A necessary relation algebra for mereotopology

Ivo Düntsch
School of Information and Software Engineering
University of Ulster
Newtownabbey, BT 37 0QB, N.Ireland
I.Duentsch@ulst.ac.uk

Gunther Schmidt, Michael Winter
Department of Computer Science
University of the Federal Armed Forces Munich
85577 Neubiberg, Germany
{schmidt|thrash}@informatik.unibw-muenchen.de

Abstract

We show that the basic operations of the relational calculus on a “contact relation” generate at least 25 relations in any model of the Region Connection Calculus [33], and we show how to interpret these relations in the collection of regular open sets in the two-dimensional Euclidean plane.

1 Introduction

Mereotopology is an area of qualitative spatial reasoning (QSR) which aims to develop formalisms for reasoning about spatial entities [1, 12, 30, 31]. The structures used in mereotopology consist of three parts:

1. A relational (or mereological) part,
2. An algebraic part,
3. A topological part.

The algebraic part is often an atomless Boolean algebra, or, more generally, an orthocomplemented lattice, both without smallest element.

Due to the presence of the binary relations “part-of” and “contact” in the relational part of mereotopology, composition based reasoning with binary relations has been of interest to the QSR community, and the expressive power, consistency and complexity of relational reasoning has become an object of study [2–4, 33]. The first time that the relational calculus has been mentioned in (modern) spatial reasoning was in the interpretation of the 4-intersection matrix in [19], see also [37].

It has been known for some time, that the expressiveness of reasoning with basic operations on binary relations is equal to the expressive power of the three variable fragment of first order logic [see 41, and the references therein]. Thus, it seems worthwhile to use methods of relation algebras, initiated by Tarski [40], to study contact relations in their own right, and then explore their expressive power with respect to topological domains.

The Region Connection Calculus (RCC) was introduced as a formal structure to reason about spatial entities and the relationships among them [33]. Its models are basically atomless Boolean algebras with an additional contact relation which satisfies certain axioms. A standard model of the RCC is the Boolean algebra of regular open sets of a regular connected topological space, where two such sets are in contact, if their boundaries intersect. However, these are not the only RCC models.

Gotts [22] explores how much topology can be defined by using the full first order RCC formalism. Our aim is similar: We are interested which relations can be defined with relation algebra logic (i.e. the three variable fragment of first order logic) in the algebraic setting of the RCC, interpreted in a topological context.

2 Relation algebras

The calculus of relations has been an important component of the development of logic and algebra since the middle of the nineteenth century. Since the mid-1970's it has become clear that the calculus of relations is also a fundamental conceptual and methodological tool in computer science. Some examples are program semantics [7, 34, 42], program specification [5] and derivation [6], and last but not least qualitative spatial reasoning. For a detailed overview we invite the reader to consult [9].

Let U be a nonempty set. We denote the set of all binary relations on U, i.e., the powerset of $U \times U$, by $Rel(U)$. We usually indicate the fact $\langle x, y\rangle \in R$ for $R \in Rel(U)$ by xRy. Furthermore, we define for $R, S \in Rel(U)$

(2.1) $$R \circ S \stackrel{\text{def}}{=} \{\langle x, y\rangle : (\exists z \in U) xRzSy\},$$ Composition

(2.2) $$R^{\smile} \stackrel{\text{def}}{=} \{\langle x, y\rangle : yRx\},$$ Converse

(2.3) $$xR \stackrel{\text{def}}{=} \{y : xRy\},$$ Image of x under R

(2.4) $$Ry \stackrel{\text{def}}{=} \{x : xRy\}.$$ Inverse image of y under R

We also let $1'$ be the identity relation on U, and $V = U \times U$ be the universal relation. The *full algebra of binary relations on* U is the algebra of type $\langle 2, 2, 1, 0, 0, 2, 1, 0\rangle$

$$\langle Rel(U), \cap, \cup, -, \emptyset, V, \circ, ^{\smile}, 1'\rangle.$$

We shall usually identify algebras with their base set. Every subalgebra of $Rel(U)$ is called an *algebra of binary relations* (BRA). If $\{R_i : i \in I\} \subseteq Rel(U)$, we let $\langle\{R_i : i \in I\}\rangle$ be the BRA generated by $\{R_i : i \in I\}$.

If an RA A is complete and atomic – in particular, if A is finite –, then each nonzero element is a sum of atoms, and relational composition can be described by a matrix, whose rows and columns are labelled by the atoms and an entry $\langle P, Q\rangle$ is the set of atoms contained in $P \circ Q$. If A is integral, we omit column and row $1'$.

If $\mathcal{R} = \{R_i : i \in I\}$ is a partition of V such that $\mathcal{R}$ is closed under converse, and either $R_i \subseteq 1'$ or $R_i \cap 1' = \emptyset$ for all $i \in I$, we define the *weak composition* of $\mathcal{R}$ as the mapping $\circ_w : \mathcal{R} \times \mathcal{R} \to 2^{\mathcal{R}}$ such that for all $i, j \in I$

$$S \in R_i \circ_w R_j \overset{\text{def}}{\iff} S \cap (R_i \circ R_j) \neq \emptyset. \tag{2.5}$$

Just as in the case of $\circ$, we can determine composition tables for $\circ_w$. Note that $R_i \circ R_j \subseteq R_i \circ_w R_j$; if equality holds everywhere, i.e. when $\circ = \circ_w$, we call the weak composition table *extensional*.

An abstract relation algebra (RA) is a structure

$$\langle A, +, \cdot, -, 0, 1, \circ, \breve{}, 1'\rangle$$

of type $\langle 2, 2, 1, 0, 0, 2, 1, 0\rangle$ which satisfies for all $a, b, c \in A$,

1. $\langle A, +, \cdot, -, 0, 1\rangle$ is a Boolean algebra (BA). Its induced ordering is denoted by $\leq$.
2. $\langle A, \circ, \breve{}, 1'\rangle$ is an involuted monoid, i.e.
   (a) $\langle A, \circ, 1'\rangle$ is a semigroup with identity $1'$,
   (b) $a^{\breve{}\breve{}} = a,\ (a \circ b)\breve{} = b\breve{} \circ a\breve{}$.
3. The following conditions are equivalent:

$$(a \circ b) \cdot c = 0,\ (a\breve{} \circ c) \cdot b = 0,\ (c \circ b\breve{}) \cdot a = 0. \tag{2.6}$$

The properties (2.6) are sometimes called the *complement-free Schröder-equivalences*. They are equivalent to the Schröder-equivalences introduced by Schröder in [36]

$$a \circ b \leq c \iff a\breve{} \circ -b \leq -c \iff -c \circ b\breve{} \leq -a. \tag{2.7}$$

Each BRA is an RA with the obvious operations, but not vice versa [28].

The logic of RAs is a fragment of first order logic, and the following fundamental result is due to A. Tarski [41]:

Theorem 2.1. *If $\{R_i : i \in I\} \subseteq Rel(U)$, then $\langle\{R_i : i \in I\}\rangle$ is the set of all binary relations on U which are definable in the (language of the) relational structure $\langle U, \{R_i : i \in I\}\rangle$ by first order formulas using at most three variables.*

If a, b are elements of a RA A, we define the *right residual of a and b* by

$$a \diagdown b \stackrel{\text{def}}{=} -(a\breve{} \circ -b). \tag{2.8}$$

$a \diagdown b$ is the largest relation c such that $a \circ c \leq b$.

