Relational Concepts in Social Choice

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Abstract. What now is called social choice theory has ever since attracted mathematicians — not least several Nobel laureates — who try to capture the comparison relations expressed and to aggregate them. Their results are often referred to nowadays. The purpose of this paper is to make point-free relation-algebraic mathematics available as a tool for the study of social choice. Thus, we provide simplification, additional systematics, more compact relation-algebraic proofs and also an access to solving such problems with programs in the language TituRel — at least for the medium sized cases.

Keywords: relation, social choice, rationalization, revealed preference.

1 Introduction

Social choice is concerned with sets of decisions expressed by individuals and tries to aggregate these to a collective decision relation. Much of this paper is some sort of a translation of the respective theory to a point-free relation-algebraic form. However, it is not simply translated, but also in a non-trivial way transferred to a shorthand form. This in turn provides new insights and enables more compact algebraic proofs. It is, however, also a valuable scientific step that may help understanding the highly involved concepts. Much of the discussion runs along [Sen70, Suz83, Wri85].

2 Relation-Algebraic Preliminaries

This section is inserted to make the paper more or less self-contained, giving [SS89, SS93, Sch11] as a general reference. We write $R : V \longrightarrow W$ if R is a relation with source V and target W, often conceived as a subset of $V \times W$. If the sets V and W are finite of size m and n, respectively, and ordered, we may consider R as a Boolean matrix with m rows and n columns; called a *homogeneous* relation when $m = n$.

We assume the reader to be familiar with the basic operations on relations, namely R^{T} (*converse*), $\overline{R}$ (*negation*), $R \cup S$ (*union*), $R \cap S$ (*intersection*), and $R ; S$ (*composition*), the predicate $R \subseteq S$ (*containment*), and the special relations[1] $\mathbb{⊥}$ (*empty relation*), $\mathbb{T}$ (*universal relation*), and $\mathbb{I}$ (*identity relation*).

[1] Suppressing indices here.

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A *heterogeneous relation algebra* is a structure that

— is a category with respect to composition “ $;$ ” and identities $\mathbb{I}$,
— has complete atomic Boolean lattices with $\cup, \cap, \overline{}, \perp\!\!\!\perp, \top\!\!\!\top, \subseteq$ as morphism sets,
— obeys rules for transposition in connection with the category and the lattice aspect just mentioned that may be stated in either one of the following two ways:

$$\begin{array}{lll} \text{Dedekind} & R;S \cap Q \subseteq (R \cap Q;S^\top);(S \cap R^\top;Q) & \text{or} \\ \text{Schröder} & R;S \subseteq Q \iff R^\top;\overline{Q} \subseteq \overline{S} \iff \overline{Q};S^\top \subseteq \overline{R}. & \end{array}$$

Residuals are often introduced via $A;B \subseteq C \iff A \subseteq \overline{\overline{C};B^\top} =: C/B$, where B is divided from C on the right side. Intersecting such residuals in $\mathtt{syq}(R,S) := \overline{R^\top;\overline{S}} \cap \overline{\overline{R}^\top;S}$, the *symmetric quotient* $\mathtt{syq}(R,S) : W \longrightarrow Z$ of two relations $R : V \longrightarrow W$ and $S : V \longrightarrow Z$ is introduced. Symmetric quotients serve the purpose of ‘column comparison’: $\left[\mathtt{syq}(R,S)\right]_{wz} = \forall v \in V : R_{vw} \leftrightarrow S_{vz}$.

The symmetric quotient is not least applied to introduce *membership relations* $\varepsilon : X \longrightarrow \mathcal{P}(X)$ between a set X and its powerset $\mathcal{P}(X)$ or $\mathbf{2}^X$. These can be characterized algebraically up to isomorphism demanding $\mathtt{syq}(\varepsilon,\varepsilon) \subseteq \mathbb{I}$ and surjectivity of $\mathtt{syq}(\varepsilon,R)$ for all R. With a membership the powerset ordering is easily described as $\Omega = \overline{\varepsilon^\top;\overline{\varepsilon}}$.

There is another point to observe, namely the transition from a *subset* $V \subseteq X$, conceived as a relation $V : X \longrightarrow \mathbb{1}$, to its counterpart *element* $e_V = \mathtt{syq}(\varepsilon, V) \subseteq \mathbf{2}^X$. It often helps if one makes this difference explicit, using membership ε in

$$\begin{array}{r} \\ \varepsilon = \\ \\ e_V^\top = \end{array}\!\!\begin{array}{c} \\ \mathrm{a} \\ \mathrm{b} \\ \mathrm{c} \\ \\ \end{array}\!\!\begin{array}{c} \begin{array}{cccccccc} \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} \end{array} \\ \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \end{pmatrix} \\ (\mathbf{0} \; \mathbf{0} \; \mathbf{0} \; \mathbf{0} \; \mathbf{0} \; \mathbf{0} \; \mathbf{1} \; \mathbf{0}) \end{array} \qquad \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{1} \end{pmatrix} = \varepsilon;e_V = V$$

3 Order versus Preference

Orderings are generalized to preference structures as they have developed over the years and partly *after* the tremendous success of the work by Kenneth Arrow and Amartya Sen. Historically, orders E or strictorders C have more or less been used at free will with the possibility in mind that with $E = C \cup \mathbb{I}$ and $C = E \cap \overline{\mathbb{I}}$ everything may be freely converted from one form to the other.

However, this is not really true; with orderings E one often looses all the consequences of the Ferrers property. A (possibly heterogeneous) relation R has the Ferrers property if $R;\overline{R}^\top;R \subseteq R$, which expresses that one may find from any situation R_{ij} and R_{km} that either R_{kj} or R_{im}; an absolutely useful condition giving rise to a plethora of consequences concerning thresholds, semiorders, and intervalorders.

Orderings do not comfortably fit into the hierarchy of order concepts (see [Sch11] Prop. 12.1) in contrast to preorders, i.e., reflexive and transitive relations. This together with other indications has persuaded us to prefer the irreflexive form — not least that irreflexive Ferrers orderings are the slightly more general concept since E Ferrers implies C Ferrers, but not vice versa.

A next problem came up when researchers started investigating preference structures as a generalization of orderings. The by now standard way is to consider a so-called weak preference relation $R =$ 'is not worse than' and derive from it strict preference P, indifference I, and incomparability J. We have collected in [Sch11] Prop. 13.9 much of the dispersed information on how these concepts are interrelated. The bijective mutual transitions $\alpha : R \mapsto (P, I, J)$ and $\beta : (P, I, J) \mapsto R$ can be given explicitly as

$$\alpha(R) := (R \cap \overline{R}^{\mathsf{T}}, R \cap R^{\mathsf{T}}, \overline{R} \cap \overline{R}^{\mathsf{T}}) \qquad \text{and} \qquad \beta(P, I, J) := P \cup I.$$

Prop. 3.1.ii justifies the 'is not worse'-idea. Because one feels that indifference should be reflexive, it gives reason to demand already R to be reflexive.

