Rectangles, Fringes, and Inverses

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Abstract. Relational composition is an associative operation; therefore semigroup considerations often help in relational algebra. We study here some less known such effects and relate them with maximal rectangles inside a relation, i.e., with the basis of concept lattice considerations. The set of points contained in precisely one maximal rectangle makes up the fringe. We show that the converse of the fringe sometimes acts as a generalized inverse of a relation. Regular relations have a generalized inverse. They may be characterized by an algebraic condition.

1 Introduction

Relation algebra has had influx from semigroup theory, but only a study in a point-free form seems to offer chances to use it in a wider range. Inverses need not exist in general; the containment-ordering of relations, however, allows to consider sub-inverses. Occasionally the greatest sub-inverse also meets the requirements of an inverse. In interesting cases as they often originate from applications, not least around variants of orderings (semiorder, interval order, block-transitive order, e.g.), an inverse is needed and it may be characterized by appropriate means from that application area. It seems that this new approach generalizes earlier ones and at the same time facilitates them. In particular, semiorder considerations in [7] get a sound algebraic basis.

2 Prerequisites

We assume much of relation algebra to be known in the environment of RelMiCS, to be found not least in our standard reference [8, 9], and concentrate on a few less known, unknown, or even new details. Already here, we announce two points: Unless explicitly stated otherwise, all our relations are possibly heterogeneous relations. When we quantify $\forall X, \exists X$, we always mean *. . . for which the construct in question is defined.*

A relation A is **difunctional**[1] if $A; A^{\mathsf{T}}; A \subseteq A$, which means that A can be written in block diagonal form by suitably rearranging rows and columns. If A is difunctional, the same obviously holds for A^{T}.

[1] In [1] called a *matching relation* or simply a *match*.

If A, R are relations, f is a mapping, and x is a point, then negation commutes with composition so that $f{;}\overline{A} = \overline{f{;}A}$ as well as $\overline{R}{;}x = \overline{R{;}x}$.

Given any two relations R, S with coinciding domain, their **left residuum** is defined as $R\backslash S := \overline{R^{\mathsf{T}}{;}\overline{S}}$, and correspondingly for P, Q with coinciding codomain their **right residuum** $Q/P := \overline{\overline{Q}{;}P^{\mathsf{T}}}$.

Combining this, we define the **symmetric quotient** $\mathtt{syq}(A,B) := \overline{A^{\mathsf{T}}{;}\overline{B}} \cap \overline{\overline{A}^{\mathsf{T}}{;}B}$ for any two relations A, B with coinciding domain. Obviously, $\mathtt{syq}(A,B) = A\backslash B \cap \overline{A}\backslash\overline{B}$. We recall several canceling formulae for the symmetric quotient: For arbitrary relations A, B, C we have

$$\begin{aligned} \mathtt{syq}(A,B){;}\mathtt{syq}(B,C) &= \mathtt{syq}(A,C) \cap \mathtt{syq}(A,B){;}\mathbb{T} \\ &= \mathtt{syq}(A,C) \cap \mathbb{T}{;}\mathtt{syq}(B,C) \ \subseteq\ \mathtt{syq}(A,C). \end{aligned}$$

If $\mathtt{syq}(A,B)$ is total, or if $\mathtt{syq}(B,C)$ is surjective, then

$$\mathtt{syq}(A,B){;}\mathtt{syq}(B,C) = \mathtt{syq}(A,C).$$

For a given relation R, we define its corresponding **row-contains preorder**[2] $\mathcal{R}(R) := \overline{\overline{R}{;}R^{\mathsf{T}}} = R/R$ and **column-is-contained preorder** $\mathcal{C}(R) := \overline{R^{\mathsf{T}}{;}\overline{R}} = R\backslash R$.

Given an ordering "$\leq_E$", resp. E, one traditionally calls the element $s \in V$ an **upper bound** of the set $U \subseteq V$ provided $\forall u \in U : u \leq_E s$. In point-free form we use the always existing — but possibly empty — set $\mathtt{ubd}_E(U) = \overline{\overline{E}^{\mathsf{T}}{;}U}$. Having this in mind, we introduce for any relation R two functionals, namely

$\mathtt{ubd}_R(X) := \overline{\overline{R}^{\mathsf{T}}{;}X}$, the **upper bound cone functional** and

$\mathtt{lbd}_R(X) := \overline{\overline{R}{;}X}$, the **lower bound cone functional**.

They are built in analogy to the construct given before, however, without assuming the relation R to be an ordering, nor need it be a homogeneous relation. The most important properties may nevertheless be shown using the Schröder equivalences.

2.1 Proposition. Given any fitting relations R, X, the following hold

i) $\mathtt{ubd}_R(\mathtt{lbd}_R(\mathtt{ubd}_R(X))) = \mathtt{ubd}_R(X)$, i.e., $\overline{\overline{R}^{\mathsf{T}}{;}\overline{\overline{R}{;}\overline{\overline{R}^{\mathsf{T}}{;}X}}} = \overline{\overline{R}^{\mathsf{T}}{;}X}$

ii) $\mathtt{lbd}_R(\mathtt{ubd}_R(\mathtt{lbd}_R(X))) = \mathtt{lbd}_R(X)$, i.e., $\overline{\overline{R}{;}\overline{\overline{R}^{\mathsf{T}}{;}\overline{\overline{R}{;}X}}} = \overline{\overline{R}{;}X}$ □

These formulae are really general, but have been studied mostly in more specialized contexts so far. We now get rid of any additional assumptions that are unnecessary and just tradition of the respective application field.

For the symmetric quotient, we once more refer to our standard reference [8, 9] and add a new result here.