Analogously, we define the *left residual $a \diagup b$ of a and b* as

$$a \diagup b \stackrel{\text{def}}{=} -(-b \circ a\breve{}). \tag{2.9}$$

$a \diagup b$ is the largest relation d such that $d \circ a \leq b$. The *symmetric quotient* $\mathrm{syq}(a, b)$ *of a and b* is defined by

$$\mathrm{syq}(a, b) \stackrel{\text{def}}{=} (a \diagdown b) \cdot (b\breve{} \diagup a\breve{}) = -(a\breve{} \circ -b) \cdot -(-a\breve{} \circ b). \tag{2.10}$$

In a BRA the residuals and the symmetric quotient of R and S can be characterised by

$$R \diagdown S = \{\langle x, y\rangle : Rx \subseteq Sy\}, \tag{2.11}$$

$$R \diagup S = \{\langle x, y\rangle : yS \subseteq xR\}, \tag{2.12}$$

$$\mathrm{syq}(R, S) = \{\langle x, y\rangle : Rx = Sy\}. \tag{2.13}$$

The following properties of the residual will be needed later:

Lemma 2.2. *[14, 32] In every RA the following holds:*

1. *$c \diagdown c$ is reflexive and transitive.*
2. *If c is reflexive and symmetric, then $(c \diagdown c)\breve{} \circ (c \diagdown c) \leq c$.*

In a BRA there is an elegant way to characterize a subset of the universe U. To this end, associate the relation

$$m \stackrel{\text{def}}{=} \{\langle x, y\rangle : x \in V, y \in M\}$$

with the subset $M \subseteq U$. It is easy to see that m is characterized by $m = V \circ m$ where V is the greatest relation over the universe U. Such a relation is called a *vector*. Analogously, a one-element subset or an element of U may be described by a vector which is univalent $m\breve{} \circ m \subseteq 1'$. These relations are called *points*.

Given an ordering P, i.e., a reflexive, transitive and antisymmetric relation, one may be interested in lower bounds $\mathrm{lb}_P(m)$ of subset characterized by a vector m. This vector is given by $\mathrm{lb}_P(m) \stackrel{\text{def}}{=} -(m \circ -P\breve{}) = m\breve{} \diagdown P\breve{} = (P \diagup m)\breve{}$. Analogously, $\mathrm{ub}_P(m) \stackrel{\text{def}}{=} -(m \circ -P) = m\breve{} \diagdown P$ is the vector of upper bounds of m. Last but not least, the relations $\mathrm{glb}_P(m) \stackrel{\text{def}}{=} \mathrm{lb}_P(m) \cap \mathrm{ub}_P(\mathrm{lb}_P(m))$ and $\mathrm{lub}_P(m) \stackrel{\text{def}}{=} \mathrm{ub}_P(m) \cap \mathrm{lb}_P(\mathrm{ub}_P(m))$ are either empty or a point, describing the greatest lower bound and the least upper bound, if they exist, respectively. More details about the relational description of orderings, extremal elements and their properties can be found in [35].

For properties of relation algebras not mentioned here, we refer the reader to [10, 24, 35], and for Boolean algebras to [25].

3 Mereology

Mereology, the study of "part-of" relations, was given a formal framework by Leśniewski [26, 27] as part of his programme to establish a paradox-free foundation of Mathematics. Clarke [11] has generalised Leśniewski's classical mereology by taking a "contact" relation C as the basic structural element. The axioms which C needs to fulfil are

(3.1) C is reflexive and symmetric,

(3.2) $Cx = Cy$ implies $x = y$.

It was shown in [14] that the extensionality axiom (3.2) may be replaced by

(3.3) $C \setminus C$ is antisymmetric

i.e.

$$\operatorname{syq}(C, C) \subseteq 1'. \tag{3.4}$$

The term "mereology" has nowadays become (almost) synonymous with the study of "part-of" and "contact" relations in QSR.

If C is a contact relation we set

(3.5) $P \stackrel{\text{def}}{=} C \setminus C,$ part of

(3.6) $PP \stackrel{\text{def}}{=} P \cap -1'.$ proper part of

Lemma 2.2 and (3.3) tell us that P is a partial order which we shall call the *part of relation* (of C). We also write $x \leq y$ instead of xPy. PP is called the *proper part of* relation.

We now define the additional relations

(3.7) $O \stackrel{\text{def}}{=} P\breve{} \circ P$ overlap

(3.8) $PO \stackrel{\text{def}}{=} O \cap -(P \cup P\breve{})$ partial overlap

(3.9) $EC \stackrel{\text{def}}{=} C \cap -O$ external contact

(3.10) $TPP \stackrel{\text{def}}{=} PP \cap (EC \circ EC)$ tangential proper part

(3.11) $NTPP \stackrel{\text{def}}{=} PP \cap -TPP$ non–tangential proper part

(3.12) $DC \stackrel{\text{def}}{=} -C$ disconnected

(3.13) $DR \stackrel{\text{def}}{=} -O$ discrete.

Given a contact relation C, we will use the definitions (3.5) – (3.13) of the relations throughout the remainder of the paper.

Mereological structures also have an algebraic part: If C is a contact relation on U and $\emptyset \neq X \subseteq U$, $x \in U$, then x is called the *fusion of* X, written as $\sum X$, if

$$(\forall y \in U)[xCy \iff (\exists z \in X)yCz]. \tag{3.14}$$

A *model of mereology* is a structure $\langle U, C, \sum \rangle$, where C is a contact relation, and the fusion $\sum$ exists for all nonempty $X \subseteq U$. If

$$C = O, \tag{3.15}$$

then $\langle U, C, \sum \rangle$ is a model of classical mereology, since "contact" C is definable by "part of" P as $C = P^{\smile} \circ P$.

Note that the definition of a model of mereology is not first order; a *weak model of mereology* is a structure $\langle U, C, + \rangle$, where C is a contact relation, and for all $x, y \in U$, the fusion $x + y$ exists.

Given a model of mereology $\langle U, C, \sum \rangle$, one can define additional operations on U as follows [11]:

(3.16)	$1 = \sum \{x : xCx\}$	Universal element
(3.17)	$x^* = \sum \{y : y(-C)x\}$	Complement
(3.18)	$\prod X = \sum \{z : zPx \text{ for all } x \in X\}$	Product

Observe that * and $\prod$ are partial operations, and that they require completeness of fusion. Biacino & Gerla [8] have shown that the models of mereology are exactly the complete orthocomplemented lattices with the 0 element removed, and

$$xCy \iff x \not\leq -y$$

Models of classical mereology arise from complete Boolean algebras B with the 0 element removed as shown in [39]; here P is the Boolean order.

4 The Region Connection Calculus

The Region Connection Calculus (RCC) was introduced in [33] as a tool for reasoning about spatial phenomena, and has since received some prominence. It uses a contact relation C which fulfils the conditions (3.1) and (3.2).

A model for the RCC consists of a base set $U = R \cup N$, where R, N are disjoint, a distinguished $1 \in R$, a unary operation $^* : R_0 \to R_0$, where $R_0 \stackrel{\text{def}}{=} R \setminus \{1\}$, a binary operation $+ : R \times R \to R$, another binary operation $\cdot : R \times R \to R \cup N$, and a binary relation C on R. In order to avoid trivialities, we assume that $|U| \geq 2$.

The RCC axioms are as follows:

RCC 1. $(\forall x \in R)xCx$

RCC 2. $(\forall x, y \in R)[xCy \Longrightarrow yCx]$

RCC 3. $(\forall x \in R)xC1$

RCC 4. $(\forall x \in R, y \in R_0)$,

(a) $xCy^* \Longleftrightarrow \neg\, xNTPPy$

(b) $xOy^* \Longleftrightarrow \neg\, xPy$

RCC 5. $(\forall x, y, z \in R)[xCy + z \Longleftrightarrow xCy$ or xCz

RCC 6. $(\forall x, y, z \in R)[xCy \cdot z \Longleftrightarrow (\exists w \in R)(wPy$ and wPz and $xCw)]$

RCC 7. $(\forall x, y \in R)[x \cdot y \in R \Longleftrightarrow xOy]$

RCC 8. If xPy and yPx, then $x = y$.

We shall in the sequel assume without loss of generality that $N = \{0\}$. Axioms RCC 1, RCC 2, RCC 5 and RCC 8 show that $\langle R, C, +\rangle$ is a weak model of mereology. It is, however, not a model of mereology in the sense of Section 3, since it has a different definition of complement: In the RCC models, each proper region x is connected to its complement x^*, which is impossible in models of mereology. It was shown in [15] and [38] that the algebraic part of an RCC model is a Boolean algebra. Each atomless Boolean algebra can be made into an RCC model by defining an appropriate contact relation [13].