Proposition 3.1. i) $P \subseteq \overline{\mathbb{I}}$ for every R.
ii) R reflexive $\implies \mathbb{I} \subseteq I$.

Proof: i) $R \cap \mathbb{I} = (R \cap \mathbb{I})^{\mathsf{T}} \subseteq R^{\mathsf{T}}$ implies $\mathbb{T} = \overline{R} \cup \overline{\mathbb{I}} \cup R^{\mathsf{T}}$ and $R \cap \overline{R}^{\mathsf{T}} \subseteq \overline{\mathbb{I}}$.
ii) $\mathbb{I} \subseteq R \implies \mathbb{I} \subseteq R \cap R^{\mathsf{T}} = I$. □

One obtains always the partition $P \cup P^{\mathsf{T}} \cup I \cup J = \mathbb{T}$ and observes that P is asymmetric, I is reflexive and symmetric, and J is irreflexive and symmetric.

Then several other concepts are defined, mentioned not least in [Suz83]. The following shows their translation into a point-free — and thus shorthand — form. As defined above, we will always have a relation R for which its asymmetric part is defined as

$$P := P(R) := R \cap \overline{R}^{\mathsf{T}}.$$

When R is agreed upon, we will use the respective shorter version. Since R is in general not an ordering, one has to investigate the following concepts anew from scratch that are concerned with cycle avoidance.

Definition 3.2. We consider the relation R and use its asymmetric part P.

i)	R **quasi-transitive**	$:\iff$	P transitive
ii)	R **acyclic**	$:\iff$	$P^+ \subseteq \overline{P}^{\mathsf{T}}$
iii)	R **acyclic**$_{\text{Sen}}$	$:\iff$	$P^+ \subseteq R$

iv)	R **consistent**	$:\Longleftrightarrow$	$P;R^* \subseteq \overline{R}^{\mathsf{T}}$
v)	P **progressively finite**	$:\Longleftrightarrow$	$\varepsilon \subseteq \mathbb{T};(\varepsilon \cap \overline{P;\varepsilon})$ □

Being progressively finite is the adequate relation-algebraic formulation that excludes an infinite run over ever new points in the same way as running into a circuit; cf. [SS93], p. 121. The condition is easily understood interpreting the right side as looking for elements of the subset from which one cannot proceed according to P to another point inside it: $\overline{P;\varepsilon}$.

In the following example, the two non-empty sets $\{3, 4\}$ and $\{1, 3, 4\}$ do not have a maximal element so that the corresponding columns in $\varepsilon \cap \overline{P;\varepsilon}$ vanish.

$$P = \begin{array}{cc} & \begin{array}{cccc}1&2&3&4\end{array}\\ \begin{array}{c}1\\2\\3\\4\end{array} & \begin{pmatrix}\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{1}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}\\ \mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}\end{pmatrix}\end{array} \qquad \varepsilon \cap \overline{P;\varepsilon} = \begin{array}{cc} & \begin{array}{cccccccccccccccc}\{\}&\{1\}&\{2\}&\{1,2\}&\{3\}&\{1,3\}&\{2,3\}&\{1,2,3\}&\{4\}&\{1,4\}&\{2,4\}&\{1,2,4\}&\{3,4\}&\{1,3,4\}&\{2,3,4\}&\{1,2,3,4\}\end{array}\\ \begin{array}{c}1\\2\\3\\4\end{array} & \begin{pmatrix}\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{1}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{1}&\mathbf{1}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{1}&\mathbf{1}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\end{pmatrix}\end{array}$$

Fig. 1. Illustrating the condition of being progressively finite; P is not

Several interdependencies follow immediately. One is in particular interested in consistent preference; that is, one does not like iterated preference with indifferences in between to result in preference in reverse direction. A lot of literature has appeared how to avoid problems of this kind.

Proposition 3.3. Let be given the situation of the preceding definition.

i)	R transitive	$\Longrightarrow$	R quasi-transitive, i.e., P transitive
ii)	R transitive	$\Longrightarrow$	R consistent
iii)	R consistent	$\Longrightarrow$	R acyclic
iv)	R quasi-transitive	$\Longrightarrow$	R acyclic
v)	R acyclic$_{\text{Sen}}$	$\Longrightarrow$	R acyclic
vi)	R acyclic$_{\text{Sen}}$	$\not\Longleftarrow$	R acyclic

Proof: i) The proof of $P;P \subseteq P$ decomposes into two parts:

$P;P \subseteq R;R \subseteq R$ since R is assumed to be transitive

$P;P = (R \cap \overline{R}^{\mathsf{T}});(R \cap \overline{R}^{\mathsf{T}}) \subseteq \overline{R}^{\mathsf{T}}$, where the latter follows via the Schröder rule and transitivity from $(R^{\mathsf{T}} \cap \overline{R});R^{\mathsf{T}} \subseteq \overline{R} \cup R^{\mathsf{T}}$.

ii) $P;R^* \subseteq \overline{R}^{\mathsf{T}} \iff R^{\mathsf{T}};R^{*\mathsf{T}} \subseteq \overline{P} = \overline{R} \cup R^{\mathsf{T}}$, which holds due to transitivity.

iii) $P^+ \subseteq R^+ = R^*;R \subseteq \overline{P}^{\mathsf{T}}$, the last step uses consistency in Schröderized form: $R^{\mathsf{T}};R^{*\mathsf{T}} \subseteq \overline{P}$.

iv) If P is transitive, acyclicity reads $P \subseteq \overline{P}^{\mathsf{T}}$. This, however, is trivially satisfied in view of the definition of P:

$P = R \cap \overline{R}^{\mathsf{T}} \subseteq \overline{R}^{\mathsf{T}} \cup R = \overline{R \cap \overline{R}^{\mathsf{T}}}^{\mathsf{T}} = \overline{P}^{\mathsf{T}}$.

v) since $P^+ \subseteq R \subseteq \overline{R}^{\mathsf{T}} \cup R = \overline{P}^{\mathsf{T}}$

vi) $R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{array}$ provides a counter-example with non-reflexive R. □

4 The Mechanics of Being Greatest

Since $R : X \longrightarrow X$ need not be an ordering, we must be very careful and avoid any informal reasoning, because much — but not all — stays the same. With $\mathtt{ubd}_R(\varepsilon) = \overline{\overline{R}^{\mathsf{T}} ; \varepsilon} : X \longrightarrow \mathbf{2}^X$, we obtain the set of upper bounds of all subsets in one hit. Upper bound points may exist, or not, and there may be one or many.