[2] In French: préordre finissant and préordre commençant; [5]

2.2 Proposition. For any fitting relations R, X, Y

$$\mathtt{syq}(\mathtt{lbd}_R(X), \mathtt{lbd}_R(\mathtt{ubd}_R(Y))) = \mathtt{syq}(\mathtt{ubd}_R(\mathtt{lbd}_R(X)), \mathtt{ubd}_R(Y)).$$

Proof: Applying $\mathtt{syq}(A, B) = \mathtt{syq}(\overline{A}, \overline{B})$ first, this expands to

$$\mathtt{syq}(\overline{R};X, \overline{R};\overline{\overline{R}^\mathsf{T};Y}) = \overline{\overline{X^\mathsf{T};\overline{R}^\mathsf{T}};\overline{R};\overline{\overline{R}^\mathsf{T};Y}} \cap \overline{X^\mathsf{T};\overline{R}^\mathsf{T};\overline{\overline{R};\overline{\overline{R}^\mathsf{T};Y}}}$$

$$\mathtt{syq}(\overline{R}^\mathsf{T};\overline{\overline{R};X}, \overline{R}^\mathsf{T};Y) = \overline{\overline{\overline{X^\mathsf{T};\overline{R}^\mathsf{T}};\overline{R}};\overline{R}^\mathsf{T};Y} \cap \overline{\overline{X^\mathsf{T};\overline{R}^\mathsf{T}};\overline{R};\overline{\overline{R}^\mathsf{T};Y}}$$

Now, the first term in the first equals the second term in the second line. The other terms may be transformed into one another, applying Prop. 2.1. $\square$

With the symmetric quotient we may characterize membership relations ε, demanding $\mathtt{syq}(\varepsilon, \varepsilon) \subseteq \mathbb{I}$ to hold as well as surjectivity $\mathtt{syq}(\varepsilon, R)$ for arbitrary relations R. Using this, the containment ordering on the powerset may be built as $\Omega := \overline{\varepsilon^\mathsf{T};\overline{\varepsilon}} = \varepsilon \backslash \varepsilon$.

3 Rectangles

For an order, e.g., we observe that every element of the set u of elements smaller than some element e is related to every element of the set v of elements greater than e. Also for equivalences and preorders, square zones in the block-diagonal have proven to be important, accompanied by possibly rectangular zones off diagonal.

3.1 Definition. Given $u \subseteq X$ and $v \subseteq Y$, together with compatible universal relations $\mathbb{T}$, we call $u;v^\mathsf{T} = u;\mathbb{T} \cap (v;\mathbb{T})^\mathsf{T}$ a **rectangular relation** or, simply, a **rectangle**[3]. We say that u, v define a **rectangle inside** R if $u;v^\mathsf{T} \subseteq R$ (or equivalently $\overline{R};v \subseteq \overline{u}$, or $\overline{R}^\mathsf{T};u \subseteq \overline{v}$). $\square$

The definitional variants obviously mean the same. Sometimes we speak correspondingly of a **rectangle containing** R if $R \subseteq u;v^\mathsf{T}$, or we say that u, v is a **rectangle outside** R if u, v is a rectangle *inside* $\overline{R}$. Note that yet another definition of a rectangle u, v inside R may be given by $u \subseteq R/v^\mathsf{T}$ and $v^\mathsf{T} \subseteq u\backslash R$.

Although not many scientists seem to be aware of this fact, a significant amount of our reasoning is concerned with "rectangles" in/of a relation. A lower bound cone of an arbitrary relation R together with its upper bound cone form a rectangle inside R. Rectangles are handled at various places from the theoretical point of view as well as from the practical side. Among the application areas are concept lattices, clustering methods, and measuring, to mention just a few seemingly unrelated ones. In most cases, rectangles are treated in the respective application environment, i.e., together with certain additional properties, so that their status as rectangles is not clearly recognized, and consequently the corresponding algebraic properties are not applied or not fully exposed.

[3] There are variant notations. In the context of bipartitioned graphs, a rectangle inside a relation is called a *block*; see, e.g. [3]. [4] speaks of *cross vectors*.

We now consider rectangles inside a relation that cannot be enlarged.

3.2 Definition. The rectangle u, v inside R is said to be **maximal**[4] if for any rectangle u', v' inside R with $u \subseteq u'$ and $v \subseteq v'$, it follows that $u = u'$ and $v = v'$. $\square$

The property of being maximal has an elegant algebraic characterisation.

3.3 Proposition. Let u, v define a rectangle inside the relation[5] R. Precisely when both, $\overline{R};v \supseteq \overline{u}$ *and* $\overline{R}^{\mathsf{T}};u \supseteq \overline{v}$, are also satisfied, there will not exist a strictly greater rectangle u', v' inside R.

Proof: Let us assume a rectangle that does not satisfy, e.g., the first inclusion: $\overline{u} \not\supseteq \overline{R};v$, so that there will exist a point $p \subseteq \overline{u} \cap \overline{\overline{R};v}$. Then $u' := u \cup p \neq u$ and $v' := v$ is a strictly greater rectangle because $p;v^{\mathsf{T}} \subseteq R$.

Consider for the opposite direction a rectangle u, v inside R satisfying the two inclusions together with a rectangle u', v' inside R such that $u \subseteq u'$ and $v \subseteq v'$. Then we may conclude with monotony and an application of the Schröder rule that $\overline{v'} \supseteq \overline{R}^{\mathsf{T}};u' \supseteq \overline{R}^{\mathsf{T}};u \supseteq \overline{v}$. This results in $v' = v$. In a similar way it is shown that $u = u'$. To sum up, u', v' can not be strictly greater than u, v. $\square$

In other words, u, v constitute a maximal rectangle inside R if and only if both, $\overline{R};v = \overline{u}$ and $\overline{R}^{\mathsf{T}};u = \overline{v}$, are satisfied. A reformulation of these conditions using residuals is $u = R/v^{\mathsf{T}}$ and $v^{\mathsf{T}} = u\backslash R$. Consider a pair of elements (x, y) related by some relation R, i.e., $x;y^{\mathsf{T}} \subseteq R$ or, equivalently, $x \subseteq R;y$. The relation $R^{\mathsf{T}};x$ is the set of all elements of the codomain side related with x. Since we started with $(x, y) \in R$, it is nonempty, i.e., $\perp\!\!\!\perp \neq y \subseteq R^{\mathsf{T}};x$.

For reasons we will accept shortly, it is advisable to use the identity $R^{\mathsf{T}};x = \overline{\overline{R}^{\mathsf{T}};x}$ which holds because negation commutes with multiplying a point from the right side. We then see that a whole rectangle — may be only a one-element relation — is contained in R. Some preference has just been given to x, so that we expect something similar to hold when starting from y.

3.4 Proposition. Every point $x;y^{\mathsf{T}} \subseteq R$ in a relation R gives rise to

i) the maximal rectangle inside R **started horizontally**

$$u_x := \overline{\overline{R};\overline{\overline{R}^{\mathsf{T}};x}} = \overline{\overline{R};R^{\mathsf{T}};x} \supseteq x, \qquad v_x := \overline{\overline{R}^{\mathsf{T}};x} = R^{\mathsf{T}};x \supseteq y$$

ii) the maximal rectangle inside R **started vertically**

$$u_y := \overline{\overline{R};y} = R;y \supseteq x, \qquad v_y := \overline{\overline{R}^{\mathsf{T}};\overline{\overline{R};y}} = \overline{\overline{R}^{\mathsf{T}};R;y} \supseteq y$$

Proof: Indeed, u_x, v_x as well as u_y, v_y are maximal rectangles inside R since they both satisfy Prop. 3.3. $\square$

[4] In case, R is a homogeneous relation, it is also called a *diclique*, preferably with $u \neq \perp\!\!\!\perp$ as well as $v \neq \perp\!\!\!\perp$ to exclude trivialities; [3].

