Linking functional programming and topology

Gunther Schmidt

Fakultät für Informatik, Universität der Bundeswehr München
85577 Neubiberg, Germany
gunther.schmidt@unibw.de

Abstract. In advanced functional programming, researchers have investigated the existential image, the power transpose, and the power relator, e.g. It will be shown how the existential image is of use when studying continuous mappings between different topologies relationally. Normally, structures are compared using homomorphisms and sometimes isomorphisms. This applies to group homomorphisms, to graph homomorphisms and many more. The technique of comparison for topological structures will be shown to be quite different. Having in mind the cryptomorphic versions of neighborhood topology, open kernel topology, open sets topology, etc., this seems important.

Lifting concepts to a relational and, thus, algebraically manipulable and shorthand form, shows that existential and inverse images must here be used for structure comparison. Applying the relational language TituRel to such topological concepts allows to study and also visualize them.

Keywords. relational mathematics, homomorphism, topology, existential image, continuity.

1 Prerequisites

We will work with heterogeneous relations and provide a general reference to [Sch11a], but also to the earlier [SS89,SS93]. Our operations are, thus, binary union "$\cup$", intersection "$\cap$", composition "$;$", unary negation "$\overline{}$", transposition or conversion "$^{\mathsf{T}}$", together with zero-ary null relations "$\perp\!\!\!\perp$", universal relations "$\top\!\!\!\top$", and identities "$\mathbb{I}$". A *heterogeneous relation algebra*

- is a category wrt. composition "$;$" and identities $\mathbb{I}$,
- has as morphism sets complete atomic boolean lattices with $\cup, \cap, \overline{}, \perp\!\!\!\perp, \top\!\!\!\top, \subseteq$,
- obeys rules for transposition $^{\mathsf{T}}$ in connection with the latter two concepts that may be stated in either one of the following two ways:

  Dedekind rule:
  $R;S \cap Q \subseteq (R \cap Q;S^{\mathsf{T}});(S \cap R^{\mathsf{T}};Q)$

  Schröder equivalences:
  $A;B \subseteq C \iff A^{\mathsf{T}};\overline{C} \subseteq \overline{B} \iff \overline{C};B^{\mathsf{T}} \subseteq \overline{A}$

The two rules are equivalent in the context mentioned. Many rules follow out of this setting; not least that mappings f may be *shunted*, i.e. that $A;f \subseteq B \iff A \subseteq B;f^{\mathsf{T}}$.

1.1 Quotient forming

Whoever has a multiplication operation is inclined to ask for division. Division of relations with common source is indeed possible to the following extent:

$R{;}X = S$ has a solution X precisely when $S \subseteq R{;}\overline{R^{\mathsf{T}}{;}\overline{S}}$,

or else when $S = R{;}\overline{R^{\mathsf{T}}{;}\overline{S}}$. Among all solutions of $R{;}X = S$ the greatest is $\overline{R^{\mathsf{T}}{;}\overline{S}}$. Often this is turned into the operation $R\backslash S := \overline{R^{\mathsf{T}}{;}\overline{S}}$ of forming the *left residuum* — as in division allegories. An illustration of the left residuum is as follows:

$$S = \begin{array}{rc} & \begin{array}{cccccccccccc} \text{Jan} & \text{Feb} & \text{Mar} & \text{Apr} & \text{May} & \text{Jun} & \text{Jul} & \text{Aug} & \text{Sep} & \text{Oct} & \text{Nov} & \text{Dec} \end{array} \\ \begin{array}{r} \text{US} \\ \text{French} \\ \text{German} \\ \text{British} \\ \text{Spanish} \end{array} & \begin{pmatrix} 0&0&0&1&0&1&1&1&0&1&0&0 \\ 1&0&0&1&0&0&1&0&0&1&0&0 \\ 1&1&0&0&1&1&0&1&0&0&0&1 \\ 1&1&0&0&0&0&1&0&1&0&1&1 \\ 0&0&0&1&0&1&1&1&0&0&0&0 \end{pmatrix} \end{array}$$

$$R = \begin{array}{rc} & \begin{array}{ccccccccccccc} \text{A} & \text{K} & \text{Q} & \text{J} & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 \end{array} \\ \begin{array}{r} \text{US} \\ \text{French} \\ \text{German} \\ \text{British} \\ \text{Spanish} \end{array} & \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&1&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&1&0&0&0&1&1&0&1&0&1&0 \\ 0&1&1&0&0&0&0&1&0&0&0&0&1 \\ 0&0&0&1&0&0&1&0&0&1&1&0&1 \end{pmatrix} \end{array}$$

$$\begin{array}{rc} & \begin{array}{cccccccccccc} \text{Jan} & \text{Feb} & \text{Mar} & \text{Apr} & \text{May} & \text{Jun} & \text{Jul} & \text{Aug} & \text{Sep} & \text{Oct} & \text{Nov} & \text{Dec} \end{array} \\ \begin{array}{r} \text{A} \\ \text{K} \\ \text{Q} \\ \text{J} \\ 10 \\ 9 \\ 8 \\ 7 \\ 6 \\ 5 \\ 4 \\ 3 \\ 2 \end{array} & \begin{pmatrix} 1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&0&0&0&0&0&1&0&0&0&0&0 \\ 1&1&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&1&0&1&1&1&0&0&0&0 \\ 1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&1&0&1&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0 \\ 1&1&1&1&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&1&0&1&0&0&0&0 \\ 0&0&0&1&0&1&1&1&0&0&0&0 \\ 1&1&0&0&1&1&0&1&0&0&0&1 \\ 0&0&0&0&0&0&1&0&0&0&0&0 \end{pmatrix} \end{array}$$

$R\backslash S$

Left residua show how columns of the relation R below the fraction backslash are contained in columns of the relation S above, i.e., some sort of subjunction.

As an often used term built upon residua, the *symmetric quotient* of two relations with common source has been introduced as $\mathtt{syq}(R,S) := \overline{R^{\mathsf{T}}{;}\overline{S}} \cap \overline{\overline{R}^{\mathsf{T}}{;}S}$. The illustration of the symmetric quotient is as follows:

$$R = \begin{array}{rc} & \begin{array}{ccccccccccccc} \text{A} & \text{K} & \text{Q} & \text{J} & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 \end{array} \\ \begin{array}{r} \text{US} \\ \text{French} \\ \text{German} \\ \text{British} \\ \text{Spanish} \end{array} & \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&1&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&1&0&0&0&1&1&0&1&0&1&0 \\ 0&1&1&0&0&0&0&1&0&0&0&0&1 \\ 0&0&0&1&0&0&1&0&0&1&1&0&1 \end{pmatrix} \end{array}$$

$$S = \begin{array}{rc} & \begin{array}{cccccccccccc} \text{Jan} & \text{Feb} & \text{Mar} & \text{Apr} & \text{May} & \text{Jun} & \text{Jul} & \text{Aug} & \text{Sep} & \text{Oct} & \text{Nov} & \text{Dec} \end{array} \\ \begin{array}{r} \text{US} \\ \text{French} \\ \text{German} \\ \text{British} \\ \text{Spanish} \end{array} & \begin{pmatrix} 0&0&0&1&0&1&1&1&0&1&0&0 \\ 1&0&0&1&0&0&1&0&0&1&0&0 \\ 1&1&0&0&1&1&0&1&0&0&0&1 \\ 1&1&0&0&0&0&1&0&1&0&1&1 \\ 0&0&0&1&0&1&1&1&0&0&0&0 \end{pmatrix} \end{array}$$

$$\begin{array}{rc} & \begin{array}{cccccccccccc} \text{Jan} & \text{Feb} & \text{Mar} & \text{Apr} & \text{May} & \text{Jun} & \text{Jul} & \text{Aug} & \text{Sep} & \text{Oct} & \text{Nov} & \text{Dec} \end{array} \\ \begin{array}{r} \text{A} \\ \text{K} \\ \text{Q} \\ \text{J} \\ 10 \\ 9 \\ 8 \\ 7 \\ 6 \\ 5 \\ 4 \\ 3 \\ 2 \end{array} & \begin{pmatrix} 0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \end{array}$$

$\mathtt{syq}(R,S)$

The symmetric quotient shows which columns of the left are equal to columns of the right relation in $\mathtt{syq}(R,S)$, with S conceived as the denominator.