Given a relation $R : X \longrightarrow X$, we also introduce $\mathtt{max}_R : X \longrightarrow \mathbf{2}^X$ as assigning *the set* of maximal elements; in the finite case, this set will always be non-empty for a non-empty set. Executing this simultaneously,

$$\mathtt{max}_R(\varepsilon) := \varepsilon \cap \overline{(R \cap \overline{R}^{\mathsf{T}}) ; \varepsilon} = \varepsilon \cap \overline{P ; \varepsilon}$$

describes columnwise those elements that belong to the set and for which it is not the case that they are in relation $R \cap \overline{R}^{\mathsf{T}}$ — i.e. strictly R-below — to any element of the set.

In much a similar way, we here conceive the $\mathtt{gre}$ to deliver *always* a result; however, the result may correspond to the empty set indicating that there is no greatest *element*. In contrast to the classical case, there may occur several greatest elements for a relation R which is not an ordering. That is, given a relation $R : X \longrightarrow X$, we type this function as $\mathtt{gre}_R : X \longrightarrow \mathbf{2}^X$. One has to intersect sets with their upper bound sets,

$$\mathtt{gre}_R(\varepsilon) = \varepsilon \cap \mathtt{ubd}_R(\varepsilon),$$

to get greatest element sets for all subsets 'columnwise' simultaneously. For the set $\{1, 4\}$ in Fig. 2, we get the result $\{4\}$, e.g. This $\{4\}$ is a subset $\subseteq X$, for which we will now consider the corresponding element in $\mathbf{2}^X$; and this executed simultaneously for all greatest element sets, resulting (see the end of Sect. 2) in a relation G, so that:

$$\mathtt{gre}_R(\varepsilon) = \varepsilon ; G \qquad \text{and} \qquad G := \mathtt{syq}(\varepsilon, \mathtt{gre}_R(\varepsilon)).$$

With this highly compact notation, we will now generalize a result best known for orderings to arbitrary R. Concerning (ii) in Prop. 4.1, one often says that for R a finite preorder in every nonempty subset S a maximal element exists.

Proposition 4.1. Let an arbitrary homogeneous relation R be given.

i) $\mathtt{gre}_R(\varepsilon) \subseteq \mathtt{max}_R(\varepsilon)$
ii) R finite preorder $\Longrightarrow$ $\varepsilon \subseteq \mathbb{T} ; \mathtt{max}_R(\varepsilon)$
iii) R preorder $\Longrightarrow$ $\mathtt{gre}_R(\varepsilon) = \mathtt{max}_R(\varepsilon) \cap \mathbb{T} ; \mathtt{gre}_R(\varepsilon)$.

Proof: i) We have to prove $\varepsilon \cap \overline{\overline{R}^{\mathsf{T}} ; \varepsilon} \subseteq \varepsilon \cap \overline{(R \cap \overline{R}^{\mathsf{T}}) ; \varepsilon}$, but this is obvious.

ii) The asymmetric part P of a finite preorder R is certainly progressively finite (does not admit cycling), so that with Def. 3.2.v $\varepsilon \subseteq \mathbb{T} ; (\varepsilon \cap \overline{P ; \varepsilon}) = \mathbb{T} ; \mathtt{max}_R(\varepsilon)$.

iii) In view of (i), only $\mathtt{max}_R(S) \subseteq \mathtt{gre}_R(S)$ needs a proof. Assume a point $x \subseteq \mathtt{gre}_R(S) = S \cap \overline{\overline{R}^{\mathsf{T}} ; S}$ to exist, which is equivalent to $x \subseteq S \subseteq R ; x$.

Now we consider an arbitrary point $z \subseteq \mathtt{max}_R(S) = S \cap \overline{(R \cap \overline{R}^{\mathsf{T}}) ; S}$, which implies $z \subseteq S \subseteq (R \cup \overline{R}^{\mathsf{T}}) ; z$.

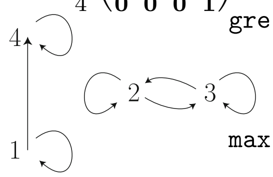
Combining all this crosswise, $z \subseteq S \subseteq R ; x$ and $x \subseteq S \subseteq (R \cup \overline{R}^{\mathsf{T}}) ; z$ where the latter implies $z \subseteq (R^{\mathsf{T}} \cup \overline{R}) ; x$. From both follows $z \subseteq \big[R \cap (\overline{R} \cup R^{\mathsf{T}})\big] ; x = \big[(R \cap \overline{R}) \cup (R \cap R^{\mathsf{T}})\big] ; x = (R \cap R^{\mathsf{T}}) ; x = I ; x$. Shunting and transposing gives $x \subseteq I ; z$, so that in total

$$z \subseteq S \subseteq R ; x \subseteq R ; I ; z \subseteq R ; R ; z \subseteq R ; z$$

due to transitivity of a preorder. This means $z \subseteq \mathtt{gre}_R(S)$. □

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ 2 & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ 3 & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ 4 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{array}$$

R



$$\mathtt{gre}_R(\varepsilon) = \begin{array}{c|cccccccccccccccc} & \{\} & \{1\} & \{2\} & \{1,2\} & \{3\} & \{1,3\} & \{2,3\} & \{1,2,3\} & \{4\} & \{1,4\} & \{2,4\} & \{1,2,4\} & \{3,4\} & \{1,3,4\} & \{2,3,4\} & \{1,2,3,4\} \\ \hline 1 & 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 2 & 0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0 \\ 3 & 0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0 \\ 4 & 0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0 \end{array}$$

$$\mathtt{max}_R(\varepsilon) = \begin{array}{c|cccccccccccccccc} 1 & 0&1&0&1&0&1&0&1&0&0&0&0&0&0&0&0 \\ 2 & 0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\ 3 & 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 4 & 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \end{array}$$

Fig. 2. $\mathtt{gre}_R(\varepsilon)$ and $\mathtt{max}_R(\varepsilon)$ using the membership relation ε

There hold further interesting formulae in case of greatest element sets.

Proposition 4.2. For every homogeneous relation R

i) $\mathtt{syq}(\mathtt{gre}_R(\varepsilon), \varepsilon) \subseteq \Omega^{\mathsf{T}}$,
ii) $\mathtt{gre}_R(\varepsilon) ; \Omega^{\mathsf{T}} \cap \varepsilon = \mathtt{gre}_R(\varepsilon)$.

