Relational Measures and Integration in Preference Modeling

Gunther Schmidt [a,*] Rudolf Berghammer [b]

[a] *Fakultät für Informatik, Universität der Bundeswehr München 85577 Neubiberg, Germany*

[b] *Institut für Informatik, Christian-Albrechts-Universität zu Kiel Olshausenstraße 40, 24098 Kiel, Germany*

Abstract

Based on a set of criteria and a measuring lattice, we introduce relational measures as generalizations of fuzzy measures. The latter have recently made their way from the interval $[0, 1] \subseteq \mathbb{R}$ to the ordinal or even to the qualitative level. We proceed further and introduce relational measures and relational integration. First ideas of this kind, but for the real-valued linear orderings stem from Choquet (1950s) and Sugeno (1970s). We generalize to not necessarily linear orders and handle it algebraically and in a point-free manner. We thus open this area of research for treatment with theorem provers which would be extremely difficult for the classical presentation of Choquet and Sugeno integrals. Our specification of the relational integral is operational. It can immediately be translated into the programming language of RelView and, hence, the tool can be used for solving practical problems.

Key words: relational measure, relational integral, Choquet integral, Sugeno integral, relation algebra, evidence and belief, plausibility measure

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1 Introduction

Mankind has developed a multitude of concepts to reason about something that is *better than* or is *more attractive than* or is *similar to* something else. Such concepts lead to an enormous bulk of formulae and interdependencies which are extensively studied in such differently shaped books as [6,9,10,16], to mention only a few.

* Corresponding author.
Email address: `gunther.schmidt@unibw.de` (Gunther Schmidt).

We start from the concept of an *order* and a *strictorder*, defined as a transitive, antisymmetric, reflexive relation or as a transitive and asymmetric relation, respectively. In earlier times it was not clear at all that orderings need not be *linear* orderings. But since the development of lattice theory in the 1930s it became more and more evident that most of our reasoning with orderings was also possible when they failed to be linear ones. So people studied fuzziness mainly along the linear order of $\mathbb{R}$ and began only later to generalize to the ordinal level: Numbers indicate the relative position of items, but no longer the magnitude of difference. Then they moved to the interval level: Numbers indicate the magnitude of difference between items, but there is no absolute zero point. Examples are attitude scales and opinion scales. We proceed even further and introduce relational measures with values in a lattice. Measures traditionally provide a basis for integration. Astonishingly, this holds true for these relational measures so that it becomes possible to introduce a concept of relational integration.

After this introduction, we sketch in Sect. 2 several approaches of Preference Modeling. Sect. 3 provides an introductory example in some detail. Then several relation-algebraic preliminaries are recalled in Sect. 4 and in addition order-theoretic functionals in Sect. 5. In Sect. 6 we introduce the concept of a relational measure, with which we perform relational integration in Sect. 7. Two ways of constructing relational measures follow in Sect. 8. The article closes with some concluding remarks.

2 Modeling Preferences

Who is about to make severe decisions will usually base these on carefully selected basic information and clean lines of reasoning. It is in general not too difficult to apply *just one* criterion and to operate according to this criterion. If several criteria must be taken into consideration, one has also to consider the all too often occurring situation that these provide contradictory information as, e.g., in “This car looks nicer, but it is much more expensive”. Social and economical sciences have developed techniques to model what takes place when decisions are to be made in an environment with a multitude of diverging criteria; see, e.g., the collection [5].

So finding decisions became abstracted to a scientific task. We may observe two lines of development. The Anglo-Saxon countries, in particular, formulated *utility theory*, in which numerical values shall indicate the intensity of some preference. Mainly in continental Europe, on the other hand side, binary relations were used to model pairwise preference; see, e.g., [8,9,12]. While the former idea allows to easily relate to statistics, the latter is based on evidence via direct comparison.

In earlier years indeed, basic information was quite often statistical in nature and expressed in real numbers. Today we have more often fuzzy, vague, rough, etc. forms of qualification. Corresponding to this observation, appropriate methods and techniques have been studied; see, e.g., [9]. But when coming from work with real numbers, one is not immediately ready to abandon monotone realizability of orders on the real axis.

In this article, we start on the other side: We assume the measuring to take place in a lattice instead of the linear order $(\mathbb{R}, \leq)$, and we employ point-free relation algebra, which shortens proofs considerably. We found out that one may then reformulate the theory of belief of, e.g., [16]. This work can then be supported in two ways: Proofs may be checked or even found with theorem provers, and practical problems my be tackled with computer help, e.g., with the relation-algebraic programming and visualization tool RelView. This article has been prepared with the help of TituRel [13,14], which is an elaborate extension of Haskell providing relations as a data type with full relational typing control and domain construction facilities. In addition, an interpreter allows to evaluate relational terms and to present them, e.g., as Boolean matrices with row and column markings.

3 Introductory Example

We first give an example of relational integration deciding for a car to be bought out of several offers. We intend to follow a set $\mathcal{C}$ of three criteria, namely *color, price*, and *speed.* They are, of course, not of equal importance for us; price, e.g., will most certainly outweigh the color of the car. Nevertheless, let the valuation with these criteria be given on an ordinal scale $\mathcal{L}$ with 5 linearly ordered values as indicated by the Boolean matrix on the left side of Fig. 3.1. (Here for simplicity, the ordering is linear, but it need not.) We name these values $1, 2, 3, 4, 5$, but do not combine this with any arithmetic; i.e., value 4 is not intended to mean two times as good as value 2. Rather, they might be described with linguistic variables as *bad, not totally bad, medium, outstanding, absolutely outstanding*; purposefully these example qualifications have not been chosen “equidistant”.

$$\begin{array}{r c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{r} \text{color} \\ \text{price} \\ \text{speed} \end{array} & \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{array} \qquad \begin{aligned} 4 = \texttt{lub}\,[\,&\texttt{glb}\,(4_{v(color)}, 4_{\mu\{c,p\}}), \\ &\texttt{glb}\,(4_{v(price)}, 4_{\mu\{c,p\}}), \\ &\texttt{glb}\,(2_{v(speed)}, 5_{\mu\{c,p,s\}})] \end{aligned}$$

Fig. 3.1 Valuation of 3 criteria integrated to value 4 with measure of Fig. 3.3

First we concentrate on the left side of Fig. 3.1. The task is to arrive at *one* overall valuation of the car out of these three. In a simple-minded approach, we might indeed conceive numbers $1, 2, 3, 4, 5 \in \mathbb{R}$ and then evaluate in a classical way the average value as $\frac{1}{3}(4+4+2) = 3.3333\ldots$, which is a value not expressible in the given scale. When considering the second example Fig. 3.2, we would arrive at the same average value although the switch from Fig. 3.1 to Fig. 3.2 between price and speed would trigger most people to decide differently.

$$\begin{array}{r@{}c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{r} \text{color} \\ \text{price} \\ \text{speed} \end{array} & \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \end{pmatrix} \end{array} \qquad \begin{aligned} 3 = \mathtt{lub}\,[&\mathtt{glb}\,(4_{v(color)}, 3_{\mu\{c,s\}}), \\ &\mathtt{glb}\,(2_{v(price)}, 5_{\mu\{c,p,s\}}), \\ &\mathtt{glb}\,(4_{v(speed)}, 3_{\mu\{c,s\}})] \end{aligned}$$

Fig. 3.2 Different valuation of these criteria integrated to value 3

With relational integration, we learn to make explicit which set of criteria to apply with which weight. To this end, we look at the right-hand sides of the two figures. There we can see from $4_{v(color)}$ that color, e.g., has been valuated with 4. From $3_{\mu\{c,s\}}$, we see those criteria that are valuated not strictly inferior to *color* forming the set $\{$*color*, *speed*$\}$ that gets value 3 by the Boolean matrix in Fig. 3.3. It is conceivable that criteria c_1, c_2 are given a low weight but the criteria set $\{c_1, c_2\}$ in conjunction a high one. This means that we introduce a **relational measure** assigning values in $\mathcal{L}$ to subsets of $\mathcal{C}$.

$$\mu = \begin{array}{l@{}c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{l} \{\} \\ \{\text{color}\} \\ \{\text{price}\} \\ \{\text{color,price}\} \\ \{\text{speed}\} \\ \{\text{color,speed}\} \\ \{\text{price,speed}\} \\ \{\text{color,price,speed}\} \end{array} & \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix} \end{array}$$

Fig. 3.3 A relational measure

For gauging purposes we demand that the empty criteria set gets assigned the least value in $\mathcal{L}$ and the full criteria set the greatest. A point to stress is that we assume values of the criteria as well as the measures of subsets of criteria as *commensurable.*

The relational measure μ should obviously be monotonic with respect to the inclusion ordering Ω on the powerset of $\mathcal{C}$ and the ordering E on $\mathcal{L}$. We do not demand continuity (additivity), however. In the example above for instance, the *price* alone is ranked of medium importance 3, higher than *speed* alone, while *color* alone is considered completely unimportant and ranks 1. However, *color* and *price* together are ranked 4, i.e., higher than the supremum of ranks for color alone and for *price* alone, etc.