It is extremely helpful that the symmetric quotient enjoys certain cancellation properties. These are far from being broadly known. Just minor side conditions have to be observed. In any of the following propositions correct typing is assumed. What is more important is that one may calculate with the symmetric quotient in a fairly traditional algebraic way. Proofs may be found in [Sch11a].

1.1 Proposition. Arbitrary relations A, B, C satisfy in analogy to $a \cdot \frac{b}{a} = b$

i) $A ; \mathtt{syq}(A,B) = B \cap \mathbb{T} ; \mathtt{syq}(A,B)$,
ii) $\mathtt{syq}(A,B)$ surjective $\implies A ; \mathtt{syq}(A,B) = B$. $\square$

The analogy holds except for the fact that certain columns are "cut out" or are annihilated when the symmetric quotient fails to be surjective — meaning that certain columns of the first relation fail to have counterparts in the second.

1.2 Proposition. Arbitrary relations A, B, C satisfy in analogy to $\frac{b}{a} \cdot \frac{c}{b} = \frac{c}{a}$

i) $$\begin{aligned}\mathtt{syq}(A,B) ; \mathtt{syq}(B,C) &= \mathtt{syq}(A,C) \cap \mathtt{syq}(A,B) ; \mathbb{T}\\ &= \mathtt{syq}(A,C) \cap \mathbb{T} ; \mathtt{syq}(B,C)\end{aligned}$$
ii) If $\mathtt{syq}(A,B)$ is total, **or** if $\mathtt{syq}(B,C)$ is surjective, then
$$\mathtt{syq}(A,B) ; \mathtt{syq}(B,C) = \mathtt{syq}(A,C).$$ $\square$

1.2 Domain construction

The relational language TituRel (see [Sch03,Sch11b]) makes use of characterizations *up to isomorphism* and bases domain constructions on these. This applies to the obvious cases of direct products (tuple forming) with projections named π, ρ and direct sums (variant handling). It then enables the construction of natural projections to a quotient modulo an equivalence and the extrusion of a subset out of its domain, so as to have both of them as "first-class citizens" among the domains considered — not just as "dependent types".

Along with the direct product, we automatically have the Kronecker product of any two relations and (when sources coincide) the fork operator for relations,

$$(R \otimes S) := \pi ; R ; \pi'^{\mathsf{T}} \cap \rho ; S ; \rho'^{\mathsf{T}} \quad \text{and} \quad (P \otimes Q) := P ; \pi'^{\mathsf{T}} \cap Q ; \rho'^{\mathsf{T}}.$$

Here, we include the *direct power*. *Any* relation ε satisfying

$$\mathtt{syq}(\varepsilon,\varepsilon) \subseteq \mathbb{I}, \quad \mathtt{syq}(\varepsilon,R) \text{ surjective for every relation } R \text{ starting in } X.$$

is called a *membership relation* and its codomain the *direct power* of X.

`DirPow x`	$\mathbf{2}^X$
`Member x`	$\varepsilon : X \longrightarrow \mathbf{2}^X$

$$\begin{tikzcd} & \mathcal{P}(X) \arrow[d, "\mathtt{syq}\,(\varepsilon{,}R)"] \\ X \arrow[ur, "\varepsilon"] \arrow[r, "R"'] & Y \end{tikzcd}$$

Above it is indicated how this is represented in the language. One will observe the 1^{st}-order style of the definition — however quantifying over relations. Classically, the characterisation of the powerset requires 2^{nd}-order.

One will also observe the fractal structure in the following example of a membership relation, together with an interesting interplay between subsets U, their ordering $\Omega := \overline{\varepsilon^{\mathsf{T}};\overline{\varepsilon}}$, and elements like e in the powerset:

$$U = \varepsilon; e \qquad e = \mathtt{syq}(\varepsilon, U)$$

$$\begin{array}{r@{\,}l} & \begin{matrix} \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} & \{d\} & \{a,d\} & \{b,d\} & \{a,b,d\} & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\} \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{pmatrix} 0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1 \\ 0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\ 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \end{array}$$

$$(0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0) = e^{\mathsf{T}}$$

$$\begin{array}{r@{\,}l} & \begin{matrix} \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} & \{d\} & \{a,d\} & \{b,d\} & \{a,b,d\} & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\} \end{matrix} \\ \begin{matrix} \{\} \\ \{a\} \\ \{b\} \\ \{a,b\} \\ \{c\} \\ \{a,c\} \\ \{b,c\} \\ \{a,b,c\} \\ \{d\} \\ \{a,d\} \\ \{b,d\} \\ \{a,b,d\} \\ \{c,d\} \\ \{a,c,d\} \\ \{b,c,d\} \\ \{a,b,c,d\} \end{matrix} & \begin{pmatrix} 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1 \\ 0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\ 0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1 \\ 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&1&0&1&0&0&0&0&0&1&0&1 \\ 0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1 \\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1 \\ 0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \end{pmatrix} \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1\\0\\0\\0\\0\\0 \end{pmatrix} \end{array}$$

Subset U and corresponding point e in the powerset via ε, Ω

Then the direct product together with the direct power allow in particular to define join and meet in the powerset concisely as

$$\mathcal{M} := \mathtt{syq}(\,(\varepsilon \otimes \varepsilon)\,, \varepsilon) \qquad \mathcal{J} := \mathtt{syq}(\,(\overline{\varepsilon} \otimes \overline{\varepsilon})\,, \overline{\varepsilon})$$

which then satisfies such nice formulae as

$$(\varepsilon \otimes \varepsilon); \mathcal{M} = \varepsilon$$
$$\mathtt{syq}(X,\ (\varepsilon \otimes \varepsilon)\,); \mathcal{M} = \mathtt{syq}(X,\ (\varepsilon \otimes \varepsilon); \mathcal{M})$$

2 Recalling concepts of topology

Now we apply the techniques mentioned for topological structures. Topology may be defined via open or closed sets, neighborhoods, transition to open kernels, etc. We show that at least the neighborhood version — in the form given by Felix Hausdorff — shows an inherently "linear" configuration, which is apt to being formulated using relations.