Proof: i) $\overline{\Omega} = \varepsilon^{\mathsf{T}} ; \overline{\varepsilon} \subseteq \varepsilon^{\mathsf{T}} ; (\overline{\varepsilon} \cup \overline{R}^{\mathsf{T}} ; \varepsilon) = \varepsilon^{\mathsf{T}} ; \overline{\mathtt{gre}_R(\varepsilon)}$
$\subseteq \varepsilon^{\mathsf{T}} ; \overline{\mathtt{gre}_R(\varepsilon)} \cup \overline{\varepsilon}^{\mathsf{T}} ; \mathtt{gre}_R(\varepsilon) = \overline{\mathtt{syq}(\varepsilon, \mathtt{gre}_R(\varepsilon))}$.

ii) This means by definition $(\varepsilon \cap \overline{\overline{R}^{\mathsf{T}} ; \varepsilon}) ; \Omega^{\mathsf{T}} \cap \varepsilon = \varepsilon \cap \overline{\overline{R}^{\mathsf{T}} ; \varepsilon}$. We will use $\varepsilon ; \Omega = \varepsilon$. Direction $\supseteq$ is clear because Ω is reflexive. For $\subseteq$, we may restrict ourselves to showing

$$\overline{\overline{R}^{\mathsf{T}} ; \varepsilon} ; \Omega^{\mathsf{T}} \subseteq \overline{\overline{R}^{\mathsf{T}} ; \varepsilon} \iff \overline{R}^{\mathsf{T}} ; \varepsilon ; \Omega \subseteq \overline{R}^{\mathsf{T}} ; \varepsilon.$$ □

Although R is not an ordering, the interpretation is not very far from the ordering case: Stepping down from some greatest element of a set via the powerset ordering Ω^{T}, but staying inside that set, one will remain in the set of greatest elements. (In case R is an order, the greatest element set would be an at most 1-element set.)

5 Preferences versus Choice Functions

Choice is considered in powersets, where one indicates the — often strictly smaller — subsets of a subset from which elements *may be chosen*. When looking at definitions in [Sen70], e.g., one will find that the author is careful in demanding non-empty argument sets to which non-empty choice sets are assigned. Suzumura [Suz83] (page 27) discussed this in detail and decided for Sen's way. In an appendix of Chapt. 2, however, he also discusses slightly more general variants.

We go here even further and start from a set X of so-called **conceivable states** of which we intend to form subsets $\varepsilon : X \longrightarrow \mathbf{2}^X$ and consider the powerset ordering $\Omega : \mathbf{2}^X \longrightarrow \mathbf{2}^X$ of these. To make the distinctions in Fig. 3 clear, we define as follows:

Definition 5.1. Consider a relation $C : \mathbf{2}^X \longrightarrow \mathbf{2}^X$ that is univalent and contracting, i.e., a function which satisfies $C \subseteq \Omega^{\mathsf{T}}$. We call C a

i)	**Sen-type choice function**	if $C \subseteq \mathbb{T};\varepsilon$	and	$C;\mathbb{T} = \varepsilon^{\mathsf{T}};\mathbb{T}$,
ii)	**Suzumura-type choice function**	if $C \subseteq \mathbb{T};\varepsilon$	and	$C;\mathbb{T} \subseteq \varepsilon^{\mathsf{T}};\mathbb{T}$,
iii)	(generalized) **choice mapping**	if $C;\mathbb{T} = \mathbb{T}$.		

In either case, one defines $\mathcal{S} := C;\mathbb{T}$ and calls $(X, \mathcal{S})$ a **choice space**. □

$$
\begin{tikzcd}
X \arrow[loop left, "R"] \arrow[rr, "\varepsilon"] & & 2^X \arrow[loop right, "C", "\Omega"'] \arrow[dl, "\mathcal{S}"] \\
& \mathbb{1} &
\end{tikzcd}
$$

Fig. 3. Typing choice functions C as opposed to weak preferences R

In this way, results in (i,ii) are assigned only to non-empty sets and results are always non-empty subsets of the argument since $C \subseteq \mathbb{T};\varepsilon$. The C in (iii) will, due to contraction, assign the empty set to the empty set.

Every Sen-type choice function is obviously a Suzumura choice function. None of the two can ever be a choice mapping which is totally defined by definition. However, both — in particular the more general Suzumura version — are easily converted to the generalized mapping.

Proposition 5.2. i) For C a choice function, $C_{\text{Gen}} := C \cup \overline{C{:}\mathbb{T}} \,;\, \overline{\varepsilon^{\mathsf{T}}{:}\mathbb{T}}^{\mathsf{T}}$ is a generalized choice mapping.

ii) Given any choice mapping C, we obtain $C_{\text{Suz}} := C \cap \mathbb{T}{:}\varepsilon$ as a Suzumura choice function.

Proof: The proof is obvious when looking at Fig. 4. □

Researchers have always been very careful to execute all the case distinctions that arise when admitting an empty choice, be it from a non-empty subset, or of the empty subset. Being an empty choice might, however, smoothly be interpreted as an abstention. We will see that when proceeding to point-free relation-algebraic handling these problems disappear and results obtain a more uniform shape.

$$C_{\text{Suz}} = \begin{array}{r} \\ \{\} \\ \{1\} \\ \{2\} \\ \{1{,}2\} \\ \{3\} \\ \{1{,}3\} \\ \{2{,}3\} \\ \{1{,}2{,}3\} \end{array} \begin{array}{c} \begin{array}{cccccccc} \rotatebox{90}{\{\}} & \rotatebox{90}{\{1\}} & \rotatebox{90}{\{2\}} & \rotatebox{90}{\{1,2\}} & \rotatebox{90}{\{3\}} & \rotatebox{90}{\{1,3\}} & \rotatebox{90}{\{2,3\}} & \rotatebox{90}{\{1,2,3\}} \end{array} \\ \begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix} \end{array} \qquad C_{\text{Gen}} = \begin{array}{r} \\ \{\} \\ \{1\} \\ \{2\} \\ \{1{,}2\} \\ \{3\} \\ \{1{,}3\} \\ \{2{,}3\} \\ \{1{,}2{,}3\} \end{array} \begin{array}{c} \begin{array}{cccccccc} \rotatebox{90}{\{\}} & \rotatebox{90}{\{1\}} & \rotatebox{90}{\{2\}} & \rotatebox{90}{\{1,2\}} & \rotatebox{90}{\{3\}} & \rotatebox{90}{\{1,3\}} & \rotatebox{90}{\{2,3\}} & \rotatebox{90}{\{1,2,3\}} \end{array} \\ \begin{pmatrix} 1&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&1&0 \\ 1&0&0&0&0&0&0&0 \end{pmatrix} \end{array}$$

Fig. 4. Toggling between Suzumura choice function and choice mapping

6 Generating Choice Functions from Preferences

Now we look at possibilities how to obtain choice functions or mappings. The frequently applied idea is to start from any relation R on a set X and let C map every subset of X to the subset of its R-greatest elements — recall that the definition of the functional $\mathtt{gre}_R(u)$ above has already sailed free from the requirement that R be an ordering.

