteq R$

iv) $(R \otimes R) \subseteq R;\mathfrak{M}^{\mathsf{T}}$, in fact an equality!

v) $R;N \subseteq N;R^{\mathsf{T}}$

vi) $R \subseteq R;R$, i.e., R is dense □

Equality for (iv) is easy to verify using (iii): $R;\mathfrak{M}^{\mathsf{T}} \subseteq (R \otimes R) = R;\pi^{\mathsf{T}} \cap R;\rho^{\mathsf{T}}$, where, e.g., $R;\mathfrak{M}^{\mathsf{T}} \subseteq R;\pi^{\mathsf{T}}$ is via shunting equivalent with $R;\mathfrak{M}^{\mathsf{T}};\pi \subseteq R$ with $\mathfrak{M}^{\mathsf{T}};\pi = \Omega$ according to [SW14] Prop. 9.2.ii.

5.12 Remark. Given any membership relation $\varepsilon : X \longrightarrow \mathbf{2}^X$, the powerset ordering Ω satisfies all the requirements for a nearness.

Proof: Again, (i,ii,iii,vi) are trivial.

iv) follows from Prop. 9.2.iv of [SW14].

v) $\Omega;N = \overline{\varepsilon^\mathsf{T};\overline{\varepsilon}};N = \overline{\varepsilon^\mathsf{T};\varepsilon} = N;\overline{\overline{\varepsilon}^\mathsf{T};\varepsilon} = N;\Omega^\mathsf{T}$ □

The nearness consisting simply of the powerset ordering is the greatest among all possible ones.

5.13 Proposition. For any given proximity Δ, the relation $R := \overline{\Delta;N}$ is a nearness.

Proof: i) is shown for the first part using Prop. 5.1.ii. It follows by symmetry for the second:

$$\overline{\varepsilon^\mathsf{T};\mathbb{T}} \subseteq R \quad\iff\quad \overline{R} \subseteq \varepsilon^\mathsf{T};\mathbb{T} \quad\iff\quad \overline{R};N \subseteq \varepsilon^\mathsf{T};\mathbb{T};N \quad\iff\quad \Delta = \overline{R;N} \subseteq \varepsilon^\mathsf{T};\mathbb{T}$$

ii) $R \subseteq \Omega \quad\iff\quad \overline{\Delta;N} \subseteq \overline{\varepsilon^\mathsf{T};\overline{\varepsilon}} \quad\iff\quad \varepsilon^\mathsf{T};\overline{\varepsilon} \subseteq \Delta;N \quad\iff\quad \varepsilon^\mathsf{T};\overline{\varepsilon};N = \varepsilon^\mathsf{T};\varepsilon \subseteq \Delta$

where the latter is guaranteed by Def. 5.1.iii.

iii) Following Prop. 5.5, we are entitled to use Def. 5.4.v, viz. $\Delta;\Omega \subseteq \Delta$, and in transposed form also $\Omega^\mathsf{T};\Delta \subseteq \Delta$.

$$\begin{aligned}
&\Omega;R;\Omega \subseteq R \quad\iff\quad \Omega;\overline{\Delta;N};\Omega \subseteq \overline{\Delta;N}\\
\iff\quad &\Omega^\mathsf{T};\Delta;N \subseteq \overline{\overline{\Delta;N};\Omega} \quad\Longleftarrow\quad \Delta;N \subseteq \overline{\overline{\Delta;N};\Omega}\\
\iff\quad &\overline{\Delta;N};\Omega \subseteq \overline{\Delta;N} \quad\iff\quad \Delta;N;\Omega^\mathsf{T} \subseteq \Delta;N \quad\iff\quad \Delta;\Omega = \Delta;N;\Omega^\mathsf{T};N \subseteq \Delta
\end{aligned}$$

iv) We prove even equality:

$$\begin{aligned}
&\overline{\Delta;N};\mathfrak{M}^\mathsf{T} = \overline{\Delta;N};N;\mathfrak{J}^\mathsf{T};\mathcal{N} \quad \text{due to Prop. 9.2.i of [SW14], De Morgan rule}\\
&= \overline{\Delta};\mathfrak{J}^\mathsf{T};\mathcal{N} = \overline{\Delta;\mathfrak{J}^\mathsf{T}};\mathcal{N}\\
&= \overline{\Delta;(\pi \cup \rho)^\mathsf{T};\mathcal{N}} = \overline{\Delta;\pi^\mathsf{T} \cup \Delta;\rho^\mathsf{T};\mathcal{N}}\\
&= \overline{\Delta;\pi^\mathsf{T};\mathcal{N} \cup \Delta;\rho^\mathsf{T};\mathcal{N}} = \overline{\Delta;N;\pi^\mathsf{T} \cup \Delta;N;\rho^\mathsf{T}} = \overline{\Delta;N;\pi^\mathsf{T}} \cap \overline{\Delta;N;\rho^\mathsf{T}} = \overline{\Delta;N};\pi^\mathsf{T} \cap \overline{\Delta;N};\rho^\mathsf{T}
\end{aligned}$$

v) $R;N = \overline{\Delta;N};N = \overline{\Delta};N;N = \overline{\Delta} = N;N;\overline{\Delta} = N;\overline{\Delta;N}^\mathsf{T} = N;R^\mathsf{T}$

vi) $\overline{\Delta} \subseteq \overline{\Delta};N;\overline{\Delta} \quad\iff\quad \overline{\Delta};N \subseteq \overline{\Delta};N;\overline{\Delta};N \quad\iff\quad \overline{\Delta;N} \subseteq \overline{\Delta;N};\overline{\Delta;N} \quad\iff\quad R \subseteq R;R$ □

In Fig. 5.5 and Fig. 5.6, we show an example of proximity and nearness.

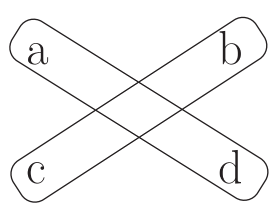


Fig. 5.5 The basis of open sets for Fig. 5.6

$$
\begin{array}{rc}
 & \begin{array}{cccccccccccccccc} \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} & \{d\} & \{a,d\} & \{b,d\} & \{a,b,d\} & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\} \end{array} \\
\begin{array}{r} \{\} \\ \{a\} \\ \{b\} \\ \{a,b\} \\ \{c\} \\ \{a,c\} \\ \{b,c\} \\ \{a,b,c\} \\ \{d\} \\ \{a,d\} \\ \{b,d\} \\ \{a,b,d\} \\ \{c,d\} \\ \{a,c,d\} \\ \{b,c,d\} \\ \{a,b,c,d\} \end{array} &
\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&1&0&1&0&1&1&1&1&1&1&1&1&1\\
0&0&1&1&1&1&1&1&0&0&1&1&1&1&1&1\\
0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&1&1&1&1&1&1&0&0&1&1&1&1&1&1\\
0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&1&1&1&1&1&1&0&0&1&1&1&1&1&1\\
0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&1&0&1&0&1&0&1&1&1&1&1&1&1&1&1\\
0&1&0&1&0&1&0&1&1&1&1&1&1&1&1&1\\
0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1
\end{pmatrix}
\end{array}
\qquad
\begin{array}{rc}
 & \begin{array}{cccccccccccccccc} \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} & \{d\} & \{a,d\} & \{b,d\} & \{a,b,d\} & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\} \end{array} \\
\begin{array}{r} \{\} \\ \{a\} \\ \{b\} \\ \{a,b\} \\ \{c\} \\ \{a,c\} \\ \{b,c\} \\ \{a,b,c\} \\ \{d\} \\ \{a,d\} \\ \{b,d\} \\ \{a,b,d\} \\ \{c,d\} \\ \{a,c,d\} \\ \{b,c,d\} \\ \{a,b,c,d\} \end{array} &
\begin{pmatrix}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1\\
0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1\\
0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix}
\end{array}
$$

Fig. 5.6 A pair of proximity and nearness based on the open set basis of Fig. 5.5

$$
\begin{array}{rc}
 & \begin{array}{cccccccccccccccc} \{\} & \{1\} & \{2\} & \{1,2\} & \{3\} & \{1,3\} & \{2,3\} & \{1,2,3\} & \{4\} & \{1,4\} & \{2,4\} & \{1,2,4\} & \{3,4\} & \{1,3,4\} & \{2,3,4\} & \{1,2,3,4\} \end{array} \\
\begin{array}{r} \{\} \\ \{1\} \\ \{2\} \\ \{1,2\} \\ \{3\} \\ \{1,3\} \\ \{2,3\} \\ \{1,2,3\} \\ \{4\} \\ \{1,4\} \\ \{2,4\} \\ \{1,2,4\} \\ \{3,4\} \\ \{1,3,4\} \\ \{2,3,4\} \\ \{1,2,3,4\} \end{array} &
\begin{pmatrix}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1\\
0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1\\
0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1\\
0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1\\
0&0&0&0&0&1&0&1&0&0&0&0&0&1&0&1\\
0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1\\
0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix}
\end{array}
\qquad
\begin{array}{rc}
 & \begin{array}{cccccccccccccccc} \{\} & \{1\} & \{2\} & \{1,2\} & \{3\} & \{1,3\} & \{2,3\} & \{1,2,3\} & \{4\} & \{1,4\} & \{2,4\} & \{1,2,4\} & \{3,4\} & \{1,3,4\} & \{2,3,4\} & \{1,2,3,4\} \end{array} \\
\begin{array}{r} \{\} \\ \{1\} \\ \{2\} \\ \{1,2\} \\ \{3\} \\ \{1,3\} \\ \{2,3\} \\ \{1,2,3\} \\ \{4\} \\ \{1,4\} \\ \{2,4\} \\ \{1,2,4\} \\ \{3,4\} \\ \{1,3,4\} \\ \{2,3,4\} \\ \{1,2,3,4\} \end{array} &
\begin{pmatrix}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1\\
0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1\\
0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix}
\end{array}
$$

Fig. 5.7 Biggest and a smaller nearness corresponding to Fig. 5.1

Nearly the same as Prop. 5.13 is possible in the other direction.

5.14 Proposition. For a given nearness R, the relation $\Delta : \mathbf{2}^X \longrightarrow \mathbf{2}^X$ defined as

$$\Delta := \overline{R;N},$$

will be a proximity.

Proof: i) $\Delta^\mathsf{T} \subseteq \Delta \quad\iff\quad \overline{R;N}^\mathsf{T} \subseteq \overline{R;N} \quad\iff\quad R;N \subseteq N;R^\mathsf{T}$, i.e. Def. 5.11.v

ii) $\Delta;\mathbb{T} = \overline{R;N};\mathbb{T} = \overline{R};N;\mathbb{T} = \overline{R};\mathbb{T} \subseteq \overline{\overline{\varepsilon^\mathsf{T};\mathbb{T}} \cup \overline{\mathbb{T};\overline{\varepsilon}}};\mathbb{T} = (\varepsilon^\mathsf{T};\mathbb{T} \cap \mathbb{T};\overline{\varepsilon});\mathbb{T} \subseteq \varepsilon^\mathsf{T};\mathbb{T}$, i.e. with Def. 5.11.i

iii) $\varepsilon^\mathsf{T};\varepsilon \subseteq \Delta = \overline{R;N} \quad\iff\quad \varepsilon^\mathsf{T};\overline{\varepsilon} = \varepsilon^\mathsf{T};\varepsilon;N \subseteq \overline{R} \quad\iff\quad R \subseteq \Omega$, i.e. Def. 5.11.ii.

iv) $\Delta{;}\,\mathfrak{J}^\mathsf{T} = \overline{R{;}N}{;}\,\mathfrak{J}^\mathsf{T}$ by the definition above
$= \overline{R{;}N{;}\,\mathfrak{J}^\mathsf{T}}$ since $\mathfrak{J}$ is a mapping
$= \overline{R{;}\,\mathfrak{M}^\mathsf{T}{;}\mathcal{N}}$ point-free De Morgan rule
$= \overline{(R \olessthan R){;}\mathcal{N}}$ using the equality in (iv)
$= \overline{(R{;}N \olessthan R{;}N)}$ since $\mathcal{N} = (N \otimes N)$ is a mapping
$= \overline{R{;}N{;}\pi^\mathsf{T} \cap R{;}N{;}\rho^\mathsf{T}}$
$= \overline{R{;}N{;}\pi^\mathsf{T}} \cup \overline{R{;}N{;}\rho^\mathsf{T}}$
$= \overline{R{;}N}{;}\pi^\mathsf{T} \cup \overline{R{;}N}{;}\rho^\mathsf{T}$
$= \overline{R{;}N}{;}(\pi \cup \rho)^\mathsf{T} = \Delta{;}(\pi \cup \rho)^\mathsf{T}$

v) We start from Def. 5.11.vi to obtain

$$R \subseteq R{;}R \quad \Longleftrightarrow \quad R{;}N \subseteq R{;}R{;}N = R{;}N{;}N{;}R{;}N \quad \Longleftrightarrow \quad \overline{\Delta} \subseteq \overline{\Delta}{;}N{;}\overline{\Delta} \qquad \square$$

5.4 Apartness and connection algebra

Others formalize the concept of a point being apart from a set of points. We reinvestigate the definition of [BStV01], which reads as follows: Assume a set X and a relation **apart** $: X \longrightarrow \mathbf{2}^X$, intended to express that a point x *is apart from a subset* u when $(x, u) \in A$ that satisfies

- $x \neq y \quad \Longrightarrow \quad \mathbf{apart}(x, \{y\})$,
- $\mathbf{apart}(x, u) \quad \Longrightarrow \quad x \notin u$,
- $\mathbf{apart}(x, u \cup v) \quad \Longleftrightarrow \quad \mathbf{apart}(x, u) \wedge \mathbf{apart}(x, v)$,
- $x \in -u \subseteq \tilde{}v \quad \Longrightarrow \quad \mathbf{apart}(x, v)$,

There follows sometimes a last point not mentioned here; it is considered interesting when dealing with specialities of constructive mathematics.