Notice, that some of the axioms above may be written in a relation algebraic manner as follows:

RCC 1'. $1' \subseteq C$

RCC 2'. $C^{\smile} \subseteq C$

RCC 8'. $\mathrm{syq}(C, C) \subseteq 1'$.

In the original RCC, the relations

$$1',\ TPP,\ TPP^{\smile},\ NTPP,\ NTPP^{\smile},\ PO,\ EC,\ DC \tag{4.1}$$

were considered base relations in a system called RCC8. Somewhat earlier, Egenhofer & Franzosa [16] arrive at a similar set of relations by purely topological considerations. Seeing that the largest element 1 is RA – definable from C, it was noted in [14] that investigation of the RCC can be restricted to the set $U = R \cap -\{1\}$, and that EC and PO split into the disjoint non–empty relations

$$ECD \stackrel{\text{def}}{=} -(PP \circ PP^{\smile} \cup PP^{\smile} \circ PP), \tag{4.2}$$

$$ECN \stackrel{\text{def}}{=} EC \cap -ECD, \tag{4.3}$$

$$PON \stackrel{\text{def}}{=} \# \cap (PP^{\smile} \circ PP) \cap (PP \circ PP^{\smile}), \tag{4.4}$$

$$POD \stackrel{\text{def}}{=} \# \cap (PP^{\smile} \circ PP) \cap -(PP \circ PP^{\smile}), \tag{4.5}$$

where $\# = -(P \cup P^{\smile})$ is the incomparability relation. It is not hard to see that

$$\begin{aligned}
xECDy &\Longleftrightarrow x = y^*, \\
xECNy &\Longleftrightarrow xECy \text{ and } x + y \neq 1, \\
xPONy &\Longleftrightarrow x\#y,\ x \cdot y \neq 0,\ x + y \neq 1, \\
xPODy &\Longleftrightarrow x\#y,\ x \cdot y \neq 0,\ x + y = 1,
\end{aligned}$$

In the sequel, we shall regard

(4.6) $$1',\ TPP,\ TPP^{\smile},\ NTPP,\ NTPP^{\smile},\ PON,\ POD,\ ECN,\ ECD,\ DC$$

as defined above as the basic relations in terms of which other relations will be defined below. The weak composition of these relations is given in Table 1; it is worth to point out that the table does not have an extensional interpretation, i.e. there is no RA whose composition is given by Table 1. Nevertheless, the base relations are the atoms of a semi-associative relation algebra in the sense of Maddux [29].

Using the relation ECD, another RCC axiom can be written in algebraic form as follows:

RCC 4'. (a) $C \circ ECD = -NTPP$
(b) $O \circ ECD = -P$

Let V be the greatest relation over R. Notice, that the property

(4.7) $$\mathrm{lub}_{(C \searrow C)}(R) \circ V = R \circ V$$

forces the algebraic part of a RCC model over R to be a complete BA without a least element since for every nonempty vector m the least upper bound $\mathrm{lub}_{(C \searrow C)}(m)$ is also nonempty. Under this assumption, the greatest element 1 is characterized by the relation $\mathrm{lub}_{(C \searrow C)}(V)$. Furthermore, if we require

(4.8) $$\mathrm{lub}_{(C \searrow C)}(R) \circ C = R \circ C \text{ for all relations } R$$

then the relation algebraic counterparts of the remaining axioms RCC 3, RCC 5, RCC 6 and RCC 7 are provable.

Lemma 4.1. *For all nonempty vectors m we have the following:*

RCC 3': $\mathrm{lub}_{(C \searrow C)}(V); C = V$,

RCC 5': $\mathrm{lub}_{(C \searrow C)}(m) \circ C = m \circ C$,

RCC 6': $\mathrm{glb}_{(C \searrow C)}(m) \circ C = \mathrm{lb}_{(C \searrow C)}(m) \circ C$,

RCC 7': $\mathrm{glb}_{(C \searrow C)}(m) \neq \emptyset \iff \mathrm{lb}_{(C \searrow C)}(m) \neq \emptyset$.

Table 1: The RCC 10 weak composition table

$\circ_w$	TPP	TPP$^{\smile}$	NTPP	NTPP$^{\smile}$	PON	POD	ECN	ECD	DC
TPP	TPP, NTPP	1', TPP, TPP$^{\smile}$, PON, ECN, DC	NTPP	TPP$^{\smile}$, NTPP$^{\smile}$, PON, ECN, DC	TPP, NTPP, PON, ECN, DC	TPP, NTPP, PON, POD, ECN, ECD	ECN, DC	ECN	DC
TPP$^{\smile}$	1', TPP, TPP$^{\smile}$, PON, POD	TPP$^{\smile}$, NTPP$^{\smile}$	TPP, NTPP, PON, POD	NTPP$^{\smile}$	TPP$^{\smile}$, NTPP$^{\smile}$, PON, POD	POD	TPP$^{\smile}$, NTPP$^{\smile}$, PON, POD, ECN, ECD	POD	TPP$^{\smile}$, NTPP$^{\smile}$, PON, ECN, DC
NTPP	NTPP	TPP, NTPP, PON, ECN, DC	NTPP	1', TPP, TPP$^{\smile}$, NTPP, NTPP$^{\smile}$, PON, ECN, DC	TPP, NTPP, PON, ECN, DC	TPP, NTPP, PON, POD, ECN, ECD, DC	DC	DC	DC
NTPP$^{\smile}$	TPP$^{\smile}$, NTPP$^{\smile}$, PON, POD	NTPP$^{\smile}$	1', TPP, TPP$^{\smile}$, NTPP, NTPP$^{\smile}$, PON, POD	NTPP$^{\smile}$	TPP$^{\smile}$, NTPP$^{\smile}$, PON, POD	POD	TPP$^{\smile}$, NTPP$^{\smile}$, PON, POD	POD	TPP$^{\smile}$, NTPP$^{\smile}$, PON, POD, ECN, ECD, DC
PON	TPP, NTPP, PON, POD	TPP$^{\smile}$, NTPP$^{\smile}$, PON, ECN, DC	TPP, NTPP, PON, POD	TPP$^{\smile}$, NTPP$^{\smile}$, PON, ECN, DC	1', TPP, TPP$^{\smile}$, NTPP, NTPP$^{\smile}$, PON, POD, ECN, ECD, DC	TPP, NTPP, PON, POD	TPP$^{\smile}$, NTPP$^{\smile}$, PON, ECN, DC	PON	TPP$^{\smile}$, NTPP$^{\smile}$, PON, ECN, DC
POD	POD	TPP$^{\smile}$, NTPP$^{\smile}$, PON, POD, ECN, ECD	POD	TPP$^{\smile}$, NTPP$^{\smile}$, PON, POD, ECN, ECD, DC	TPP$^{\smile}$, NTPP$^{\smile}$, PON, POD	1', TPP, TPP$^{\smile}$, NTPP, NTPP$^{\smile}$, PON, POD	TPP$^{\smile}$, NTPP$^{\smile}$	TPP$^{\smile}$, NTPP$^{\smile}$	NTPP$^{\smile}$
ECN	TPP, NTPP, PON, POD, ECN, ECD	ECN, DC	TPP, NTPP, PON, POD	DC	TPP, NTPP, PON, ECN, DC	TPP, NTPP	1', TPP, TPP$^{\smile}$, PON, ECN, DC	TPP	TPP$^{\smile}$, NTPP$^{\smile}$, PON, ECN, DC
ECD	POD	ECN	POD	DC	PON	TPP, NTPP	TPP$^{\smile}$	1'	NTPP$^{\smile}$
DC	TPP, NTPP, PON, ECN, DC	DC	TPP, NTPP, PON, POD, ECN, ECD, DC	DC	TPP, NTPP, PON, ECN, DC	NTPP	TPP, NTPP, PON, ECN, DC	NTPP	1', TPP, TPP$^{\smile}$, NTPP, NTPP$^{\smile}$, PON, ECN, C

Proof. RCC 3' and RCC 5' follow from 4.8. Notice, that we have

$$(*) \quad \mathrm{glb}_{(C\setminus C)}(m) = \mathrm{lub}_{(C\setminus C)}(\mathrm{lb}_{(C\setminus C)}(m)).$$

A proof may be found in [35]. RCC 6' and RCC 7' follow from $(*)$ and 4.8 resp. 4.7. □

Notice, that the inclusion $\supseteq$ in 4.8 can be proven.