[5] We assume a finite representable relation algebra satisfying the point axiom.

These two may coincide, a case to be handled soon. One will find out that — although R has again not been defined as an ordering — the construct is similar to those defining upper bound sets and lower bound sets of upper bound sets.

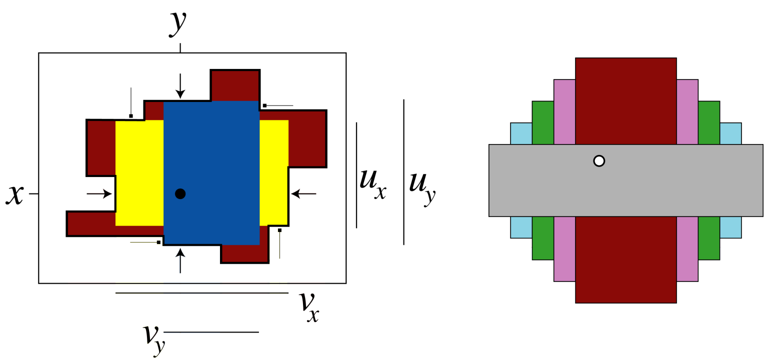

Fig. 1 Points contained in maximal rectangles

In Fig. 1, let the left relation R in question be the "non-white" area, inside which we consider an arbitrary pair (x, y) of elements related by R. To illustrate the pair (u_x, v_x), let the point (x, y) first slide inside R horizontally over the maximum distance v_x, limited as indicated by $\rightarrow \leftarrow$. Then move the full subset v_x as far as possible inside R vertically, obtaining u_x, and thus, the light-shaded rectangle. Symbols like ▟ indicate where the light grey-shaded rectangle cannot be enlarged in vertical direction.

In much the same way, slide the point (x, y) on column y as far as possible inside R, obtaining u_y, limited by $\downarrow$ and $\uparrow$. This vertical interval is then moved horizontally inside R as far as possible resulting in v_y and in the dark-shaded rectangle, confined by ▪–.

Observe, that the maximal rectangles need not be coherent in the general case; nor need there be just two. The example on the right of Fig. 1, where the relation considered is assumed to be precisely the union of all rectangles, shows a point contained in five maximal rectangles. What will also become clear is that with those obtained by looking for the maximum horizontal or vertical extensions first, one gets extreme cases.

As already announced, we now study the circumstances under which a point (x, y) is *contained in exactly one* maximal rectangle.

3.5 Proposition. A pair (x, y) of points related by R is contained in exactly one maximal rectangle inside R precisely when $x;y^{\mathsf{T}} \subseteq R \cap \overline{R;\overline{R}^{\mathsf{T}};R}$.

Proof: If there is just one maximal rectangle for $x;y^{\mathsf{T}} \subseteq R$, the extremal rectangles according to Prop. 3.4.i,ii will coincide. The proof then uses

$$\overline{\overline{R};\overline{\overline{R}^{\mathsf{T}};x}} \supseteq \overline{\overline{R};y} \iff x;y^{\mathsf{T}} \subseteq \overline{R;\overline{R}^{\mathsf{T}};R} \qquad \square$$

Important concepts concerning relations depend heavily on rectangles. For example, a decomposition into a set of maximal rectangles, or even dicliques, provides an efficient way of storing information in a database; see, e.g., [3].

3.6 Proposition. Given any relation R, the following constructs determine the set of all maximal rectangles — including the trivial ones with one side empty and the other side full. Let ε be the membership relation starting from the domain side and ε' the corresponding one from the codomain side. Let Ω, Ω' be the corresponding powerset orderings. The construct

$$\Lambda := \mathtt{syq}(\overline{\varepsilon}, \overline{R} ; \varepsilon') \cap \mathtt{syq}(\overline{R}^{\mathsf{T}} ; \varepsilon, \overline{\varepsilon'}) \quad \text{or, equivalently,}$$
$$\Lambda := \mathtt{syq}(\varepsilon, \mathtt{lbd}_R(\varepsilon')) \cap \mathtt{syq}(\mathtt{ubd}_R(\varepsilon'), \varepsilon')$$

serves to relate $1:1$ the row sets to the column sets of the maximal rectangles.

Proof: Using $\varepsilon, \varepsilon'$, apply the condition Prop. 3.3 for a maximal rectangle simultaneously to all rows, or columns, respectively.

It is easy to convince oneself that Λ is a matching, i.e., satisfies $\Lambda^{\mathsf{T}} ; \Lambda \subseteq \mathbb{I}$ and $\Lambda ; \Lambda^{\mathsf{T}} \subseteq \mathbb{I}$. We show one of the cases using cancellation of the symmetric quotient together with the characterization of the membership relation ε':

$$\Lambda^{\mathsf{T}} ; \Lambda = \big(\mathtt{syq}(\overline{\varepsilon}, \overline{R} ; \varepsilon') \cap \mathtt{syq}(\overline{R}^{\mathsf{T}} ; \varepsilon, \overline{\varepsilon'})\big)^{\mathsf{T}} ; \big(\mathtt{syq}(\overline{\varepsilon}, \overline{R} ; \varepsilon') \cap \mathtt{syq}(\overline{R}^{\mathsf{T}} ; \varepsilon, \overline{\varepsilon'})\big)$$
$$\subseteq \mathtt{syq}(\overline{\varepsilon'}, \overline{R}^{\mathsf{T}} ; \varepsilon) ; \mathtt{syq}(\overline{R}^{\mathsf{T}} ; \varepsilon, \overline{\varepsilon'}) \subseteq \mathtt{syq}(\overline{\varepsilon'}, \overline{\varepsilon'}) = \mathtt{syq}(\varepsilon', \varepsilon') = \mathbb{I} \qquad \square$$

Now we consider those rows/columns that participate in a maximal rectangle and extrude the respective rows/columns with ι to inject the subset described by the vector $\Lambda ; \mathbb{T}$ and ι' to inject the subset described by the vector $\Lambda^{\mathsf{T}} ; \mathbb{T}$. This allows us to define the two versions of the concept lattice based on the powerset orderings

$$\text{left concept lattice} := \iota ; \Omega ; \iota^{\mathsf{T}} \qquad \text{right concept lattice} := \iota' ; \Omega' ; \iota'^{\mathsf{T}}.$$

The two, sometimes referred to as lattice of *extent*, or *intent* resp., are $1:1$ related by the matching $\lambda := \iota ; \Lambda ; \iota'^{\mathsf{T}}$.