We lift the idea of the preceding definition so as to obtain a point-free version.

5.15 Definition. Assume a set X and a relation $A : X \longrightarrow \mathbf{2}^X$. This relation will then be called an **apartness**, provided

i) $\overline{\mathbb{I}}{;}\sigma \subseteq A$,   with the singleton injection $\sigma := \mathtt{syq}(\mathbb{I}, \varepsilon)$,

ii) $A \subseteq \overline{\varepsilon}$,

iii) $A{;}\,\mathfrak{J}^\mathsf{T} = (A \olessthan A) = A{;}\pi^\mathsf{T} \cap A{;}\rho^\mathsf{T}$,

iv) $A{;}N{;}\Omega \subseteq A{;}N$ or better $A{;}\Omega^\mathsf{T} \subseteq A$. $\square$

In view of Prop. 5.17, one should discuss whether it is wise to postulate instead of equality in (iii) — and less restrictive — only "$\subseteq$". With the following remark we establish the complemented membership relation ε as a most trivial example of an apartness.

5.16 Proposition. Given any membership $\varepsilon : X \longrightarrow \mathbf{2}^X$, its complement $A := \overline{\varepsilon}$ is an apartness.

Proof: i) $\overline{\mathbb{I}};\sigma = \overline{\mathbb{I}};\mathtt{syq}(\mathbb{I},\varepsilon) = \overline{\mathbb{I}};\mathtt{syq}(\overline{\mathbb{I}},\overline{\varepsilon}) \subseteq \overline{\varepsilon} = A$

ii) by definition

iii) $A;\mathfrak{J}^{\mathsf{T}} = \overline{\varepsilon};\mathfrak{J}^{\mathsf{T}} = \overline{\varepsilon;\mathfrak{J}^{\mathsf{T}}} = \overline{\varepsilon;\pi^{\mathsf{T}} \cup \varepsilon;\rho^{\mathsf{T}}} = \overline{\varepsilon};\pi^{\mathsf{T}} \cap \overline{\varepsilon};\rho^{\mathsf{T}} = A;\pi^{\mathsf{T}} \cap A;\rho^{\mathsf{T}}$ using Prop. 9.1.iii of [SW14]

iv) $A;N;\Omega;N = \overline{\varepsilon};N;\Omega;N = \varepsilon;\Omega;N = \varepsilon;N = \overline{\varepsilon} = A$ □

Also the complement of an Aumann contact in the form as obtained in Prop. 4.2 always comes close to an apartness.

5.17 Proposition. Given any Aumann contact relation $C : X \longrightarrow \mathbf{2}^X$, its complement modified to

$$A := \overline{C} \cup \overline{\mathbb{I}};\sigma$$

is *nearly* an apartness, i.e., only with "$\subseteq$" in (iii).

Proof: i) is satisfied by construction.

ii) By definition of contact, $\varepsilon \subseteq C$; furthermore $\overline{\mathbb{I}};\mathtt{syq}(\mathbb{I},\varepsilon) = \overline{\mathbb{I}};\mathtt{syq}(\overline{\mathbb{I}},\overline{\varepsilon}) \subseteq \overline{\varepsilon}$.

iii) Without loss of generality, we confine ourselves to proving $A;\mathfrak{J}^{\mathsf{T}} \subseteq A;\pi^{\mathsf{T}}$, which means via shunting $A;\mathfrak{J}^{\mathsf{T}};\pi \subseteq A$ and by Prop. 9.2.iii of [SW14] $A;\Omega^{\mathsf{T}} \subseteq A$, that is $(\overline{C} \cup \overline{\mathbb{I}};\sigma);\Omega^{\mathsf{T}} \subseteq \overline{C} \cup \overline{\mathbb{I}};\sigma$.

We start with property Def. 4.1.ii for contact $C^{\mathsf{T}};\overline{C} \subseteq \varepsilon^{\mathsf{T}};\overline{C} \subseteq \varepsilon^{\mathsf{T}};\overline{\varepsilon} = \overline{\Omega}$, transpose this to $\overline{C}^{\mathsf{T}};C \subseteq \overline{\Omega}^{\mathsf{T}}$and apply the Schröder rule to finally obtain $\overline{C};\Omega^{\mathsf{T}} \subseteq \overline{C}$.

Furthermore

$$\overline{\mathbb{I}};\sigma;\Omega^{\mathsf{T}} = \overline{\mathbb{I}};(\sigma \cup \overline{\mathbb{T};\varepsilon}) \quad \text{Prop. 4.2.iii of [SW14]}$$
$$= \overline{\mathbb{I}};\sigma \cup \overline{\mathbb{I}};\overline{\mathbb{T};\varepsilon} \subseteq \overline{\mathbb{I}};\sigma \cup \overline{\mathbb{T};\varepsilon} \subseteq \overline{\mathbb{I}};\sigma \cup \overline{C} \quad \text{Def. 4.1.i}$$

iv) $A;\Omega^{\mathsf{T}} = (\overline{C} \cup \overline{\mathbb{I}};\sigma);\Omega^{\mathsf{T}} = \overline{C};\Omega^{\mathsf{T}} \cup \overline{\mathbb{I}};\sigma;\Omega^{\mathsf{T}} = \overline{C};\Omega^{\mathsf{T}} \cup \overline{\mathbb{I}};(\sigma \cup \overline{\mathbb{T};\varepsilon})$ using Prop. 4.2.iii of [SW14]. This should now be contained in $\overline{C} \cup \overline{\mathbb{I}};\sigma = A$. The middle part is obvious. The last follows with

$$\overline{\mathbb{I}};\overline{\mathbb{T};\varepsilon} \subseteq \overline{\mathbb{T};\varepsilon} \subseteq \overline{C}$$

The first uses the contact property

$$C^{\mathsf{T}};\overline{C} \subseteq \varepsilon^{\mathsf{T}};\overline{C} \subseteq \varepsilon^{\mathsf{T}};\overline{\varepsilon} = \overline{\Omega}$$
$$\iff \quad \overline{C}^{\mathsf{T}};C \subseteq \overline{\Omega}^{\mathsf{T}} \quad \text{transposed}$$
$$\iff \quad \overline{C};\Omega^{\mathsf{T}} \subseteq \overline{C} \quad \text{Schröder rule}$$

□

The reverse will only be satisfied in specific situations, not in general.

5.18 Remark. Given an apartness $A : X \longrightarrow \mathbf{2}^X$ on a set X with at least two elements (algebraically: with $\overline{\mathbb{I}};\mathbb{T} = \mathbb{T}$), its complement $C := \overline{A}$ need not form an Aumann contact relation.

We would have to show $\varepsilon \subseteq C \subseteq \mathbb{T};\varepsilon$ and $C^{\mathsf{T}};\overline{C} \subseteq \varepsilon^{\mathsf{T}};\overline{C}$ of which the first inclusion is trivial in view of Def. 5.15.ii.

For the second, we start with

$$\begin{aligned}
&\overline{C} \cup \mathbb{T};\varepsilon = A \cup \mathbb{T};\varepsilon \supseteq A;\Omega^{\mathsf{T}} \cup \mathbb{T};\varepsilon \quad \text{Def. 5.15.v}\\
&\supseteq \overline{\mathbb{I}};\sigma;\Omega^{\mathsf{T}} \cup \mathbb{T};\varepsilon \quad \text{Def. 5.15.i}\\
&= \overline{\mathbb{I}};(\sigma \cup \overline{\mathbb{T};\varepsilon}) \cup \mathbb{T};\varepsilon \supseteq \overline{\mathbb{I}};\overline{\mathbb{T};\varepsilon} \cup \mathbb{T};\varepsilon \quad \text{Lemma 4.2.ii of [SW14]}\\
&= \overline{\mathbb{I}};\overline{\mathbb{T};\varepsilon} \cup \overline{\mathbb{I}};\mathbb{T};\varepsilon \quad \text{condition above}\\
&= \overline{\mathbb{I}};(\overline{\mathbb{T};\varepsilon} \cup \mathbb{T};\varepsilon)\\
&= \overline{\mathbb{I}};\mathbb{T} = \mathbb{T} \quad \text{condition above}
\end{aligned}$$

Concerning the third, we have only $A;\Omega^{\mathsf{T}} = A;\overline{\overline{\varepsilon}^{\mathsf{T}};\varepsilon} \subseteq A$, which doesn't suffice to establish

$$\overline{A}^{\mathsf{T}};A \subseteq \varepsilon^{\mathsf{T}};A \quad\iff\quad \overline{\varepsilon^{\mathsf{T}};A};A^{\mathsf{T}} \subseteq A^{\mathsf{T}} \quad\iff\quad A;\overline{A^{\mathsf{T}};\varepsilon} \subseteq A \qquad \square$$

In total, there seems to be a strong indication that one should not postulate Def. 5.15.i. Whenever one has an apartness A' without, one may add $A := A' \cup \overline{\mathbb{I}};\mathtt{syq}(\mathbb{I},\varepsilon)$ and will get an apartness-like relation again. The neighborhoods for A, A', however, are differently interesting. While those for A' produce interesting topologies, those for A don't.

$$\begin{array}{c} \\ \mathrm{a}\\ \mathrm{b}\\ \mathrm{c}\\ \mathrm{d}\end{array}
\begin{array}{c}
\begin{array}{cccccccccccccccc}
\{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} & \{d\} & \{a,d\} & \{b,d\} & \{a,b,d\} & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\}
\end{array}\\
\begin{pmatrix}
1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0\\
1&1&0&0&1&0&0&0&1&0&0&0&1&0&0&0\\
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&0&1&0&1&0&0&0&0&0&0&0&0&0
\end{pmatrix}
\end{array}$$

Fig. 5.8 Complement of an Aumann contact as standard example of an apartness without (i,iii)

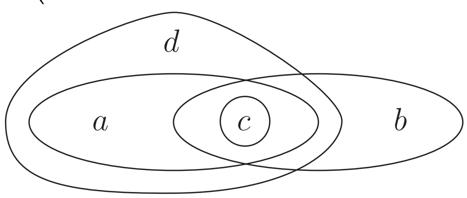
The underlying Aumann contact is obviously non-topological. We now show apartnesses stemming from Aumann contact relations and suggest to discuss the relevance of properties (i) and (iii).

5.19 Example.

$$A' = \begin{array}{c} \\ \mathrm{a}\\ \mathrm{b}\\ \mathrm{c}\\ \mathrm{d}\end{array}
\begin{array}{c}
\begin{array}{cccccccccccccccc}
\{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} & \{d\} & \{a,d\} & \{b,d\} & \{a,b,d\} & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\}
\end{array}\\
\begin{pmatrix}
1&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0\\
1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0
\end{pmatrix}
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccccccc}
\{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} & \{d\} & \{a,d\} & \{b,d\} & \{a,b,d\} & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\}
\end{array}\\
\begin{pmatrix}
1&0&1&0&1&0&1&0&1&0&0&0&0&0&0&0\\
1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0\\
1&1&1&0&0&0&0&0&1&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0
\end{pmatrix}
\end{array} = A$$

apartness properties for A':
$\overline{\mathbb{I}};\sigma \subseteq A'$: False
$A';\mathcal{J}^{\mathsf{T}} = (A' \oslash A')$: True
neighborhood $\mathcal{U} := \overline{A'};\overline{\overline{A'}^{\mathsf{T}}};\overline{\varepsilon} =$

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccccccccccccccc} \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} & \{d\} & \{a,d\} & \{b,d\} & \{a,b,d\} & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\} \end{array} \\ \begin{pmatrix} 0&0&0&0&0&1&0&1&0&0&0&0&0&1&0&1 \\ 0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1 \\ 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1 \end{pmatrix} \end{array}$$



apartness properties for $A = A' \cup \overline{\mathbb{I}};\sigma$:
$\overline{\mathbb{I}};\sigma \subseteq A$: True
$A;\mathcal{J}^{\mathsf{T}} = (A \oslash A)$: False
neighborhood $\mathcal{U} := \overline{A};\overline{\overline{A}^{\mathsf{T}}};\overline{\varepsilon} =$

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccccccccccccccc} \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} & \{d\} & \{a,d\} & \{b,d\} & \{a,b,d\} & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\} \end{array} \\ \begin{pmatrix} 0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1 \\ 0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\ 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \end{pmatrix} \end{array}$$

apparently equal to ε !