5 Basic relational properties

In this section, we shall collect some properties of the relations listed in (4.6), which follow from the RCC axioms. These will be used in the next Section for the definition of the relation algebra. We commence with several basic connections which were already proved in [14].

Lemma 5.1. *1. $1' \subseteq NTPP^{\smile} \circ NTPP$, i.e. for all z there is some x with $xNTPPz$.*

2. $ECN = TPP \circ ECD$, i.e. $xECNz \iff xTPPz^*$.

3. If $xDCz$, then $xTPP(x+z)$.

4. $xNTPPz$ and $yNTPPz \iff (x+y)NTPPz$.

5. If $xNTPPz$, then $x^* \cdot zTPPz$.

Our second lemma deals with compositions of P, DC, TPP and $NTPP$.

Lemma 5.2. *1. $DC \circ P^{\smile} \subseteq DC$, i.e. $xDCy$ and $z \leq y$ imply $xDCz$.*

2. $NTPP = ECD \circ NTPP^{\smile} \circ ECD$, i.e. $xNTPPy \iff y^*NTPPx^*$.

3. $P \circ NTPP \leq NTPP$, i.e. $x \leq y$ and $yNTPPz$ imply $xNTPPz$.

4. $NTPP \circ TPP = NTPP$.

5. $NTPP \circ P = NTPP$.

6. $TPP \circ NTPP = NTPP$.

7. $1' \leq NTPP \circ NTPP^{\smile}$, i.e. for all x there is some z with $xNTPPz$.

Proof. 1. Consider the following computation:

$$\begin{aligned} DC \circ P^{\smile} \subseteq DC &\iff -C \circ -(-C^{\smile} \circ C) \subseteq -C && \text{by definition of } DC \text{ and } P \\ &\iff -C^{\smile} \circ C \subseteq -C^{\smile} \circ C. && \text{by 2.7} \end{aligned}$$

2. Consider the following computation:

$$\begin{aligned} NTPP &= -(C \circ ECD) && \text{by RCC 4'a} \\ &= ECD \circ -(ECD \circ C) \circ ECD && \text{since } ECD \text{ is a bijection} \\ &= ECD \circ -(C \circ ECD)^{\smile} \circ ECD && \text{since } C \text{ and } ECD \text{ are symmetric} \\ &= ECD \circ NTPP^{\smile} \circ ECD. && \text{by RCC 4'a} \end{aligned}$$

3. Let $xPyNTPPz$, and assume that $\neg\, xNTPPz$. Then we have xCz^* and $\neg\, yCz^*$ by RCC 4a. It follows that xCz^*DCy holds, i.e. $x(C^{\smile} \circ -C)y$ and hence $\neg xPy$ by the definition of P. But this is a contradiction.

4. "$\subseteq$": We prove the stronger assumption

$$(*) \quad NTPP \circ P \subseteq NTPP.$$

Let $xNTPPyPz$ and assume that $\neg xNTPPz$. Then we have xCz^* by RCC 4a. On the other hand, $xNTPPy$ implies $xDCy^*$. Since $y \le z$ and therefore $z^* \le y^*$ holds we conclude $xDCz^*$ by 1, a contradiction.

"$\supseteq$": Let $xNTPPz$. With 5.1(1) choose some $wNTPPx^* \cdot z$, and set $y = w^* \cdot z$. Then we have

$$\begin{aligned} wNTPPx^* \cdot z &\Longrightarrow wNTPPx^* && \text{by } (*) \\ &\Longrightarrow xNTPPw^* && \text{by 5.2(2)} \\ &\Longrightarrow xDCw && \text{by RCC 4a} \\ &\Longrightarrow xDCw \text{ and } xDCz^* && \text{by } xNTPPz \text{ and RCC 4a} \\ &\Longrightarrow xDC(w + z^*) && \text{by RCC 5} \\ &\Longrightarrow xNTPP(w + z^*)^* && \text{by RCC 4a} \\ &\Longrightarrow xNTPPy. && \text{Definition of } y \end{aligned}$$

Furthermore, $wNTPPx^* \cdot z$ implies $wNTPPz$, and by 5.1(5) we get $y = w^* \cdot zTPPz$.

5. "$\subseteq$" was already shown in 4.$(*)$ and "$\supseteq$" follows from 4.

6. "$\subseteq$": This follows from 3.

"$\supseteq$": Let $xNTPPz$. With 5.1(1) choose some $yNTPPx^* \cdot z$. 5. gives us $yNTPPz$ and $yNTTPx^*$. Using 5.1(4) and RCC 4a we get $x + yNTPPz$ and $yDCx$. Together we conclude $xTPPx + yNTPPz$ by 5.1(2).

7. Let $x \in U$. By Lemma 5.11, there is some $y \in U$ such that $yNTPPx^*$, and by 2 above, $xNTPPy^*$. □

Our next lemma exhibits some new arithmetical properties involving the algebraic operations.

Lemma 5.3. *1.* $xNTPPy$ *and* $xNTPPz \Longleftrightarrow xNTPPy \cdot z$.

2. $ECD \circ DC = NTPP\breve{}$, *i.e.* $x^*DCz \iff zNTPPx$.

3. $PON \circ ECD = PON$, *i.e.* $xPONz \iff xPONz^*$.

4. $TPP\breve{} \circ ECD = POD \cap -(ECD \circ NTPP)$.

5. $xECN \circ TPPz \iff xECNx^* \cdot zTPPz$.

6. $xTPP\breve{} \circ TPPz \iff xTPP\breve{}x \cdot zTPPz$.

7. *If* $x \cdot z \neq 0$ *then* $x - (TPP\breve{} \circ TPP)z \iff x \cdot zNTPPx$ *or* $x \cdot zNTPPz$.

8. $xTPP \circ TPP\breve{}z \iff xTPP(x+z)TPP\breve{}z$.

9. *If* $yNTPP(x+z)$ *and* $yDCz$ *then* $yNTPPx$.

Proof. 1. This follows from 5.1(4) and 5.2(2).

2. Consider the following computation:

$$
\begin{aligned}
x(ECD \circ DC)z &\iff x^*DCz \\
&\iff zDCx^* && \text{by RCC 2} \\
&\iff zNTPPx. && \text{by RCC 4a} \\
&\iff xNTPP\breve{}z.
\end{aligned}
$$

3. "$\Longrightarrow$": Suppose $xPONz$. Then the definition of PON implies

$$x \not\leq z, \tag{5.1}$$

$$z \not\leq x, \tag{5.2}$$

$$x \cdot z \neq 0, \tag{5.3}$$

$$x + z \neq 1. \tag{5.4}$$

We have to prove (5.1)-(5.4) for z^* instead of z. Consider the following computation:

$$
\begin{aligned}
z^* \leq x &\Longrightarrow x + z \geq z^* + z = 1, \text{ contradicting (5.4)}, \\
x \leq z^* &\Longrightarrow x \cdot z \leq z^* \cdot z = 0, \text{ contradicting (5.3)}, \\
x \cdot z^* = 0 &\Longrightarrow x \leq z, \text{ contradicting (5.1)}, \\
x + z^* = 1 &\Longrightarrow z \leq x, \text{ contradicting (5.2)}.
\end{aligned}
$$

"$\Longleftarrow$" is shown analogously.

4. "$\subseteq$": First, we show $TPP\breve{} \circ ECD \subseteq POD$. Suppose z^*TPPx. The we have

$$
\begin{aligned}
&x \not\leq z, && \text{since } x \leq z \text{ implies } z^* \leq x^*\text{, a contradiction} \\
&z \not\leq x, && \text{since } z \leq x \text{ is a contradiction} \\
&x \cdot z \neq \emptyset, && \text{since } x \cdot z = \emptyset \text{ implies } x \leq z^*\text{, a contradiction} \\
&x + z = 1, && \text{since } z^* \leq x
\end{aligned}
$$

and hence $xPODz$. Furthermore, consider the following computation:

$$\begin{aligned}
&TPP^{\smile} \circ ECD \cap ECD \circ NTPP \\
&\qquad = TPP^{\smile} \circ ECD \cap ECD \circ ECD \circ NTPP^{\smile} \circ ECD && \text{by 5.2(2)} \\
&\qquad = (TPP^{\smile} \cap NTPP^{\smile}) \circ ECD && \text{since } ECD \text{ is a bijection} \\
&\qquad = 0,
\end{aligned}$$

which shows $TPP^{\smile} \circ ECD \subseteq -(ECD \circ NTPP)$.