Definition 6.1. Given any homogeneous relation R, not necessarily an order or a preorder, we call $C := \mathtt{syq}(\mathtt{gre}_R(\varepsilon), \varepsilon)$ its **corresponding choice mapping** and speak of the corresponding choice function F of

i) **Suzumura-type** if $F = C \cap \mathbb{T}{:}\varepsilon$,
ii) **Sen-type** if $F = C \cap \mathbb{T}{:}\varepsilon$ and in addition $F{:}\mathbb{T} = \varepsilon^{\mathsf{T}}{:}\mathbb{T}$. □

The claim for C to be a mapping needs a proof which is given below as Prop. 6.2.i,ii. The side conditions in Def. 6.1.i,ii seem slightly artificial.

The typical investigation is now to look at R and try to guarantee certain favourable properties of C; that it generates a Sen-type choice function, e.g. Such work has a great tradition, and we cannot report much of it; in particular, because we have changed part of the foundation in moving to choice mappings.

$$
\begin{array}{c|ccc}
 & 1 & 2 & 3 \\ \hline
1 & \mathbf{1} & \mathbf{1} & \mathbf{1} \\
2 & \mathbf{1} & \mathbf{1} & \mathbf{1} \\
3 & \mathbf{0} & \mathbf{0} & \mathbf{1}
\end{array}
\qquad
\begin{array}{c|cccccccc}
 & \{\} & \{1\} & \{2\} & \{1,2\} & \{3\} & \{1,3\} & \{2,3\} & \{1,2,3\} \\ \hline
\{\} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\{1\} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\{2\} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\{1,2\} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\{3\} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\{1,3\} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\{2,3\} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\{1,2,3\} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{array}
\qquad
\begin{array}{c|cccccccc}
 & \{\} & \{1\} & \{2\} & \{1,2\} & \{3\} & \{1,3\} & \{2,3\} & \{1,2,3\} \\ \hline
\{\} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\{1\} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\{2\} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\{1,2\} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\{3\} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\{1,3\} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\{2,3\} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\{1,2,3\} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{array}
$$

Fig. 5. R, its corresponding Sen-type choice function, and choice mapping

We will soon see that we have dropped conditions for reasons of simplicity and uniformity. The latter idea is very much supported by the following proposition. By the way, (iv) of Prop. 6.2 has in [Sen70] been termed 'property α'.

Proposition 6.2. Let be given a homogeneous relation R and its corresponding choice mapping $C = \mathtt{syq}(\mathtt{gre}_R(\varepsilon), \varepsilon)$. Then

i) C is indeed a mapping, i.e., total and univalent,
ii) $C \subseteq \Omega^{\mathsf{T}}$,
iii) $\varepsilon ; C^{\mathsf{T}} = \mathtt{gre}_R(\varepsilon)$,
iv) $\varepsilon ; C^{\mathsf{T}} = \varepsilon ; C^{\mathsf{T}} ; \Omega^{\mathsf{T}} \cap \varepsilon$.

Proof: i) C is a mapping by definition; cf. [Sch11] Def. 7.13.

ii) is the statement of Prop. 4.2.i.

iii) $\varepsilon ; C^{\mathsf{T}} = \varepsilon ; \left[\mathtt{syq}(\mathtt{gre}_R(\varepsilon), \varepsilon)\right]^{\mathsf{T}} = \varepsilon ; \mathtt{syq}(\varepsilon, \mathtt{gre}_R(\varepsilon)) = \mathtt{gre}_R(\varepsilon)$ according to [Sch11] Prop. 7.14.

iv) We start with $\subseteq$: The first containment is trivial since Ω is reflexive, while the second is a consequence of (ii). For $\supseteq$, we start with

$$
\begin{array}{ll}
\varepsilon ; \Omega = \varepsilon, \quad \text{which implies} & (\varepsilon \cap \overline{R}^{\mathsf{T}} ; \varepsilon) ; \Omega \subseteq \overline{R}^{\mathsf{T}} ; \varepsilon ; \Omega = \overline{R}^{\mathsf{T}} ; \varepsilon = \overline{R}^{\mathsf{T}} ; \varepsilon \cup \overline{\varepsilon} \\
\iff & (\varepsilon \cap \overline{\overline{R}^{\mathsf{T}}} ; \varepsilon) ; \Omega^{\mathsf{T}} \subseteq \overline{\varepsilon} \cup \overline{\overline{R}^{\mathsf{T}}} ; \varepsilon \\
\iff & \mathtt{gre}_R(\varepsilon) ; \Omega^{\mathsf{T}} \cap \varepsilon = (\varepsilon \cap \overline{\overline{R}^{\mathsf{T}} ; \varepsilon}) ; \Omega^{\mathsf{T}} \cap \varepsilon \subseteq \overline{\overline{R}^{\mathsf{T}} ; \varepsilon} \\
\implies & \varepsilon ; C^{\mathsf{T}} ; \Omega^{\mathsf{T}} \cap \varepsilon = \mathtt{gre}_R(\varepsilon) ; \Omega^{\mathsf{T}} \cap \varepsilon \subseteq \varepsilon \cap \overline{\overline{R}^{\mathsf{T}} ; \varepsilon} = \mathtt{gre}_R(\varepsilon) = \varepsilon ; C^{\mathsf{T}} \qquad \square
\end{array}
$$

Normally, several groups of conditions are assembled and then the proof is given that R defines a Sen-type choice function. We will here proceed the other way round and first formulate the condition on C aimed at.

Proposition 6.3. A choice mapping C corresponding to R will have a corresponding Sen-type choice function precisely when the following condition on R is satisfied

$$\varepsilon \subseteq \mathbb{T};(\varepsilon \cap \overline{\overline{R}^{\mathsf{T}};\varepsilon}).$$

Proof: The Sen condition on C is that it assigns non-empty subsets to non-empty argument sets, i.e., $C^{\mathsf{T}} \cap \mathbb{T};\varepsilon \subseteq \varepsilon^{\mathsf{T}};\mathbb{T}$, which one will verify looking at Fig. 6, derived from Fig. 5.