4 Fringes

The points contained in just one maximal rectangle inside a relation R play an important rôle, so that we introduce a notation for them.

4.1 Definition. For arbitrary R we define its $\mathtt{fringe}(R) := R \cap \overline{R ; \overline{R}^{\mathsf{T}} ; R}$. $\square$

A first inspection shows that $\mathtt{fringe}(R^{\mathsf{T}}) = [\mathtt{fringe}(R)]^{\mathsf{T}}$. The concept of a fringe has unexpectedly many applications. We announce already here that every fringe will turn out to be difunctional, and thus enjoys a powerful "geometric characterization as a (possibly partial) block-diagonal". As a first example for this, we mention that the fringe of an ordering E is the identity, since

$$\mathtt{fringe}(E) = E \cap \overline{E ; \overline{E}^{\mathsf{T}} ; E} = E \cap \overline{\overline{E}^{\mathsf{T}} ; E} = E \cap \overline{\overline{E}}^{\mathsf{T}} = E \cap E^{\mathsf{T}} = \mathbb{I}.$$

We are accustomed to use the identity $\mathbb{I}$. For heterogeneous relations there is none; often in such cases, the fringe takes over and may be made similar use of.

The fringe of the strict order C is always contained in its Hasse relation $H := C \cap \overline{C^2}$ since C is irreflexive. The existence of a non-empty fringe heavily depends on finiteness or at least discreteness. The following resembles a result of Michael Winter [10]. Let us for a moment call C a **dense** relation if it satisfies $C;C = C$. An example is obviously the relation "<" on the real numbers. This strict order is transitive $C;C \subseteq C$, but satisfies also $C \subseteq C;C$, meaning that whatever element relationship one chooses, e.g., $3.7 < 3.8$, one will find an element in between, $3.7 < 3.75 < 3.8$. To be a dense relation implies that the Hasse relation will be empty. A dense linear strict ordering has an empty fringe. We show in the subsequent sections that the fringe of a relation is central for difunctional, Ferrers, and block-transitive relations.

Now we present a plexus of formulae that are heavily interrelated. The fringe gives rise to "partial equivalences" or symmetric idempotents, closely resembling row and column equivalence $\Xi(R) := \mathtt{syq}(R^\mathsf{T}, R^\mathsf{T}) = \mathtt{syq}(\overline{\overline{R};R^\mathsf{T}}^\mathsf{T}, \overline{\overline{R};R^\mathsf{T}}^\mathsf{T})$ and $\Psi(R) := \mathtt{syq}(R, R) = \mathtt{syq}(\overline{\overline{R}^\mathsf{T};R}^\mathsf{T}, \overline{\overline{R}^\mathsf{T};R}^\mathsf{T})$.

4.2 Definition. For an arbitrary relation R and its fringe $f := \mathtt{fringe}(R) = R \cap \overline{R;\overline{R}^\mathsf{T};R}$, we define

i) $\Xi_F(R) := f;f^\mathsf{T}$, the $\mathtt{fringe}$**-partial row equivalence**

ii) $\Psi_F(R) := f^\mathsf{T};f$, the $\mathtt{fringe}$**-partial column equivalence** $\square$

We recall that the fringe collects those entries of a relation R that are contained in precisely one maximal rectangle. The fringe may also be obtained with the symmetric quotient from the row-contains-preorder and the relation in question:

4.3 Proposition. For an arbitrary relation R, the fringe and the row-contains-preorder $\mathcal{R}(R)$, satisfy

$$\mathtt{fringe}(R) = \mathtt{syq}(\mathcal{R}(R), R)$$

Proof: We expand $\mathtt{fringe}$, $\mathtt{syq}$, $\mathcal{R}(R)$, and apply trivial operations to obtain

$$R \cap \overline{R;\overline{R}^\mathsf{T};R} = \overline{R;\overline{R}^\mathsf{T};R} \cap \overline{\overline{R;\overline{R}^\mathsf{T}};\overline{R}}$$

It remains, thus, to finally apply that $\overline{R} = \overline{R;\overline{R}^\mathsf{T}};\overline{R}$. $\square$

Thus, we are allowed to make use of cancellation formulae from Sect. 2 for the symmetric quotient. We show that to a certain extent the row equivalence $\Xi(R)$ may be substituted by $\Xi_F(R)$; both coincide as long as the fringe is total. They may be different, but only in the way that a square diagonal block of the $\mathtt{fringe}$-partial row equivalence is either equal to the one in $\Xi(R)$, or empty.

4.4 Proposition. For an arbitrary relation R and its fringe $f := \mathtt{fringe}(R)$ the $\mathtt{fringe}$-partial row resp. column equivalences satisfy the following:

i) $\Xi_F(R) = \Xi(R) \cap f;\mathbb{T}$

ii) $\Xi(R);f = \Xi_F(R);f = f;f^\mathsf{T};f = f \qquad f = f;f^\mathsf{T};f = f;\Psi_F(R) = f;\Psi(R)$

iii) $f^\mathsf{T};\Xi(R);f \subseteq \Psi(R)$

iv) $\Xi_F(R);R \subseteq R;f^\mathsf{T};R \subseteq R$ and $R;\Psi_F(R) \subseteq R;f^\mathsf{T};R \subseteq R.$

Proof: i)

$$\begin{aligned}
&\Xi_F(R) = f;f^\mathsf{T} && \text{Def. 4.2}\\
&= \mathtt{syq}(\overline{\overline{R};R^\mathsf{T}}, R);\mathtt{syq}(R, \overline{\overline{R};R^\mathsf{T}}) && \text{Prop. 4.3}\\
&= \mathtt{syq}(\overline{\overline{R};R^\mathsf{T}}, \overline{\overline{R};R^\mathsf{T}}) \cap \mathtt{syq}(\overline{\overline{R};R^\mathsf{T}}, R);\mathbb{T} && \text{cancellation property}\\
&= \Xi(\overline{R}) \cap f;\mathbb{T} = \Xi(R) \cap f;\mathbb{T} && \text{definition of } \Xi(R)
\end{aligned}$$

ii) The definition of $\Xi(R)$ together with Prop. 4.3 show that

$$\Xi(R);f = \Xi(\overline{R});f = \mathtt{syq}(\overline{\overline{R};R^\mathsf{T}}, \overline{\overline{R};R^\mathsf{T}});\mathtt{syq}(\overline{\overline{R};R^\mathsf{T}}, R) \subseteq \mathtt{syq}(\overline{\overline{R};R^\mathsf{T}}, R) = f,$$

applying cancellation again. Then we may proceed with $f;f^\mathsf{T};f = \Xi_F(R);f \subseteq$

$$\begin{aligned}
&\subseteq \Xi(R);f && \text{according to (i)}\\
&\subseteq f && \text{see above}\\
&\subseteq f;f^\mathsf{T};f && A \subseteq A;A^\mathsf{T};A \text{ for every } A
\end{aligned}$$

obtaining equality everywhere in between.