We recall that a set X endowed with a system $\mathcal{U}(p)$ of subsets for every $p \in X$ — called neighborhoods — is a **topological structure**, provided

i) $p \in U$ for every neighborhood $U \in \mathcal{U}(p)$
ii) If $U \in \mathcal{U}(p)$ and $V \supseteq U$, then $V \in \mathcal{U}(p)$
iii) If $U_1, U_2 \in \mathcal{U}(p)$, then $U_1 \cap U_2 \in \mathcal{U}(p)$ and $X \in \mathcal{U}(p)$
iv) For every $U \in \mathcal{U}(p)$ there is a $V \in \mathcal{U}(p)$ so that $U \in \mathcal{U}(y)$ for all $y \in V$.

The same shall now be expressed with membership ε, conceiving $\mathcal{U}$ as a relation

$$\varepsilon : X \longrightarrow \mathbf{2}^X \quad \text{and} \quad \mathcal{U} : X \longrightarrow \mathbf{2}^X.$$

At other occasions, it has been shown that condition (iv), e.g., can semi-formally be lifted step by step to a relational form:

For every $U \in \mathcal{U}(p)$ there is a $V \in \mathcal{U}(p)$ such that $U \in \mathcal{U}(y)$ for all $y \in V$.

$$\forall p, U : U \in \mathcal{U}(p) \;\rightarrow\; \big(\exists V : V \in \mathcal{U}(p) \wedge \big(\forall y : y \in V \rightarrow U \in \mathcal{U}(y)\big)\big)$$

$$\forall p, U : \mathcal{U}_{pU} \;\rightarrow\; \big(\exists V : \mathcal{U}_{pV} \wedge \big(\forall y : \varepsilon_{yV} \rightarrow \mathcal{U}_{yU}\big)\big)$$

$$\forall p, U : \mathcal{U}_{pU} \;\rightarrow\; \big(\exists V : \mathcal{U}_{pV} \wedge \overline{\exists y : \varepsilon_{yV} \wedge \overline{\mathcal{U}_{yU}}}\big)$$

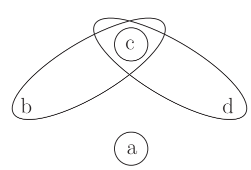
$$\forall p, U : \mathcal{U}_{pU} \;\rightarrow\; \big(\exists V : \mathcal{U}_{pV} \wedge \overline{\varepsilon^{\mathsf{T}};\overline{\mathcal{U}}}_{VU}\big)$$

$$\forall p, U : \mathcal{U}_{pU} \;\rightarrow\; \big(\mathcal{U};\overline{\varepsilon^{\mathsf{T}};\overline{\mathcal{U}}}\big)_{pU}$$

$$\mathcal{U} \subseteq \mathcal{U};\overline{\varepsilon^{\mathsf{T}};\overline{\mathcal{U}}}$$

One could see how the lengthy verbose or the predicate logic formula is traced back to a “lifted” relational version free of quantifiers, that employs a residuum. Such algebraic versions should be preferred in many respects. They support proof assisting systems and may be written down in the language TituRel so as to evaluate terms built with them and, e.g., visualize concepts of this paper. An example of a neighborhood topology $\mathcal{U}$ and the basis of its open sets:

$$\mathcal{U} = \begin{array}{c} \\ \text{a} \\ \text{b} \\ \text{c} \\ \text{d} \end{array} \begin{array}{c} \begin{array}{cccccccccccccccc} \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} & \{d\} & \{a,d\} & \{b,d\} & \{a,b,d\} & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\} \end{array} \\ \begin{pmatrix} 0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1 \\ 0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1 \\ 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix} \end{array}$$



This, together with a transfer of the other properties to the relational level, and using ε derived from the source of $\mathcal{U}$ gives rise to the lifting of the initial Hausdorff definition, thus making it point-free as in:

2.1 Definition. A relation $\mathcal{U} : X \longrightarrow \mathbf{2}^X$ will be called a **neighborhood topology** if the following properties are satisfied:

i) $\mathcal{U}{:}\mathbb{T} = \mathbb{T}$ and $\mathcal{U} \subseteq \varepsilon$,
ii) $\mathcal{U}{:}\Omega \subseteq \mathcal{U}$,
iii) $(\mathcal{U} \otimes \mathcal{U}){:}\mathcal{M} \subseteq \mathcal{U}$,
iv) $\mathcal{U} \subseteq \mathcal{U}{:}\overline{\varepsilon^{\mathsf{T}}{:}\overline{\mathcal{U}}}$. □

Correspondingly, lifting may be executed for various other topology concepts. We start with the mapping to open kernels, assuming $\Omega := \overline{\varepsilon^{\mathsf{T}}{:}\overline{\varepsilon}}$ to represent the powerset ordering.

2.2 Definition. We call a relation $\mathcal{K} : \mathbf{2}^X \longrightarrow \mathbf{2}^X$ a **mapping-to-open-kernel topology**, if

i) $\mathcal{K}$ is a kernel forming, i.e., $\mathcal{K} \subseteq \Omega^{\mathsf{T}}$, $\Omega{:}\mathcal{K} \subseteq \mathcal{K}{:}\Omega$, $\mathcal{K}{:}\mathcal{K} \subseteq \mathcal{K}$,
ii) $\varepsilon{:}\mathcal{K}^{\mathsf{T}}$ is total,
iii) $(\mathcal{K} \otimes \mathcal{K}){:}\mathcal{M} = \mathcal{M}{:}\mathcal{K}$. □

Conditions (i) obviously request that $\mathcal{K}$ maps to subsets of the original one, is isotonic, and is idempotent. Condition (iii) requires $\mathcal{K}$ and $\mathcal{M}$ to commute: One may obtain kernels of an arbitrary pair of subsets first and then form their intersection, or, equivalently, start intersecting these subsets and then getting the kernel.

2.3 Proposition. The following operations are inverses of one another:

i) Given any neighborhood topology $\mathcal{U}$, the construct $\mathcal{K} := \mathtt{syq}(\mathcal{U}, \varepsilon)$ is a kernel-mapping topology.
ii) Given any kernel-mapping topology $\mathcal{K}$, the construct $\mathcal{U} := \varepsilon{:}\mathcal{K}^{\mathsf{T}}$ results in a neighborhood topology.

We cannot give the full proof for reasons of space, but indicate a part of it: The $\mathcal{K}$ defined in (i) is certainly a mapping, due to cancellation $\mathcal{K}^{\mathsf{T}}{:}\mathcal{K} \subseteq \mathtt{syq}(\varepsilon, \varepsilon) = \mathbb{I}$, and, since forming the symmetric quotient with ε on the right side of $\mathtt{syq}$ gives a total relation by definition of a membership relation.

$$\mathcal{U}(\mathcal{K}(\mathcal{U})) = \varepsilon{:}[\mathtt{syq}(\mathcal{U}, \varepsilon)]^{\mathsf{T}} = \varepsilon{:}\mathtt{syq}(\varepsilon, \mathcal{U}) = \mathcal{U} \quad \text{since } \mathtt{syq}(\varepsilon, X) \text{ is surjective}$$
$$\mathcal{K}(\mathcal{U}(\mathcal{K})) = \mathtt{syq}(\varepsilon{:}\mathcal{K}^{\mathsf{T}}, \varepsilon) = \mathcal{K}{:}\mathtt{syq}(\varepsilon, \varepsilon) = \mathcal{K}{:}\mathbb{I} = \mathcal{K} \quad \text{since } \mathcal{K} \text{ is a mapping}$$