$\square$

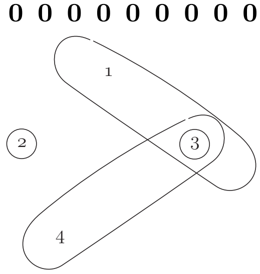
5.20 Example.

$$A' = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccccccccccccc} \{\} & \{1\} & \{2\} & \{1,2\} & \{3\} & \{1,3\} & \{2,3\} & \{1,2,3\} & \{4\} & \{1,4\} & \{2,4\} & \{1,2,4\} & \{3,4\} & \{1,3,4\} & \{2,3,4\} & \{1,2,3,4\} \end{array} \\ \begin{pmatrix} 1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0 \\ 1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0 \\ 1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0 \end{pmatrix} \end{array} \begin{array}{c} \begin{array}{cccccccccccccccc} \{\} & \{1\} & \{2\} & \{1,2\} & \{3\} & \{1,3\} & \{2,3\} & \{1,2,3\} & \{4\} & \{1,4\} & \{2,4\} & \{1,2,4\} & \{3,4\} & \{1,3,4\} & \{2,3,4\} & \{1,2,3,4\} \end{array} \\ \begin{pmatrix} 1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0 \\ 1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0 \\ 1&1&1&0&0&0&0&0&1&0&0&0&0&0&0&0 \\ 1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0 \end{pmatrix} \end{array} = A$$

apartness properties for A':
$\overline{\mathbb{I}};\sigma \subseteq A'$: False
$A';\mathcal{J}^{\mathsf{T}} = (A' \oslash A')$: True

neighborhood $\mathcal{U} := \overline{A'};\overline{\overline{A'}^{\mathsf{T}}};\overline{\varepsilon} =$

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccccccccccccc} \{\} & \{1\} & \{2\} & \{1,2\} & \{3\} & \{1,3\} & \{2,3\} & \{1,2,3\} & \{4\} & \{1,4\} & \{2,4\} & \{1,2,4\} & \{3,4\} & \{1,3,4\} & \{2,3,4\} & \{1,2,3,4\} \end{array} \\ \begin{pmatrix} 0&0&0&0&0&1&0&1&0&0&0&0&0&1&0&1 \\ 0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\ 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix} \end{array}$$



apartness properties for $A = A' \cup \overline{\mathbb{I}};\sigma$:
$\overline{\mathbb{I}};\sigma \subseteq A$: True
$A;\mathcal{J}^{\mathsf{T}} = (A \oslash A)$: False

neighborhood $\mathcal{U} := \overline{A};\overline{\overline{A}^{\mathsf{T}}};\overline{\varepsilon} =$

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccccccccccccc} \{\} & \{1\} & \{2\} & \{1,2\} & \{3\} & \{1,3\} & \{2,3\} & \{1,2,3\} & \{4\} & \{1,4\} & \{2,4\} & \{1,2,4\} & \{3,4\} & \{1,3,4\} & \{2,3,4\} & \{1,2,3,4\} \end{array} \\ \begin{pmatrix} 0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1 \\ 0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\ 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \end{pmatrix} \end{array}$$

apparently equal to ε !

$\square$

5.21 Example.

$$A' = \begin{array}{r@{\,}c} & \begin{array}{*{32}{c}} \{\} & \{1\} & \{2\} & \{1,2\} & \{3\} & \{1,3\} & \{2,3\} & \{1,2,3\} & \{4\} & \{1,4\} & \{2,4\} & \{1,2,4\} & \{3,4\} & \{1,3,4\} & \{2,3,4\} & \{1,2,3,4\} & \{5\} & \{1,5\} & \{2,5\} & \{1,2,5\} & \{3,5\} & \{1,3,5\} & \{2,3,5\} & \{1,2,3,5\} & \{4,5\} & \{1,4,5\} & \{2,4,5\} & \{1,2,4,5\} & \{3,4,5\} & \{1,3,4,5\} & \{2,3,4,5\} & \{1,2,3,4,5\} \end{array} \\ \begin{array}{c}1\\2\\3\\4\\5\end{array} & \begin{pmatrix} 1&0&1&0&1&0&1&0&1&0&1&0&0&0&0&0&1&0&1&0&0&0&0&0&1&0&1&0&0&0&0&0 \\ 1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&1&0&0&0&0&0&1&0&1&0&0&0&0&0&1&0&1&0&0&0&0&0&1&0&1&0&0&0&0&0 \\ 1&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \end{array}$$

$$A = A' \cup \overline{\mathbb{I}}{;}\sigma = \begin{array}{r@{\,}c} & \begin{array}{*{32}{c}} \{\} & \{1\} & \{2\} & \{1,2\} & \{3\} & \{1,3\} & \{2,3\} & \{1,2,3\} & \{4\} & \{1,4\} & \{2,4\} & \{1,2,4\} & \{3,4\} & \{1,3,4\} & \{2,3,4\} & \{1,2,3,4\} & \{5\} & \{1,5\} & \{2,5\} & \{1,2,5\} & \{3,5\} & \{1,3,5\} & \{2,3,5\} & \{1,2,3,5\} & \{4,5\} & \{1,4,5\} & \{2,4,5\} & \{1,2,4,5\} & \{3,4,5\} & \{1,3,4,5\} & \{2,3,4,5\} & \{1,2,3,4,5\} \end{array} \\ \begin{array}{c}1\\2\\3\\4\\5\end{array} & \begin{pmatrix} 1&0&1&0&1&0&1&0&1&0&1&0&0&0&0&0&1&0&1&0&0&0&0&0&1&0&1&0&0&0&0&0 \\ 1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&1&1&0&0&0&0&0&1&0&1&0&0&0&0&0&1&0&1&0&0&0&0&0&1&0&1&0&0&0&0&0 \\ 1&1&1&0&1&0&1&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \end{array}$$

apartness properties for A':
$\overline{\mathbb{I}}{;}\sigma \subseteq A'$: False
$A'{;}\mathcal{J}^{\mathsf{T}} = (A' \oslash A')$: False

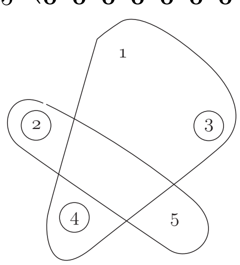
apartness properties for A:
$\overline{\mathbb{I}}{;}\sigma \subseteq A$: True
$A{;}\mathcal{J}^{\mathsf{T}} = (A \oslash A)$: False

neighborhood $\mathcal{U} := \overline{A'}{;}\overline{\overline{A'}^{\mathsf{T}}}{;}\overline{\varepsilon} =$

$$\begin{array}{r@{\,}c} & \begin{array}{*{32}{c}} \{\} & \{1\} & \{2\} & \{1,2\} & \{3\} & \{1,3\} & \{2,3\} & \{1,2,3\} & \{4\} & \{1,4\} & \{2,4\} & \{1,2,4\} & \{3,4\} & \{1,3,4\} & \{2,3,4\} & \{1,2,3,4\} & \{5\} & \{1,5\} & \{2,5\} & \{1,2,5\} & \{3,5\} & \{1,3,5\} & \{2,3,5\} & \{1,2,3,5\} & \{4,5\} & \{1,4,5\} & \{2,4,5\} & \{1,2,4,5\} & \{3,4,5\} & \{1,3,4,5\} & \{2,3,4,5\} & \{1,2,3,4,5\} \end{array} \\ \begin{array}{c}1\\2\\3\\4\\5\end{array} & \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1 \\ 0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\ 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \end{pmatrix} \end{array}$$

neighborhood $\mathcal{U} := \overline{A}{;}\overline{\overline{A}^{\mathsf{T}}}{;}\overline{\varepsilon} =$

$$\begin{array}{r@{\,}c} & \begin{array}{*{32}{c}} \{\} & \{1\} & \{2\} & \{1,2\} & \{3\} & \{1,3\} & \{2,3\} & \{1,2,3\} & \{4\} & \{1,4\} & \{2,4\} & \{1,2,4\} & \{3,4\} & \{1,3,4\} & \{2,3,4\} & \{1,2,3,4\} & \{5\} & \{1,5\} & \{2,5\} & \{1,2,5\} & \{3,5\} & \{1,3,5\} & \{2,3,5\} & \{1,2,3,5\} & \{4,5\} & \{1,4,5\} & \{2,4,5\} & \{1,2,4,5\} & \{3,4,5\} & \{1,3,4,5\} & \{2,3,4,5\} & \{1,2,3,4,5\} \end{array} \\ \begin{array}{c}1\\2\\3\\4\\5\end{array} & \begin{pmatrix} 0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1 \\ 0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\ 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \end{pmatrix} \end{array}$$



apparently equal to ε !

$\square$

Looking at these examples, an observation is immediate: x is apart from a set U, if there exists an open set $V \supseteq U$ that does not contain x. In this form, it reminds us of a separability axiom like $T_0, T_1 \ldots$

Connection algebra

Some other concepts have also been studied: Boolean contact algebras. Their definition in [GW14] is as follows:

5.22 Definition. A relation $C : \mathbf{2}^X \longrightarrow \mathbf{2}^X$, defined besides membership ε, powerset containment Ω and projections from pairs π, ρ, is called a **Boolean contact algebra** when

i) $\mathbb{T}{;}\varepsilon \supseteq C$,

ii) $\mathbb{I} \cap \mathbb{T}{;}\varepsilon \subseteq C$,

iii) $C^{\mathsf{T}} \subseteq C$,

iv) $C{;}\Omega \subseteq C$,

v) $C{;}\,\mathfrak{J}^{\mathsf{T}} \subseteq C{;}\pi^{\mathsf{T}} \cup C{;}\rho^{\mathsf{T}}$,

vi) $\mathtt{syq}(C, C) \subseteq \mathbb{I}$,

vii) $C{;}N{;}C \subseteq C$,

viii) $N \cap \mathbb{T}{;}\varepsilon \cap \mathbb{T}{;}\overline{\varepsilon} \subseteq C$.

This obviously subsumes under our general theme, which we will, however, not elaborate here. One should consider all these, nearness, proximity, apartness, etc. as cryptomorphic concepts, thus avoiding to study them in separate axiomatizations over and over again.

6 Simplicial Complexes

This section is intended to show how one might work relationally also for algebraic topology. We give a glimpse on simplicial complexes, usually subsumed under that topic.

Siegel writes in [Sie79] about his former Frankfurt colleague Max Dehn solving the 3. Hilbert Problem: "We know that the areas of two given triangles can be proved equal by means of elementary geometry, i.e., without resorting to integral calculus or other limit processes. The question remained as to whether the same were possible for 3-dimensional figures; specifically, whether the volume of a tetrahedron could be rigorously defined without taking limits. This was one of the famous unsolved problems in mathematics posed by Hilbert at the international congress of mathematicians in Paris in 1900; Dehn was the first to have solved one of the Hilbert problems. The answer to the problem was in the negative, for Dehn showed that the theory of volume could not be developed on the basis of elementary geometry alone." Dehn has simply constructed two equally voluminous polyhedra that he proved not to be *zerlegungsgleich* nor *ergänzungsgleich*, i.e. not equal by cutting it into pieces and recombining.

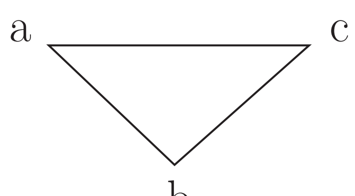
This remark has been inserted in order to prevent us from all too simplistic reasoning. Another hint in that direction are the four articles by Oskar Perron [Per40b, Per40c, Per40a, Per41], the titles of which seem astonishing.