The condition slightly modified is $C^{\mathsf{T}};(\mathbb{I} \cap \mathbb{T};\varepsilon) = C^{\mathsf{T}} \cap \mathbb{T};\varepsilon \subseteq \varepsilon^{\mathsf{T}};\mathbb{T}$. Using the Schröder rule, since C is a mapping, and using Prop. 6.2.iii, we get

$$C;\overline{\varepsilon^{\mathsf{T}};\mathbb{T}} = \overline{C;\varepsilon^{\mathsf{T}};\mathbb{T}} = \overline{[\mathtt{gre}_R(\varepsilon)]^{\mathsf{T}};\mathbb{T}} \subseteq \overline{\mathbb{I} \cap \mathbb{T};\varepsilon}.$$

Negating, transposing, and expanding $\mathtt{gre}$ gives

$$\mathbb{I} \cap \mathbb{T};\varepsilon \subseteq \mathbb{T};(\varepsilon \cap \overline{\overline{R}^{\mathsf{T}};\varepsilon}),$$

from which we obtain the final result as

$$\varepsilon \subseteq \mathbb{T};\varepsilon = \mathbb{T};(\mathbb{I} \cap \mathbb{T};\varepsilon) \subseteq \mathbb{T};\mathbb{T};(\varepsilon \cap \overline{\overline{R}^{\mathsf{T}};\varepsilon}) = \mathbb{T};(\varepsilon \cap \overline{\overline{R}^{\mathsf{T}};\varepsilon}).$$ □

$$\begin{array}{c} \\ \{\} \\ \{1\} \\ \{2\} \\ \{1,2\} \\ \{3\} \\ \{1,3\} \\ \{2,3\} \\ \{1,2,3\} \end{array}
\begin{array}{c}
\begin{array}{cccccccc} \{\} & \{1\} & \{2\} & \{1,2\} & \{3\} & \{1,3\} & \{2,3\} & \{1,2,3\} \end{array} \\
\begin{pmatrix}
1&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&0&1&1&1&1\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0
\end{pmatrix} \\
C^{\mathsf{T}}
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} \{\} & \{1\} & \{2\} & \{1,2\} & \{3\} & \{1,3\} & \{2,3\} & \{1,2,3\} \end{array} \\
\begin{pmatrix}
0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&1
\end{pmatrix} \\
\mathbb{I} \cap \mathbb{T};\varepsilon = \mathbb{I} \cap \varepsilon^{\mathsf{T}};\mathbb{T}
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} \{\} & \{1\} & \{2\} & \{1,2\} & \{3\} & \{1,3\} & \{2,3\} & \{1,2,3\} \end{array} \\
\begin{pmatrix}
0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1
\end{pmatrix} \\
\varepsilon^{\mathsf{T}};\mathbb{T}
\end{array}$$

Fig. 6. Condition on a choice mapping to lead to a (Sen) choice function

Once we are in this position, we may look for combinations of the widely known conceivable properties of R that satisfy this requirement; e.g., being reflexive and/or connex, and/or transitive etc. The homogeneous relation R will be called **connex** provided $\mathbb{T} = R \cup R^{\mathsf{T}}$; it is thus reflexive and complete, the latter meaning $\overline{\mathbb{I}} = R \cup R^{\mathsf{T}}$.

Proposition 6.4. Whenever R is connex, and quasi-transitive on a finite set, the corresponding choice mapping $C := \mathtt{syq}(\mathtt{gre}_R(\varepsilon), \varepsilon)$ will give rise to a Sen-type choice function.

Proof: Following Prop. 6.3, we have to prove

$$R \cup R^{\mathsf{T}} = \mathbb{T}, \quad P;P \subseteq P \qquad \Longrightarrow \qquad \varepsilon \subseteq \mathbb{T};(\varepsilon \cap \overline{\overline{R}^{\mathsf{T}};\varepsilon}).$$

Since $\overline{R}^{\mathsf{T}} \subseteq R$ and thus $P = R \cap \overline{R}^{\mathsf{T}} = \overline{R}^{\mathsf{T}}$, this means $\varepsilon \subseteq \mathbb{T}\!;(\varepsilon \cap \overline{P;\varepsilon})$. However, this is the condition for being progressively finite according to Def. 3.2; and indeed, as a transitive and by construction asymmetric relation on a finite set, P is a strictorder, and thus progressively finite. $\square$

Traditionally, many more such results are proved, usually with page-long free-style proofs. The one above written in full, in contrast, may be proof-checked.

Often the criterion is acyclicity.

Proposition 6.5. Let R be a finite connex relation. Then the corresponding choice mapping $C := \mathtt{syq}(\mathtt{gre}_R(\varepsilon), \varepsilon)$ will give rise to a Sen-type choice function provided R is acyclic$_{\text{Sen}}$.

Proof: As in the preceding proof, we get $\overline{R}^{\mathsf{T}} \subseteq R$ from connexity, and thus $P = R \cap \overline{R}^{\mathsf{T}} = \overline{R}^{\mathsf{T}}$, so that we have to prove $\varepsilon \subseteq \mathbb{T}\!;(\varepsilon \cap \overline{P;\varepsilon})$. We use that being progressively finite is equivalent with being circuit-free $P^+ \subseteq \overline{\mathbb{I}}$ in case of finiteness; cf. [SS93] Prop. 6.3.2.

Now we proceed assuming P not to be circuit-free. Then there exists a finite at least 2-element sequence of points $x_1, x_2, \ldots x_{n+1} = x_1$ such that $x_i \subseteq P;x_{i+1}$, counting the indices cyclically modulo n. With Sen-acyclicity $P^+ \subseteq R$, we obtain that they are all mutually related $x_i \subseteq R;x_j$ for $i, j = 1, \ldots n$; and therefore also $x_i \subseteq I;x_j$. This is a contradiction, because P, P^{T}, I, J form a disjunction. $\square$

$$R = \begin{array}{r c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{r} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix} \end{array} \qquad C = \begin{array}{r c} & \begin{array}{cccccccccccccccc} \{\} & \{1\} & \{2\} & \{1,2\} & \{3\} & \{1,3\} & \{2,3\} & \{1,2,3\} & \{4\} & \{1,4\} & \{2,4\} & \{1,2,4\} & \{3,4\} & \{1,3,4\} & \{2,3,4\} & \{1,2,3,4\} \end{array} \\ \begin{array}{r} \{\} \\ \{1\} \\ \{2\} \\ \{1,2\} \\ \{3\} \\ \{1,3\} \\ \{2,3\} \\ \{1,2,3\} \\ \{4\} \\ \{1,4\} \\ \{2,4\} \\ \{1,2,4\} \\ \{3,4\} \\ \{1,3,4\} \\ \{2,3,4\} \\ \{1,2,3,4\} \end{array} & \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \end{array}$$

Fig. 7. A homogeneous relation R determining a generalized choice mapping C

The above matrices visualize forming the choice mapping. (One should remember that Sen ususally presents the matrix of an ordering with the greatest element down to the least.) Obviously, R is not an ordering. One will recognize that there is no greatest element in the set $\{1, 3\}$ resulting in assigning the empty set via C. On the other hand, the set $\{4\}$ is at the same time the set of greatest elements of $\{1, 4\}$ and $\{4\}$.