It remains the obligation to prove

$$\begin{array}{lcl}
\mathcal{U};\mathbb{T} = \mathbb{T}\,, & & \mathcal{K} \subseteq \Omega^{\mathsf{T}}, \\
\mathcal{U} \subseteq \varepsilon, & & \Omega;\mathcal{K} \subseteq \mathcal{K};\Omega, \\
\mathcal{U};\Omega \subseteq \mathcal{U}, & \iff & \mathcal{K};\mathcal{K} \subseteq \mathcal{K}, \\
(\mathcal{U} \otimes \mathcal{U})\,;\mathcal{M} \subseteq \mathcal{U}, & & \varepsilon;\mathcal{K}^{\mathsf{T}};\mathbb{T} = \mathbb{T}\,, \\
\mathcal{U} \subseteq \mathcal{U};\overline{\varepsilon^{\mathsf{T}};\overline{\mathcal{U}}}. & & (\mathcal{K} \otimes \mathcal{K});\mathcal{M} = \mathcal{M};\mathcal{K}.
\end{array}$$

A third form of a topology definition runs as follows:

2.4 Definition. A binary vector $\mathcal{O}_V$ along $\mathbf{2}^X$ will be called an **open set topology** provided

$$\begin{array}{rl}
\text{i)} & \mathtt{syq}(\varepsilon, \mathbb{L}) \subseteq \mathcal{O}_V \quad \mathtt{syq}(\varepsilon, \mathbb{T}) \subseteq \mathcal{O}_V, \\
\text{ii)} & v \subseteq \mathcal{O}_V \quad \Longrightarrow \quad \mathtt{syq}(\varepsilon, \varepsilon;v) \subseteq \mathcal{O}_V \quad \text{for all vectors } v \subseteq \mathbf{2}^X, \\
\text{iii)} & \mathcal{M}^{\mathsf{T}};(\mathcal{O}_V \otimes \mathcal{O}_V) \subseteq \mathcal{O}_V.
\end{array}$$

□

With (i), $\mathbb{T}$ and $\mathbb{L}$ are declared to be open. The vector v in (ii) determines a set of open sets conceived as points in the powerset. It is demanded that their union be open again. According to (iii), intersetion (meet $\mathcal{M}$) applied to two (i.e., finitely many) open sets must be open.

One may also study the membership restricted to open sets $\varepsilon_{\mathcal{O}} := \varepsilon \cap \mathbb{T};\mathcal{O}_V^{\mathsf{T}}$.

All these topology concepts are cryptomorphic — as could be expected. The transitions below may be written down in TituRel so as to achieve the version intended. In particular, $\mathcal{O}_V$ and $\mathcal{O}_D$ are distinguished, although they are very similar, namely "diagonal matrix" vs. "column vector" to characterize a subset.

$$\mathcal{U} \quad \mapsto \quad \mathcal{K} := \mathtt{syq}(\mathcal{U}, \varepsilon) : \mathbf{2}^X \longrightarrow \mathbf{2}^X$$

$$\mathcal{K} \quad \mapsto \quad \mathcal{U} := \varepsilon;\mathcal{K}^{\mathsf{T}} : X \longrightarrow \mathbf{2}^X.$$

$$\mathcal{O}_D \quad \mapsto \quad \mathcal{U} := \varepsilon;\mathcal{O}_D;\Omega$$

$$\mathcal{O}_D \quad \mapsto \quad \mathcal{O}_V := \mathcal{O}_D;\mathbb{T}$$

$$\mathcal{K}, \mathcal{U}, \mathcal{O}_V \quad \mapsto \quad \mathcal{O}_D := \mathbb{I} \cap \overline{\varepsilon^{\mathsf{T}};\overline{\mathcal{U}}} = \mathbb{I} \cap \mathcal{O}_V;\mathbb{T} = \mathcal{K}^{\mathsf{T}};\mathcal{K}$$

One may, thus, obtain the same topology in different forms as it is shown below for the example given before Definition 2.1:

$$
\begin{array}{cc}
 & \begin{array}{cccccccccccccccc} \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} & \{d\} & \{a,d\} & \{b,d\} & \{a,b,d\} & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\} \end{array} \\
\begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix}
0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1 \\
0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\
0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\
0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1
\end{pmatrix}
\end{array}
$$

$$
\begin{array}{cc}
 & \begin{array}{cccccccccccccccc} \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} & \{d\} & \{a,d\} & \{b,d\} & \{a,b,d\} & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\} \end{array} \\
\begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix}
0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1 \\
0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1 \\
0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1
\end{pmatrix}
\end{array}
$$

$$
\begin{array}{cc}
 & \begin{array}{cccccccccccccccc} \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} & \{d\} & \{a,d\} & \{b,d\} & \{a,b,d\} & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\} \end{array} \\
\begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix}
0&1&0&0&0&1&0&1&0&0&0&0&0&1&0&1 \\
0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1 \\
0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1
\end{pmatrix}
\end{array}
$$

$$
\begin{array}{cc}
 & \begin{array}{cccccccccccccccc} \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} & \{d\} & \{a,d\} & \{b,d\} & \{a,b,d\} & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\} \end{array} \\
\begin{array}{r} \{\} \\ \{a\} \\ \{b\} \\ \{a,b\} \\ \{c\} \\ \{a,c\} \\ \{b,c\} \\ \{a,b,c\} \\ \{d\} \\ \{a,d\} \\ \{b,d\} \\ \{a,b,d\} \\ \{c,d\} \\ \{a,c,d\} \\ \{b,c,d\} \\ \{a,b,c,d\} \end{array} & \begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix}
\end{array}
$$

ε $\qquad \mathcal{U}$ $\qquad \varepsilon_{\mathcal{O}} := \varepsilon \cap \mathbb{T} ; \mathcal{O}_V^{\mathsf{T}} = \varepsilon ; \mathcal{K} \cap \varepsilon$ $\qquad \mathcal{K} := \mathtt{syq}(\mathcal{U}, \varepsilon)$ indicating $\mathcal{O}_D$ as diagonal

Open sets as membership and open kernel mapping

By the way, there exists also a kernel-forming that doesn't lead to a topology; it is not intersection-closed as can be seen from the subsets $\{a, b\}$ and $\{b, d\}$ with intersection $\{d\}$:

$$
\mathcal{K} = \begin{array}{cc}
 & \begin{array}{cccccccccccccccc} \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} & \{d\} & \{a,d\} & \{b,d\} & \{a,b,d\} & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\} \end{array} \\
\begin{array}{r} \{\} \\ \{a\} \\ \{b\} \\ \{a,b\} \\ \{c\} \\ \{a,c\} \\ \{b,c\} \\ \{a,b,c\} \\ \{d\} \\ \{a,d\} \\ \{b,d\} \\ \{a,b,d\} \\ \{c,d\} \\ \{a,c,d\} \\ \{b,c,d\} \\ \{a,b,c,d\} \end{array} & \begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0 \\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix}
\end{array}
$$

3 Continuity

For a mathematical structure, one routinely defines its structure-preserving mappings. Traditionally, this is handled under the name of a homomorphism; it may be defined for relational structures as well as for algebraic ones (i.e., those where structure is described by *mappings* as for groups, e.g.) in more or less the same standard way; it is available for a homogeneous as well as a heterogeneous structure.

One might naively be tempted to study also the comparison of topologies with the concept of homomorphism; however, this doesn't work.