As now the valuations according to the criteria as well as the valuation according to the relative measuring of the criteria are given, we may proceed as visualized on the right-hand sides of Fig. 3.1 and Fig. 3.2. We run through the criteria and always look for two items: their corresponding value and in addition for the value of that subset of criteria *assigning equal or higher values*. Then we determine the greatest lower bound for the two values. From the list thus obtained, the least upper bound is taken. The two examples above show how by simple evaluation along this concept, one will arrive at the overall values 4 or 3, respectively. This results from the fact that in the second case only such rather unimportant criteria as color and speed assign the higher values.

The effect is counterrunning: Low values of criteria as for *speed* in Fig. 3.3 are intersected with rather high μ-values as *many* criteria give higher scores and μ is monotonic. Highest values of criteria as for color or speed in Fig. 3.2 are intersected with the μ-value of a small or even one-element criteria set; i.e., with a rather small one. In total we find that here are two operations applied in a way we already know from matrix multiplication: a "sum" operator, `lub` or $\vee$, following the application of a "product" operator, `glb` or $\wedge$.

This example gave a first idea of how relational integration works and how it may be useful. Introducing a relational measure and using it for integration serves an important purpose: Concerns are now separated. One may design the criteria and the measure in a design phase prior to polling. Only then shall the questionnaire be filled, or the voters be polled. The procedure of coming to an overall valuation is now just computation and should no longer lead to quarrels.

4 Relation-Algebraic Preliminaries

As we cannot present all the prerequisites on relation algebra, we give [15] as a general reference for handling relations as Boolean matrices and subsets of a set as Boolean vectors. We write $R : V \longrightarrow W$ if R is a relation with domain V and range W, i.e., a subset of $V \times W$. If the sets V and W of R's *type* $V \longrightarrow W$ are finite and of size m and n, respectively, we may consider R as a Boolean matrix with m rows and n columns. This Boolean matrix interpretation is well suited for many purposes and also used by RelView, [1,3], to depict relations. We assume the reader to be familiar with the basic operations on relations, viz. R^{T} (*transposition*), $\overline{R}$ (*complement*), $R \cup S$ (*union*), $R \cap S$ (*intersection*), and $R;S$ (*composition*), the predicate $R \subseteq S$ (*inclusion*), and the special relations $\perp\!\!\!\perp$ (*empty relation*), $\top\!\!\!\top$ (*universal relation*), and $\mathbb{I}$ (*identity relation*) and with the most prominent rule, namely $R;S \subseteq T$ if and only if $R^{\mathsf{T}};\overline{T} \subseteq \overline{S}$, the so-called Schröder rule.

We will often use index notation $R_{i,j}$ instead of $(i,j) \in R$. For mappings R, it is more comfortable to write $R(i)$ for the one-element-set of those elements j that satisfy $(i,j) \in R$. If sets are concerned, we will write v_i instead of $i \in v$.

By $\mathtt{syq}(R,S) := \overline{R^{\mathsf{T}};\overline{S}} \cap \overline{\overline{R}^{\mathsf{T}};S}$, the *symmetric quotient* $\mathtt{syq}(R,S) : W \longrightarrow Z$ of two relations $R : V \longrightarrow W$ and $S : V \longrightarrow Z$ is defined. Many properties of this construct can be found in [15]. Especially, we have for all $y \in W$ and $z \in Z$ that $\mathtt{syq}(R,S)_{y,z}$ if and only if for all $x \in V$ the two relationships $R_{x,y}$ and $S_{x,z}$ are equivalent.

There are some relational possibilities to model sets. Our first modeling uses *vectors*, which are relations v with $v = v;\mathbb{T}$, i.e., "row-constant". Since for a vector the range is irrelevant, we consider in the following mostly vectors $v : V \longrightarrow \mathbb{1}$ with a specific singleton set $\mathbb{1} = \{\perp\}$ as range. Such a vector can be considered as a Boolean column vector, and it *represents* a subset of its domain V. A non-empty vector v is said to be a *point* if $v;v^{\mathsf{T}} \subseteq \mathbb{I}$, i.e., v is *injective*. This means that it represents a singleton subset of its domain or an element from it if we identify a singleton set with the only element it contains.

As a second way to model sets is to conceive them as partial diagonal relations. We will apply the relation-level equivalents of the set-theoretic symbol "$\in$", i.e., *membership-relations* $\varepsilon : V \longrightarrow \mathcal{P}(V)$ between a base set V and its powerset $\mathcal{P}(V)$. These specific relations are defined by demanding for all $x \in V$ and $W \in \mathcal{P}(V)$ that $\varepsilon_{x,W}$ if and only if $x \in W$. A Boolean matrix implementation of membership relations requires exponential space. However, in [2] an implementation using reduced ordered binary decision diagrams (ROBDDs) is given, the number of nodes of which is linear in the size of the base set.

Based on the relation $\varepsilon : V \longrightarrow \mathcal{P}(V)$, relation-algebraic specifications of many set-theoretic constructions can be established, not least the following:

$$\Omega = \overline{\varepsilon^{\mathsf{T}};\overline{\varepsilon}} \qquad 0_\Omega = \mathtt{syq}(\varepsilon, \mathbb{\perp\!\!\!\perp}) \qquad 1_\Omega = \mathtt{syq}(\varepsilon, \mathbb{T})$$

Point-wise reasoning shows for all $W, Z \in \mathcal{P}(V)$ that $\Omega_{W,Z}$ if and only if $W \subseteq Z$. Hence, $\Omega : \mathcal{P}(V) \longrightarrow \mathcal{P}(V)$ relation-algebraically specifies set inclusion on the powerset $\mathcal{P}(V)$. In the same manner we see that the two points $0_\Omega : \mathcal{P}(V) \longrightarrow \mathbb{1}$ and $1_\Omega : \mathcal{P}(V) \longrightarrow \mathbb{1}$ represent the empty set $\emptyset$ and the universum V, respectively, as elements of the powerset $\mathcal{P}(V)$.

5 Order-Theoretic Functionals

A relation $E : V \longrightarrow V$ is a partial order relation if and only if $\mathbb{I} \subseteq E$ (reflexivity), $E \cap E^{\mathsf{T}} \subseteq \mathbb{I}$ (antisymmetry), and $E;E \subseteq E$ (transitivity). In view of later applications we ask for bounds and extremal elements with respect to such an ordering. We define relational functionals dependent on E and a further relation $R : V \longrightarrow W$ as follows:

Let an order relation E be given on a set V. An element e is called an *upper bound* (also: *majorant*) of the subset of V characterized by the vector u of V

provided $\forall x \in u : E_{xe}$. From the predicate logic version, we easily derive a relation-algebraic formulation as $e \subseteq \overline{\overline{E}^{\mathsf{T}} ; u}$, so that we introduce the order-theoretic functional $\mathtt{ubd}_E(u) := \overline{\overline{E}^{\mathsf{T}} ; u}$ to return the possibly empty vector of all upper bounds. Analogously, we have the set of lower bounds $\mathtt{lbd}_E(u) := \overline{\overline{E} ; u}$. More generally, we define

$$\mathtt{lbd}_E(R) = \overline{\overline{E} ; R}, \qquad \mathtt{ubd}_E(R) = \overline{\overline{E}^{\mathsf{T}} ; R},$$

$$\mathtt{gre}_E(R) = R \cap \mathtt{ubd}_E(R), \qquad \mathtt{glb}_E(R) = \mathtt{gre}_E(\mathtt{lbd}_E(R)).$$

Transposing the ordering relation in the above relational functionals yields relation-algebraic specifications for the least element $\mathtt{lea}_E(R) = \mathtt{gre}_{E^{\mathsf{T}}}(R)$, and for the least upper bound $\mathtt{lub}_E(R) = \mathtt{glb}_{E^{\mathsf{T}}}(R)$.

As a tradition, a vector is often a column vector. In many cases, however, a row vector would be more convenient. We decided to introduce a variant denotation for order-theoretic functionals working on row vectors:

$$\mathtt{lubR}_E(X) := [\mathtt{lub}_E(X^{\mathsf{T}})]^{\mathsf{T}},$$

etc. We are here concerned with lattice orderings E only, for which we introduce notation for least and greatest elements as $0_E = \mathtt{glb}_E(\mathbb{T}), 1_E = \mathtt{lub}_E(\mathbb{T})$.

The following is an important connection between the membership relation ε, the powerset ordering Ω and the respective least upper bounds:

5.1 Proposition. If ε is the membership relation and Ω the corresponding subset inclusion, the following equations hold for arbitrary relations X:

i) $\varepsilon ; \overline{\varepsilon^{\mathsf{T}} ; X} = \overline{X}$ and $\overline{\varepsilon} ; \overline{\overline{\varepsilon}^{\mathsf{T}} ; X} = \overline{X}$,

ii) $\mathtt{lub}_\Omega(X) = \mathtt{syq}(\varepsilon, \varepsilon ; X)$.