"$\supseteq$": Suppose $xPODz$ and $x^* - NTPPz$. Then we conclude $z^* \lneq x$ since $x + z = 1$ and $x \cdot z \neq \emptyset$. Furthermore, we have z^*TPPx because z^*NTPPx implies x^*NTPPz by 5.2(2), a contradiction.

5. We only have to show "$\Longrightarrow$". Let $xECNyTPPz$. First we have

$$\begin{aligned}
xECNy &\Longleftrightarrow yECNx && \text{symmetry of } ECN \\
&\Longleftrightarrow yTPPx^* && \text{by 5.1(2)} \\
&\Longrightarrow y \leq x^* \cdot z. && \text{since } yTPPz
\end{aligned}$$

Assume $xNTPPx + z^*$. Then we conclude

$$\begin{aligned}
xNTPPx + z^* &\Longleftrightarrow xDCx^* \cdot z && \text{by RCC 4a} \\
&\Longrightarrow xDCy && \text{by 5.2(1)} \\
&\Longleftrightarrow xNTPPy^*. && \text{by RCC 4a}
\end{aligned}$$

But $xECNy$ gives us $xTPPy^*$ by 5.1(2), a contradiction. Since $x \leq x + z^*$ we conlude $xTPPx + z^*$ and using 5.1(2) again $xECNx^* \cdot z$. Assume $x^* \cdot zNTPPz$. Then we aim at $yNTPPz$ by 5.2(3), contradicting $yTPPz$. Since $x^* \cdot z \leq z$ we get $x^* \cdot zTPPz$.

6. Again, we only have to show "$\Longrightarrow$". Let $xTPP^{\smile}yTPPz$. Then we have $y \leq x \cdot z$. Assume $x \cdot zNTPPx$. We conclude $yNTPPx$ by 5.2(3), a contradiction.

7. "$\Longrightarrow$": Let $x \cdot z \gneq 0$. Suppose w.l.o.g that $x \cdot yTPPx$ holds. The hypothesis $x - (TPP^{\smile} \circ TPP)z$ implies that for all y,

$$yTPPx \Longrightarrow y - TPPz.$$

Thus, $x \cdot z(-TPP)z$ and hence $x \cdot zNTPPz$ since $x \cdot z \leq z$.

"$\Longleftarrow$": Suppose w.l.o.g. that $x \cdot zNTPPz$ holds. Furthermore, assume that we have $xTPP^{\smile}yTPPz$ for some $y \in U$. Then we get $y \leq x \cdot z$ and by 5.2(3) $yNTPPz$, a contradiction.

8. Similarly to 6.

9. Consider the following computation:

$$
\begin{aligned}
yNTPP(x+z) \text{ and } yDCz &\iff x^* \cdot z^* NTPPy^* \text{ and } zNTPPy^* && \text{by 5.2(2) and RCC 4a}\\
&\iff (x^* \cdot y^* + z)NTPPy^* && \text{by 5.1(4)}\\
&\iff yNTPP(z^* \cdot (x+z)) && \text{by 5.2(2)}\\
&\iff yNTPPz^* \cdot x\\
&\Longrightarrow yNTPPx, && \text{by 5.2(5)}
\end{aligned}
$$

which finishes the proof. □

The last lemma deals with some new relation algebraic properties of the relations listed in (4.6).

Lemma 5.4. *1.* $ECN \circ TPP \circ ECD = TPP \circ TPP^{\smile}$.

2. $(ECN \circ TPP)^{\smile} \circ ECD = TPP^{\smile} \circ TPP$.

3. $TPP \circ TPP^{\smile} \circ ECD = ECN \circ TPP$.

4. $TPP^{\smile} \circ TPP \circ ECD = (ECN \circ TPP)^{\smile}$.

5. $-(ECN \circ TPP) \circ ECD = -(TPP \circ TPP^{\smile})$.

6. $-(ECN \circ TPP)^{\smile} \circ ECD = -(TPP^{\smile} \circ TPP)$.

7. $-(TPP \circ TPP^{\smile}) \circ ECD = -(ECN \circ TPP)$.

8. $-(TPP^{\smile} \circ TPP) \circ ECD = -(ECN \circ TPP)^{\smile}$.

Proof. 1. Consider the following computation:

$$
\begin{aligned}
ECN \circ TPP \circ ECD &= TPP \circ ECD \circ ECN && \text{by 5.1(2)}\\
&= TPP \circ (ECN \circ ECD)^{\smile} && \text{since } ECN \text{ and } ECD \text{ are symmetric}\\
&= TPP \circ (TPP \circ ECD \circ ECD)^{\smile} && \text{by 5.1(2)}\\
&= TPP \circ (TPP^{\smile}. && \text{since } ECD \text{ is a bijection}
\end{aligned}
$$

2. Consider the following computation:

$$
\begin{aligned}
(ECN \circ TPP)^{\smile} \circ ECD &= TPP^{\smile} \circ ECN \circ ECD && \text{since } ECN \text{ is symmetric}\\
&= TPP^{\smile} \circ TPP \circ ECD \circ ECD && \text{by 5.1(2)}\\
&= TPP^{\smile} \circ TPP. && \text{since } ECD \text{ is a bijection}
\end{aligned}
$$

3.-8. Follow from 1. and 2. since ECD is a bijection. □

Table 2: Atoms of $\mathfrak{A}$

$1'$	
$TPPA$	$= TPP \cap (ECN \circ TPP)$
$TPPA^{\smile}$	$= TPP^{\smile} \cap (ECN \circ TPP)^{\smile}$
$TPPB$	$= TPP \cap -(ECN \circ TPP)$
$TPPB^{\smile}$	$= TPP^{\smile} \cap -(ECN \circ TPP)^{\smile}$
$NTPP$	
$NTPP^{\smile}$	
$PONXA1$	$= PON \cap (ECN \circ TPP) \cap (ECN \circ TPP)^{\smile} \cap (TPP \circ TPP^{\smile}) \cap (TPP^{\smile} \circ TPP)$
$PONXA2$	$= PON \cap (ECN \circ TPP) \cap (ECN \circ TPP)^{\smile} \cap (TPP \circ TPP^{\smile}) \cap -(TPP^{\smile} \circ TPP)$
$PONXB1$	$= PON \cap (ECN \circ TPP) \cap (ECN \circ TPP)^{\smile} \cap -(TPP \circ TPP^{\smile}) \cap (TPP^{\smile} \circ TPP)$
$PONXB2$	$= PON \cap (ECN \circ TPP) \cap (ECN \circ TPP)^{\smile} \cap -(TPP \circ TPP^{\smile}) \cap -(TPP^{\smile} \circ TPP)$
$PONYA1$	$= PON \cap -(ECN \circ TPP) \cap (ECN \circ TPP)^{\smile} \cap (TPP \circ TPP^{\smile}) \cap (TPP^{\smile} \circ TPP)$
$PONYA2$	$= PON \cap -(ECN \circ TPP) \cap (ECN \circ TPP)^{\smile} \cap (TPP \circ TPP^{\smile}) \cap -(TPP^{\smile} \circ TPP)$
$PONYA1^{\smile}$	$= PON \cap (ECN \circ TPP) \cap -(ECN \circ TPP)^{\smile} \cap (TPP \circ TPP^{\smile}) \cap (TPP^{\smile} \circ TPP)$
$PONYA2^{\smile}$	$= PON \cap (ECN \circ TPP) \cap -(ECN \circ TPP)^{\smile} \cap (TPP \circ TPP^{\smile}) \cap -(TPP^{\smile} \circ TPP)$
$PONYB$	$= PON \cap -(ECN \circ TPP) \cap (ECN \circ TPP)^{\smile} \cap -(TPP \circ TPP^{\smile})$
$PONYB^{\smile}$	$= PON \cap (ECN \circ TPP) \cap -(ECN \circ TPP)^{\smile} \cap -(TPP \circ TPP^{\smile})$
$PONZ$	$= PON \cap -(ECN \circ TPP) \cap -(ECN \circ TPP)^{\smile}$
$PODYA$	$= POD \cap -(ECD \circ NTPP) \cap (TPP^{\smile} \circ TPP)$
$PODYB$	$= POD \cap -(ECD \circ NTPP) \cap -(TPP^{\smile} \circ TPP)$
$PODZ$	$= ECD \circ NTPP$
$ECNA$	$= ECN \cap (TPP \circ TPP^{\smile})$
$ECNB$	$= ECN \cap -(TPP \circ TPP^{\smile})$
ECD	
DC	