The continuity condition turns out to be a mixture of going forward and backwards as we will see. We recall the standard definition of continuity.

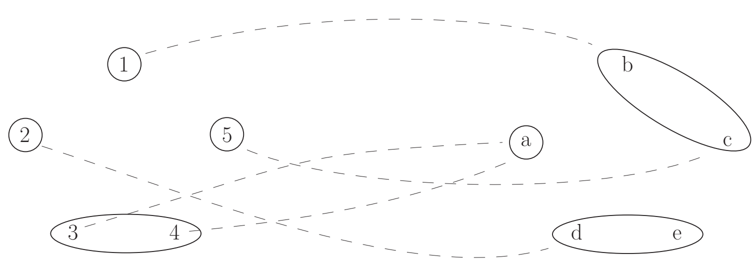
$$\begin{tikzcd}
2^X & 2^{X'} \arrow[l, "\vartheta_{f^\mathsf{T}}"'] \\
X \arrow[u, "\mathcal{U}"] \arrow[r, "f"'] & X' \arrow[u, "\mathcal{U}'"']
\end{tikzcd}$$

For two given neighborhood topologies $\mathcal{U}, \mathcal{U}'$ on sets X, X', one calls a mapping $f : X \longrightarrow X'$

f **continuous** $:\Longleftrightarrow$ For every point $p \in X$ and every $U' \in \mathcal{U}'(f(p))$, there exists a $U \in \mathcal{U}(p)$ such that $f(U) \subseteq U'$.

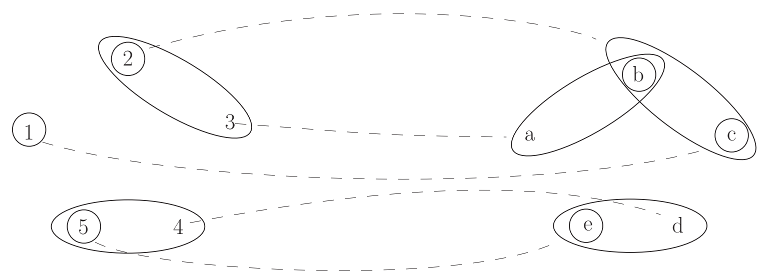
A first example of a continuous mapping shows two open set bases, arranged as columns of matrices R_1, R_2, and the mapping f:

$$R_1 = \begin{array}{cc} & \begin{array}{cccc}\text{alpha}&\text{beta}&\text{gamma}&\text{delta}\end{array} \\ \begin{array}{c}1\\2\\3\\4\\5\end{array} & \begin{pmatrix}\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}\\\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}\\\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}\end{pmatrix}\end{array} \qquad f = \begin{array}{cc} & \begin{array}{ccccc}\text{a}&\text{b}&\text{c}&\text{d}&\text{e}\end{array} \\ \begin{array}{c}1\\2\\3\\4\\5\end{array} & \begin{pmatrix}\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}\\\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}\end{pmatrix}\end{array} \qquad R_2 = \begin{array}{cc} & \begin{array}{ccc}\text{alpha2}&\text{beta2}&\text{gamma2}\end{array} \\ \begin{array}{c}\text{a}\\\text{b}\\\text{c}\\\text{d}\\\text{e}\end{array} & \begin{pmatrix}\mathbf{1}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{1}&\mathbf{0}\\\mathbf{0}&\mathbf{1}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{1}\\\mathbf{0}&\mathbf{0}&\mathbf{1}\end{pmatrix}\end{array}$$



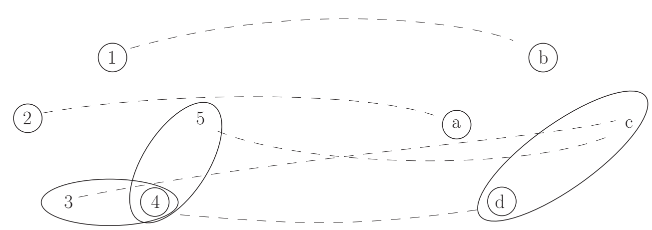
The following is another example of a continuous mapping. Again two open set bases are arranged as columns of matrices R_1, R_2 and shown together with the mapping f:

$$R_1 = \begin{array}{c@{}c} & \begin{array}{ccccc} \rotatebox{90}{alpha} & \rotatebox{90}{beta} & \rotatebox{90}{gamma} & \rotatebox{90}{delta} & \rotatebox{90}{epsilon} \end{array} \\ \begin{array}{c}1\\2\\3\\4\\5\end{array} & \begin{pmatrix} \mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{1}&\mathbf{1}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{1} \end{pmatrix} \end{array} \qquad f = \begin{array}{c@{}c} & \begin{array}{ccccc} \rotatebox{90}{a} & \rotatebox{90}{b} & \rotatebox{90}{c} & \rotatebox{90}{d} & \rotatebox{90}{e} \end{array} \\ \begin{array}{c}1\\2\\3\\4\\5\end{array} & \begin{pmatrix} \mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1} \end{pmatrix} \end{array} \qquad R_2 = \begin{array}{c@{}c} & \begin{array}{cccccc} \rotatebox{90}{alpha2} & \rotatebox{90}{beta2} & \rotatebox{90}{gamma2} & \rotatebox{90}{delta2} & \rotatebox{90}{epsilon2} & \rotatebox{90}{eta2} \end{array} \\ \begin{array}{c}\mathrm{a}\\\mathrm{b}\\\mathrm{c}\\\mathrm{d}\\\mathrm{e}\end{array} & \begin{pmatrix} \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}\\ \mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{1}&\mathbf{0}\\ \mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}\\ \mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{1} \end{pmatrix} \end{array}$$



Now follows a third example of a continuous mapping, using the same style.

$$R_1 = \begin{array}{c@{}c} & \begin{array}{ccccc} \rotatebox{90}{alpha} & \rotatebox{90}{beta} & \rotatebox{90}{gamma} & \rotatebox{90}{delta} & \rotatebox{90}{epsilon} \end{array} \\ \begin{array}{c}1\\2\\3\\4\\5\end{array} & \begin{pmatrix} \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}\\ \mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}\\ \mathbf{1}&\mathbf{1}&\mathbf{1}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0} \end{pmatrix} \end{array} \qquad f = \begin{array}{c@{}c} & \begin{array}{cccc} \rotatebox{90}{a} & \rotatebox{90}{b} & \rotatebox{90}{c} & \rotatebox{90}{d} \end{array} \\ \begin{array}{c}1\\2\\3\\4\\5\end{array} & \begin{pmatrix} \mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}\\ \mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}\\ \mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0} \end{pmatrix} \end{array} \qquad R_2 = \begin{array}{c@{}c} & \begin{array}{cccc} \rotatebox{90}{alpha2} & \rotatebox{90}{beta2} & \rotatebox{90}{gamma2} & \rotatebox{90}{delta2} \end{array} \\ \begin{array}{c}\mathrm{a}\\\mathrm{b}\\\mathrm{c}\\\mathrm{d}\end{array} & \begin{pmatrix} \mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}\\ \mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}\\ \mathbf{0}&\mathbf{1}&\mathbf{1}&\mathbf{0} \end{pmatrix} \end{array}$$



According to our general policy, we should try to lift the continuity definition to a point-free relational level. However, one soon sees that this requires that we need the concept of an existential and of an inverse image.