Proof: " $\subseteq$ " of (i) follows with the Schröder rule. It remains to prove " $\supseteq$ ":

$$\begin{aligned}
\overline{X} &= \varepsilon ; \mathtt{syq}(\varepsilon, \overline{X}) && \text{(4.4.2.ii) of [15]} \\
&= \varepsilon ; (\overline{\overline{\varepsilon}^{\mathsf{T}} ; \overline{X}} \cap \overline{\varepsilon^{\mathsf{T}} ; \overline{\overline{X}}}) && \text{by definition of } \mathtt{syq} \\
&= \varepsilon ; (\ldots \cap \overline{\varepsilon^{\mathsf{T}} ; X}) && \text{double negation} \\
&\subseteq \varepsilon ; \overline{\varepsilon^{\mathsf{T}} ; X} && \text{by monotonicity}
\end{aligned}$$

The second case is handled only slightly differently.

$$\begin{aligned}
\text{(ii)}\ \mathtt{syq}(\varepsilon, \varepsilon ; X) &= \overline{\overline{\varepsilon}^{\mathsf{T}} ; \varepsilon ; X} \cap \overline{\varepsilon^{\mathsf{T}} ; \overline{\varepsilon ; X}} && \text{definition of } \mathtt{syq} \\
&= \overline{\overline{\varepsilon}^{\mathsf{T}} ; \varepsilon ; X} \cap \overline{\varepsilon^{\mathsf{T}} ; \overline{\varepsilon} ; \overline{\overline{\varepsilon}^{\mathsf{T}} ; \varepsilon ; X}} && \text{second of (i)} \\
&= \overline{\overline{\Omega}^{\mathsf{T}} ; X} \cap \overline{\overline{\Omega} ; \overline{\overline{\Omega}^{\mathsf{T}} ; X}} && \text{definition of } \Omega \\
&= \mathtt{lub}_\Omega(X) && \text{definition of } \mathtt{lub}_\Omega \qquad \square
\end{aligned}$$

6 Relational Measures

Assume the following basic situation of Fig. 6.1 with a set $\mathcal{C}$ of so-called criteria and a measuring lattice $\mathcal{L}$. Depending on the application envisaged, $\mathcal{C}$ may also be interpreted as a set of players in a cooperative game, of attributes, of experts, or of voters in an opinion polling problem. This includes the setting with $\mathcal{L}$ being the interval $[0,1] \subseteq \mathbb{R}$ or a linear ordering for measuring. We consider a (relational) measure generalizing the concept of a fuzzy measure (or capacité in French origin) assigning via μ measures in $\mathcal{L}$ for subsets of $\mathcal{C}$.

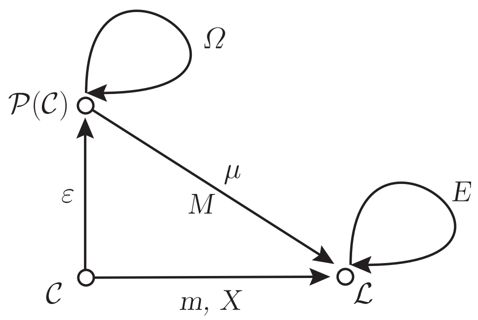

Fig. 6.1 Basic situation for relational integration

The relation ε is the membership relation between $\mathcal{C}$ and its powerset $\mathcal{P}(\mathcal{C})$. The measures envisaged will be called μ, other relations will be denoted as M. Valuations according to the criteria will be X or m depending on the context.

$E =$

	(low,lazy)	(medium,lazy)	(low,fair)	(high,lazy)	(medium,fair)	(low,good)	(high,fair)	(medium,good)	(low,bulldozer)	(high,good)	(medium,bulldozer)	(high,bulldozer)
(low,lazy)	1	1	1	1	1	1	1	1	1	1	1	1
(medium,lazy)	0	1	0	1	1	0	1	1	0	1	1	1
(low,fair)	0	0	1	0	1	1	1	1	1	1	1	1
(high,lazy)	0	0	0	1	0	0	1	0	0	1	0	1
(medium,fair)	0	0	0	0	1	0	1	1	0	1	1	1
(low,good)	0	0	0	0	0	1	0	1	1	1	1	1
(high,fair)	0	0	0	0	0	0	1	0	0	1	0	1
(medium,good)	0	0	0	0	0	0	0	1	0	1	1	1
(low,bulldozer)	0	0	0	0	0	0	0	0	1	0	1	1
(high,good)	0	0	0	0	0	0	0	0	0	1	0	1
(medium,bulldozer)	0	0	0	0	0	0	0	0	0	0	1	1
(high,bulldozer)	0	0	0	0	0	0	0	0	0	0	0	1

Fig. 6.2 Ordering E of the value lattice $\mathcal{L}$ represented as a matrix

As a running example assume the task to assess persons according to their intellectual abilities as well as according to the workload they achieve to master.

Figs. 6.2 and 6.3 show the value ordering as a matrix and as a graph.

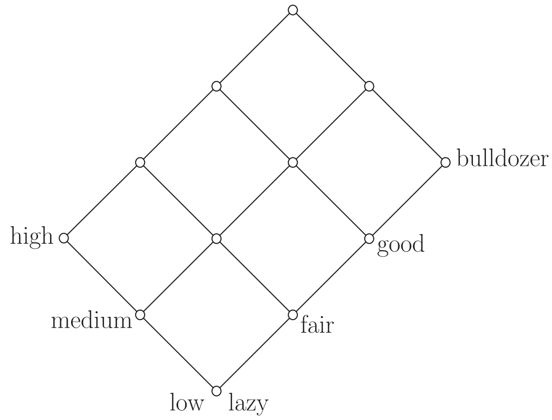

Fig. 6.3 Hasse diagram of the ordering E of the value lattice $\mathcal{L}$

6.1 Definition. Suppose a set of criteria $\mathcal{C}$ to be given together with some lattice $\mathcal{L}$, ordered by E, into which every subset of these criteria shall mapped. Let Ω be the inclusion ordering on $\mathcal{P}(\mathcal{C})$. We call a mapping $\mu : \mathcal{P}(\mathcal{C}) \to \mathcal{L}$ a **belief mapping**, or shorter but a bit sloppily a **(relational) measure**, provided

- $\Omega;\mu \subseteq \mu;E$, meaning that μ is isotonic wrt. to the orderings Ω and E,
- $\mu^\mathsf{T};0_\Omega = 0_E$, meaning that the empty subset of $\mathcal{P}(\mathcal{C})$ is mapped to the least element of $\mathcal{L}$,
- $\mu^\mathsf{T};1_\Omega = 1_E$, meaning that the full subset of $\mathcal{P}(\mathcal{C})$ is mapped to the greatest element of $\mathcal{L}$. □

A (relational) measure for $s \in \mathcal{P}(\mathcal{C})$, i.e., $\mu(s)$ when the classical notation of a mapping is used, or $\mu^\mathsf{T};s$ when written in relation form, may be interpreted as the weight of importance we attribute to the combination s of criteria. It should not be mixed up with a probability. The latter would require the setting $\mathcal{L} = [0,1] \subseteq \mathbb{R}$ and in addition that μ is continuous.

Here we face a serious notational problem. On the one hand, we would like to be as close to what is known in Analysis concerning measures and integration. On the other hand side, many difficult problems disappear when switching to the discrete or even finite case. The term *belief mapping* seems too much related to a specific application. The term *measure* is shorter. Above, it lacks, however, continuity when compared with traditional measure and integration theory. As the term *Bayesian measure* has already been used in the literature, implying that continuity further restricts a measure, we feel free to speak of a *measure* when meaning a *belief mapping*. In Sect. 7, when introducing integration, we face a similar problem.

Many ideas of this type have been collected by Glenn Shafer [16] under the heading *theory of evidence*, calling μ a *belief function*. Using it, he explained a basis of rational behaviour. We attribute certain weights to evidence, but do not explain in which way. These weights shall in our case be lattice-ordered. This alone gives us reason to rationally decide this or that way. Real-valued belief functions have numerous applications in artificial intelligence, expert systems, approximate reasoning, knowledge extraction from data, and Bayesian Networks.

In the extreme case, we have complete ignorance expressed by the so-called **vacuous belief mapping**, defined by

$$\mu_0(s) = \begin{cases} 0_E & \text{if } \mathcal{C} \neq s, \\ 1_E & \text{if } \mathcal{C} = s. \end{cases}$$

On the other side, we may completely overspoil our trust expressed by what we may call a **light-minded belief mapping**, defined by

$$\mu_1(s) = \begin{cases} 0_E & \text{if } 0_\Omega = s, \\ 1_E & \text{otherwise.} \end{cases}$$

To an arbitrary non-empty set of criteria, the light-minded belief mapping attributes all the components of trust or belief.

The definition above does *not* demand continuity (sometimes called additivity) for the measure. Concerning additivity, the example of Glenn Shafer [16] is when one is wondering whether a Ming vase is a genuine one or a fake. We have to put the full amount of our belief on the disjunction "*genuine* or *fake*" as one of the alternatives will certainly be the case. But the amount of trust we are willing to put on the alternatives may in both cases be very small as we have only tiny hints for being genuine, but also very tiny hints for being a fake.

With the idea of probability, we could not cope so easily with the ignorance just mentioned. Probability does not allow one to withhold belief from a proposition without *according the withheld amount of belief to the negation.* When thinking on the Ming vase in terms of probability we would have to attribute p to *genuine* and $1 - p$ to *fake*.

Sometimes, however, we will have lattice-continuous measures, for which case we provide the following definition.