6.1 Simplices

Several aspects of topology have been treated successfully using simplicial complexes. It seems that part of this can also be handled relationally. A non-oriented simplex is simply a finite set X with all subsets of it declared to be simplices. One then studies properties of the descent from a simplex of size n to all its subsets of size $n - 1$. The Hasse relation of the powerset ordering Ω

$$H := C \cap \overline{C{;}C} \quad \text{with} \quad C := \overline{\mathbb{I}} \cap \Omega$$

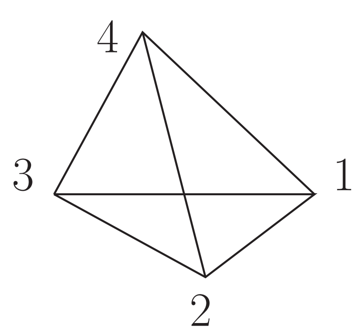
is obviously helpful. Its converse H^T leads from a subset precisely to subsets of one element less. An example is given in Fig. 6.1 with the set $X := \{a, b, c\}$ of which all subsets are considered as being simplices.



$$\begin{array}{cc} & \begin{matrix} \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} \end{matrix} \\ \begin{matrix} \{\} \\ \{a\} \\ \{b\} \\ \{a,b\} \\ \{c\} \\ \{a,c\} \\ \{b,c\} \\ \{a,b,c\} \end{matrix} & \begin{pmatrix} 1&1&1&1&1&1&1&1 \\ 0&1&0&1&0&1&0&1 \\ 0&0&1&1&0&0&1&1 \\ 0&0&0&1&0&0&0&1 \\ 0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&1&0&1 \\ 0&0&0&0&0&0&1&1 \\ 0&0&0&0&0&0&0&1 \end{pmatrix} \end{array} \qquad \begin{array}{cc} & \begin{matrix} \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} \end{matrix} \\ \begin{matrix} \{\} \\ \{a\} \\ \{b\} \\ \{a,b\} \\ \{c\} \\ \{a,c\} \\ \{b,c\} \\ \{a,b,c\} \end{matrix} & \begin{pmatrix} 0&1&1&0&1&0&0&0 \\ 0&0&0&1&0&1&0&0 \\ 0&0&0&1&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&1&1&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0 \end{pmatrix} \end{array}$$

Fig. 6.1 Powerset ordering Ω and its Hasse relation H

While Ω has the relatively obvious fractal generation $\Omega_0 = \begin{pmatrix} 1 \end{pmatrix}$, $\Omega_{n+1} = \begin{pmatrix} \Omega_n & \Omega_n \\ \perp\!\!\!\perp & \Omega_n \end{pmatrix}$, the corresponding fractal generation of H is less immediate: $H_0 = \begin{pmatrix} 0 \end{pmatrix}$, $H_{n+1} = \begin{pmatrix} H_n & \mathbb{I} \\ \perp\!\!\!\perp & H_n \end{pmatrix}$. Another example of a simplex is provided with Fig. 6.2.



$$H = \begin{array}{cc} & \begin{matrix} \{\} & \{1\} & \{2\} & \{1,2\} & \{3\} & \{1,3\} & \{2,3\} & \{1,2,3\} & \{4\} & \{1,4\} & \{2,4\} & \{1,2,4\} & \{3,4\} & \{1,3,4\} & \{2,3,4\} & \{1,2,3,4\} \end{matrix} \\ \begin{matrix} \{\} \\ \{1\} \\ \{2\} \\ \{1,2\} \\ \{3\} \\ \{1,3\} \\ \{2,3\} \\ \{1,2,3\} \\ \{4\} \\ \{1,4\} \\ \{2,4\} \\ \{1,2,4\} \\ \{3,4\} \\ \{1,3,4\} \\ \{2,3,4\} \\ \{1,2,3,4\} \end{matrix} & \begin{pmatrix} 0&1&1&0&1&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&1&0&1&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&1&0&0&1&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&1&1&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&1&1&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \end{array}$$

Fig. 6.2 Hasse relation H of the powerset order of a 3-dimensional simplex

6.2 Orientation

The next idea is to attach to all the so far non-oriented simplices some orientation and to study how the descent mentioned behaves with regard to orientation.

Convention for the representation of oriented simplices: For all the lower-dimensional simplices we demand that their tuples always be oriented according to the baseorder of the set X. An exception from this rule is made for the maximum-dimensional simplices: Since we usually give them as an input when studying some example, we accept for them also the orientation as given in the input. □

Fig. 6.3 illustrates this convention. We have typed $(2,1,3)$ providing an orientation indicated with the rotational arrow. In the cases of 1-dimensional arrows we always assume $(1,2)$, $(2,3)$, and $(1,3)$ etc. Obviously, (1,3) agrees with (2,1,3) in orientation, but (1,2) does not.

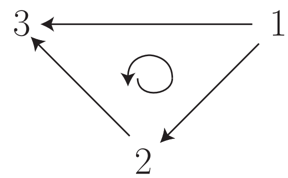


Fig. 6.3 Boundaries of a 2-dimensional oriented simplex

Based on this observation, transition to the boundary shall now be subdivided into two parts, which we call the positive as well as the negative side.

Interlude An early attempt in this regard stems from classical homology of simplicial complexes: There, one is usually given an (additive) Abelian group G, and has to consider linear mappings sending the set of all oriented simplices into G. A mapping sending the n-dimensional oriented simplices into G is called an n-**chain** $\mathcal{C}_n$, provided $\mathcal{C}_n(-S) = -\mathcal{C}_n(S)$ for positively and negatively oriented versions of any simplex S. One is normally not interested in the values of these mappings beyond the combinatorial effect of applying a **boundary operator** to chains.

The boundary operator ∂ is a linear functional sending n-chains to $(n-1)$-chains. Since the boundary operator on chains is assumed to be linear, it need only to be defined for simplexes. If the n-simplex $(x_0,\ldots,x_n)$ gets by $\mathcal{C}$ assigned the value $g \in G$, we will for the moment denote this as $(x_0,\ldots,x_n)_g$. The definition of ∂ is then given showing to which lower-dimensional simplexes it contributes, written as a formal sum

$$(x_0,\ldots,x_n)_g \longmapsto \sum_i (-1)^i\,(x_0,\ldots,x_{i-1},\quad \texttt{x}_\texttt{i}\ \texttt{deleted!!!},\quad x_{i+1},\ldots,x_n)_g.$$

This suffices as a definition, since every chain may be decomposed down to the values it assigns to the single simplices. It means in particular that ∂ maps the value g assigned to (a,b,c) as

$$(a,b,c)_g \longmapsto (b,c)_g - (a,c)_g + (a,b)_g$$

and correspondingly $(a,b)_g \longmapsto (b)_g - (a)_g$. The main theorem then says that $\partial(\partial(x)) = 0$. To understand this result, we observe in this example how the contributions develop

$$\begin{aligned}(a,b,c)_g &\longmapsto (b,c)_g - (a,c)_g + (a,b)_g\\ &\longmapsto [(c)_g - (b)_g] - [(c)_g - (a)_g] + [(b)_g - (a)_g] = 0,\end{aligned}$$

regardless of how $\mathcal{C}$ is actually defined, just following from the assumed linearity of ∂. □

We take our visualization from Fig. 6.14 and give a fairly "arbitrary" chain with group G equal to $\mathbb{Z}$ in Fig. 6.4. It shows the result of applying the boundary operation twice to a 2-chain getting a 0-chain assigning always 0. What homology is intended to do using all this group theory is to keep track of the relative situations of the oriented simplices involved.

Working relationally, we are not in a position to *subtract* as above. We can, however, do some accounting or book-keeping of positive as well as of negative orientations and finally show that both sides result in the same.

	(4,0,3)	(1,0,4)	(4,3,2)	(2,3,1)	(2,1,0)	
	17	2	29	-3	5	2-chain

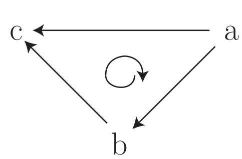
	(1,4)	(0,4)	(0,3)	(3,4)	(2,4)	(2,3)	(1,3)	(1,2)	(0,2)	(0,1)	
from (4,0,3)		-17	17	17							
(1,0,4)	-2	2								-2	
(4,3,2)				-29	29	-29					
(2,3,1)						-3	3	-3			
(2,1,0)								-5	5	-5	
Sum	-2	-15	17	-12	29	-32	3	-8	5	-7	1-chain

	(0)	(1)	(2)	(3)	(4)	
from (1,4)		2			-2	
(0,4)	15				-15	
(0,3)	-17			17		
(3,4)				12	-12	
(2,4)			-29		29	
(2,3)			32	-32		
(1,3)		-3		3		
(1,2)		8	-8			
(0,2)	-5		5			
(0,1)	7	-7				
Sum	0	0	0	0	0	0-chain

Fig. 6.4 Applying the boundary operation ∂ twice to a 2-chain

Considering Fig. 6.4, we have in mind the subgroup of cycles, defined as having boundary 0 as well as the subgroup of boundaries, characterized as images of higher-dimensional chains. The quotient "cycles/boundaries" establishes the famous homology concept.

In a way corresponding to the boundary ∂, the converse $B := H^\mathsf{T}$ of H shall now be partitioned as in Fig. 6.5. This gives a boundary operation assigning to every simplex the set of all the oriented simplices that consist of precisely one element less and are oriented as described above: Positive boundaries of (a, b, c) are (b, c) and (a, b), while (a, c) is considered a negative one.



$$
B^P = \begin{array}{r|cccccccc|}
 & \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} \\
\hline
\{\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{a\} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{b\} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{a,b\} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\{c\} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{a,c\} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\{b,c\} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\{a,b,c\} & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0
\end{array}
\qquad
B^M = \begin{array}{r|cccccccc|}
 & \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} \\
\hline
\{\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{a\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{b\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{a,b\} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{c\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{a,c\} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{b,c\} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\{a,b,c\} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}
$$

Fig. 6.5 Positive and negative boundary operation $H^\mathsf{T} = B^P \cup B^M$

Linear ordering of the powerset When given an ordering $E : X \longrightarrow X$ on a baseset, one may wish to find an ordering $F : \mathbf{2}^X \longrightarrow \mathbf{2}^X$ on its powerset that respects E in some way. For comparison think of the pair of two ordered sets for which we are accustomed to work with the lexicographic ordering which is monotonic wrt. the first projection. This brought forward the study of Egli-Milner orderings when working on semantics of nondeterminism and powerdomains. However, in nearly all cases these turned out to be just preorders even if E was a linear order, and one had trouble to handle the empty set appropriately; see Chapt. 19 of [Sch11].

In the following, we show how it is indeed possible to obtain a linear ordering on the powerset using relational means. We start with the linear baseorder

$$E = \begin{array}{cc} & \begin{array}{ccc} \mathrm{a} & \mathrm{b} & \mathrm{c} \end{array} \\ \begin{array}{c} \mathrm{a} \\ \mathrm{b} \\ \mathrm{c} \end{array} & \begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix} \end{array},$$

consider its Hasse relation H_E and evaluate its decreasing sequence of points as

$$e_1 := \overline{\overline{E}^{\mathsf{T}} ; \mathbb{T}} = \begin{array}{c} \mathrm{a} \\ \mathrm{b} \\ \mathrm{c} \end{array} \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{pmatrix} \quad e_2 := H_E ; e_1 = \begin{array}{c} \mathrm{a} \\ \mathrm{b} \\ \mathrm{c} \end{array} \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \end{pmatrix} \quad e_3 := H_E ; e_2 = \begin{array}{c} \mathrm{a} \\ \mathrm{b} \\ \mathrm{c} \end{array} \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}.$$

Herefrom, we get sets "above" as

$$v_1 := \varepsilon^{\mathsf{T}} e_1 = \begin{array}{c} \{\} \\ \{\mathrm{a}\} \\ \{\mathrm{b}\} \\ \{\mathrm{a,b}\} \\ \{\mathrm{c}\} \\ \{\mathrm{a,c}\} \\ \{\mathrm{b,c}\} \\ \{\mathrm{a,b,c}\} \end{array} \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \end{pmatrix} \quad v_2 := \varepsilon^{\mathsf{T}} e_2 = \begin{array}{c} \{\} \\ \{\mathrm{a}\} \\ \{\mathrm{b}\} \\ \{\mathrm{a,b}\} \\ \{\mathrm{c}\} \\ \{\mathrm{a,c}\} \\ \{\mathrm{b,c}\} \\ \{\mathrm{a,b,c}\} \end{array} \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{1} \end{pmatrix} \quad v_3 := \varepsilon^{\mathsf{T}} e_3 = \begin{array}{c} \{\} \\ \{\mathrm{a}\} \\ \{\mathrm{b}\} \\ \{\mathrm{a,b}\} \\ \{\mathrm{c}\} \\ \{\mathrm{a,c}\} \\ \{\mathrm{b,c}\} \\ \{\mathrm{a,b,c}\} \end{array} \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{1} \end{pmatrix}.$$