3.1 Existential and inverse image

The lifting of a relation R to a corresponding relation ϑ_R on the powerset level has been called its existential image; cf. [Bd96]. (There exist also the power transpose Λ_R and the power relator ζ_R.)

$$\begin{tikzcd}
X \arrow[r, "R"] \arrow[d, "\varepsilon"'] \arrow[dr, bend left, "\Lambda"] & Y \arrow[d, "\varepsilon'"] \\
\mathbf{2}^X \arrow[r, "\vartheta"] \arrow[r, bend right=40, "\zeta"'] & \mathbf{2}^Y
\end{tikzcd}$$

Assuming an arbitrary relation $R : X \longrightarrow Y$ with membership relations $\varepsilon : X \longrightarrow \mathbf{2}^X$ and $\varepsilon' : Y \longrightarrow \mathbf{2}^Y$ on either side one calls

$$\vartheta := \vartheta_R := \mathtt{syq}(R^\mathsf{T};\varepsilon, \varepsilon') = \overline{\varepsilon^\mathsf{T};R;\overline{\varepsilon'}} \cap \overline{\overline{\varepsilon^\mathsf{T};R};\varepsilon'},$$

its **existential image**. The **inverse image** is obtained when taking the existential image of the transposed relation.

It turns out, according to [Bd96,Sch11a], that ϑ is

- (lattice-)continuous wrt. the powerset orders $\Omega = \overline{\varepsilon^\mathsf{T};\overline{\varepsilon}}$,
- multiplicative: $\vartheta_{Q;R} = \vartheta_Q;\vartheta_R$,
- preserves identities: $\vartheta_{\mathbb{I}_X} = \mathbb{I}_{\mathbf{2}^X}$,
- R may be re-obtained from ϑ_R as $R = \overline{\varepsilon;\vartheta_R;\overline{\varepsilon'}^\mathsf{T}}$.

It also satisfies, according to [dRE98,Sch11a], the following simulation property. R and its existential image as well as its inverse image **simulate** each other via $\varepsilon, \varepsilon'$:

$$\varepsilon^\mathsf{T};R = \vartheta_R;\varepsilon'^\mathsf{T} \qquad \varepsilon'^\mathsf{T};R^\mathsf{T} = \vartheta_{R^\mathsf{T}};\varepsilon^\mathsf{T}.$$

$$\begin{tikzcd}
2^X \arrow[loop left, "\Omega"] \arrow[r, shift left, "\vartheta_R"] \arrow[r, bend right=50, "\zeta"'] & 2^{X'} \arrow[l, shift left, "\vartheta_{R^\mathsf{T}}"] \arrow[loop right, "\Omega'"] \\
X \arrow[u, "\varepsilon"] \arrow[r, "R"'] & X' \arrow[u, "\varepsilon'"']
\end{tikzcd}$$

$$
R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c}
\begin{array}{cccc} \text{a} & \text{b} & \text{c} & \text{d} \end{array} \\
\begin{pmatrix}
0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0
\end{pmatrix}
\end{array}
\qquad
\vartheta_R = \begin{array}{r|cccccccccccccccc}
 & \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} & \{d\} & \{a,d\} & \{b,d\} & \{a,b,d\} & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\} \\
\hline
\{\} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{1\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\{2\} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{1,2\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\{3\} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{1,3\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\{2,3\} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{1,2,3\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\{4\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{1,4\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\{2,4\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{1,2,4\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\{3,4\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\{1,3,4\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\{2,3,4\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\{1,2,3,4\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\{5\} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{1,5\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\{2,5\} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{1,2,5\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\{3,5\} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{1,3,5\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\{2,3,5\} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{1,2,3,5\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\{4,5\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\{1,4,5\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\{2,4,5\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\{1,2,4,5\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\{3,4,5\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\{1,3,4,5\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\{2,3,4,5\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\{1,2,3,4,5\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

The inverse image is obviously not the transpose of the existential image.

$$
\vartheta_{R^\mathsf{T}} = \begin{array}{r|cccccccccccccccccccccccccccccccc}
 & \{\} & \{1\} & \{2\} & \{1,2\} & \{3\} & \{1,3\} & \{2,3\} & \{1,2,3\} & \{4\} & \{1,4\} & \{2,4\} & \{1,2,4\} & \{3,4\} & \{1,3,4\} & \{2,3,4\} & \{1,2,3,4\} & \{5\} & \{1,5\} & \{2,5\} & \{1,2,5\} & \{3,5\} & \{1,3,5\} & \{2,3,5\} & \{1,2,3,5\} & \{4,5\} & \{1,4,5\} & \{2,4,5\} & \{1,2,4,5\} & \{3,4,5\} & \{1,3,4,5\} & \{2,3,4,5\} & \{1,2,3,4,5\} \\
\hline
\{\} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{a\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{b\} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{a,b\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{c\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{a,c\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{b,c\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{a,b,c\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{d\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{a,d\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\{b,d\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{a,b,d\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\{c,d\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\{a,c,d\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\{b,c,d\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\{a,b,c,d\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

The existential image and the inverse image also satisfy formulae with respect to the powerset orderings:

i) $\Omega' \mathbin{;} \vartheta_{f^\mathsf{T}} \subseteq \vartheta_{f^\mathsf{T}} \mathbin{;} \Omega$ if f is a mapping,
ii) $\Omega \mathbin{;} \vartheta_{f^\mathsf{T}}^\mathsf{T} = \vartheta_f \mathbin{;} \Omega'$ if f is a mapping.

The proof cannot be given in the present limited environment.

3.2 Lifting the continuity condition

With the inverse image, we will manage to lift the continuity definition to a point-free relational level.

3.1 Definition. Consider two neighborhood topologies $\mathcal{U} : X \longrightarrow \mathbf{2}^X$ and $\mathcal{U}' : X' \longrightarrow \mathbf{2}^{X'}$ as well as a mapping $f : X \longrightarrow X'$. We call

$$f \quad \mathcal{U}\textbf{-continuous} \quad :\Longleftrightarrow \quad f;\mathcal{U}';\vartheta_{f^{\mathsf{T}}} \subseteq \mathcal{U} \qquad \square$$

The semi-formal development of the point-free version out of the predicate-logic form is rather tricky — and too long to be included here in full length. It is interesting to observe that one must not quantify over subsets U, V. One should always restrict to quantify over *elements in the powerset* u, v.

For every $p \in X$, every $V \in \mathcal{U}'(f(p))$, there is a $U \in \mathcal{U}(p)$ so that $f(U) \subseteq V$.

$$\begin{aligned}
&\forall p \in X : \forall V \in \mathcal{U}'(f(p)) : \exists U \in \mathcal{U}(p) : f(U) \subseteq V\\
&\forall p \in X : \forall v \in \mathbf{2}^{X'} : \mathcal{U}'_{f(p),v} \longrightarrow \big(\exists u : \mathcal{U}_{p,u} \wedge f^{\mathsf{T}};\varepsilon;u \subseteq \varepsilon';v\big)\\
&\quad \ldots\\
&\forall p : \forall u : \big(\exists q : \exists v : f_{p,q} \wedge \mathcal{U}'_{q,v} \wedge [\vartheta_{f^{\mathsf{T}}}]_{v,u}\big) \longrightarrow \mathcal{U}_{p,u}\\
&\forall p : \forall u : [f;\mathcal{U}';\vartheta_{f^{\mathsf{T}}}]_{p,u} \longrightarrow \mathcal{U}_{p,u}\\
&f;\mathcal{U}';\vartheta_{f^{\mathsf{T}}} \subseteq \mathcal{U}
\end{aligned}$$

The equivalent version $f;\mathcal{U}' \subseteq \mathcal{U};\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}}$ is obtained by shunting the mapping $\vartheta_{f^{\mathsf{T}}}$.