6.2 Definition. Given our basic situation, we call the relational measure μ

i) a **Bayesian measure** if it is lattice-continuous, i.e.,

$$\mathtt{lub}_E(\mu^\mathsf{T} ; s) = \mu^\mathsf{T} ; \mathtt{lub}_\Omega(s)$$

for all subsets $s \subseteq \mathcal{P}(\mathcal{C})$, or else, for all sets of subsets of $\mathcal{C}$,

ii) a **simple support mapping** focused on U valued with v, if U is a non-empty subset $U \subseteq \mathcal{C}$ and $v \in \mathcal{L}$ an element such that

$$\mu(s) = \begin{cases} 0_E & \text{if } s \not\supseteq U, \\ v & \text{if } \mathcal{C} \neq s \supseteq U, \\ 1_E & \text{if } \mathcal{C} = s. \end{cases}$$ □

In particular, μ_1 is Bayesian while μ_0 is not. In the real-valued environment, the condition for a Bayesian measure is: additive when non-overlapping. Lattice-continuity incorporates two concepts, namely additivity

$$\mu^\mathsf{T};(s_1 \cup s_2) = \mathtt{lub}_E(\mu^\mathsf{T};s_1 \cup \mu^\mathsf{T};s_2)$$

and sending 0_Ω to 0_E.

For the linearly ordered real-valued case, Dempster [7] found a way of combining measures in a form closely related to conditional probability. It shows a way of adjusting opinion in the light of new evidence. We have re-modeled this for the relational case. One should be aware of how a measure behaves on upper and lower cones:

6.3 Proposition. Measures satisfy $\mu = \mathtt{lubR}_E(\Omega^\mathsf{T};\mu)$ and $\mu = \mathtt{glbR}_E(\Omega;\mu)$.

Proof: μ satisfies $\Omega^\mathsf{T};\overline{\mu;E} = \overline{\mu;E}$, from which $\supseteq$ is trivial. But also $\subseteq$ as this is with the Schröder rule equivalent with $\Omega;\mu;E \subseteq \mu;E$, which follows from monotonicity of μ and transitivity of E. The rest is now easy:

$$\begin{aligned}
\mathtt{lubR}_E(\Omega^\mathsf{T};\mu) &= (\mathtt{lub}_E(\mu^\mathsf{T};\Omega))^\mathsf{T} && \text{definition of } \mathtt{lubR} \\
&= \overline{\Omega^\mathsf{T};\mu;\overline{E}} \cap \overline{\overline{\Omega^\mathsf{T};\mu;\overline{E}};\overline{E}^\mathsf{T}} && \text{definition of } \mathtt{lub}\text{, transposing} \\
&= \overline{\Omega^\mathsf{T};\overline{\mu;E}} \cap \overline{\overline{\Omega^\mathsf{T};\overline{\mu;E}};\overline{E}^\mathsf{T}} && \text{as } \mu \text{ is a mapping} \\
&= \overline{\overline{\mu;E}} \cap \overline{\overline{\overline{\mu;E}};\overline{E}^\mathsf{T}} && \text{see above} \\
&= \mu;E \cap \mu;\overline{E;\overline{E}^\mathsf{T}} && \text{double negations, } \mu \text{ is a mapping} \\
&= \mu;E \cap \mu;E^\mathsf{T} && \text{as } \overline{E}^\mathsf{T} = E;\overline{E}^\mathsf{T} \text{ for an ordering } E \\
&= \mu;(E \cap E^\mathsf{T}) = \mu;\mathbb{I} = \mu && E \text{ is antisymmetric}
\end{aligned}$$ □

When one has in addition to μ got further evidence from a second measure μ', one will intersect the upper cones resulting in a possibly smaller cone positioned higher up and take its greatest lower bound and, thus, define

$$\mu \oplus \mu' := \mathtt{glbR}_E(\mu;E \cap \mu';E) = \mathtt{lubR}_E(\mu \cup \mu').$$

One might, however, also look where μ and μ' agree, and thus intersect the lower bound cones resulting in a possibly smaller cone positioned deeper down and take its least upper bound and, thus, define

$$\mu \otimes \mu' := \mathtt{lubR}_E(\mu;E^\top \cap \mu';E^\top) = \mathtt{glbR}_E(\mu \cup \mu').$$

The definitions allow the variant forms indicated as, e.g.,

$$\begin{aligned}\mathtt{glbR}_E(\mu;E \cap \mu';E) &= \mathtt{glbR}_E(\overline{\mu;\overline{E}} \cap \overline{\mu';\overline{E}}) = \mathtt{glbR}_E(\overline{(\mu \cup \mu');\overline{E}}) \\ &= \mathtt{glbR}_E(\mathtt{ubdR}_E(\mu \cup \mu')) = \mathtt{lubR}_E(\mu \cup \mu').\end{aligned}$$

We show, that we indeed arrive at an algebraic structure with these constructs.

6.4 Proposition. If the measures μ, μ' are given, $\mu \oplus \mu'$ as well as $\mu \otimes \mu'$ are measures again. Both operations are commutative and associative. The vacuous belief mapping μ_0 is the null element while the light-minded belief mapping μ_1 is the unit element among measures:

$$\mu \oplus \mu_0 = \mu \qquad \mu \otimes \mu_1 = \mu \qquad \mu \otimes \mu_0 = \mu_0$$

Proof: The least element must be sent to the least element. This result is prepared using in the second and fourth step that 0_Ω is a point.

$$\begin{aligned}\mathtt{lbd}_E([\mu;E \cap \mu';E]^\top);0_\Omega &= \overline{\overline{\overline{E}};[\mu;E \cap \mu';E]^\top;0_\Omega} && \text{by definition of } \mathtt{lbd} \\ &= \overline{\overline{\overline{E}};[\mu;E \cap \mu';E]^\top;0_\Omega} && \text{(4.2.4.iii) of [15]} \\ &= \overline{\overline{\overline{E}};[E^\top;\mu^\top \cap E^\top;\mu'^\top];0_\Omega} && \text{transposition} \\ &= \overline{\overline{\overline{E}};[E^\top;\mu^\top;0_\Omega \cap E^\top;\mu'^\top;0_\Omega]} && \text{(4.2.4.ii) of [15]} \\ &= \overline{\overline{\overline{E}};[E^\top;0_E \cap E^\top;0_E]} && \text{definition of measure} \\ &= \overline{\overline{\overline{E}};\mathbb{T}} = \mathtt{lbd}(\mathbb{T}) = 0_E && E \text{ is a complete lattice}\end{aligned}$$

Now we obtain the desired equality as follows

$$\begin{aligned}(\mu \oplus \mu')^\top;0_\Omega &= \mathtt{glb}_E([\mu;E \cap \mu';E]^\top);0_\Omega \\ &= (\,\mathtt{lbd}_E([\mu;E \cap \mu';E]^\top) \cap \mathtt{ubd}(\mathtt{lbd}_E([\mu;E \cap \mu';E]^\top));0_\Omega \\ &= \mathtt{lbd}_E([\mu;E \cap \mu';E]^\top);0_\Omega \cap \overline{\overline{E}^\top;\mathtt{lbd}_E([\mu;E \cap \mu';E]^\top)};0_\Omega \\ &= 0_E \cap \overline{\overline{E}^\top;\mathtt{lbd}_E([\mu;E \cap \mu';E]^\top);0_\Omega} \\ &= 0_E \cap \overline{\overline{E}^\top;0_E} = 0_E \cap \mathtt{ubd}(0_E) = 0_E \cap \mathbb{T} = 0_E\end{aligned}$$

As μ, μ' are measures, we have that $\mu^\top;1_\Omega = 1_E$ and also $\mu'^\top;1_\Omega = 1_E$. In both cases, the cone above the image is simply 1_E, and so also their intersection as well as the greatest lower bound thereof is 1_E.

The other less difficult parts of the proof are left to the reader. $\square$

7 Relational Integration

Assume now that for all the criteria $\mathcal{C}$ a valuation has taken place with values in $\mathcal{L}$. With the following construction, we arrive at an overall valuation by rational means, for which μ shall be the guideline.