6 A necessary relation algebra

We are now ready to describe the relation algebra $\mathfrak{A}$ which is a subalgebra of every BRA generated by the contact relation of any RCC model. The relations we are going to consider are shown in Table 2. The definitions of the relations $PONYB$, $PONZ$ and $PODZ$ give rise to some simple questions answered by the next lemma.

Lemma 6.1. *1.* $PONZ \subseteq TPP \circ TPP^{\smile}$.

2. $PONYB \subseteq TPP^{\smile} \circ TPP$.

3. $PONZ \subseteq TPP^{\smile} \circ TPP$.

4. $PODZ \subseteq POD$.

5. $PODZ \subseteq TPP^{\smile} \circ TPP$.

Proof. 1. Let $xPONZz$ and assume $x(-(TPP \circ TPP^{\smile}))z$. By 5.3(8) this is equivalent to $x - TPP(x+z)$ or $z - TPP(x+z)$. Assume w.l.o.g. that $x - TPP(x+z)$ holds. Since $x \lneq x+z$ we conclude $xNTPP(x+z)$. Furthermore, we have

$$\begin{aligned} xNTPP(x+z) &\Longrightarrow xDC(x+z)^* && \text{by RCC 4a} \\ &\Longrightarrow xTPP(x+x^* \cdot z^*) && \text{by 5.1(3)} \\ &\Longleftrightarrow xTPP(x+z^*) \\ &\Longrightarrow xECNx^* \cdot z, && \text{by 5.1(2)} \end{aligned}$$

contradicting $x(-(ECN \circ TPP))z$ by 5.3(5).

2. We will show that $(PONYB \cap -(TPP^{\smile} \circ TPP)) \circ ECD = \emptyset$. This implies $PONYB \cap -(TPP^{\smile} \circ TPP) = \emptyset$ since ECD is a bijection, and finally, $PONYB \subseteq TPP^{\smile} \circ TPP$. The property above is proved by

$$\begin{aligned} &(PONYB \cap -(TPP^{\smile} \circ TPP)) \circ ECD \\ &\quad = PONYB \circ ECD \cap -(TPP^{\smile} \circ TPP) \circ ECD && \text{since } ECD \text{ is a bijection} \\ &\quad = PONYB \circ ECD \cap -(ECN \circ TPP)^{\smile} && \text{by 5.4(8)} \\ &\quad = PON \circ ECD \cap (ECN \circ TPP)^{\smile} \circ ECD \\ &\qquad \cap -(ECN \circ TPP) \circ ECD \cap -(TPP \circ TPP^{\smile}) \circ ECD \\ &\qquad \cap -(ECN \circ TPP)^{\smile} && \text{since } ECD \text{ is a bijection} \\ &\quad = PON \cap (TPP^{\smile} \circ TPP) \cap -(TPP \circ TPP^{\smile}) \\ &\qquad \cap -(ECN \circ TPP) \cap -(ECN \circ TPP)^{\smile} && \text{by 5.4(5)- (7) and 5.3(7)} \\ &\quad = PONZ \cap (TPP^{\smile} \circ TPP) \cap -(TPP \circ TPP^{\smile}) \\ &\quad = \emptyset. && \text{by 1} \end{aligned}$$

3. Similarly to 2., we show $(PONZ \cap -(TPP \circ TPP^{\smile})) \circ ECD = \emptyset$. This property is proved by

$$\begin{aligned} &(PONZ \cap -(TPP \circ TPP^{\smile})) \circ ECD \\ &\quad = PONZ \circ ECD \cap -(TPP \circ TPP^{\smile}) \circ ECD && \text{since } ECD \text{ is a bijection} \\ &\quad = PONZ \circ ECD \cap -(ECN \circ TPP) && \text{by 5.4(7)} \\ &\quad = PON \circ ECD \cap -(ECN \circ TPP) \circ ECD \\ &\qquad \cap -(ECN \circ TPP)^{\smile} \circ ECD \cap -(ECN \circ TPP) && \text{since } ECD \text{ is a bijection} \\ &\quad = PON \cap -(TPP \circ TPP^{\smile}) \cap -(ECN \circ TPP) \\ &\qquad \cap -(TPP^{\smile} \circ TPP) && \text{by 5.4(5),(2) and 5.3(7)} \\ &\quad \subseteq PONYB \cap -(TPP^{\smile} \circ TPP) \\ &\quad = \emptyset. && \text{by 2} \end{aligned}$$

4. Suppose x^*NTPPz which implies $x^* \leq z$. Obviously, we have

$$x \not\leq z, \quad z \not\leq x, \quad x \cdot z \neq 0, \quad x + z = 1,$$

and hence $xPODz$.

5. Again, suppose x^*NTPPz which implies $x \cdot z \neq 0$. Now, assume $x \cdot zNTPPx$. Then we conclude

$$\begin{aligned} x \cdot zNTPPx &\iff x^*NTPPx^* + z^* && \text{by 5.2(2)} \\ &\iff x^*NTPPx^*, && \text{by 5.3(9) since } x^*DCz^* \iff x^*NTPPz \end{aligned}$$

a contradiction. Analogously, it follows that $x \cdot zNTPPz$ is impossible. Together we have $xTPP\breve{}\,x \cdot zTPPz$ and hence $PODZ \subseteq TPP\breve{} \circ TPP$. □

We now state our main result.

Theorem 6.2. *The set $\mathcal{R}$ of relations given in Table 2 is the set of atoms of a relation algebra $\mathfrak{A}$.*

Proof. We have to show following [23]:

1. The relations are pairwise disjoint, their union is $U \times U$.
2. $\mathcal{R}$ is closed under taking converses, and either $R \subseteq 1'$ or $R \cap 1' = \emptyset$ for all $R \in \mathcal{R}$.
3. Each relation is non-empty.
4. The composition of any two of them is a union of elements of $\mathcal{R}$.

1. Lemma 6.1 gives us

$$\begin{array}{rll} TPPA, TPPB & \text{is a partition of} & TPP, \\ ECNA, ECNB & \text{is a partition of} & ECN, \\ PONXA1 – PONZ, & \text{is a partition of} & PON, \\ PODYA, PODYB, PODZ & \text{is a partition of} & POD, \end{array}$$

which proves 1.

2. This is obvious from the definitions.

3. We shall indicate elements of U which are in the relations of Table 4.6. Notice, that we have the following.

(a) $TPPA \circ ECD = ECNA$.

(b) $TPPB \circ ECD = ECNB$.

(c) $TPPA\breve{} \circ ECD = PODYA$.

(d) $TPPB\breve{} \circ ECD = PODYB$.

(e) $PONXA1 \circ ECD = PONXA1$.

(f) $PONXA2 \circ ECD = PONYA1^{\smile}$.

(g) $PONXB1 \circ ECD = PONYA1$.

(h) $PONXB2 \circ ECD = PONZ$.

(i) $PONYA1 \circ ECD = PONXB1$.

(j) $PONYA2 \circ ECD = PONYB^{\smile}$.