3.3 Remark on comparison of structures in general

Comparison of structures via homomorphisms or structure-preserving mappings is omnipresent in mathematics and theoretical computer science, be it for groups, lattices, modules, graphs, or others. Most of these follow a general schema.

$$\begin{tikzcd}
Y_1 \arrow[r, "\Psi"] & Y_2 \\
X_1 \arrow[u, "R_1"] \arrow[r, "\Phi"'] & X_2 \arrow[u, "R_2"']
\end{tikzcd}$$

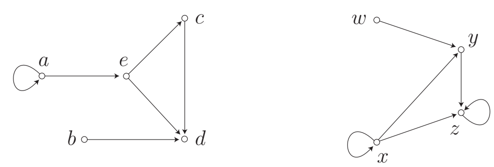
Two "structures" of whatever kind shall be given by a relation $R_1 : X_1 \longrightarrow Y_1$ and a relation $R_2 : X_2 \longrightarrow Y_2$. With mappings $\Phi : X_1 \longrightarrow X_2$ and $\Psi : Y_1 \longrightarrow Y_2$ they shall be compared, and we may ask whether these mappings transfer the first structure "sufficiently nice" into the second one.

The standard mechanism is to call the pair Φ, Ψ a **homomorphism** from R_1 to R_2, if $R_1 ; \Psi \subseteq \Phi ; R_2$. The two Φ, Ψ constitute an **isomorphism**, if Φ, Ψ as well as $\Phi^\mathsf{T}, \Psi^\mathsf{T}$ are homomorphisms.

If any two elements x, y are related by R_1, so are their images $\Phi(x), \Psi(y)$ by R_2:

$$\forall x \in X_1 : \forall y \in Y_1 : (x, y) \in R_1 \to (\Phi(x), \Psi(y)) \in R_2.$$

This concept is also suitable for relational structures; it works in particular for a graph homomorphism Φ, Φ — meaning $X_1 = X_2$, e.g. — as in the following example of a graph homomorphism, i.e., a homomorphism of a non-algebraic structure.



$$R_1 = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \!\! \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} \end{pmatrix} \end{array} \qquad \Phi = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \!\! \begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \end{pmatrix} \end{array} \qquad R_2 = \begin{array}{c} \\ w \\ x \\ y \\ z \end{array} \!\! \begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix} \end{array}$$

We recall the rolling of homomorphisms when Φ, Ψ are mappings as in

$$R_1 ; \Psi \subseteq \Phi ; R_2 \iff R_1 \subseteq \Phi ; R_2 ; \Psi^\mathsf{T} \iff \Phi^\mathsf{T} ; R_1 \subseteq R_2 ; \Psi^\mathsf{T} \iff \Phi^\mathsf{T} ; R_1 ; \Psi \subseteq R_2$$

If relations Φ, Ψ are not mappings, one cannot fully execute this rolling; there remain different forms of (bi-)simulations as explicated in [dRE98].

This is where the continuity condition fails. One cannot "roll" in this way and has just the two forms given above.

3.4 Cryptomorphy of the continuity conditions

Once we have the lifted relation-algebraic form for a neighborhood topology that uses the inverse image, we will imediately extend it to the other topology versions.

3.2 Definition. Given sets X, X' with topologies, we consider a mapping $f : X \longrightarrow X'$ together with its inverse image $\vartheta_{f^\mathsf{T}} : \mathbf{2}^{X'} \longrightarrow \mathbf{2}^X$. Then we say that f is

i) **$\mathcal{K}$-continuous** $:\iff$ $\mathcal{K}_2^{\mathsf{T}} ; \vartheta_{f^{\mathsf{T}}} \subseteq \overline{\varepsilon_2{}^{\mathsf{T}} ; f^{\mathsf{T}} ; \overline{\varepsilon_1}} ; \mathcal{K}_1^{\mathsf{T}}$,
ii) **$\mathcal{O}_D$continuous** $:\iff$ $\mathcal{O}_{D2} ; \vartheta_{f^{\mathsf{T}}} \subseteq \vartheta_{f^{\mathsf{T}}} ; \mathcal{O}_{D1}$,
iii) **$\mathcal{O}_V$-continuous** $:\iff$ $\vartheta_{f^{\mathsf{T}}}^{\mathsf{T}} ; \mathcal{O}'_V \subseteq \mathcal{O}_V$,
iv) **$\varepsilon_{\mathcal{O}}$-continuous** $:\iff$ $f ; \varepsilon_{\mathcal{O}_2} ; \vartheta_{f^{\mathsf{T}}} \subseteq \varepsilon_{\mathcal{O}_1}$. $\square$

The easiest access is to the open sets version: Inverse images of open sets have to be open again. Continuity with regard to kernel mapping is an ugly condition.

All these versions of continuity can be shown to be equivalent, so that there is an obligation to prove f is **$\mathcal{U}$-continuous** $\iff$

f is **$\mathcal{K}$-continuous** $\iff$ f is **$\mathcal{O}_D$-continuous** $\iff$
f is **$\mathcal{O}_V$-continuous** $\iff$ f is **$\varepsilon_{\mathcal{O}}$-continuous**

For economy of proof, we formulate this slightly differently. Then an immediate equivalence is proved, followed by a long cyclic proof.

3.3 Proposition. The various continuity conditions mean essentially the same:

i) **$\mathcal{U}$-continuous** $\iff$ **$\mathcal{K}$-continuous**
ii) **$\mathcal{U}$-continuous** $\implies$ **$\mathcal{O}_D$-continuous**
iii) **$\mathcal{O}_D$-continuous** $\implies$ **$\mathcal{O}_V$-continuous**
iv) **$\mathcal{O}_V$-continuous** $\implies$ **$\varepsilon_{\mathcal{O}}$-continuous**
v) **$\varepsilon_{\mathcal{O}}$-continuous** $\implies$ **$\mathcal{U}$-continuous**