7.1 Definition. We assume as in Fig. 6.1 a relational measure μ, a membership relation ε, and a lattice order E. Furthermore, we suppose a mapping $X : \mathcal{C} \longrightarrow \mathcal{L}$ that indicates the values assigned to the criteria. We define the **relational integral** by

$$(R)\int X \circ \mu \ := \mathtt{lubR}_E(\mathbb{T} ; \mathtt{glbR}_E[(X \cup \mathtt{syq}(X ; E^{\mathsf{T}} ; X^{\mathsf{T}}, \varepsilon) ; \mu)]) \qquad \square$$

The idea behind this integral is as follows: From the valuation of any criterion proceed to all higher valuations and from these back to those criteria that assigned such higher values. With $X ; E ; X^{\mathsf{T}}$, the transition from all the criteria to the set of criteria are given. Now a symmetric quotient is needed in order

$$\mu = \begin{array}{r} \\ \{\} \\ \{\text{Abe}\} \\ \{\text{Bob}\} \\ \{\text{Abe,Bob}\} \\ \{\text{Carl}\} \\ \{\text{Abe,Carl}\} \\ \{\text{Bob,Carl}\} \\ \{\text{Abe,Bob,Carl}\} \\ \{\text{Don}\} \\ \{\text{Abe,Don}\} \\ \{\text{Bob,Don}\} \\ \{\text{Abe,Bob,Don}\} \\ \{\text{Carl,Don}\} \\ \{\text{Abe,Carl,Don}\} \\ \{\text{Bob,Carl,Don}\} \\ \{\text{Abe,Bob,Carl,Don}\} \end{array} \begin{array}{c} \begin{array}{cccccccccccc} \rotatebox{90}{(low,lazy)} & \rotatebox{90}{(medium,lazy)} & \rotatebox{90}{(low,fair)} & \rotatebox{90}{(high,lazy)} & \rotatebox{90}{(medium,fair)} & \rotatebox{90}{(low,good)} & \rotatebox{90}{(high,fair)} & \rotatebox{90}{(medium,good)} & \rotatebox{90}{(low,bulldozer)} & \rotatebox{90}{(high,good)} & \rotatebox{90}{(medium,bulldozer)} & \rotatebox{90}{(high,bulldozer)} \end{array} \\ \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{pmatrix} \end{array}$$

Fig. 7.1 Example measure

to comprehend all these sets to elements of the powerset. (To this end, the converse is needed.) Once the sets are elements of the powerset, the measure μ may be applied. As already shown in the initial example, we have now the value of the respective criterion and in addition the valuation of the criteria set. From the two, we form the greatest lower bound. So in total, we have lower bounds for all the criteria. These are combined in one set multiplying the universal relation from the left side. Finally, the least upper bound is taken.

We are now in a position to understand why gauging $\mu^{\mathsf{T}}; 1_\Omega = 1_E$ is necessary for μ, or "greatest element is sent to greatest element". Consider, e.g., the special case of an X with all criteria assigning the same value. We certainly expect the relational integral to precisely deliver this value regardless of the measure chosen. But this might not be the case if a measure should assign too small a value to the full set.

7.2 Example. We continue our running example of Sect. 6 and provide the following highly non-continuous measure of Fig. 7.1.

Here, e.g., $\mu(\text{Abe}) = (\text{high}, \text{lazy})$ and $\mu(\text{Bob}) = (\text{medium}, \text{fair})$, with supremum $(\text{high}, \text{fair})$ but in excess to this, μ assigns $\mu(\text{Abe}, \text{Bob}) = (\text{high}, \text{good})$.

$$
\begin{array}{c}
\begin{array}{rc}
& \begin{array}{cccccccccccc} \rotatebox{90}{(low,lazy)} & \rotatebox{90}{(medium,lazy)} & \rotatebox{90}{(low,fair)} & \rotatebox{90}{(high,lazy)} & \rotatebox{90}{(medium,fair)} & \rotatebox{90}{(low,good)} & \rotatebox{90}{(high,fair)} & \rotatebox{90}{(medium,good)} & \rotatebox{90}{(low,bulldozer)} & \rotatebox{90}{(high,good)} & \rotatebox{90}{(medium,bulldozer)} & \rotatebox{90}{(high,bulldozer)} \end{array} \\
X_1 = \begin{array}{r} \text{Abe} \\ \text{Bob} \\ \text{Carl} \\ \text{Don} \end{array} & \begin{pmatrix} \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0} \\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0} \\ \mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0} \\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0} \end{pmatrix} \\
X_2 = \begin{array}{r} \text{Abe} \\ \text{Bob} \\ \text{Carl} \\ \text{Don} \end{array} & \begin{pmatrix} \mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0} \\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0} \\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0} \\ \mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0} \end{pmatrix} \\
(R)\int X_1 \circ \mu = & (\mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{1}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}) \\
(R)\int X_2 \circ \mu = & (\mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{1}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0})
\end{array}
\end{array}
$$

Fig. 7.2 Two relational integrations

Fig. 7.2 shows two valuations X_1, X_2 and then the relational integrals computed with the TituRel system. One can see that the supremum of the valuations $2, 5, 6, 3$ according to X_2, e.g., is 8. Nevertheless, the integral assigns only 5 meaning — with obvious abbreviations A for Abe and so on —

$$
\begin{aligned}
(medium, fair) = \mathtt{lub}_E\,(&\mathtt{glb}_E(X_2(A), \mu\{A, B\}), \mathtt{glb}_E(X_2(B), \mu\{B\}), \\
&\mathtt{glb}_E(X_2(C), \mu\{C\}), \mathtt{glb}_E(X_2(D), \mu\{B, C, D\}). \quad \square
\end{aligned}
$$

The considerations of this section originate from a free re-interpretation by the present authors of concepts for work in $[0, 1] \subseteq \mathbb{R}$. In [4] e.g., the **Sugeno**

integral operator is explained as

$$M_{S,\mu}(x_1 \ldots, x_m) = (S)\int x \circ \mu = \bigvee_{i=1}^{m} [x_i \wedge \mu(A_i)]$$

and the **Choquet integral** operator as

$$M_{C,\mu}(x_1, \ldots, x_m) = (C)\int x \circ \mu = \sum_{i=1}^{m} [(x_i - x_{i-1}) \cdot \mu(A_i)]$$

In both cases the elements of the vector $(x_1, \ldots, x_m)$, and parallel to this, the criteria set $\mathcal{C} = \{C_1, \ldots, C_m\}$ have been reordered each time such that

$$0 = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_m \leq x_{m+1} = 1 \quad \text{and} \quad \mu(A_i) = \mu(C_i, \ldots, C_m).$$

What we have introduced earlier in the present chapter as a relational integral has indeed been designed looking at the Choquet and the Sugeno integral above. Only a careful analysis, however, will identify the underlying common idea. As we have indeed some sort of a valued summation, we decided for calling this an integral, although many severe problems of integrability do not show up for relations, not least when these are finite.

The concept of Choquet integral has first been introduced for a real-valued context in [4] and later used by Michio Sugeno [17]. These integrals have nice properties for aggregation: They are continuous, non-decreasing, and stable under certain interval preserving transformations. Not least do they reduce to the weighted arithmetic mean as soon as they become additive.

8 Defining Relational Measures

Relational measures as used in relational integrals may be given directly, which is, however, a costly task as a powerset is involved all of whose elements need values. Therefore, they are usually constructed in some other way. We are going to discuss two methods: first we investigate measures originating from direct valuation of criteria, and secondly the measures originating from a body of evidence.

Let a **direct valuation** of the criteria be given as any relation m between $\mathcal{C}$ and $\mathcal{L}$. Although it is allowed to be contradictory and non-univalent, we provide for a way of defining a relational measure based on it. This will happen via the following constructs

$$\sigma(m) := \overline{\varepsilon^{\mathsf{T}} ; m ; \overline{E}} \qquad \pi(\mu) := \overline{\varepsilon ; \mu ; \overline{E}^{\mathsf{T}}},$$

which very obviously satisfy the Galois correspondence requirement

$$m \subseteq \pi(\mu) \quad \Longleftrightarrow \quad \mu \subseteq \sigma(m).$$

They satisfy $\sigma(m;E^\mathsf{T}) = \sigma(m)$ and $\pi(\mu;E) = \pi(\mu)$, so that in principle only lower, respectively upper, cones occur as arguments. Applying $\overline{W;E} = \overline{W;E};E^\mathsf{T}$, we get

$$\sigma(m);E = \overline{\varepsilon^\mathsf{T};m;\overline{E}};E = \overline{\varepsilon^\mathsf{T};m;\overline{E};E^\mathsf{T}};E = \overline{\varepsilon^\mathsf{T};m;\overline{E}} = \sigma(m),$$

so that images of σ are always upper cones — and thus best described by their greatest lower bounds $\mathtt{glbR}_E(\sigma(m))$.

8.1 Proposition. Given any relation $m : \mathcal{C} \to \mathcal{L}$, the construct

$$\mu_m := \mu_0 \oplus \mathtt{glbR}_E(\sigma(m))$$

forms a relational measure, the so-called **possibility measure**.