iii)

$$\begin{aligned}
&f^\mathsf{T};\Xi(R);f \subseteq f^\mathsf{T};f && \text{see above}\\
&= \Psi_F(R) && \text{by Def. 4.2}\\
&\subseteq \Psi(R) && \text{applying (i) to } R^\mathsf{T}.
\end{aligned}$$

iv)

$$\begin{aligned}
&R;f^\mathsf{T};R &&\\
&= R;[\mathtt{syq}(\overline{\overline{R};R^\mathsf{T}}, R)]^\mathsf{T};R && \text{Prop. 4.3}\\
&= R;\mathtt{syq}(R, \overline{\overline{R};R^\mathsf{T}});R && \text{transposing a symmetric quotient}\\
&\subseteq \overline{\overline{R};R^\mathsf{T}};R && \text{cancelling the symmetric quotient}\\
&\subseteq R && \text{which holds for every relation}
\end{aligned}$$

The rest is then simple because $\Xi_F(R) = f;f^\mathsf{T} \subseteq R;f^\mathsf{T}$. □

Anticipating Def. 5.1, we may say that f^T is always a subinverse of R. We have already seen in (i) that $\Xi_F(R)$ is nearly an equivalence. When in (iv) equality holds, $\Xi_F(R);R = R$, we may expect important consequences, since then something as a congruence is established.

The following proposition relates the fringe of the row-contains-preorder with the row equivalence.

4.5 Proposition. We have for every relation R, that

$$\mathtt{fringe}(\mathcal{R}(R)) = \mathtt{fringe}(\overline{\overline{R};R^\mathsf{T}}) = \mathtt{syq}(R^\mathsf{T}, R^\mathsf{T}) = \Xi(R),$$

$$\mathtt{fringe}(\mathcal{C}(R)) = \mathtt{fringe}(\overline{\overline{R}^\mathsf{T};R}) = \mathtt{syq}(R, R) = \Psi(R).$$

Proof: In both cases, only the equality in the middle is important because the rest is just expansion of definitions. Thus reduced, the first identity, e.g., requires to prove that

$$\overline{\overline{R};R^\mathsf{T}} \cap \overline{\overline{\overline{R};R^\mathsf{T}};R;\overline{R}^\mathsf{T};\overline{\overline{R};R^\mathsf{T}}} = \overline{\overline{R};R^\mathsf{T}} \cap \overline{R;\overline{R}^\mathsf{T}}.$$

The first term on the left equals the first on the right. In addition, the second terms are equal, which is not seen so easily, but also trivial. $\square$

The fringe may indeed be important because it is intimately related with difunctionality: For arbitrary R, the construct $\mathtt{fringe}(R)$ is difunctional and a relation R is difunctional precisely when $R = \mathtt{fringe}(R)$. Also: Forming the fringe turns out to be an idempotent operation, i.e., $\mathtt{fringe}(\mathtt{fringe}(R)) = \mathtt{fringe}(R)$.

5 Inverses

Fringes and difunctionality are related to the following concepts of inverses. Inverses are defined for real-valued matrices in linear algebra or for numerical problems. We introduce here similar definitions for relations using the same names. They will provide deeper insight into the structure of a difunctional relation.

5.1 Definition. Let some relation A be given. The relation G is called

i) a **sub-inverse** of A if
  $A;G;A \subseteq A$.
ii) a **generalized inverse** of A if
  $A;G;A = A$.
iii) a **Thierrin-Vagner inverse** of A if the following two conditions hold
  $A;G;A = A, \qquad G;A;G = G.$
iv) a **Moore-Penrose inverse** of A if the following four conditions hold
  $A;G;A = A, \qquad G;A;G = G, \qquad (A;G)^{\mathsf{T}} = A;G, \qquad (G;A)^{\mathsf{T}} = G;A.$

The relation R is called **regular**, if it has a generalized inverse. Due to the symmetric situation in case of a Thierrin-Vagner inverse G of A, the two relations A, G are also simply called **inverses** of each other. $\square$

In a number of situations semigroup theory is applicable to relations. Some of these ideas stem from [4] and are here reconsidered from the relational side. A sub-inverse will always exist since $\bot\!\!\!\bot$ satisfies the requirement. With two sub-inverses G, G' also their union $G \cup G'$ is obviously a sub-inverse so that one will ask which is the greatest.

5.2 Proposition. $\overline{R;\overline{R}^{\mathsf{T}};R}^{\mathsf{T}}$ is the greatest subinverse of R.

Proof: Assuming an arbitrary sub-inverse X of R, it satisfies by definition $R;X;R \subseteq R$, which is equivalent with

$$\iff \quad X^{\mathsf{T}};R^{\mathsf{T}};\overline{R} \subseteq \overline{R} \quad \iff \quad R;\overline{R}^{\mathsf{T}};R \subseteq \overline{X}^{\mathsf{T}} \quad \iff \quad X \subseteq \overline{R;\overline{R}^{\mathsf{T}};R}^{\mathsf{T}} \qquad \square$$

A generalized inverse is not uniquely determined: As an example assume a homogeneous $\mathbb{T}$. It has at least the generalized inverses $\mathbb{I}$ and $\mathbb{T}$. With generalized inverses G_1, G_2 also $G_1 \cup G_2$ is a generalized inverse. There will, thus, exist a greatest one — if any. Regular relations, i.e., those with existing generalized inverse, may precisely be characterized by the following containment which is in fact an equation:

5.3 Proposition. R regular $\iff$ $R \subseteq R;\overline{R;\overline{R}^\top;R}^\top;R$.

Proof: If R is regular, there exists an X with $R;X;R = R$. It is, therefore, a sub-inverse and so $X \subseteq \overline{R;\overline{R}^\top;R}^\top$ according to Prop. 5.2. Then

$$R = R;X;R \subseteq R;\overline{R;\overline{R}^\top;R}^\top;R.$$

Specializing $X := \overline{R;\overline{R}^\top;R}^\top$ in the proof of Prop. 5.2, we have already seen that $R;\overline{R;\overline{R}^\top;R}^\top;R \subseteq R$ for arbitrary R. □

We will learn in Def. 7.1, that every block-transitive relation is regular in this sense; see Prop. 7.3.