Proof: i) $f ; \mathcal{U}_2 ; \vartheta_{f^{\mathsf{T}}} \subseteq \mathcal{U}_1 = \varepsilon_1 ; \mathcal{K}_1^{\mathsf{T}}$ assumption and expansion of $\mathcal{U}_1$
$\iff$ $f ; \varepsilon_2 ; \mathcal{K}_2^{\mathsf{T}} ; \vartheta_{f^{\mathsf{T}}} ; \mathcal{K}_1 \subseteq \varepsilon_1$ expanding $\mathcal{U}_2$ and shunting
$\iff$ $\varepsilon_2{}^{\mathsf{T}} ; f^{\mathsf{T}} ; \overline{\varepsilon_1} \subseteq \overline{\mathcal{K}_2^{\mathsf{T}} ; \vartheta_{f^{\mathsf{T}}} ; \mathcal{K}_1}$ Schröder rule
$\iff$ $\mathcal{K}_2^{\mathsf{T}} ; \vartheta_{f^{\mathsf{T}}} ; \mathcal{K}_1 \subseteq \overline{\varepsilon_2{}^{\mathsf{T}} ; f^{\mathsf{T}} ; \overline{\varepsilon_1}}$ negated
$\iff$ $\mathcal{K}_2^{\mathsf{T}} ; \vartheta_{f^{\mathsf{T}}} \subseteq \overline{\varepsilon_2{}^{\mathsf{T}} ; f^{\mathsf{T}} ; \overline{\varepsilon_1}} ; \mathcal{K}_1^{\mathsf{T}}$ shunting again

ii) $\overline{\varepsilon_2^{\mathsf{T}} ; \overline{\mathcal{U}_2}} \subseteq \overline{\varepsilon_2^{\mathsf{T}} ; f^{\mathsf{T}} ; f ; \overline{\mathcal{U}_2}} = \overline{\varepsilon_2^{\mathsf{T}} ; f^{\mathsf{T}} ; \overline{f ; \mathcal{U}_2}} = \overline{\vartheta_{f^{\mathsf{T}}} ; \varepsilon_1^{\mathsf{T}} ; \overline{f ; \mathcal{U}_2}}$
$\subseteq \overline{\vartheta_{f^{\mathsf{T}}} ; \varepsilon_1^{\mathsf{T}} ; \overline{\mathcal{U}_1 ; \vartheta_{f^{\mathsf{T}}}^{\mathsf{T}}}} = \vartheta_{f^{\mathsf{T}}} ; \overline{\varepsilon_1^{\mathsf{T}} ; \overline{\mathcal{U}_1}} ; \vartheta_{f^{\mathsf{T}}}^{\mathsf{T}}$
$\implies$ $\mathcal{O}_{D2} = \mathbb{I} \cap \overline{\varepsilon_2^{\mathsf{T}} ; \overline{\mathcal{U}_2}} \subseteq \vartheta_{f^{\mathsf{T}}} ; \vartheta_{f^{\mathsf{T}}}^{\mathsf{T}} \cap \vartheta_{f^{\mathsf{T}}} ; \overline{\varepsilon_1^{\mathsf{T}} ; \overline{\mathcal{U}_1}} ; \vartheta_{f^{\mathsf{T}}}^{\mathsf{T}}$
$= \vartheta_{f^{\mathsf{T}}} (\mathbb{I} \cap \overline{\varepsilon_1^{\mathsf{T}} ; \overline{\mathcal{U}_1}}) ; \vartheta_{f^{\mathsf{T}}}^{\mathsf{T}} = \vartheta_{f^{\mathsf{T}}} \mathcal{O}_{D1} ; \vartheta_{f^{\mathsf{T}}}^{\mathsf{T}}$

iii) $\vartheta_{f^{\mathsf{T}}}^{\mathsf{T}} ; \mathcal{O}_{V_2} = \vartheta_{f^{\mathsf{T}}}^{\mathsf{T}} ; \mathcal{O}_{D2} ; \mathbb{T} = \vartheta_{f^{\mathsf{T}}}^{\mathsf{T}} ; \mathcal{O}_{D2}^{\mathsf{T}} ; \mathbb{T} \subseteq \mathcal{O}_{D1}^{\mathsf{T}} ; \vartheta_{f^{\mathsf{T}}}^{\mathsf{T}} ; \mathbb{T} = \mathcal{O}_{D1} ; \vartheta_{f^{\mathsf{T}}}^{\mathsf{T}} ; \mathbb{T} \subseteq \mathcal{O}_{D1} ; \mathbb{T} = \mathcal{O}_{V_1}$

iv) $f ; \varepsilon_{\mathcal{O}_2} ; \vartheta_{f^{\mathsf{T}}} = f ; (\varepsilon_2 \cap \mathbb{T} ; \mathcal{O}_{V_2}^{\mathsf{T}}) ; \vartheta_{f^{\mathsf{T}}} = (f ; \varepsilon_2 \cap f ; \mathbb{T} ; \mathcal{O}_{V_2}^{\mathsf{T}}) ; \vartheta_{f^{\mathsf{T}}}$
$= (\varepsilon_1 ; \vartheta_{f^{\mathsf{T}}}^{\mathsf{T}} \cap \mathbb{T} ; \mathcal{O}_{V_2}^{\mathsf{T}}) ; \vartheta_{f^{\mathsf{T}}}$
$= \varepsilon_1 \cap \mathbb{T} ; \mathcal{O}_{V_2}^{\mathsf{T}} ; \vartheta_{f^{\mathsf{T}}}$ [Sch11a, Prop. 5.4]

$\subseteq \varepsilon_1 \cap \mathbb{T} ; \mathcal{O}^{\mathsf{T}}_{V_1} = \varepsilon_{\mathcal{O}_1}$

v) $f ; \mathcal{U}_2 ; \vartheta_{f^{\mathsf{T}}} = f ; \varepsilon_{\mathcal{O}_2} ; \Omega_2 ; \vartheta_{f^{\mathsf{T}}}$
$\subseteq f ; \varepsilon_{\mathcal{O}_2} ; \vartheta_{f^{\mathsf{T}}} ; \Omega_1$
$\subseteq \varepsilon_{\mathcal{O}_1} ; \Omega_1$ assumption
$= \mathcal{U}_1$

4 Conclusion

This article is part of a more extended ongoing research concerning relational methods in topology and in programming. Other attempts are directed towards simplicial complexes, e.g., for pretzels with several holes, the projective plane, or knot decompositions. An important question is whether it is possible to decide orientability, e.g., of a manifold *without* working on it globally. Compare this with the classic philosophers problem. Modelling the actions of the dining philosophers is readily available. One will be able to work on the state space based on 10 philosophers or 15. However, this doesn't scale up, so that local work is necessary. This work is intended to enhance such studies concerning communication and protocols.

Acknowledgement. The author gratefully acknowledges fruitful email discussions with Michael Winter.

Literatur

Bd96. Richard S. Bird and Oege de Moor. *Algebra of Programming.* Prentice-Hall International, 1996.

dRE98. Willem-Paul de Roever and Kai Engelhardt. *Data Refinement: Model-Oriented Proof Methods and their Comparison.* Number 47 in Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 1998.

Sch03. Gunther Schmidt. Relational Language. Technical Report 2003-05, Fakultät für Informatik, Universität der Bundeswehr München, 2003. 101 pages, `http://mucob.dyndns.org:30531/~gs/Papers/LanguageProposal.html`.

Sch11a. Gunther Schmidt. *Relational Mathematics*, volume 132 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, 2011. ISBN 978-0-521-76268-7, 584 pages.

Sch11b. Gunther Schmidt. TituRel: Sprache für die Relationale Mathematik. Technical Report 132, Arbeitsberichte des Instituts für Wirtschaftsinformatik, Universität Münster, 2011. 11 pages, `http://mucob.dyndns.org:30531/~gs/Papers/Raesfeld2011ExtendedAbstract.pdf`.

SS89. Gunther Schmidt and Thomas Ströhlein. *Relationen und Graphen.* Mathematik für Informatiker. Springer-Verlag, 1989. ISBN 3-540-50304-8, ISBN 0-387-50304-8.

SS93. Gunther Schmidt and Thomas Ströhlein. *Relations and Graphs — Discrete Mathematics for Computer Scientists.* EATCS Monographs on Theoretical Computer Science. Springer-Verlag, 1993. ISBN 3-540-56254-0, ISBN 0-387-56254-0.