Proof: The relation $F := \mathtt{glbR}_E(\sigma(m))$ is a mapping by construction (not necessarily a measure!) since E is a lattice, so that according to (4.2.3) of [15] $R \subseteq S;F^\mathsf{T}$ if and only if $R;F \subseteq S$. First, we disregard gauging with μ_0 which is only introduced to make sure that $F^\mathsf{T};1_\Omega = 1_E$. Then

$$\begin{aligned}
F^\mathsf{T};0_\Omega &= \mathtt{glbR}_E(\sigma(m))^\mathsf{T};0_\Omega \\
&= \mathtt{glb}_E(\sigma(m)^\mathsf{T});0_\Omega \\
&= \mathtt{glb}_E(\sigma(m)^\mathsf{T};0_\Omega) && \text{as } 0_\Omega \text{ is a point} \\
&= \mathtt{glb}_E(\overline{\overline{E}^\mathsf{T};m^\mathsf{T};\varepsilon};0_\Omega) \\
&= \mathtt{glb}_E(\overline{\overline{E}^\mathsf{T};m^\mathsf{T};\varepsilon;0_\Omega}) && \text{as } 0_\Omega \text{ is a point} \\
&= \mathtt{glb}_E(\overline{\overline{E}^\mathsf{T};m^\mathsf{T};\perp\!\!\!\perp}) \\
&= \mathtt{glb}_E(\top\!\!\!\top) = 0_E
\end{aligned}$$

Now we prove monotony $\Omega;F \subseteq F;E$.

$$\begin{aligned}
\Omega;F &= \Omega;\mathtt{glbR}_E(\sigma(m)) \\
&= \Omega;\mathtt{glbR}_E(\overline{\varepsilon^\mathsf{T};m;\overline{E}}) \\
&= \Omega;[\mathtt{glb}_E(\overline{\overline{E}^\mathsf{T};m^\mathsf{T};\varepsilon})]^\mathsf{T} \\
&= \Omega;[\mathtt{glb}_E(\mathtt{ubd}(m^\mathsf{T};\varepsilon))]^\mathsf{T} \\
&= \Omega;[\mathtt{lub}_E(m^\mathsf{T};\varepsilon)]^\mathsf{T} \\
&\subseteq \Omega;[\mathtt{ubd}_E(m^\mathsf{T};\varepsilon)]^\mathsf{T} \\
&= \Omega;\sigma(m) = \sigma(m) && \text{see below} \\
&\subseteq \sigma(m);E \\
&= \mathtt{glbR}_E(\sigma(m));E \\
&= F;E && \text{with (3.3.9.ii) of [15]}
\end{aligned}$$

We have used $\Omega;\sigma(m) = \sigma(m)$; the only interesting part may be proved thus:

$$
\begin{array}{rrcll}
 & \Omega;\sigma(m) & \subseteq & \sigma(m) & \\
\Longleftrightarrow & \overline{\varepsilon^{\mathsf T};\overline{\varepsilon}};\sigma(m) & \subseteq & \sigma(m) & \text{definition of } \Omega \\
\Longleftrightarrow & \overline{\overline{\varepsilon^{\mathsf T}};\varepsilon};\varepsilon^{\mathsf T};m;\overline{E} & \subseteq & \varepsilon^{\mathsf T};m;\overline{E} & \text{definition of } \sigma \text{ and Schröder rule} \\
\Longleftarrow & \overline{\overline{\varepsilon^{\mathsf T}};\varepsilon};\varepsilon^{\mathsf T} & \subseteq & \varepsilon^{\mathsf T} & \text{Schröder rule} \qquad \Box
\end{array}
$$

Possibility measures need not be Bayesian. Addition of the vacuous belief mapping μ_0 is again necessary for gauging purposes. In case m is a mapping, the situation becomes even nicer. From

$$
\begin{array}{rll}
\pi(\sigma(m;E^{\mathsf T})) & = \pi(\sigma(m)) & \\
 & = \overline{\varepsilon;\overline{\varepsilon^{\mathsf T};m;\overline{E}};\overline{E}^{\mathsf T}} & \\
 & = \overline{\overline{\overline{m;\overline{E}}};\overline{E}^{\mathsf T}} & \text{see Prop. 5.1} \\
 & = m;\overline{\overline{\overline{\overline{E}}};\overline{E}^{\mathsf T}} & \text{as } m \text{ was assumed to be a mapping} \\
 & = m;\overline{E;\overline{E}^{\mathsf T}} = m;E^{\mathsf T} &
\end{array}
$$

we see that this is an adjunction on cones. The lower cones $m;E^{\mathsf T}$ in turn are one-to-one represented by their least upper bounds $\mathtt{lubR}_E(m;E^{\mathsf T})$.

$$
m = \begin{array}{r} \\ \text{Abe} \\ \text{Bob} \\ \text{Carl} \\ \text{Don} \end{array}
\begin{array}{c}
\begin{array}{cccccccccccc}
\rotatebox{90}{(low,lazy)} & \rotatebox{90}{(medium,lazy)} & \rotatebox{90}{(low,fair)} & \rotatebox{90}{(high,lazy)} & \rotatebox{90}{(medium,fair)} & \rotatebox{90}{(low,good)} & \rotatebox{90}{(high,fair)} & \rotatebox{90}{(medium,good)} & \rotatebox{90}{(low,bulldozer)} & \rotatebox{90}{(high,good)} & \rotatebox{90}{(medium,bulldozer)} & \rotatebox{90}{(high,bulldozer)}
\end{array} \\
\begin{pmatrix}
0&1&1&0&0&0&0&0&0&0&0&0\\
0&1&1&1&1&0&0&0&0&0&0&0\\
1&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0
\end{pmatrix}
\end{array}
$$

$$
\mu_m = \begin{array}{r}
\{\} \\ \{\text{Abe}\} \\ \{\text{Bob}\} \\ \{\text{Abe,Bob}\} \\ \{\text{Carl}\} \\ \{\text{Abe,Carl}\} \\ \{\text{Bob,Carl}\} \\ \{\text{Abe,Bob,Carl}\} \\ \{\text{Don}\} \\ \{\text{Abe,Don}\} \\ \{\text{Bob,Don}\} \\ \{\text{Abe,Bob,Don}\} \\ \{\text{Carl,Don}\} \\ \{\text{Abe,Carl,Don}\} \\ \{\text{Bob,Carl,Don}\} \\ \{\text{Abe,Bob,Carl,Don}\}
\end{array}
\begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix}
$$

Fig. 8.1 Possibility measure μ_m derived from a direct valuation relation m

The following proposition exhibits that a Bayesian measure is a rather special case, namely more or less directly determined as a possibility measure for a direct valuation via a mapping m. Fig. 8.1 shows an example. One may proceed from m to the measure according to Prop. 8.1 or vice versa according to Prop. 8.2.

8.2 Proposition. Let μ be a Bayesian measure and $\iota := \mathtt{syq}(\mathbb{I}, \varepsilon)$, the mapping injecting singletons into the powerset. Then $m_\mu := \iota; \mu$ is that direct valuation for which $\mu = \mu_{m_\mu}$.

Proof: The remarkable property of ι is that $\iota^\mathsf{T}; \iota \subseteq \mathbb{I}$ characterizes the atoms of the powerset ordering Ω. The following calculation uses in the second step that elements in the powerset are the union of all their singleton subsets.

$$
\mu_B = \begin{array}{r c}
 & \begin{array}{cccccccccccc} \rotatebox{90}{(low,lazy)} & \rotatebox{90}{(medium,lazy)} & \rotatebox{90}{(low,fair)} & \rotatebox{90}{(high,lazy)} & \rotatebox{90}{(medium,fair)} & \rotatebox{90}{(low,good)} & \rotatebox{90}{(high,fair)} & \rotatebox{90}{(medium,good)} & \rotatebox{90}{(low,bulldozer)} & \rotatebox{90}{(high,good)} & \rotatebox{90}{(medium,bulldozer)} & \rotatebox{90}{(high,bulldozer)} \end{array} \\
\begin{array}{r}
\{\} \\ \{\text{Abe}\} \\ \{\text{Bob}\} \\ \{\text{Abe,Bob}\} \\ \{\text{Carl}\} \\ \{\text{Abe,Carl}\} \\ \{\text{Bob,Carl}\} \\ \{\text{Abe,Bob,Carl}\} \\ \{\text{Don}\} \\ \{\text{Abe,Don}\} \\ \{\text{Bob,Don}\} \\ \{\text{Abe,Bob,Don}\} \\ \{\text{Carl,Don}\} \\ \{\text{Abe,Carl,Don}\} \\ \{\text{Bob,Carl,Don}\} \\ \{\text{Abe,Bob,Carl,Don}\}
\end{array} &
\left(\begin{array}{cccccccccccc}
1&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&1&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&1&0&0&0&0&0 \\
0&0&0&0&0&0&1&0&0&0&0&0 \\
0&0&1&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&1&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&1&0&0&0&0&0 \\
0&0&0&0&0&0&1&0&0&0&0&0 \\
0&0&0&0&0&1&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&1&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&1&0&0 \\
0&0&0&0&0&0&0&0&0&1&0&0 \\
0&0&0&0&0&1&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&1&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&1&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&1
\end{array}\right)
\end{array}
$$

$$
m_{\mu_B} = \begin{array}{r c}
\begin{array}{r} \text{Abe} \\ \text{Bob} \\ \text{Carl} \\ \text{Don} \end{array} &
\left(\begin{array}{cccccccccccc}
0&0&0&0&1&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&1&0&0&0&0&0 \\
0&0&1&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&1&0&0&0&0&0&0
\end{array}\right)
\end{array}
$$

Fig. 8.2 Bayesian measure μ_B with corresponding direct valuation m_{μ_B}

$$
\begin{array}{ll}
\mu^\mathsf{T} = \mu^\mathsf{T}; \mathbb{I} & \\
\quad = \mu^\mathsf{T}; \mathtt{lub}_\Omega(\iota^\mathsf{T}; \iota; \Omega) & \\
\quad = \mathtt{lub}_E(\mu^\mathsf{T}; \iota^\mathsf{T}; \iota; \Omega) & \text{continuity of the Bayesian measure } \mu \\
\quad = [\mathtt{lubR}_E(\Omega^\mathsf{T}; \iota^\mathsf{T}; \iota; \mu)]^\mathsf{T} & \\
\quad = [\mathtt{lubR}_E(\varepsilon^\mathsf{T}; \iota; \mu)]^\mathsf{T} & \text{as } \varepsilon = \iota; \Omega
\end{array}
$$

$$
\begin{aligned}
&= \mathtt{lub}_E(\mu^\mathsf{T};\iota^\mathsf{T};\varepsilon)\\
&= \mathtt{glb}_E(\mathtt{ubd}_E(\mu^\mathsf{T};\iota^\mathsf{T};\varepsilon))\\
&= \mathtt{glb}_E(\overline{\overline{E}^\mathsf{T};\mu^\mathsf{T};\iota^\mathsf{T};\varepsilon})\\
&= \mathtt{glb}_E(\overline{\varepsilon^\mathsf{T};\iota;\mu;\overline{E}^\mathsf{T}})\\
&= [\mathtt{glbR}_E(\sigma(\iota;\mu))]^\mathsf{T} \qquad \square
\end{aligned}
$$

We illustrate the construction of the underlying direct valuation m_{μ_B} for a Bayesian measure μ_B with Fig. 8.2.