7 Rationalization Conceived as a Galois Correspondence

Since it is always a promising situation when one finds some Galois correspondence, we mention here the following result. So far, however, we have not had the opportunity to look for all its possible consequences.

Proposition 7.1. There exists a Galois correspondence between the R- and the C-side. It concerns arbitrary relations R and C, the latter contained in Ω^{T}, and looks as follows

$$\pi(C) \subseteq R \quad \Longleftrightarrow \quad C \subseteq \sigma(R)$$

with $\quad \sigma(R) := \overline{\overline{\mathtt{gre}_R(\varepsilon)}^{\mathsf{T}};\varepsilon} \quad$ and $\quad \pi(C) := \varepsilon;C;\varepsilon^{\mathsf{T}}$.

Proof: We will use that $C;\varepsilon^{\mathsf{T}} \subseteq \varepsilon^{\mathsf{T}}$, which is trivial because we have $C \subseteq \Omega^{\mathsf{T}}$.

$$\begin{array}{ll}
& C \subseteq \sigma(R) = \overline{\overline{\mathtt{gre}_R(\varepsilon)}^{\mathsf{T}};\varepsilon} \\
\Longleftrightarrow & \overline{\mathtt{gre}_R(\varepsilon)}^{\mathsf{T}};\varepsilon \subseteq \overline{C} \\
\Longleftrightarrow & C;\varepsilon^{\mathsf{T}} \subseteq \left[\mathtt{gre}_R(\varepsilon)\right]^{\mathsf{T}} = \varepsilon^{\mathsf{T}} \cap \overline{\varepsilon^{\mathsf{T}};\overline{R}} \\
\Longleftrightarrow & C;\varepsilon^{\mathsf{T}} \subseteq \overline{\varepsilon^{\mathsf{T}};\overline{R}} \\
\Longleftrightarrow & \varepsilon^{\mathsf{T}};\overline{R};\varepsilon \subseteq \overline{C} \\
\Longleftrightarrow & \overline{R}^{\mathsf{T}};\varepsilon;C \subseteq \overline{\varepsilon} \\
\Longleftrightarrow & \varepsilon;C^{\mathsf{T}};\varepsilon^{\mathsf{T}} \subseteq R^{\mathsf{T}} \\
\Longleftrightarrow & \varepsilon;C;\varepsilon^{\mathsf{T}} \subseteq R
\end{array}$$

□

This correspondence seems to be related with rationalization.

Definition 7.2. We consider some choice function $C : \mathbf{2}^X \longrightarrow \mathbf{2}^X$. A relation $R : X \longrightarrow X$ is said to **rationalize** C if $\varepsilon;C^{\mathsf{T}} = \mathtt{gre}_R(\varepsilon)$. If such an R exists, C is called a **rational** choice. If this R is in addition an ordering, C is called a **fully rational** choice. □

Should the choice C have been constructed starting from some relation R, this underlying R will obviously rationalize C, since, according to [Sch11] Prop. 7.14, $X = \varepsilon;\mathtt{syq}(\varepsilon, X)$ for every X. But there may exist other rationalizing relations, not least via the above Galois mechanism. There are more Cs than Rs, so that one may hope for an adjunction.

8 Revealing a Preference Out of a Choice Function

Rationalization asks whether an executed choice C has followed some 'rational' criterion R. While we have so far defined a choice function starting from an arbitrary relation R, we will now go in reverse direction and try to reveal (i.e., extract) such a relation R from an arbitrary choice function C.

Definition 8.1. For every choice function $C : \mathbf{2}^X \longrightarrow \mathbf{2}^X$, we define the following $R : X \longrightarrow X$, calling it the

i) **revealed preference** $R_C := \varepsilon; C; \varepsilon^{\mathsf{T}}$,

ii) **revealed strict preference** $R^*_C := (\varepsilon; C \cap \overline{\varepsilon}); \varepsilon^{\mathsf{T}}$. □

In [Suz83], (ii) is written as $R^*_C = \bigcup_{S \in \mathcal{S}} \left[C(S) \times \{S \setminus C(S)\}\right]$, and explained with *$x$ is R^*_C-preferred to y if and only if x is chosen and y could have been chosen but was actually rejected from some $S \in \mathcal{S}$.* (Order reversed!)

It is certainly an important case when the revealed R can somehow re-determine the C one has been starting from.

The following is mentioned in order to demonstrate that the construct of a choice *mapping* — as opposed to the choice *functions* — is indeed a profitable idea.

Proposition 8.2. Def. 8.1.i delivers the same relation, regardless of whether formed of a choice mapping, its corresponding Sen-type choice function, or its corresponding Suzumura-type choice function, i.e.,

$$\varepsilon; C; \varepsilon^{\mathsf{T}} = \varepsilon; (C \cap \mathbb{T}; \varepsilon); \varepsilon^{\mathsf{T}}.$$

Proof: For the Suzumura-case (as well as for the Sen-case which has an additional condition on C), we apply two times obvious matrix product formulae, which say, e.g., that annihilating columns of the second factor is equivalent to annihilating these columns in a product:

$$\varepsilon; (C \cap \mathbb{T}; \varepsilon); \varepsilon^{\mathsf{T}} = (\varepsilon; C \cap \mathbb{T}; \varepsilon); \varepsilon^{\mathsf{T}} = \varepsilon; C; (\varepsilon^{\mathsf{T}} \cap [\mathbb{T}; \varepsilon]^{\mathsf{T}}) = \varepsilon; C; \varepsilon^{\mathsf{T}}$$ □

We are, thus, again enabled to go back and forth between relations $R : X \longrightarrow X$ and relations $C : \mathbf{2}^X \longrightarrow \mathbf{2}^X$ with Def. 6.1 and Def. 8.1. The question immediately arises, to which extent a revealed R_C obtained from a C which is obtained from R resembles the original relation. We have indicated this idea with the Galois correspondence above. For reasons of time and manpower, it has not yet been made a central point of our investigation. In any case, the main question is, to what extent going forth and back again comes close to an identity. It is answered below.

Proposition 8.3. i) For any R, the R_C obtained from its corresponding choice mapping satisfies $R_C \subseteq R$.

ii) In addition: R reflexive implies equality $R_C = R$.