5.4 Proposition. If R is a regular relation, its maximum Thierrin-Vagner inverse is $\overline{R;\overline{R}^\top;R}^\top;R;\overline{R;\overline{R}^\top;R}^\top =: TV$.

Proof: Evaluation of $TV;R;TV = TV$ and $R;TV;R = R$ using Prop. 5.3 with equality shows that TV is indeed a Thierrin-Vagner inverse. Any Thierrin-Vagner inverse G is in particular a sub-inverse, so that $G \subseteq \overline{R;\overline{R}^\top;R}^\top$ which implies $G = G;R;G \subseteq TV$. □

A well-known result on Moore-Penrose inverses shall be recalled:

5.5 Theorem. Moore-Penrose inverses are uniquely determined if they exist.

Proof: Assume two Moore-Penrose inverses G, H of A to be given. Then we may proceed as follows: $G = G;A;G = G;G^\top;A^\top = G;G^\top;A^\top;H^\top;A^\top = G;G^\top;A^\top;A;H = G;A;G;A;H = G;A;H = G;A;H;A;H = G;A;A^\top;H^\top;H = A^\top;G^\top;A^\top;H^\top;H = A^\top;H^\top;H = H;A;H = H$. □

These concepts will now be related with permutations and difunctionality.

5.6 Theorem. For a relation A, the following are equivalent:

i) A has a Moore-Penrose inverse.
ii) A has $A^\top$ as its Moore-Penrose inverse.
iii) A is difunctional.
iv) Any two rows (or columns) of A are either disjoint or identical.
v) There exist permutations P, Q such that $P;A;Q$ has block-diagonal form with (not necessarily square) diagonal entries $B_i = \mathbb{T}$.

Proof: of the key step (i)$\Longrightarrow$(ii): $G = G;A;G \subseteq G;A;A^\top;A;G = A^\top;G^\top;A^\top;A;G = (A;G;A)^\top;A;G = A^\top;A;G = A^\top;G^\top;A^\top = (A;G;A)^\top = A^\top$ and, deduced symmetrically, $A \subseteq G^\top$. □

6 Ferrers Relations

We have seen that a difunctional relation corresponds to a partial block diagonal relation. So the question arose as to whether there was a counterpart of a linear order with rectangular block-shaped matrices. In this context, the Ferrers property of a relation is studied.

6.1 Definition. We say that a relation A is **Ferrers** if $A;\overline{A}^{\mathsf{T}};A \subseteq A$. □

The meaning of the algebraic condition has often been visualized and interpreted. It is at first sight not at all clear that the matrix representing A may — due to Ferrers property — be written in staircase (or echelon) block form after suitably rearranging rows and columns independently.

If R is Ferrers, then so are R^{T}, $\overline{R}$, $\overline{R}^{\mathsf{T}};R$, $\overline{R};R^{\mathsf{T}}$, and $R;\overline{R}^{\mathsf{T}};R$. A relation R is Ferrers precisely when $\mathcal{R}(R)$ is connex or when $\mathcal{C}(R)$ is connex[6]:

$$R;\overline{R}^{\mathsf{T}};R \subseteq R \quad\Longleftrightarrow\quad \overline{R};R^{\mathsf{T}};\overline{R} \subseteq \overline{R} \quad\Longleftrightarrow\quad R;\overline{R}^{\mathsf{T}} \subseteq \overline{\overline{R};R^{\mathsf{T}}}$$

We now prove several properties of a Ferrers relation that make it attractive for purposes of modeling preferences etc. An important contribution to this comes from a detailed study of the behaviour of the fringe[7].

6.2 Proposition. For a finite Ferrers relation R, the following statements hold, in which we abbreviate $f := \mathtt{fringe}(R)$:

i) The construct $R;\overline{R}^{\mathsf{T}}$ is a progressively bounded semi-connex strict order.

ii) There exists a natural number $k \geq 0$ that gives rise to a strictly increasing exhaustion as
$\perp\!\!\!\perp = (R;\overline{R}^{\mathsf{T}})^k \subsetneqq (R;\overline{R}^{\mathsf{T}})^{k-1} \subsetneqq \ \dots\ \subsetneqq R;\overline{R}^{\mathsf{T}};R;\overline{R}^{\mathsf{T}} \subsetneqq R;\overline{R}^{\mathsf{T}}$

iii) $R;\overline{R}^{\mathsf{T}} = f;\overline{R}^{\mathsf{T}}, \qquad \overline{R}^{\mathsf{T}};R = \overline{R}^{\mathsf{T}};f, \qquad R;\overline{R}^{\mathsf{T}};R = f;\overline{R}^{\mathsf{T}};f$

iv) R allows a disjoint decomposition as
$R = \mathtt{fringe}(R) \cup \mathtt{fringe}(R;\overline{R}^{\mathsf{T}};R) \cup \dots \cup \mathtt{fringe}((R;\overline{R}^{\mathsf{T}})^k;R)$ for some $k \geq 0$

v) R allows a disjoint decomposition as
$R = \mathtt{fringe}(R) \cup \mathtt{fringe}(f;\overline{R}^{\mathsf{T}};f) \cup \dots \cup \mathtt{fringe}((f;\overline{R}^{\mathsf{T}})^k;f)$ for some $k \geq 0$

vi) R allows a disjoint decomposition as $R = f \cup f;\overline{R}^{\mathsf{T}};f$

vii) R allows an exhaustion as
$\perp\!\!\!\perp = (f;\overline{R}^{\mathsf{T}})^k;f \subsetneqq (f;\overline{R}^{\mathsf{T}})^{k-1};f \ \subsetneqq\ \dots\ \subsetneqq f;\overline{R}^{\mathsf{T}};f;\overline{R}^{\mathsf{T}};f \subsetneqq f;\overline{R}^{\mathsf{T}};f \subsetneqq R$

Proof: i) and ii) We start the following chain of inclusions from the right applying recursively that R is Ferrers:

$$\perp\!\!\!\perp = (R;\overline{R}^{\mathsf{T}})^k \subseteq (R;\overline{R}^{\mathsf{T}})^{k-1} \subseteq \dots \subseteq R;\overline{R}^{\mathsf{T}};R;\overline{R}^{\mathsf{T}} \subseteq R;\overline{R}^{\mathsf{T}}$$

---

[6] A relation A is connex if $\top\!\!\!\top = A \cup A^{\mathsf{T}}$; it is semi-connex if $\overline{\mathbb{I}} \subseteq A \cup A^{\mathsf{T}}$.