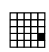
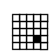
These equalities are a consequence of 5.1(2), 5.3(3),(4) and 5.4. Therefore, it is sufficient to show that the relations $TPPA, TPPB, PONXA1, PONXA2, PONXB1, PONXB2, PONYA1$ and $PONYA2$ are non-empty. To this end, we will use a configuration given by the Figure 1; this is only an indication in a familiar model. First of all, we want to show that this configuration emerges in every model of RCC with $U \neq \emptyset$. Note, that by 5.1(1) each such model must be infinite.

Let $1 \neq s \in U$ be given. Furthermore, using 5.1(1) let $tNTPPs^*$ and $wNTPP(s+t)^*$. Then we have

$sDCt,$	by RCC 4a
$sDCw,$	by RCC 4a and 5.2(1)
$tDCw,$	by RCC 4a and 5.2(1)
$s + t + w \lneq 1.$	since $(s+t) + w = 1 \Rightarrow w \geq (s+t)^*$, a contradiction

Again, using 5.1(1) let $aNTPPs,\ bNTPPa, dNTPPt$ and $cNTPPa^* \cdot s$. Since $a^* \cdot s \leq s$ we have $cNTPPs$ by 5.2(5). Furthermore, $cNTPPa^* \cdot s$ implies $cDCa + s^*$, and hence $cDCa$ by 5.2(1).

The required elements and their properties are listed in Table 3 on the following page. Proofs are straightforward and left to the reader. Using 5.3(6),(7) and (9), we conclude our assumption.

4. We have generated a composition table, and have checked that Table 4 on page 20 represents a relation algebra. Both was done with a program written in the functional language GOFER. To end up with a compact description, we have coded the sets of atoms by a 5×5-matrix given below. The table should be read as follows. The weak composition of $PODYB$  and $PODYA$  is , i.e. equal to union of the relations

$$TPPA, TPPB, TPPB^{\smile}, PONXA1, PONXB1, PONYA1, PONYA1^{\smile},$$
$$PONYB, PONYB^{\smile}, PONZ, PODZ.$$

$1'$	$TPPA$	$TPPA^{\smile}$	$TPPB$	$TPPB^{\smile}$
$NTPP$	$NTPP^{\smile}$	$PONXA1$	$PONXA2$	$PONXB1$
$PONXB2$	$PONYA1$	$PONYA2$	$PONYA1^{\smile}$	$PONYA2^{\smile}$
$PONYB$	$PONYB^{\smile}$	$PONZ$	$PODYA$	$PODYB$
$PODZ$	$ECNA$	$ECNB$	ECD	DC

This completes the proof. $\square$

Figure 1: $sDCt$, $sDCw$, $tDCw$, $s + t + w \lneq 1$, $aNTPPs$, $bNTPPa$, $cNTPPs$, $cDCa$, $dNTPPt$

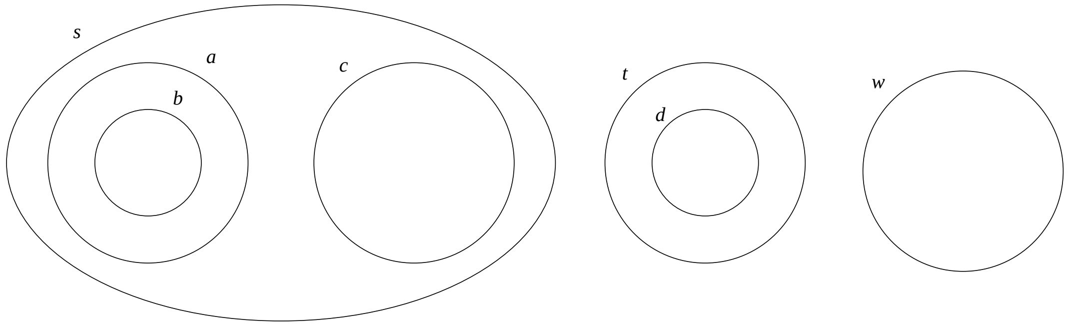
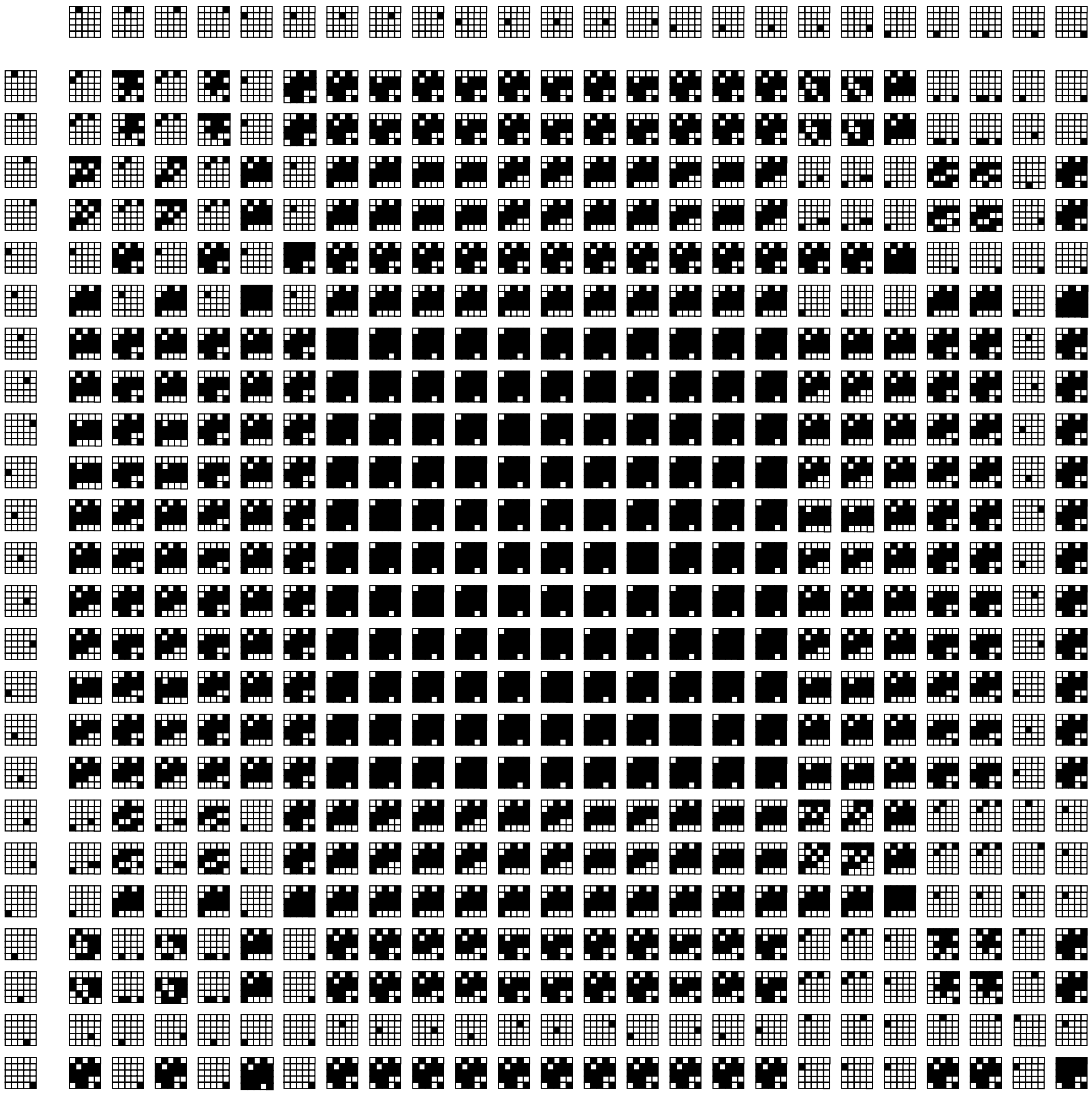