This allows us to form

$$q_1 := \overline{v_1} ; v_1^{\mathsf{T}} = \begin{array}{cc} & \begin{array}{cccccccc} \{\} & \{\mathrm{a}\} & \{\mathrm{b}\} & \{\mathrm{a,b}\} & \{\mathrm{c}\} & \{\mathrm{a,c}\} & \{\mathrm{b,c}\} & \{\mathrm{a,b,c}\} \end{array} \\ \begin{array}{c} \{\} \\ \{\mathrm{a}\} \\ \{\mathrm{b}\} \\ \{\mathrm{a,b}\} \\ \{\mathrm{c}\} \\ \{\mathrm{a,c}\} \\ \{\mathrm{b,c}\} \\ \{\mathrm{a,b,c}\} \end{array} & \begin{pmatrix} 0&0&0&0&1&1&1&1 \\ 0&0&0&0&1&1&1&1 \\ 0&0&0&0&1&1&1&1 \\ 0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix} \end{array} \quad q_2 := \overline{v_2} ; v_2^{\mathsf{T}} \cap \overline{q_1}^{\mathsf{T}} = \begin{array}{cc} & \begin{array}{cccccccc} \{\} & \{\mathrm{a}\} & \{\mathrm{b}\} & \{\mathrm{a,b}\} & \{\mathrm{c}\} & \{\mathrm{a,c}\} & \{\mathrm{b,c}\} & \{\mathrm{a,b,c}\} \end{array} \\ \begin{array}{c} \{\} \\ \{\mathrm{a}\} \\ \{\mathrm{b}\} \\ \{\mathrm{a,b}\} \\ \{\mathrm{c}\} \\ \{\mathrm{a,c}\} \\ \{\mathrm{b,c}\} \\ \{\mathrm{a,b,c}\} \end{array} & \begin{pmatrix} 0&0&1&1&0&0&1&1 \\ 0&0&1&1&0&0&1&1 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&1 \\ 0&0&0&0&0&0&1&1 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix} \end{array}$$

$$q_3 := \overline{v_3} ; v_3^{\mathsf{T}} \cap \overline{q_1}^{\mathsf{T}} \cap \overline{q_2}^{\mathsf{T}} = \begin{array}{cc} & \begin{array}{cccccccc} \{\} & \{\mathrm{a}\} & \{\mathrm{b}\} & \{\mathrm{a,b}\} & \{\mathrm{c}\} & \{\mathrm{a,c}\} & \{\mathrm{b,c}\} & \{\mathrm{a,b,c}\} \end{array} \\ \begin{array}{c} \{\} \\ \{\mathrm{a}\} \\ \{\mathrm{b}\} \\ \{\mathrm{a,b}\} \\ \{\mathrm{c}\} \\ \{\mathrm{a,c}\} \\ \{\mathrm{b,c}\} \\ \{\mathrm{a,b,c}\} \end{array} & \begin{pmatrix} 0&1&0&1&0&1&0&1 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&1&0&1 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&1 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0 \end{pmatrix} \end{array},$$

out of which we finally obtain

$$F = \begin{array}{cc} & \begin{array}{cccccccc} \{\} & \{\mathrm{a}\} & \{\mathrm{b}\} & \{\mathrm{a,b}\} & \{\mathrm{c}\} & \{\mathrm{a,c}\} & \{\mathrm{b,c}\} & \{\mathrm{a,b,c}\} \end{array} \\ \begin{array}{c} \{\} \\ \{\mathrm{a}\} \\ \{\mathrm{b}\} \\ \{\mathrm{a,b}\} \\ \{\mathrm{c}\} \\ \{\mathrm{a,c}\} \\ \{\mathrm{b,c}\} \\ \{\mathrm{a,b,c}\} \end{array} & \begin{pmatrix} 1&1&1&1&1&1&1&1 \\ 0&1&1&1&1&1&1&1 \\ 0&0&1&1&1&1&1&1 \\ 0&0&0&1&1&1&1&1 \\ 0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&1&1&1 \\ 0&0&0&0&0&0&1&1 \\ 0&0&0&0&0&0&0&1 \end{pmatrix} \end{array} = \mathbb{I} \cup q_1 \cup q_2 \cup q_3.$$

This is — as of yet — an ugly iteration that should be simplified; but it shows that we have $F = \mathbb{I} \cup q_1 \cup q_2 \cup q_3$ finally evaluated by a relational construction out of E in a way comparable with a lexicographic ordering for a product of linear orders.

Quite obviously, E, F satisfy $E{;}\varepsilon = \varepsilon{;}F$. Since F may recursively be generated as

$$F_0 := (\mathbf{1}) \qquad F_{n+1} := \begin{pmatrix} F_n & \top\!\!\top \\ \perp\!\!\perp & \mathbf{1} \end{pmatrix},$$

which may be proved over this recursion:

$$E_1 = (\mathbf{1}) \quad \varepsilon_1 = (\mathbf{0}\ \mathbf{1}) \quad F_1 = \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \qquad E_{n+1} = \begin{pmatrix} E_n & \top\!\!\top \\ \perp\!\!\perp & \mathbf{1} \end{pmatrix} \quad \varepsilon_{n+1} = \begin{pmatrix} \varepsilon_n & \varepsilon_n \\ \perp\!\!\perp & \top\!\!\top \end{pmatrix}$$

$$E_{n+1}{;}\varepsilon_{n+1} = \begin{pmatrix} E_n{;}\varepsilon_n & \top\!\!\top \\ \perp\!\!\perp & \top\!\!\top \end{pmatrix} = \begin{pmatrix} \varepsilon_n{;}F_n & \top\!\!\top \\ \perp\!\!\perp & \top\!\!\top \end{pmatrix} = \varepsilon_{n+1}{;}F_{n+1}$$

Using F, it is possible to evaluate these boundary operators B^P, B^M from H and the order E of the baseset X in the following way. It follows directly the idea stemming from homotopy theory. A first contribution to positive boundaries is given by taking rowwise the greatest elements of H^T according to F:

$$B_1^P = \mathtt{greR}_F(H^\mathsf{T}) = \begin{array}{cc} & \begin{matrix} \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} \end{matrix} \\ \begin{matrix} \{\} \\ \{a\} \\ \{b\} \\ \{a,b\} \\ \{c\} \\ \{a,c\} \\ \{b,c\} \\ \{a,b,c\} \end{matrix} & \begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&1&0 \end{pmatrix} \end{array}$$

$$B_1^M = \mathtt{greR}_F(H^\mathsf{T} \cap \overline{B_1^P}) = \begin{array}{cc} & \begin{matrix} \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} \end{matrix} \\ \begin{matrix} \{\} \\ \{a\} \\ \{b\} \\ \{a,b\} \\ \{c\} \\ \{a,c\} \\ \{b,c\} \\ \{a,b,c\} \end{matrix} & \begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0 \end{pmatrix} \end{array}$$

$$B_2^P = \mathtt{greR}_F(H^\mathsf{T} \cap \overline{B_1^P \cup B_1^M}) = \begin{array}{cc} & \begin{matrix} \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} \end{matrix} \\ \begin{matrix} \{\} \\ \{a\} \\ \{b\} \\ \{a,b\} \\ \{c\} \\ \{a,c\} \\ \{b,c\} \\ \{a,b,c\} \end{matrix} & \begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&1&0&0&1&0 \end{pmatrix} \end{array} \qquad \text{etc.}$$

The idea how to proceed is evident. Already at this early point we have stability of this example iteration with $H^\mathsf{T} = B^P \cup B^M$, where $B^P = B_2^P$ and $B^M = B_1^M$.

6.3 Simplicial complexes

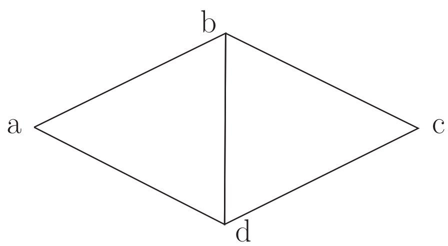
A simplicial complex in topology is usually defined on a set X of which subsets are declared to be simplices. Whenever a simplex is identified, all its subsets have to be simplices again[3]. One

[3]Should X be non-finite, one usually demands that every element be contained in only a finite number of subsets: locally finite.

then studies in particular the descent from one simplex of size n to all its subsimplices of size $n-1$. This leads us to conceive a simplicial complex on a set X as a vector s along its powerset $\mathbf{2}^X$. It must be down-closed, i.e., $\Omega ; s \subseteq s$ (with Ω the powerset ordering). We are interested in subsets of precisely one element less, so that we again work with the converse of the Hasse relation H of the powerset ordering. A not yet oriented example is provided with Fig. 6.6.

Later, we will forget the vector s and restrict the boundary relation correspondingly omitting rows and columns; see e.g. Fig. 6.7.

We intend to give orientation not just to a single simplex, but to a whole simplicial complex and start with a most trivial example of Fig. 6.7. The two triangles will be said to have the same orientation because the vertical arrow gets a counter-running orientation from the orientations of the two triangles. This resembles the idea that then the vertical arrow might be removed, leaving us with a common circuit orientation.



$$
\begin{array}{r|cccccccccccccccc|}
 & \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} & \{d\} & \{a,d\} & \{b,d\} & \{a,b,d\} & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\} \\
\{\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{a\} & 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{b\} & 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{a,b\} & 0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{c\} & 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{a,c\} & 0&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\
\{b,c\} & 0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0 \\
\{a,b,c\} & 0&0&0&1&0&1&1&0&0&0&0&0&0&0&0&0 \\
\{d\} & 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{a,d\} & 0&1&0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\
\{b,d\} & 0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0 \\
\{a,b,d\} & 0&0&0&1&0&0&0&0&0&1&1&0&0&0&0&0 \\
\{c,d\} & 0&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0 \\
\{a,c,d\} & 0&0&0&0&0&1&0&0&0&1&0&0&1&0&0&0 \\
\{b,c,d\} & 0&0&0&0&0&0&1&0&0&0&1&0&1&0&0&0 \\
\{a,b,c,d\} & 0&0&0&0&0&0&0&1&0&0&0&1&0&1&1&0
\end{array}
\qquad
\begin{array}{r|c|}
\{\} & 1 \\
\{a\} & 1 \\
\{b\} & 1 \\
\{a,b\} & 1 \\
\{c\} & 1 \\
\{a,c\} & 0 \\
\{b,c\} & 1 \\
\{a,b,c\} & 0 \\
\{d\} & 1 \\
\{a,d\} & 1 \\
\{b,d\} & 1 \\
\{a,b,d\} & 1 \\
\{c,d\} & 1 \\
\{a,c,d\} & 0 \\
\{b,c,d\} & 1 \\
\{a,b,c,d\} & 0
\end{array}
$$

Fig. 6.6 Set-theoretic concept of a simplicial complex with boundary operator H^T and s

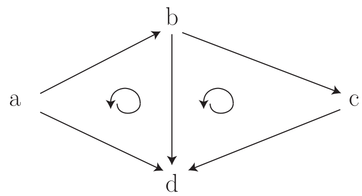
6.1 Definition (*Oriented boundary operators*)**.** For any finite set X consider the powerset ordering $\Omega : \mathbf{2}^X \longrightarrow \mathbf{2}^X$ and the converse B of its Hasse relation H. When a disjoint partition $B = B^P \cup B^M$ is given that satisfies

$$B^P ; B^P \cup B^M ; B^M = B^P ; B^M \cup B^M ; B^P,$$

we will be speak of **oriented boundary operators**. The relations B^P and B^M will in this case be called the positively (plus), resp. negatively (minus), oriented boundary operator. □

Every oriented simplex imposes an orientation on its bounding simplices, for instance running (c,b,d) means running along (c,b), (b,d), and (d,c). We have, however, agreed upon orienting the lower-dimensional simplices according to the baseorder, so that the first and the last contradict the orientations of (b,c) and (c,d) but the middle one, (b,d), agrees. Orientations may thus agree or may disagree, so that we have chosen to define the disjoint partition $H^\mathsf{T} = B^P \cup B^M$ indicating agreement resp. non-agreement.

$$\begin{array}{c|c} & \begin{array}{cccccccccccc} \{\} & \{c\} & \{d\} & \{a\} & \{b\} & \{b,c\} & \{c,d\} & \{b,d\} & \{a,d\} & \{a,b\} & \{c,b,d\} & \{d,b,a\} \end{array} \\ \begin{array}{r} \{\} \\ \{c\} \\ \{d\} \\ \{a\} \\ \{b\} \\ \{b,c\} \\ \{c,d\} \\ \{b,d\} \\ \{a,d\} \\ \{a,b\} \\ \{c,b,d\} \\ \{d,b,a\} \end{array} & \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0 \end{pmatrix} \end{array} = B^P$$



$$B^M = \begin{array}{c|c} & \begin{array}{cccccccccccc} \{\} & \{c\} & \{d\} & \{a\} & \{b\} & \{b,c\} & \{c,d\} & \{b,d\} & \{a,d\} & \{a,b\} & \{c,b,d\} & \{d,b,a\} \end{array} \\ \begin{array}{r} \{\} \\ \{c\} \\ \{d\} \\ \{a\} \\ \{b\} \\ \{b,c\} \\ \{c,d\} \\ \{b,d\} \\ \{a,d\} \\ \{a,b\} \\ \{c,b,d\} \\ \{d,b,a\} \end{array} & \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&1&0&0 \end{pmatrix} \end{array}$$

Fig. 6.7 Oriented simplicial complex with boundary operators $B^P, B^M : \mathbf{2}^X \longrightarrow \mathbf{2}^X$

Unfortunately, we have to pay attention also to whether the original maximum-dimensional simplex is positively oriented or not. To cope with orientation in a general fashion, one will consider the simplices in a two-fold form, namely as positively as well as negatively oriented.