One will find out that m_{μ_B} of Fig. 8.2 may also be obtained from the m of Fig. 8.1, taking rowwise least upper bounds according to the ordering E of Fig. 6.3. This way just a few of the many relational measures will be found.

Using direct valuations, one may also give another characterization of being Bayesian, namely that the whole measure is fully determined by the values it assigns to singleton subsets.

We need not care for adding the vacuous belief, as we have been starting from a Bayesian measure which means that the value 1_E of the full set will be the least upper bound of all the values of the singletons.

Now we switch to the definition of measures that are obtained from a body of evidence.

In a similar way, we may derive relational measures out of some arbitrary relation between $\mathcal{P}(\mathcal{C})$ and $\mathcal{L}$. Although this relation is allowed to be non-univalent, we provide for a way of defining two measures based on it — which may coincide.

8.3 Definition. Let our general setting be given.

i) A **body of evidence** is an arbitrary relation $M : \mathcal{P}(\mathcal{C}) \longrightarrow \mathcal{L}$, restricted by the requirement that
$$M^\mathsf{T};0_\Omega \subseteq 0_E.$$

ii) When the body of evidence M is in addition a mapping, we speak — following [16] — of a **basic probability assignment**. □

Assume now that trust, belief, or probability has been assigned somehow. The measure to be defined shall resemble rational behaviour, so that we will reason as follows: If I dare saying that occurrence of $A \subseteq \mathcal{C}$ deserves my trust to the amount $M(A)$, then $A' \subseteq A \subseteq \mathcal{C}$ deserves at least this amount of trusting as it occurs whenever A occurs. I might, however, not be willing to consider that $A'' \subseteq \mathcal{C}$ with $A \subseteq A''$ deserves to be trusted with the same amount as there is a chance that it occurs not so often.

In Fig. 8.3, a body of evidence is provided for our running example; it is a specific one in as far as it is univalent. It is, however, not a basic probability assignment as it is not total.

We should be aware that the basic probability assignment is meant to assign something to a set regardless of what is assigned to its proper subsets. The condition $M^{\mathsf{T}} ; 0_\Omega \subseteq 0_E$ expresses that M either does not assign any belief to the empty set or it assigns just 0_E.

$$
M := \begin{array}{r}
\\
\{\} \\ \{\text{Abe}\} \\ \{\text{Bob}\} \\ \{\text{Abe,Bob}\} \\ \{\text{Carl}\} \\ \{\text{Abe,Carl}\} \\ \{\text{Bob,Carl}\} \\ \{\text{Abe,Bob,Carl}\} \\ \{\text{Don}\} \\ \{\text{Abe,Don}\} \\ \{\text{Bob,Don}\} \\ \{\text{Abe,Bob,Don}\} \\ \{\text{Carl,Don}\} \\ \{\text{Abe,Carl,Don}\} \\ \{\text{Bob,Carl,Don}\} \\ \{\text{Abe,Bob,Carl,Don}\}
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccc}
\rotatebox{90}{(low,lazy)} & \rotatebox{90}{(medium,lazy)} & \rotatebox{90}{(low,fair)} & \rotatebox{90}{(high,lazy)} & \rotatebox{90}{(medium,fair)} & \rotatebox{90}{(low,good)} & \rotatebox{90}{(high,fair)} & \rotatebox{90}{(medium,good)} & \rotatebox{90}{(low,bulldozer)} & \rotatebox{90}{(high,good)} & \rotatebox{90}{(medium,bulldozer)} & \rotatebox{90}{(high,bulldozer)}
\end{array} \\
\begin{pmatrix}
\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\
\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}
\end{pmatrix}
\end{array}
$$

Fig. 8.3 A body of evidence

Now a construction similar to that for direct valuation becomes possible, introducing

$$\sigma'(M) := \overline{\Omega^{\mathsf{T}} ; M ; \overline{E}} \qquad \pi'(\mu) := \overline{\Omega ; \mu ; \overline{E}^{\mathsf{T}}},$$

which again satisfies the Galois correspondence requirement

$$M \subseteq \pi'(\mu) \iff \mu \subseteq \sigma'(M).$$

Obviously $\sigma'(M ; E^{\mathsf{T}}) = \sigma'(M)$ and $\pi'(\mu ; E) = \pi'(\mu)$, so that in principle only upper (E) and lower (E^{T}) cones, respectively, are connected. But again applying $\overline{W ; E} = \overline{W ; E} ; E^{\mathsf{T}}$, we get

$$\sigma'(M) ; E = \overline{\Omega^{\mathsf{T}} ; M ; \overline{E}} ; E = \overline{\Omega^{\mathsf{T}} ; M ; \overline{E} ; E^{\mathsf{T}}} ; E = \overline{\Omega^{\mathsf{T}} ; M ; \overline{E}} = \sigma'(M),$$

so that images of σ' are always upper cones — and thus best described by their greatest lower bounds $\mathtt{glbR}_E(\sigma'(M))$.

$$
\begin{aligned}
\mathtt{glbR}_E(\sigma'(M)) &= \mathtt{glbR}_E(\overline{\Omega^{\mathsf{T}};M;\overline{E}})\\
&= [\mathtt{glb}_E(\overline{\Omega^{\mathsf{T}};M;\overline{E}}^{\mathsf{T}})]^{\mathsf{T}}\\
&= [\mathtt{glb}_E(\overline{\overline{E}^{\mathsf{T}};M^{\mathsf{T}};\Omega})]^{\mathsf{T}}\\
&= [\mathtt{glb}_E(\mathtt{ubd}(M^{\mathsf{T}};\Omega))]^{\mathsf{T}}\\
&= [\mathtt{lub}_E(M^{\mathsf{T}};\Omega)]^{\mathsf{T}}\\
&= \mathtt{lubR}_E(\Omega^{\mathsf{T}};M)
\end{aligned}
$$

which — up to gauging by adding μ_0 — leads us to the following definition with proposition.

8.4 Proposition. Should some body of evidence M be given, there exist two relational measures closely resembling M, the

i) **belief measure** $\mu_{\text{belief}}(M) := \mu_0 \oplus \mathtt{lubR}_E(\Omega^{\mathsf{T}};M)$ and the
ii) **plausibility measure** $\mu_{\text{plausibility}}(M) := \mu_0 \oplus \mathtt{lubR}_E(\Omega^{\mathsf{T}};(\Omega \cap \overline{\Omega};\mathbb{T});M)$,

In general, the belief measure assigns values not exceeding those of the plausibility measure, i.e., $\mu_{\text{belief}}(M) \subseteq \mu_{\text{plausibility}}(M);E^{\mathsf{T}}$. $\square$

$$
\begin{array}{r}
\\
\{\}\\ \{\text{Abe}\}\\ \{\text{Bob}\}\\ \{\text{Abe,Bob}\}\\ \{\text{Carl}\}\\ \{\text{Abe,Carl}\}\\ \{\text{Bob,Carl}\}\\ \{\text{A,B,C}\}\\ \{\text{Don}\}\\ \{\text{Abe,Don}\}\\ \{\text{Bob,Don}\}\\ \{\text{A,B,D}\}\\ \{\text{Carl,Don}\}\\ \{\text{A,C,D}\}\\ \{\text{B,C,D}\}\\ \{\text{A,B,C,D}\}
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccc}
\text{(low,lazy)} & \text{(medium,lazy)} & \text{(low,fair)} & \text{(high,lazy)} & \text{(medium,fair)} & \text{(low,good)} & \text{(high,fair)} & \text{(medium,good)} & \text{(low,bulldozer)} & \text{(high,good)} & \text{(medium,bulldozer)} & \text{(high,bulldozer)}
\end{array}\\
\begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix}\\
\mu_{\text{belief}}(M)
\end{array}
\quad
\begin{array}{c}
\begin{array}{cccccccccccc}
\text{(low,lazy)} & \text{(medium,lazy)} & \text{(low,fair)} & \text{(high,lazy)} & \text{(medium,fair)} & \text{(low,good)} & \text{(high,fair)} & \text{(medium,good)} & \text{(low,bulldozer)} & \text{(high,good)} & \text{(medium,bulldozer)} & \text{(high,bulldozer)}
\end{array}\\
\begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix}\\
\mu_{\text{plausibility}}(M)
\end{array}
$$

Fig. 8.4 Belief measure and plausibility measure for M of Fig. 8.3

The belief measure adds information to the extent that all evidence of subsets with an evidence attached is incorporated. Another idea leads to the plausibility measure. Given a set s, one considers sets with non-empty intersection with s; then one assumes that all their evidences might flow into the respective intersection and, therefore, determines the least upper bound of all these.