[7] By the way: [6] and a whole chapter of [7] are devoted to the "holes" or "hollows" and "noses" that show up in this context; see Fig. 2.

Finiteness guarantees that it will eventually be stationary, i.e., $(R;\overline{R}^{\mathsf{T}})^{k+1} = (R;\overline{R}^{\mathsf{T}})^k$. This means in particular that the condition $Y \subseteq (R;\overline{R}^{\mathsf{T}});Y$ holds for $Y := (R;\overline{R}^{\mathsf{T}})^k$. The construct $R;\overline{R}^{\mathsf{T}}$ is obviously transitive and irreflexive, so that it is in combination with finiteness also progressively finite. According to Sect. 6.3 of [8, 9], this means that $Y = (R;\overline{R}^{\mathsf{T}})^k = \mathbb{⊥}$.

iii) $R;\overline{R}^{\mathsf{T}} = ((R \cap \overline{R;\overline{R}^{\mathsf{T}};R}) \cup (R \cap R;\overline{R}^{\mathsf{T}};R));\overline{R}^{\mathsf{T}}$

$= (f \cup R;\overline{R}^{\mathsf{T}};R);\overline{R}^{\mathsf{T}}$	since R is Ferrers
$= f;\overline{R}^{\mathsf{T}} \cup R;\overline{R}^{\mathsf{T}};R;\overline{R}^{\mathsf{T}}$	
$= f;\overline{R}^{\mathsf{T}} \cup (f;\overline{R}^{\mathsf{T}} \cup R;\overline{R}^{\mathsf{T}};R;\overline{R}^{\mathsf{T}});R;\overline{R}^{\mathsf{T}}$	applied recursively
$= f;\overline{R}^{\mathsf{T}} \cup f;\overline{R}^{\mathsf{T}};R;\overline{R}^{\mathsf{T}} \cup R;\overline{R}^{\mathsf{T}};R;\overline{R}^{\mathsf{T}};R;\overline{R}^{\mathsf{T}}$	
$= f;\overline{R}^{\mathsf{T}} \cup R;\overline{R}^{\mathsf{T}};R;\overline{R}^{\mathsf{T}};R;\overline{R}^{\mathsf{T}}$	since also $\overline{R}^{\mathsf{T}}$ is Ferrers
$= \ldots = f;\overline{R}^{\mathsf{T}} \cup \mathbb{⊥}$	see (ii)

The other proofs are left to the reader. □

It is mainly this effect which enables us to arrive at the results that follow. First, we observe how successively discarding fringes leaves a decreasing sequence of relations; strictly decreasing when finite or at least not dense. Ferrers relations may, although possibly heterogeneous, in many respects be considered as similar to a linear (strict)ordering.

The following proposition is a classic (with a very slight generalization concerning surjectivity not being demanded); it may not least be found in [2] and also with a completely different point-free proof in [9]. The idea of the proof presented here is a constructive one, which means that one may write the constructs down in the language TituRel and immediately run this as a program. The reason is that the constructs are generic ones that are uniquely characterized, so that a standard realization for interpretation is possible.

6.3 Proposition. Let $R : X \longrightarrow Y$ be a finite relation.

R Ferrers $\iff$ There exist mappings f, g and a linear strict order C such that $R = f;C;g^{\mathsf{T}}$.

Proof: "$\Longleftarrow$" follows relatively easily using several times that mappings may slip below a negation from the left without affecting the result, and that C is Ferrers.

"$\Longrightarrow$" Let R be Ferrers. There may exist empty rows or columns in R or not. To care for this in a general form, we enlarge the domain to $X + \mathbb{1}$ and the codomain to $\mathbb{1} + Y$ and consider the relation $R' := \iota_X^{\mathsf{T}};R;\kappa_Y$. In R', there will definitely exist at least one empty row and at least one empty column. It is intuitively clear — and easy to demonstrate — that also R' is Ferrers.

The relation R' has been constructed so that $\overline{R'}$ is both, total and surjective. Observe, that $\overline{R}$ in the upper right sub-rectangle of Fig. 2 would *not* have been surjective. As in general $R = f \cup f;\overline{R}^{\mathsf{T}};f$ according to Prop. 6.2.vi, also $\texttt{fringe}(\overline{R'})$ is necessarily total and surjective. As fringes are always difunctional, $\texttt{fringe}(\overline{R'})$

is a block diagonal, which will — after quotient forming — provide us with the matching λ.

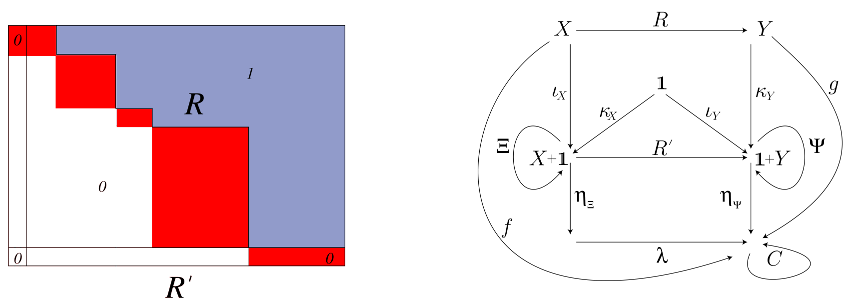

Fig. 2 Constructing a Ferrers decomposition

We introduce row equivalence $\Xi(R') := \mathtt{syq}(R'^{\mathsf{T}}, R'^{\mathsf{T}})$ as well as column equivalence $\Psi(R') := \mathtt{syq}(R', R')$ of R' together with the corresponding natural projections which we call η_Ξ, η_Ψ. We define

$$\begin{aligned}
\lambda &:= \eta_\Xi^{\mathsf{T}} ; \mathtt{fringe}(\overline{R'}) ; \eta_\Psi \\
f &:= \iota_X ; \eta_\Xi ; \lambda \\
g &:= \kappa_Y ; \eta_\Psi \\
C &:= \lambda^{\mathsf{T}} ; \eta_\Xi^{\mathsf{T}} ; R' ; \eta_\Psi
\end{aligned}$$

Now a proof is achievable requiring no case distinctions which are impossible prior to having interpreted the relation in question “with a matrix”. □

7 Block-Transitive Relations

Concepts that we already know for an order or a strict order shall now be studied generalized to a heterogeneous environment in which also multiple rows or columns may occur. The starting point is a Ferrers relation. We have seen how it can in many respects be compared with a linear (strict)order. Is it possible to obtain in such a generalized case similar results for a not necessarily linear strict order? Proceeding strictly algebraically, this will indeed be found.