The proper relational tool for such a consideration is the extrusion of the *full* subset obtaining its injection $\theta := \theta_{\mathbb{T}} : \Theta \longrightarrow \mathbf{2}^X$. We thus have the set Θ of negatively oriented versions for all the simplices, thereby generating for the simplex (a, b, c), e.g., its negatively oriented version $(a, b, c) \!\!\rightarrowtail$, so that symbolically[4] $\theta((a, b, c)\!\!\rightarrowtail) = (a, b, c)$. We recall that as an injection, the extrusion mapping θ satisfies $\quad \theta^{\mathsf{T}}; \theta = \mathbb{I}_{\mathbf{2}^X}, \quad \theta; \theta^{\mathsf{T}} = \mathbb{I}_{\Theta}$.

In the following, we form the direct sum of these two copies introducing the injections

$$\iota : \mathbf{2}^X \longrightarrow \mathbf{2}^X + \Theta \quad \text{and} \quad \kappa : \Theta \longrightarrow \mathbf{2}^X + \Theta,$$

$$\begin{tikzcd} & 2^X + \Theta & \\ 2^X \arrow[ur, "\iota"] & & \Theta \arrow[ul, "\kappa"'] \arrow[ll, "\theta"] \end{tikzcd}$$

and thus having all positively as well as all negatively oriented simplices in one set. The relation

$S_{\Updownarrow} := \iota^{\mathsf{T}}; \theta^{\mathsf{T}}; \kappa \cup \kappa^{\mathsf{T}}; \theta; \iota$

obviously regulates the transition to the differently oriented counterpart; see Fig. 6.8.

Using this basic configuration, we will now define matrices of relations to express, e.g., in submatrix position (1,2) that we go with $B^M; \theta^{\mathsf{T}}$ from a positively oriented simplex to its negative boundary. This together with the two boundary operators gives four relations, positive/negative versus positive/negative. However, instead of using the former relations B^P, B^M of Fig. 6.7, we embed them in the new configuration as

	positive	negative
positive	B^P	$B^M; \theta^{\mathsf{T}}$
negative	$\theta; B^M$	$\theta; B^P; \theta^{\mathsf{T}}$

which can be seen in Fig. 6.9, where $B^P, B^M; \theta^{\mathsf{T}}, \theta; B^M, \theta; B^P; \theta^{\mathsf{T}}$ are shown as separate relations.

Some immediate consequences follow when we conceive the matrix $\mathcal{B}$ and compute its square:

[4]For technical reasons shown in the matrices with curly brackets.

$$\mathcal{B}=\begin{pmatrix} B^P & B^M;\theta^\top \\ \theta;B^M & \theta;B^P;\theta^\top \end{pmatrix} \quad \mathcal{B}^2=\begin{pmatrix} B^P;B^P\cup B^M;\theta^\top;\theta;B^M & B^P;B^M;\theta^\top\cup B^M;\theta^\top;\theta;B^P;\theta^\top \\ \theta;B^M;B^P\cup\theta;B^P;\theta^\top;\theta;B^M & \theta;B^M;B^M;\theta^\top\cup\theta;B^P;\theta^\top;\theta;B^P;\theta^\top \end{pmatrix}$$
$$=\begin{pmatrix} B^P;B^P\cup B^M;B^M & (B^P;B^M\cup B^M;B^P);\theta^\top \\ \theta;(B^M;B^P\cup B^P;B^M) & \theta;(B^M;B^M\cup B^P;B^P);\theta^\top \end{pmatrix}$$

$$S_{\Updownarrow}=\begin{pmatrix} \perp\!\!\!\perp & \theta^\top \\ \theta & \perp\!\!\!\perp \end{pmatrix} = \begin{array}{r c}
 & \begin{array}{cccccccccccccccccccccccc} \{\} & \{c\} & \{d\} & \{a\} & \{b\} & \{b,c\} & \{c,d\} & \{b,d\} & \{a,d\} & \{a,b\} & \{c,b,d\} & \{d,b,a\} & \{\}\rightarrow & \{c\}\rightarrow & \{d\}\rightarrow & \{a\}\rightarrow & \{b\}\rightarrow & \{b,c\}\rightarrow & \{c,d\}\rightarrow & \{b,d\}\rightarrow & \{a,d\}\rightarrow & \{a,b\}\rightarrow & \{c,b,d\}\rightarrow & \{d,b,a\}\rightarrow \end{array} \\
\begin{array}{r} \{\} \\ \{c\} \\ \{d\} \\ \{a\} \\ \{b\} \\ \{b,c\} \\ \{c,d\} \\ \{b,d\} \\ \{a,d\} \\ \{a,b\} \\ \{c,b,d\} \\ \{d,b,a\} \\ \{\}\rightarrow \\ \{c\}\rightarrow \\ \{d\}\rightarrow \\ \{a\}\rightarrow \\ \{b\}\rightarrow \\ \{b,c\}\rightarrow \\ \{c,d\}\rightarrow \\ \{b,d\}\rightarrow \\ \{a,d\}\rightarrow \\ \{a,b\}\rightarrow \\ \{c,b,d\}\rightarrow \\ \{d,b,a\}\rightarrow \end{array} &
\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}
\end{array}$$

Fig. 6.8 The immediate switch $S_{\Updownarrow}$ toggling positive/negative for Fig. 6.7

One will easily recognize that the two matrices in the diagonal as well as the two outside are equal — up to their indications via $\theta,\theta^\top$. The negatively oriented outer region $(2,1,3)\!\rightarrow$, for instance, has via $\theta;B^M$ the positive boundary $(1,2)$.

We could see the orientation of the simplicial complex considered and how it switches. In algebraic topology, one would use the chains with values in a group. Their values might annihilate one another resulting after double application of the boundary operator bringing the result 0, which is here reflected by delivering the same result in two different ways.

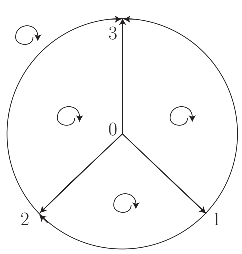
Size restriction Examples will soon get big, so that we are interested in a less spacious representation, for which we choose to concentrate on the largest dimension and the one below. Since mainly the maximum-dimensional simplices are interesting, we will later — referring back to Fig. 6.7 —, concentrate on just

$$B^P_{(2,1)}=\begin{array}{r c} & \begin{array}{ccccc} \{b,c\} & \{c,d\} & \{b,d\} & \{a,d\} & \{a,b\} \end{array} \\ \begin{array}{r} \{c,b,d\} \\ \{d,b,a\} \end{array} & \begin{pmatrix} 0&0&1&0&0 \\ 0&0&0&1&0 \end{pmatrix} \end{array} \qquad B^M_{(2,1)}=\begin{array}{r c} & \begin{array}{ccccc} \{b,c\} & \{c,d\} & \{b,d\} & \{a,d\} & \{a,b\} \end{array} \\ \begin{array}{r} \{c,b,d\} \\ \{d,b,a\} \end{array} & \begin{pmatrix} 1&1&0&0&0 \\ 0&0&1&0&1 \end{pmatrix} \end{array},$$

on which all the rest depends more or less trivially.

$$
\begin{array}{rc}
 & \begin{array}{ccccccccccccccc} \{\} & \{3\} & \{0\} & \{1\} & \{2\} & \{2,3\} & \{0,3\} & \{1,3\} & \{0,1\} & \{0,2\} & \{1,2\} & \{2,1,3\} & \{2,3,0\} & \{0,3,1\} & \{0,1,2\} \end{array} \\
\begin{array}{r} \{\} \\ \{3\} \\ \{0\} \\ \{1\} \\ \{2\} \\ \{2,3\} \\ \{0,3\} \\ \{1,3\} \\ \{0,1\} \\ \{0,2\} \\ \{1,2\} \\ \{2,1,3\} \\ \{2,3,0\} \\ \{0,3,1\} \\ \{0,1,2\} \end{array} &
\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&1&0&0&0&0
\end{pmatrix}
\end{array}
\quad
\begin{array}{c}
\begin{array}{ccccccccccccccc} \{\}\!\rightarrow & \{3\}\!\rightarrow & \{0\}\!\rightarrow & \{1\}\!\rightarrow & \{2\}\!\rightarrow & \{2,3\}\!\rightarrow & \{0,3\}\!\rightarrow & \{1,3\}\!\rightarrow & \{0,1\}\!\rightarrow & \{0,2\}\!\rightarrow & \{1,2\}\!\rightarrow & \{2,1,3\}\!\rightarrow & \{2,3,0\}\!\rightarrow & \{0,3,1\}\!\rightarrow & \{0,1,2\}\!\rightarrow \end{array} \\
\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0
\end{pmatrix}
\end{array}
$$

$$
\begin{array}{rc}
\begin{array}{r} \{\}\!\rightarrow \\ \{3\}\!\rightarrow \\ \{0\}\!\rightarrow \\ \{1\}\!\rightarrow \\ \{2\}\!\rightarrow \\ \{2,3\}\!\rightarrow \\ \{0,3\}\!\rightarrow \\ \{1,3\}\!\rightarrow \\ \{0,1\}\!\rightarrow \\ \{0,2\}\!\rightarrow \\ \{1,2\}\!\rightarrow \\ \{2,1,3\}\!\rightarrow \\ \{2,3,0\}\!\rightarrow \\ \{0,3,1\}\!\rightarrow \\ \{0,1,2\}\!\rightarrow \end{array} &
\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0
\end{pmatrix}
\end{array}
\quad
\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&1&0&0&0&0
\end{pmatrix}
$$



$$\mathcal{B} = \begin{pmatrix} B^P & B^M {;} \theta^\top \\ \theta {;} B^M & \theta {;} B^P {;} \theta^\top \end{pmatrix}$$

Fig. 6.9 Boundary operations evaluated for all directions positive/negative

6.4 Orientability of a simplicial complex

When asking for orientability of a whole simplicial complex, it must, however, be observed that, e.g.,

$(0,3,1)$ has via B^P boundary $(0,3)$ and
$(0,3,1)\!\rightarrow$ has via $\theta {;} B^P {;} \theta^\top$ boundary $(0,3)\!\rightarrow$,

thus in opposite orientation, so that $(0,3,1)$ goes via $B^P {;} \theta^\top {;} \theta {;} B^{P^\top} {;} \theta^\top = B^P {;} B^{P^\top} {;} \theta^\top$ to its inverse $(0,3,1)\!\rightarrow$. This immediate change of orientation is uninteresting; we will only be interested in long-range changes.

The basic idea is therefore to consider two *different* simplices as having the same orientation when their coinciding boundaries have opposite orientation — as already mentioned. For this, we will first observe that the inversion matrix $S_{\Updownarrow}$ satisfies

$$\mathcal{B};S_{\Updownarrow} = S_{\Updownarrow};\mathcal{B} \qquad \mathcal{B}^2;S_{\Updownarrow} = S_{\Updownarrow};\mathcal{B}^2 \qquad {S_{\Updownarrow}}^2 = \mathbb{I}_{\mathbf{2}^X+\Theta}$$

and then define some sort of an adjacency Γ derived from $\mathcal{B}$, namely

$$\mathcal{B}^\mathsf{T} = \begin{pmatrix} B^{P^\mathsf{T}} & B^{M^\mathsf{T}};\theta^\mathsf{T} \\ \theta;B^{M^\mathsf{T}} & \theta;B^{P^\mathsf{T}};\theta^\mathsf{T} \end{pmatrix} \qquad\qquad S_{\Updownarrow};\mathcal{B}^\mathsf{T} = \begin{pmatrix} B^{M^\mathsf{T}} & B^{P^\mathsf{T}};\theta^\mathsf{T} \\ \theta;B^{P^\mathsf{T}} & \theta;B^{M^\mathsf{T}};\theta^\mathsf{T} \end{pmatrix}$$

$$\Gamma := \overline{S_{\Updownarrow}} \cap \mathcal{B};S_{\Updownarrow};\mathcal{B}^\mathsf{T} = \begin{pmatrix} B^P;B^{M^\mathsf{T}} \cup B^M;B^{P^\mathsf{T}} & \left[\overline{\mathbb{I}} \cap (B^P;B^{P^\mathsf{T}} \cup B^M;B^{M^\mathsf{T}})\right];\theta^\mathsf{T} \\ \theta;\left[\overline{\mathbb{I}} \cap (B^M;B^{M^\mathsf{T}} \cup B^P;B^{P^\mathsf{T}})\right] & \theta;(B^M;B^{P^\mathsf{T}} \cup B^P;B^{M^\mathsf{T}});\theta^\mathsf{T} \end{pmatrix}$$

A comparison with the former double application $\partial_1(\partial_2(a,b,c))$ shows how $B^P;B^P \cup B^M;B^M$ as well as $B^P;B^M \cup B^M;B^P$ lead to the same — as opposed to making their difference 0.