Proof: i) $R_C = \varepsilon ; C ; \varepsilon^\mathsf{T} = \varepsilon ; [\mathtt{gre}_R(\varepsilon)]^\mathsf{T} = \varepsilon ; (\varepsilon^\mathsf{T} \cap \overline{\varepsilon^\mathsf{T} ; \overline{R}}) \subseteq \varepsilon ; \overline{\varepsilon^\mathsf{T} ; \overline{R}} \subseteq R$.

ii) This proof, which we omit, seems to need pointwise consideration. □

Fig. 8 gives an example for being unequal when R is not reflexive: Not even C resembles R in an adequate way.

$$R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{array} \qquad C = \begin{array}{c} \\ \{\} \\ \{1\} \\ \{2\} \\ \{1,2\} \\ \{3\} \\ \{1,3\} \\ \{2,3\} \\ \{1,2,3\} \end{array} \begin{array}{c} \begin{array}{cccccccc} \{\} & \{1\} & \{2\} & \{1,2\} & \{3\} & \{1,3\} & \{2,3\} & \{1,2,3\} \end{array} \\ \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{array} \qquad R_C = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{array}$$

Fig. 8. $R_C \subsetneqq R$

9 Axiomatization of Choice

Once choice functions are established, researchers usually proceed to the characterization of desirable properties of choice. Many famous people have contributed to this idea and the interdependency of all these conceivable axioms has widely been investigated.

One usually starts with certain intuitively clear and appealing postulates and looks in which way these may be satisfied or not. Impossibility theorems are well known that destroy any hope for choice mechanisms that follow simple axiomatizations. It seems that highly complicated ones are necessary.

Cycles of preference are counter-intuitive. Demanding transivity, they are excluded, but this is often considered too hard a condition; so indifference is admitted. We recall postulates that are intended to prohibit cycles. See, e.g., Prop. 6.5.

Definition 9.1. Assume a choice function C and revealed preferences thereof.

i) An H-**cycle** from some point x to x is given when $\left[R_C^* ; (R_C)^+\right]_{xx}$.
ii) An SH-**cycle** from some point x to x is given when $\left[R_C ; (R_C^*)^+\right]_{xx}$. □

The following axioms for the revealed R are often demanded to avoid cycles.

Definition 9.2. We consider the revealed preferences of some choice function.

i) **Houthakker's axiom of revealed preference** (HOA) demands that there be no H-cycle, i.e., $(R_C)^+ \subseteq \overline{R_C^*}^{\mathsf{T}}$.
ii) The **strong axiom of revealed preference** (SA) demands that there be no SH-cycle, i.e., $(R_C^*)^+ \subseteq \overline{R_C}^{\mathsf{T}}$.
iii) The **weak axiom of revealed preference** (WA) demands that there be no 2-step cycle, i.e., $R_C^* \subseteq \overline{R_C}^{\mathsf{T}}$. □

Corresponding axioms for the choice functions themselves have also been formulated and the interrelationship has been discussed.

Definition 9.3. We consider the choice function C as well as its revealed preference together with the membership relation. We will speak of the

i) **strong congruence axiom** SCA if $\quad \varepsilon;C \cap R_C^{+\mathsf{T}};\varepsilon \subseteq \varepsilon$,
ii) **weak congruence axiom** WCA if $\quad \varepsilon;C \cap R_C^{\mathsf{T}};\varepsilon \subseteq \varepsilon$. □

We have seen on several occasions that we need not explicitly mention $\mathcal{S} := C;\mathbb{T}$ every time. Not least Prop. 8.2 has shown that the empty rows of C or those that are non-empty, but assign an empty choice may be neglected without affecting the overall structure. Having this in mind, we consider, e.g., the weak congruence axiom (WCA). In [Suz83], it is presented as

$$\forall S \in \mathcal{S} : [x \in S \,\&\, \{\exists y \in C(S) : (x, y) \in R_C\}] \to x \in C(S).$$

Firstly, quantification over x is not mentioned. Another typical flaw of such considerations is that, in this case, the $S \in \mathcal{S}$ appears — without making this visible — as a subset that may contain elements and also as an element over which quantification may run. Let us denote the element in the powerset corresponding to S as e. (We also remember that our ordering is transposed compared with [Suz83].)

$$\forall x : \forall e : \left[\varepsilon_{xe} \wedge \{\exists y : (\varepsilon;C^{\mathsf{T}})_{ye} \wedge (R_C)_{yx}\}\right] \to (\varepsilon;C^{\mathsf{T}})_{xe}$$
$$\forall x : \forall e : \left[\overline{\varepsilon}_{xe} \vee \overline{\exists y : (\varepsilon;C^{\mathsf{T}})_{ye} \wedge (R_C)_{yx}}\right] \vee (\varepsilon;C^{\mathsf{T}})_{xe}$$
$$\forall x : \forall e : \left[\overline{\varepsilon} \cup \overline{R_C^{\mathsf{T}};\varepsilon;C^{\mathsf{T}}}\right]_{xe} \vee (\varepsilon;C^{\mathsf{T}})_{xe}$$
$$\varepsilon \cap R_C^{\mathsf{T}};\varepsilon;C^{\mathsf{T}} \subseteq \varepsilon;C^{\mathsf{T}}$$
$$\varepsilon;C \cap R_C^{\mathsf{T}};\varepsilon \subseteq \varepsilon$$

At last, the function C has been multiplied from the right side, using a standard formula. In analogy follows the strong congruence axiom (SCA).

We mention the following well-known implications without giving full proofs.

Proposition 9.4

i) HOA $\iff$ SCA
ii) HOA $\implies$ SA $\implies$ WA
iii) WA $\iff$ WCA

Proof: ii) is trivial since $R_C^* \subseteq R_C$.

iii) Condition WA demands for the revealed strict preference $R^*_C = (\varepsilon;C \cap \overline{\varepsilon});\varepsilon^\mathsf{T}$

$$R^*_C = (\varepsilon;C \cap \overline{\varepsilon});\varepsilon^\mathsf{T} \subseteq \overline{R_C}^\mathsf{T}$$
$$\iff R^\mathsf{T}_C;\varepsilon \subseteq \overline{\varepsilon;C} \cup \varepsilon$$
$$\iff \varepsilon;C \cap R^\mathsf{T}_C;\varepsilon \subseteq \varepsilon, \qquad \text{i.e., WCA} \qquad \square$$

10 Concluding Remark

This text is certainly just a first step directed towards a study of social choice using relations and towards computational social choice. Lifting to a point-free relation-algebraic treatment, we have achieved several goals. Firstly, this is a shorthand notation that facilitates work at least for the initiated. Secondly, we got rid of many case distinctions necessary in Sen's or Suzumura's approach; not least are relational proofs more easily computer-checkable. Scientific progress by this article may also be found in the unification of the choice concepts and in relating them to formally manipulable formulae such as being progressively finite, etc. Finally, writing all this down — as it has indeed been done — in the relational reference language TituRel, an immediate execution on a computer became possible, at least for moderately sized tasks.

Many more attempts allow a relational approach, not least centered around the Gibbard paradox with its standard rights rules; cf. [Wri85]. One may study the Arrow or the Chernoff Axiom relationally and many more as well as a lot of Pareto modelling.

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