Table 3: Elements

Relation	x	z	$x \cdot z$	$x + z$	$x^* \cdot z$	$x \cdot z^*$
$TPPA$	$a^* \cdot s$	$s + t$			$a + t$	
$TPPB$	s	$s + t$			t	
$PONXA1$	$a + t$	$d + s$	$a + d$	$s + t$	$a^* \cdot s$	$d^* \cdot t$
$PONXA2$	$a^* \cdot s$	$a + c + t$	c	$s + t$	$a + t$	$a^* \cdot c^* \cdot s$
$PONXB1$	$a^* \cdot s + b$	$a + c$	$b + c$	s	$b^* \cdot a$	$a \dot{} c^* \cdot s$
$PONXB2$	$a^* \cdot s$	$a + c$	c	s	a	$a^* \cdot c^* \cdot s$
$PONYA1$	$s + t$	$a^* \cdot s + w$	$a^* \cdot s$	$s + t + w$	w	$c + t$
$PONYA2$	s	$a + t$	a	$s + t$	t	$a^* \cdot s$

Relation	$xTPPx + z$	$x + zTPP\breve{}z$	$xTPP\breve{}x \cdot z$	$x \cdot zTPPz$	$xECNx^* \cdot z$	$x^* \cdot zTPPz$	$zECNx \cdot z^*$	$x \cdot z^*TPPx$
$TPPA$					+	+		
$TPPB$					-	+		
$PONXA1$	+	+	+	+	+	+	+	+
$PONXA2$	+	+	-	+	+	+	+	+
$PONXB1$	+	-	+	+	+	+	+	+
$PONXB2$	+	-	-	+	+	+	+	+
$PONYA1$	+	+	+	+	-	+	+	+
$PONYA2$	+	+	-	+	-	+	+	+

Table 4: The composition table of $\mathfrak{A}$



6.1 Topological properties

Suppose that $\langle X, \tau \rangle$ is a topological space. If $x \subseteq X$, we denote the closure of x by $cl(x)$, and its interior by $int(x)$. The *fringe* or *boundary* $Fr(x)$ of x is the set $cl(x) \cap -int(x)$. x is called *regular open, if* $x = int(cl(x))$. It is well known that the collection $RO(X)$ of regular open sets is a complete

Boolean algebra under set inclusion where for $v, w \in RO(X)$,

$$v + w = int(cl(v \cup w)), \tag{6.1}$$

$$v \cdot w = v \cap w, \tag{6.2}$$

$$v^* = int(-v). \tag{6.3}$$

The space $\langle X, \tau\rangle$ is called *regular* or T_3 space, if points can be separated by disjoint open sets, and if for each $a \in X$ and each closed set x not containing a, there are disjoint open sets w, v such that $a \in w,\ x \subseteq v$. $\langle X, \tau\rangle$ is called *connected* if the only open-closed (clopen) sets are X and $\emptyset$.

For properties of topological spaces not mentioned here, we invite the reader to consult [20].

As shown in [21], a standard RCC model is the complete Boolean algebra $RO(X)$ of regular open sets of a connected regular topological space $\langle X, \tau\rangle$, where for $x, y \in RO(X)$

$$xCy \overset{\text{def}}{\iff} cl(x) \cap cl(y) \neq \emptyset. \tag{6.4}$$

Theorem 6.3 gives the topological properties of the base relations and the building blocks of the others, from which the properties of the atoms can easily be derived.

Theorem 6.3. *Let B be an atomless subalgebra of $RO(X)$ and C be the connection relation of* (6.4) *defined on $U = B \cap -\{\emptyset, X\}$. Furthermore, let $x, y, z \in U,\ x \neq y$. Then,*

$$xTPPy \iff x \subsetneq y, Fr(x) \cap Fr(y) \neq \emptyset \tag{6.5}$$

$$xNTPPy \iff cl(x) \subsetneq y \tag{6.6}$$

$$xPONy \iff x \not\subseteq y, y \not\subseteq x, x \cap y \neq \emptyset, cl(x) \cup cl(y) \neq X \tag{6.7}$$

$$xPODy \iff x \not\subseteq y, y \not\subseteq x, x \cap y \neq \emptyset, cl(x) \cup cl(y) = X \tag{6.8}$$

$$xECNy \iff x \cap y = \emptyset, cl(x) \cap cl(y) \neq \emptyset, cl(x) \cup cl(y) \neq X \tag{6.9}$$

$$xECDy \iff x \cap y = \emptyset, cl(x) \cap cl(y) \neq \emptyset, cl(x) \cup cl(y) = X \tag{6.10}$$

$$xDCy \iff cl(x) \cap cl(y) = \emptyset \tag{6.11}$$

$$xECN \circ TPPy \iff Fr(x) \cap Fr(-x \cap z) \neq \emptyset, Fr(z) \cap Fr(-x \cap z) \neq \emptyset, cl(x) \cup cl(y) \neq X \tag{6.12}$$

$$xTPP \circ TPP\breve{\ }y \iff Fr(x) \cap Fr(int(cl(x \cup z))) \neq \emptyset, Fr(z) \cap Fr(int(cl(x \cup z))) \neq \emptyset \tag{6.13}$$

$$xTPP\breve{\ } \circ TPPy \iff Fr(x) \cap Fr(x \cap z) \neq \emptyset, Fr(z) \cap Fr(x \cap z) \neq \emptyset \tag{6.14}$$

$$xECD \circ NTPPy \iff x \cup y = X. \tag{6.15}$$

Proof. All equivalences are straightforward applications of the definitions of the Boolean operations given in (6.1) – (6.3) on page 21, and the properties of the relations given in Lemma 5.1. ☐

We would like to close this Section with an RCC model which has different properties than the one on the full algebra $RO(X)$. Let K be the collection of sets of the form

$$K(a, b) = \begin{cases} \{p \in \mathbb{R}^2 : a \lneq |p| \lneq b\}, & \text{if } 0 \neq a, \\ \{p \in \mathbb{R}^{\cdot}|p| \lneq b\}, & \text{if } a = 0. \end{cases}$$

where $a \in \mathbb{R}, b \in \mathbb{R} \cup \{\infty\}$, and $|p|$ is the Euclidian distance of $p \in \mathbb{R}^2$ to $(0,0)$). We also extend the ordering of $\mathbb{R}$ and set $a \lneq \infty$ for all $a \in \mathbb{R}$. Let B be the set of all finite unions of elements of K including $\emptyset$. Then B is a subalgebra of $RO(\mathbb{R}^2)$, generated by the open disks with centre at the origin; by a result of [13], $\langle B, C\rangle$ is a model of the RCC, where C is defined by (6.4).

Now, consider $x = K(0,1)$. We want to show that there is no $y \in U = B \setminus \{\mathbb{R}^2, \emptyset\}$ with $xTPPAy$. Every element y of U with $xTPPy$ is of the form $x \cup \{K(a,b) : 1 \lneq a\}$. Since $-x \cdot y = \{K(a,b) : 1 < a\}$ and $\{K(a,b) : 1 \lneq a\}$ is disconnected to x, we conclude that $xTPPBy$. It follows that the BRA generated by C on this domain is not integral. On the other hand, it is not hard to see that in $RO(\mathbb{R})$, $1' \subseteq TPPA \circ TPPA\breve{}$, so this situation cannot happen there.

7 Summary and Outlook

We have shown that each relation algebra generated by the contact relation of an RCC model contains an integral algebra $\mathfrak{A}$ with 25 atoms as a subalgebra. Thus, the expressiveness of the RCC axioms in the 3-variable fragment of first order logic is much greater than the original eight RCC base relations (which are, basically, the possible relations of a pair of circles) might suggest. We have also given a topological interpretation of the atoms of $\mathfrak{A}$.

We have not yet found a representation of $\mathfrak{A}$, and the problem is open as to whether there is an RCC model with $\mathfrak{A}$ as its associated BRA. In particular, we do not know, if $\mathfrak{A}$ is the BRA generated by C on a standard model $RO(X)$.

All RCC models that we know fulfil

$$NTPP \subseteq NTPP \circ NTPP, \tag{7.1}$$

and the question remains, whether this is always true.

It seems also worthwhile to compare the expressivity of RA logic with that of the 9-intersection model of Egenhofer & Herring [17], which is based only on topological properties.

Another promising area of research is to consider the expressive power of relational structures more general than BRAs, for example, those, in which the associativity of the composition is relaxed [29]. Egenhofer & Rodríguez [18] have given a spatial interpretation of such a structure.

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