$$\begin{array}{c} \\ \{2,1,3\} \\ \{2,3,0\} \\ \{0,3,1\} \\ \{0,1,2\} \end{array}
\begin{array}{c} \begin{array}{cccc} \{2,1,3\} & \{2,3,0\} & \{0,3,1\} & \{0,1,2\} \end{array} \\ \begin{pmatrix} 0&1&1&1\\ 1&0&1&1\\ 1&1&0&1\\ 1&1&1&0 \end{pmatrix} \end{array}
\quad
\begin{array}{c} \begin{array}{cccc} \{2,1,3\}\uparrow & \{2,3,0\}\uparrow & \{0,3,1\}\uparrow & \{0,1,2\}\uparrow \end{array} \\ \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix} \end{array}
\qquad
\begin{array}{c} \\ \{2,1,3\} \\ \{2,3,0\} \\ \{0,3,1\} \\ \{0,1,2\} \end{array}
\begin{array}{c} \begin{array}{cccc} \{2,1,3\} & \{2,3,0\} & \{0,3,1\} & \{0,1,2\} \end{array} \\ \begin{pmatrix} 1&1&1&1\\ 1&1&1&1\\ 1&1&1&1\\ 1&1&1&1 \end{pmatrix} \end{array}
\quad
\begin{array}{c} \begin{array}{cccc} \{2,1,3\}\uparrow & \{2,3,0\}\uparrow & \{0,3,1\}\uparrow & \{0,1,2\}\uparrow \end{array} \\ \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix} \end{array}$$

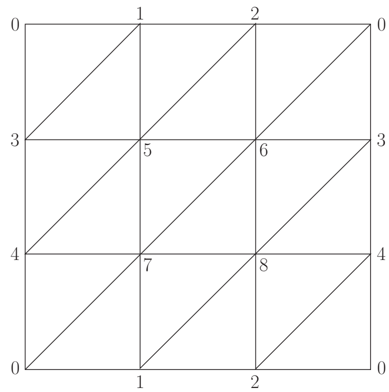
$$\begin{array}{c} \\ \{2,1,3\}\rightarrow \\ \{2,3,0\}\rightarrow \\ \{0,3,1\}\rightarrow \\ \{0,1,2\}\rightarrow \end{array}
\begin{array}{c} \begin{array}{cccc} \{2,1,3\} & \{2,3,0\} & \{0,3,1\} & \{0,1,2\} \end{array} \\ \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix} \end{array}
\quad
\begin{array}{c} \begin{array}{cccc} \{2,1,3\}\uparrow & \{2,3,0\}\uparrow & \{0,3,1\}\uparrow & \{0,1,2\}\uparrow \end{array} \\ \begin{pmatrix} 0&1&1&1\\ 1&0&1&1\\ 1&1&0&1\\ 1&1&1&0 \end{pmatrix} \end{array}
\qquad
\begin{array}{c} \\ \{2,1,3\}\rightarrow \\ \{2,3,0\}\rightarrow \\ \{0,3,1\}\rightarrow \\ \{0,1,2\}\rightarrow \end{array}
\begin{array}{c} \begin{array}{cccc} \{2,1,3\} & \{2,3,0\} & \{0,3,1\} & \{0,1,2\} \end{array} \\ \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix} \end{array}
\quad
\begin{array}{c} \begin{array}{cccc} \{2,1,3\}\uparrow & \{2,3,0\}\uparrow & \{0,3,1\}\uparrow & \{0,1,2\}\uparrow \end{array} \\ \begin{pmatrix} 1&1&1&1\\ 1&1&1&1\\ 1&1&1&1\\ 1&1&1&1 \end{pmatrix} \end{array}$$

Fig. 6.10 Maximum dimension parts of Γ of Fig. 6.9 and its reflexive transitive closure Γ^*

A slightly bigger example is shown in Fig. 6.11.

6.2 Definition. A simplicial complex will be called **orientable** if $\quad \Gamma^* \subseteq \overline{\mathsf{switch}}$. □

Any long-range adjacency must never switch orientation of a simplex.



$$
\begin{array}{rcc}
 & \begin{array}{cccccccccccccccccc}\{2,0,4\}&\{2,4,8\}&\{1,2,8\}&\{1,8,7\}&\{0,1,7\}&\{0,7,4\}&\{8,4,3\}&\{8,3,6\}&\{7,8,6\}&\{7,6,5\}&\{4,7,5\}&\{4,5,3\}&\{6,3,0\}&\{6,0,2\}&\{5,6,2\}&\{5,2,1\}&\{3,5,1\}&\{3,1,0\}\end{array} & \begin{array}{cccccccccccccccccc}\{2,0,4\}\to&\{2,4,8\}\to&\{1,2,8\}\to&\{1,8,7\}\to&\{0,1,7\}\to&\{0,7,4\}\to&\{8,4,3\}\to&\{8,3,6\}\to&\{7,8,6\}\to&\{7,6,5\}\to&\{4,7,5\}\to&\{4,5,3\}\to&\{6,3,0\}\to&\{6,0,2\}\to&\{5,6,2\}\to&\{5,2,1\}\to&\{3,5,1\}\to&\{3,1,0\}\to\end{array} \\
\begin{array}{r}\{2,0,4\}\\\{2,4,8\}\\\{1,2,8\}\\\{1,8,7\}\\\{0,1,7\}\\\{0,7,4\}\\\{8,4,3\}\\\{8,3,6\}\\\{7,8,6\}\\\{7,6,5\}\\\{4,7,5\}\\\{4,5,3\}\\\{6,3,0\}\\\{6,0,2\}\\\{5,6,2\}\\\{5,2,1\}\\\{3,5,1\}\\\{3,1,0\}\end{array} &
\begin{pmatrix}
0&1&0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&0\\
1&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&1&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&1\\
1&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&1&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&1&0&0&0&1&0&0&0&0&0\\
0&0&0&1&0&0&0&1&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&1&0&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0&0&1&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&1&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&1\\
1&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&1&0&1&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&0&1\\
0&0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&1&0
\end{pmatrix} &
\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix} \\
\begin{array}{r}\{2,0,4\}\to\\\{2,4,8\}\to\\\{1,2,8\}\to\\\{1,8,7\}\to\\\{0,1,7\}\to\\\{0,7,4\}\to\\\{8,4,3\}\to\\\{8,3,6\}\to\\\{7,8,6\}\to\\\{7,6,5\}\to\\\{4,7,5\}\to\\\{4,5,3\}\to\\\{6,3,0\}\to\\\{6,0,2\}\to\\\{5,6,2\}\to\\\{5,2,1\}\to\\\{3,5,1\}\to\\\{3,1,0\}\to\end{array} &
\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix} &
\begin{pmatrix}
0&1&0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&0\\
1&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&1&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&1\\
1&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&1&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&1&0&0&0&1&0&0&0&0&0\\
0&0&0&1&0&0&0&1&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&1&0&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0&0&1&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&1&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&1\\
1&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&1&0&1&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&0&1\\
0&0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&1&0
\end{pmatrix}
\end{array}
$$

Fig. 6.11 Torus — after identifying equally named vertices

The situation is different for the following example of a simplicial complex describing the triangulation of a Moebius tape.

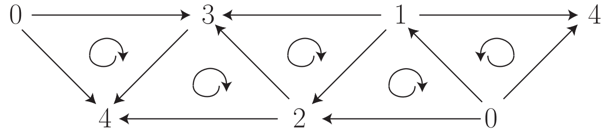
$$
\begin{array}{r|cccccccccccccccccc|cccccccccccccccccc}
 & \{2,0,4\} & \{2,4,8\} & \{1,2,8\} & \{1,8,7\} & \{0,1,7\} & \{0,7,4\} & \{8,4,3\} & \{8,3,6\} & \{7,8,6\} & \{7,6,5\} & \{4,7,5\} & \{4,5,3\} & \{6,3,0\} & \{6,0,2\} & \{5,6,2\} & \{5,2,1\} & \{3,5,1\} & \{3,1,0\} & \{2,0,4\}{\to} & \{2,4,8\}{\to} & \{1,2,8\}{\to} & \{1,8,7\}{\to} & \{0,1,7\}{\to} & \{0,7,4\}{\to} & \{8,4,3\}{\to} & \{8,3,6\}{\to} & \{7,8,6\}{\to} & \{7,6,5\}{\to} & \{4,7,5\}{\to} & \{4,5,3\}{\to} & \{6,3,0\}{\to} & \{6,0,2\}{\to} & \{5,6,2\}{\to} & \{5,2,1\}{\to} & \{3,5,1\}{\to} & \{3,1,0\}{\to} \\
\hline
\{2,0,4\} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{2,4,8\} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{1,2,8\} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{1,8,7\} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{0,1,7\} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{0,7,4\} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{8,4,3\} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{8,3,6\} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{7,8,6\} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{7,6,5\} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{4,7,5\} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{4,5,3\} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{6,3,0\} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{6,0,2\} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{5,6,2\} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{5,2,1\} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{3,5,1\} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{3,1,0\} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\{2,0,4\}{\to} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\{2,4,8\}{\to} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\{1,2,8\}{\to} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\{1,8,7\}{\to} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\{0,1,7\}{\to} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\{0,7,4\}{\to} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\{8,4,3\}{\to} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\{8,3,6\}{\to} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\{7,8,6\}{\to} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\{7,6,5\}{\to} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\{4,7,5\}{\to} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\{4,5,3\}{\to} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\{6,3,0\}{\to} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\{6,0,2\}{\to} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\{5,6,2\}{\to} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\{5,2,1\}{\to} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\{3,5,1\}{\to} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\{3,1,0\}{\to} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}
$$

Fig. 6.12 Transitive closure for Γ of Fig. 6.11

$$
\begin{array}{r|ccccc|ccccc}
 & \{1,0,4\} & \{4,0,3\} & \{4,3,2\} & \{2,3,1\} & \{2,1,0\} & \{1,0,4\}{\to} & \{4,0,3\}{\to} & \{4,3,2\}{\to} & \{2,3,1\}{\to} & \{2,1,0\}{\to} \\
\hline
\{1,0,4\} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\{4,0,3\} & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{4,3,2\} & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{2,3,1\} & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\{2,1,0\} & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hline
\{1,0,4\}{\to} & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
\{4,0,3\}{\to} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
\{4,3,2\}{\to} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
\{2,3,1\}{\to} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
\{2,1,0\}{\to} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}
\qquad
\begin{array}{r|ccccc|ccccc}
 & \{1,0,4\} & \{4,0,3\} & \{4,3,2\} & \{2,3,1\} & \{2,1,0\} & \{1,0,4\}{\to} & \{4,0,3\}{\to} & \{4,3,2\}{\to} & \{2,3,1\}{\to} & \{2,1,0\}{\to} \\
\hline
\{1,0,4\} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\{4,0,3\} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\{4,3,2\} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\{2,3,1\} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\{2,1,0\} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
\{1,0,4\}{\to} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\{4,0,3\}{\to} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\{4,3,2\}{\to} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\{2,3,1\}{\to} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\{2,1,0\}{\to} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}
$$

Fig. 6.13 Adjacency Γ and Γ^* for the Moebius tape — restricted to maximum dimension

$$
\begin{array}{c|ccccccccccccccccccccc|ccccccccccccccccccccc}
 & \{\} & \{4\} & \{3\} & \{2\} & \{0\} & \{1\} & \{1,4\} & \{0,4\} & \{0,3\} & \{3,4\} & \{2,4\} & \{2,3\} & \{1,3\} & \{1,2\} & \{0,2\} & \{0,1\} & \{1,0,4\} & \{4,0,3\} & \{4,3,2\} & \{2,3,1\} & \{2,1,0\} & \{\}\to & \{4\}\to & \{3\}\to & \{2\}\to & \{0\}\to & \{1\}\to & \{1,4\}\to & \{0,4\}\to & \{0,3\}\to & \{3,4\}\to & \{2,4\}\to & \{2,3\}\to & \{1,3\}\to & \{1,2\}\to & \{0,2\}\to & \{0,1\}\to & \{1,0,4\}\to & \{4,0,3\}\to & \{4,3,2\}\to & \{2,3,1\}\to & \{2,1,0\}\to \\
\hline
\{\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{4\} & 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{3\} & 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{2\} & 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{0\} & 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{1\} & 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{1,4\} & 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{0,4\} & 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{0,3\} & 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{3,4\} & 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 & 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{2,4\} & 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 & 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{2,3\} & 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 & 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{1,3\} & 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{1,2\} & 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{0,2\} & 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{0,1\} & 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{1,0,4\} & 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&0&0&0 \\
\{4,0,3\} & 0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{4,3,2\} & 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0 \\
\{2,3,1\} & 0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\
\{2,1,0\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0 \\
\hline
\{\}\to & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{4\}\to & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 & 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{3\}\to & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 & 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{2\}\to & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 & 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{0\}\to & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 & 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{1\}\to & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 & 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{1,4\}\to & 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 & 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{0,4\}\to & 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 & 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{0,3\}\to & 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 & 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{3,4\}\to & 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 & 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{2,4\}\to & 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 & 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{2,3\}\to & 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 & 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{1,3\}\to & 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 & 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{1,2\}\to & 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 & 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{0,2\}\to & 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 & 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{0,1\}\to & 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{1,0,4\}\to & 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&0&0&0 & 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{4,0,3\}\to & 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0 \\
\{4,3,2\}\to & 0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0 \\
\{2,3,1\}\to & 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0 \\
\{2,1,0\}\to & 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0
\end{array}
$$