7.1 Definition. A relation R is called **block-transitive** if either one of the following equivalent conditions holds, expressed via its fringe $f := \mathtt{fringe}(R)$

i) $R \subseteq f ; \mathbb{T}$ and $R \subseteq \mathbb{T} ; f$,

ii) $R \subseteq f ; \mathbb{T} ; f$,

iii) $R = \Xi_F ; R ; \Psi_F$. □

The proof of the equivalence of the variants is left to the reader. Being block-transitive is mainly a question of how big the fringe is. The fringe must be big enough so as to “span” the given relation R with its rectangular closure.

For this concept, Michael Winter had originally, see [10], coined the property to be of *order-shape*. We do not use this word here because it may cause misunderstanding: We had always been careful to distinguish an order from a strict order; they have different definitions, that both overlap in being transitive. In what follows, we will see that — in a less consistent way — definitions may share the property of being block-transitive.

The following shows the most specialized examples of a block-transitive relation:

7.2 Proposition. A difunctional relation R as well as a finite Ferrers relation R are necessarily block-transitive.

Proof: The first result is trivial since $R; R^{\mathsf{T}}; R \subseteq R \Longleftrightarrow R; \overline{R}^{\mathsf{T}}; R \subseteq \overline{R}$, so that $R = \mathtt{fringe}(R)$. For the second, we abbreviate $f := \mathtt{fringe}(R)$. According to Prop. 6.2.iv, we have $R = f \cup f; \overline{R}^{\mathsf{T}}; f$, so that $R \subseteq f; \mathbb{T}$ as well as $R \subseteq \mathbb{T}; f$. □

This is in contrast to $\mathrm{I\!R}, <$ which is Ferrers but not block-transitive, simply since its fringe has already been shown to be empty.

7.3 Proposition. For an arbitrary block-transitive relation R we again abbreviate $f := \mathtt{fringe}(R)$ and prove:

i) $R; f^{\mathsf{T}}; R = R$, i.e., f^{T} is a generalized inverse of R

ii) $R; f^{\mathsf{T}}$ and $f^{\mathsf{T}}; R$ are transitive

Proof: i) From $R \subseteq f; \mathbb{T}$, we deduce with row equivalence Ξ and Prop. 4.4.i

$$R = R \cap f; \mathbb{T} = \Xi; R \cap f; \mathbb{T} = (\Xi \cap f; \mathbb{T}); R = \Xi_F; R = f; f^{\mathsf{T}}; R \subseteq R; f^{\mathsf{T}}; R$$

The reverse direction is satisfied for every relation according to Prop. 4.4.iv.

ii) $R; f^{\mathsf{T}}; R; f^{\mathsf{T}} = R; f^{\mathsf{T}}$ using (i) □

We now introduce block-transitive kernels and Ferrers closures.

7.4 Definition. For a relation R, we define using its fringe f the **block-transitive kernel** as

$$\mathtt{btk}(R) := R \cap f; \mathbb{T} \cap \mathbb{T}; f = f; f^{\mathsf{T}}; R; f^{\mathsf{T}}; f.$$ □

7.5 Proposition. For every relation R, the fringe does not change when reducing R to its block-transitive kernel; i.e.,

$$f = \mathtt{fringe}(R \cap f; \mathbb{T} \cap \mathbb{T}; f) \quad \text{for } f := \mathtt{fringe}(R)$$ □

The proof of this statement is too lengthy and ugly to be presented. It employs hardly more than Boolean algebra, but with terms running in opposite directions, so that it is probably not easy for the reader to find it for himself.

7.6 Proposition. Every finite block-transitive relation R has a **Ferrers closure**, i.e., a Ferrers relation $F \supseteq R$ but still satisfying $\mathtt{fringe}(F) = \mathtt{fringe}(R)$.

Proof: The idea for this proof is rather immediate; its execution, though, is technically complicated: Do the quotient forming according to the fringe-partial equivalences throwing rows and columns with empty row/column of f together in one class. Divide these congruences out and apply afterwards what is called the Szpilrajn-extension or topological sorting. □

For block-transitive relations, also a factorization result similar to Prop. 6.3 may be proved, which cannot be presented for reasons of space.

8 Concluding Remark

We have tried to base known and new concepts on maximal rectangles inside a relation. The elegant relational characterization of these together with the intuitive interpretation of a fringe facilitated access to semigroup concepts, e.g., and also allowed to generalize some. Block-transitive relations constitute a novel concept that may turn out to be the method of choice in preference modeling. They are more general than semiorders or interval orders, but still allow an algebraic treatment.

References

1. Doignon, J.-P., and Falmagne, J.-C. Matching Relations and the Dimensional Structure of Social Sciences. *Math. Soc. Sciences 7* (1984), 211–229.
2. Ducamp, A., and Falmagne, J.-C. Composite Measurement. *J. Math. Psychology 6* (1969), 359–390.
3. Haralick, R. M. The diclique representation and decomposition of binary relations. *J. ACM 21* (1974), 356–366.
4. Kim, K. H. *Boolean Matrix Theory and Applications*, vol. 70 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, New York – Basel, 1982.
5. Monjardet, B. Axiomatiques et proprietés des quasi-ordres. *Mathematiques et Sciences Humaines 16*, 63 (1978), 51–82.
6. Pirlot, M. Synthetic description of a semiorder. *Discrete Appl. Mathematics 31* (1991), 299–308.
7. Pirlot, M., and Vincke, P. *Semiorders — Properties, Representations, Applications*, vol. 36 of *Theory and Decision Library, Series B: Mathematical and Statistical Methods*. Kluwer Academic Publishers, 1997.
8. Schmidt, G., and Ströhlein, T. *Relationen und Graphen*. Mathematik für Informatiker. Springer-Verlag, 1989. ISBN 3-540-50304-8, ISBN 0-387-50304-8.
9. Schmidt, G., and Ströhlein, T. *Relations and Graphs — Discrete Mathematics for Computer Scientists*. EATCS Monographs on Theoretical Computer Science. Springer-Verlag, 1993. ISBN 3-540-56254-0, ISBN 0-387-56254-0.
10. Winter, M. Decomposing Relations Into Orderings. In *Proc. of the Internat. Workshop RelMiCS '7 and 2nd Internat. Workshop on Applications of Kleene Algebra, in combination with a workshop of the COST Action 274: TARSKI. Revised Selected Papers; ISBN 3-540-22145-X, 279 pages* (2004), R. Berghammer, B. Möller, and G. Struth, Eds., no. 3051 in Lect. Notes in Comput. Sci., Springer-Verlag, pp. 261–272.