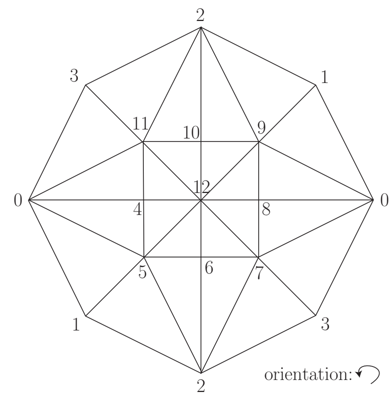


$$\mathcal{B} = \begin{pmatrix} B^P & B^M ; \theta^\mathsf{T} \\ \theta ; B^M & \theta ; B^P ; \theta^\mathsf{T} \end{pmatrix}$$

Fig. 6.14 Boundary functions evaluated for all directions positive/negative of a Moebius tape

Also for the projective plane Fig. 6.15, we obtain $\Gamma^* = \mathbb{T}$, which is, however, not shown.

$$
\begin{array}{rc}
 & \begin{array}{cccccccccccccccccccccccc} \{3,0,7\} & \{2,3,7\} & \{2,7,6\} & \{2,6,5\} & \{1,2,5\} & \{0,1,5\} & \{7,0,8\} & \{7,8,12\} & \{6,7,12\} & \{5,6,12\} & \{5,12,4\} & \{0,5,4\} & \{8,0,9\} & \{12,8,9\} & \{12,9,10\} & \{12,10,11\} & \{4,12,11\} & \{0,4,11\} & \{9,0,1\} & \{9,1,2\} & \{10,9,2\} & \{11,10,2\} & \{11,2,3\} & \{0,11,3\} \end{array} \\
\begin{array}{r} \{3,0,7\} \\ \{2,3,7\} \\ \{2,7,6\} \\ \{2,6,5\} \\ \{1,2,5\} \\ \{0,1,5\} \\ \{7,0,8\} \\ \{7,8,12\} \\ \{6,7,12\} \\ \{5,6,12\} \\ \{5,12,4\} \\ \{0,5,4\} \\ \{8,0,9\} \\ \{12,8,9\} \\ \{12,9,10\} \\ \{12,10,11\} \\ \{4,12,11\} \\ \{0,4,11\} \\ \{9,0,1\} \\ \{9,1,2\} \\ \{10,9,2\} \\ \{11,10,2\} \\ \{11,2,3\} \\ \{0,11,3\} \end{array} &
\begin{pmatrix}
0&1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0
\end{pmatrix}
\end{array}
$$



$$
\begin{array}{rc}
 & \begin{array}{cccccccccccccccccccccccc} \{3,0,7\} & \{2,3,7\} & \{2,7,6\} & \{2,6,5\} & \{1,2,5\} & \{0,1,5\} & \{7,0,8\} & \{7,8,12\} & \{6,7,12\} & \{5,6,12\} & \{5,12,4\} & \{0,5,4\} & \{8,0,9\} & \{12,8,9\} & \{12,9,10\} & \{12,10,11\} & \{4,12,11\} & \{0,4,11\} & \{9,0,1\} & \{9,1,2\} & \{10,9,2\} & \{11,10,2\} & \{11,2,3\} & \{0,11,3\} \end{array} \\
\begin{array}{r} \{3,0,7\}\rightarrow \\ \{2,3,7\}\rightarrow \\ \{2,7,6\}\rightarrow \\ \{2,6,5\}\rightarrow \\ \{1,2,5\}\rightarrow \\ \{0,1,5\}\rightarrow \\ \{7,0,8\}\rightarrow \\ \{7,8,12\}\rightarrow \\ \{6,7,12\}\rightarrow \\ \{5,6,12\}\rightarrow \\ \{5,12,4\}\rightarrow \\ \{0,5,4\}\rightarrow \\ \{8,0,9\}\rightarrow \\ \{12,8,9\}\rightarrow \\ \{12,9,10\}\rightarrow \\ \{12,10,11\}\rightarrow \\ \{4,12,11\}\rightarrow \\ \{0,4,11\}\rightarrow \\ \{9,0,1\}\rightarrow \\ \{9,1,2\}\rightarrow \\ \{10,9,2\}\rightarrow \\ \{11,10,2\}\rightarrow \\ \{11,2,3\}\rightarrow \\ \{0,11,3\}\rightarrow \end{array} &
\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}
\end{array}
$$

Fig. 6.15 Γ_{11} and Γ_{21} of the projective plane — after identifying equally named vertices

The following example shows the well-known 2-pretzel with its triangulation.

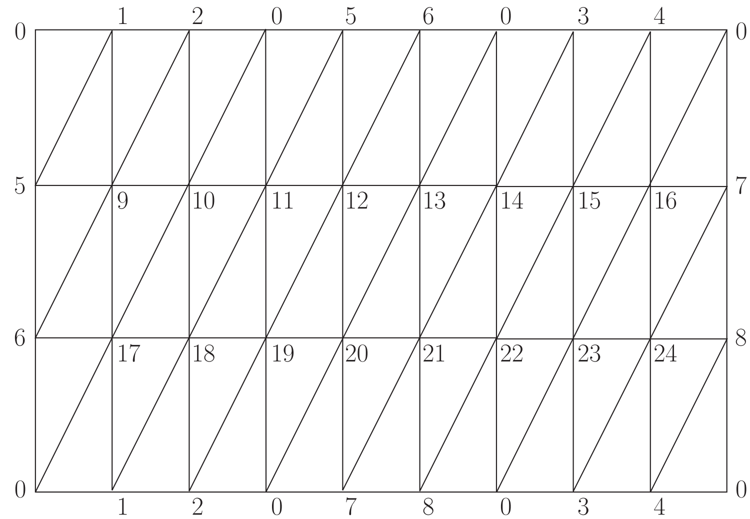


Fig. 6.16 A triangulation of the 2-hole-pretzel

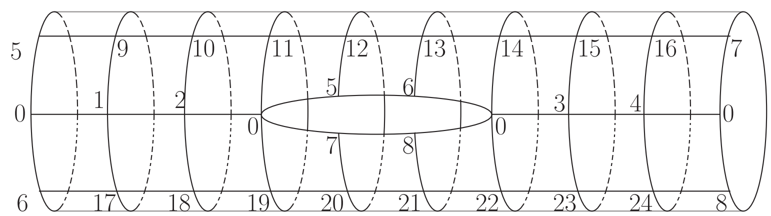
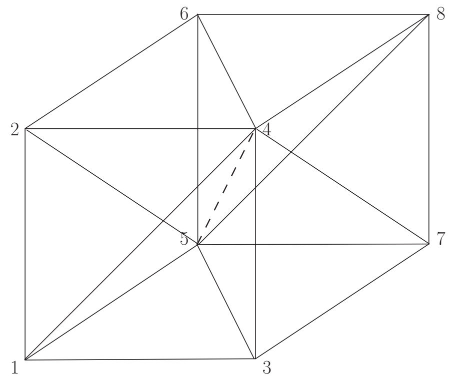


Fig. 6.17 Partial folding to obtain the 2-hole-pretzel

After this first folding of Fig. 6.16 to Fig. 6.17, further identifications are conceivable. When one identifies the two 0's of the middle ellipse, two tangent holes will appear, into which the so far open left and right ends of the "pipe" may be glued, ending with the pretzel announced. It turns out that this pretzel is indeed orientable. Since we had expected that, we do not show the respective relations here.

One will observe that the triangulation given for the 3-dimensional cube is fully determined by the dashed space diagonal and the square diagonals emanating from its endpoints, and is, thus, far from symmetric.



$$
\begin{array}{r c c}
 & \begin{array}{cccccc} \{3,4,5,7\} & \{1,3,4,5\} & \{4,5,7,8\} & \{4,5,6,8\} & \{2,4,5,6\} & \{1,2,4,5\} \end{array} & \begin{array}{cccccc} \{3,4,5,7\}\to & \{1,3,4,5\}\to & \{4,5,7,8\}\to & \{4,5,6,8\}\to & \{2,4,5,6\}\to & \{1,2,4,5\}\to \end{array} \\
\begin{array}{r} \{3,4,5,7\} \\ \{1,3,4,5\} \\ \{4,5,7,8\} \\ \{4,5,6,8\} \\ \{2,4,5,6\} \\ \{1,2,4,5\} \end{array} & \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}
\end{array}
$$

$$
\begin{array}{r c c}
 & \begin{array}{cccccc} \{3,4,5,7\} & \{1,3,4,5\} & \{4,5,7,8\} & \{4,5,6,8\} & \{2,4,5,6\} & \{1,2,4,5\} \end{array} & \begin{array}{cccccc} \{3,4,5,7\}\to & \{1,3,4,5\}\to & \{4,5,7,8\}\to & \{4,5,6,8\}\to & \{2,4,5,6\}\to & \{1,2,4,5\}\to \end{array} \\
\begin{array}{r} \{3,4,5,7\} \\ \{1,3,4,5\} \\ \{4,5,7,8\} \\ \{4,5,6,8\} \\ \{2,4,5,6\} \\ \{1,2,4,5\} \end{array} & \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}
\end{array}
$$

$$
\begin{array}{r c c}
 & \begin{array}{cccccc} \{3,4,5,7\} & \{1,3,4,5\} & \{4,5,7,8\} & \{4,5,6,8\} & \{2,4,5,6\} & \{1,2,4,5\} \end{array} & \begin{array}{cccccc} \{3,4,5,7\}\to & \{1,3,4,5\}\to & \{4,5,7,8\}\to & \{4,5,6,8\}\to & \{2,4,5,6\}\to & \{1,2,4,5\}\to \end{array} \\
\begin{array}{r} \{3,4,5,7\}\to \\ \{1,3,4,5\}\to \\ \{4,5,7,8\}\to \\ \{4,5,6,8\}\to \\ \{2,4,5,6\}\to \\ \{1,2,4,5\}\to \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}
\end{array}
$$

$$
\begin{array}{r c c}
 & \begin{array}{cccccc} \{3,4,5,7\} & \{1,3,4,5\} & \{4,5,7,8\} & \{4,5,6,8\} & \{2,4,5,6\} & \{1,2,4,5\} \end{array} & \begin{array}{cccccc} \{3,4,5,7\}\to & \{1,3,4,5\}\to & \{4,5,7,8\}\to & \{4,5,6,8\}\to & \{2,4,5,6\}\to & \{1,2,4,5\}\to \end{array} \\
\begin{array}{r} \{3,4,5,7\}\to \\ \{1,3,4,5\}\to \\ \{4,5,7,8\}\to \\ \{4,5,6,8\}\to \\ \{2,4,5,6\}\to \\ \{1,2,4,5\}\to \end{array} & \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}
\end{array}
$$

Fig. 6.18 3-dimensional cube triangulated, with Γ, Γ^* indicating orientability

Therefore, one will easily convince oneself, that the number of simplices required to represent an n-cube increases as a factorial!

We obviously have the chance to handle also higher-dimensional examples in this way. This might bring the possibility for work in knot theory! The idea is to tesselate a part of the 3-dimensional space to the extent that a given knot may be represented in it. When considering a knot as a closed file or wire, i.e., an image of mapping the unit circle into $\mathbb{R}^3$, "represented properly" would mean that it never touches simplices of dimension ≤ 1 and the intersection of the file with a 3-simplex should never consist of more than one connected component. One may hope that this enables us to compute. A knot will then be just a sequence of 3-simplices with common adversely oriented boundary. The sequence will indicate over which bounding subsimplex the wire or file of the knot runs.

The most trivial "unknotted" knot consisting of just a circle around the dashed space diagonal would then be represented as the cyclic sequence

$$(1,3,4,5), (3,4,5,7), (4,5,7,8), (4,5,6,8)\!\!\to, (2,4,5,6)\!\!\to, (1,2,4,5)\!\!\to$$

of simplexes.

7 Concluding Remarks

In the present work, we have for the first time presented a thorough relational and algebraic treatment covering the broad range of such concepts as topology, proximity, nearness, contact, closure, and finally simplicial complexes. Much of the impetus to execute all these computations came from the intention to sharpen the relational tools. In the mean time, we have reached a status from which it seems possible to classify what can be achieved relationally and what not.

Yet another stimulation for this research was the possibility to compute. For several of the topics mentioned, it seems that problems may be solved in practice. The implementation of relational methods as with RelView `http://www.informatik.uni-kiel.de/~progsys/relview/` has gained substantial power. The merely term calculating TituRel system `http://mucob.dyndns.org:30531/~gs/TituRel/indexTituRel.html` proved versatile enough to underpin all the formulae with the examples presented.

Another topic that more or less obviously lends itself to being treated relationally are matroids and their exchange property. It would be highly desirable to find a point-free relational form of the respective axioms, which are in the literature mostly given with some counting arguments.

It has been a particular concern to identify those topics where one inevitably has to use *points*, in the relation-algebraic sense, and where one may get along without. The evasion to pointwise reasoning, much in the same way as in the sharpness problem in the early 1980ies, could widely be avoided.

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