The plausibility measure collects those pieces of evidence that do *not* indicate trust against occurrence of the event or non-void parts of it. The belief as well as the plausibility measure more or less precisely determine their original body of evidence.

8.5 Proposition. Should the body of evidence be concentrated on singleton sets only, the belief and the plausibility measure will coincide.

Proof: We recall the singleton injection ι and abbreviate $a = \iota^{\mathsf{T}};\iota$, the partial diagonal describing the atoms of the powerset ordering Ω. That M is concentrated on arguments which are singleton sets means that $M = a;M$. For Ω and a one can prove $(\Omega \cap \overline{\Omega};\mathbb{T});a = a$ as the only other element less or equal to an atom, namely the least one, has been cut out via $\overline{\Omega}$. Then

$$\begin{aligned}
\Omega^{\mathsf{T}};(\Omega \cap \overline{\Omega};\mathbb{T});M &= \Omega^{\mathsf{T}};(\Omega \cap \overline{\Omega};\mathbb{T});a;M && M = a;M\\
&= \Omega^{\mathsf{T}};a;M && \text{see above}\\
&= \Omega^{\mathsf{T}};M && \text{again since } M = a;M
\end{aligned}$$ □

One should compare this result with the former one assuming m to be a mapping putting $m := \varepsilon;M$. One may also try to go in reverse direction, namely from a measure back to a body of evidence.

8.6 Definition. Let some measure μ be given and define strict subset containment $C := \overline{\mathbb{I}} \cap \Omega$. We introduce two basic probability assignments, namely

i) $A_\mu := \mathtt{lubR}_E(C^{\mathsf{T}};\mu)$, its **purely additive part**,
ii) $J_\mu := \mu_1 \otimes (\mu \cap \overline{\mathtt{lubR}_E(C^{\mathsf{T}};\mu)})$, its **jump part**. □

As an example, the purely additive part A_μ of the μ of Fig. 7.1 would assign in line {Abe,Bob} the value {high,fair} only as μ({Abe}) = {high,lazy} and μ({Bob}) = {medium,fair}. In excess to this, μ assigns {high,good}, and is, thus, not additive or Bayesian. For A_μ we have taken only what could have been computed already by summing up the values attached to *strictly* smaller subsets. In J_μ the excess of μ to A_μ is collected. In the procedure for J_μ not least all the values attached to atoms of the lattice will be preserved. This comes due to the fact that from an atom only one step down according to C is possible. The value for the least element is, however, the least element of $\mathcal{L}$. Multiplication with μ_1 serves the purpose that rows full of **0**'s be converted to rows with the least element 0_E attached as a value.

The purely additive part is 0_E for atoms and for 0_Ω. It is not a measure. The pure jump part first shows what is assigned to atoms; in addition, it identifies where more than the least upper bound of assignments to proper subsets is assigned. It is not a measure.

Now some arithmetic on these parts is possible, not least providing the insight that a measure decomposes into an additive part and a jump part.

8.7 Proposition. Given the present setting, we have

i) $A_\mu \oplus J_\mu = \mu$.
ii) $\mu_{\text{belief}}(J_\mu) = \mu$.

Proof: i) We may disregard multiplication with μ_1. It is introduced only for some technical reason: It converts empty rows to rows with 0_E assigned. This is necessary when adding, i.e., intersecting two upper cones and determining their greatest lower bound. Now, they will not be empty. In total, we have obviously

$$\mu;E = A_\mu;E \cap J_\mu;E$$

so that the greatest lower bounds will coincide.

ii) As the effect of gauging is restricted to arguments $0_\Omega, 1_\Omega$, we may handle these separately. We have that μ as well as μ_{belief} are measures, so that both will deliver results $0_E, 1_E$ for $0_\Omega, 1_\Omega$ regardless of how J_μ is defined.

We concentrate on the jump part.

$$\begin{aligned} J_\mu &= \mu_1 \otimes (\mu \cap \overline{\mathtt{lubR}_E(C^\mathsf{T};\mu)}) \\ &= \mathtt{lubR}_E(\mu_1;E^\mathsf{T} \cap (\mu \cap \overline{\mathtt{lubR}_E(C^\mathsf{T};\mu)});E^\mathsf{T}) \\ &= \mathtt{lubR}_E((\mu \cap \overline{\mathtt{lubR}_E(C^\mathsf{T};\mu)});E^\mathsf{T}) \\ &= \mathtt{lubR}_E((\mu \cap \overline{\mathtt{lubR}_E(C^\mathsf{T};\mu)})) \qquad \text{as always } \mathtt{lubR}_E(X;E^\mathsf{T}) = \mathtt{lubR}_E(X) \end{aligned}$$

As the case 0_Ω has already been handled, we have for all arguments $x \neq 1_\Omega$ that $\mu_1^\mathsf{T} x = 1_E$. This means that in this area $\overline{\mu_1;E^\mathsf{T}} = \mathbb{T}$, so that we may start a case analysis. As μ is a mapping, $\mu \cap \overline{\mathtt{lubR}_E(C^\mathsf{T};\mu)}$ is necessarily univalent. In the area where it is defined, the $\mathtt{lubR}_E$ of it will coincide with μ. Where it is not defined, the value of μ has been cut out by $\overline{\mathtt{lubR}_E(C^\mathsf{T};\mu)}$ which means $\mu \subseteq \mathtt{lubR}_E(C^\mathsf{T};\mu)$. This is a mapping contained in a mapping giving rise to an equality in this area. □

In the real-valued case, this result is not surprising at all as one may always decompose into a left-continuous part and a jump part. In Fig. 8.5, we determine the additive and the jump part for our running example.

In view of these results it seems promising to investigate in which way also concepts such as commonality, consonance, necessity measures, focal sets, and cores may be found in the relational approach. This seems particularly interesting as also the concepts of De Morgan triples have been transferred to the pointfree relational form. We leave this to future research.

As long as the set $\mathcal{C}$ of criteria is comparatively small, it seems possible to work with $\mathcal{P}(\mathcal{C})$ and, thus, to take into consideration specific combinations of criteria. As the size of $\mathcal{C}$ increases so as to handle a voting-type or polling-type problem, one will soon handle voters on an equal basis — at least in democracies. This means that the measure applied must not attribute different values to differently chosen n-element sets, e.g. That the values for an n-element set is different from the value attached to an $(n+1)$-element set, will probably be accepted.

$$
\begin{array}{r}
 \\
\{\} \\ \{\text{Abe}\} \\ \{\text{Bob}\} \\ \{\text{Abe,Bob}\} \\ \{\text{Carl}\} \\ \{\text{Abe,Carl}\} \\ \{\text{Bob,Carl}\} \\ \{\text{A,B,C}\} \\ \{\text{Don}\} \\ \{\text{Abe,Don}\} \\ \{\text{Bob,Don}\} \\ \{\text{A,B,D}\} \\ \{\text{Carl,Don}\} \\ \{\text{A,C,D}\} \\ \{\text{B,C,D}\} \\ \{\text{A,B,C,D}\}
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccc}
\text{(low,lazy)} & \text{(medium,lazy)} & \text{(low,fair)} & \text{(high,lazy)} & \text{(medium,fair)} & \text{(low,good)} & \text{(high,fair)} & \text{(medium,good)} & \text{(low,bulldozer)} & \text{(high,good)} & \text{(medium,bulldozer)} & \text{(high,bulldozer)}
\end{array} \\
\begin{pmatrix}
\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\
\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\
\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\
\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}\\
\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}
\end{pmatrix} \\
A_\mu
\end{array}
\quad
\begin{array}{c}
\begin{array}{cccccccccccc}
\text{(low,lazy)} & \text{(medium,lazy)} & \text{(low,fair)} & \text{(high,lazy)} & \text{(medium,fair)} & \text{(low,good)} & \text{(high,fair)} & \text{(medium,good)} & \text{(low,bulldozer)} & \text{(high,good)} & \text{(medium,bulldozer)} & \text{(high,bulldozer)}
\end{array} \\
\begin{pmatrix}
\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}\\
\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\
\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\
\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\
\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\
\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\
\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\
\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}
\end{pmatrix} \\
J_\mu
\end{array}
$$

Fig. 8.5 Measure of Fig. 7.1, decomposed into additive part and jump part

As a result, the technique to define the measure will be based on *operations* in $\mathcal{L}$ alone. In total: instead of a measure on $\mathcal{P}(\mathcal{C})$ we work with an operation on values of $\mathcal{L}$. This motivates the introduction of triangular norms, extensively used in, e.g., [9,11].

9 Concluding Remark

There exists a bulk of literature around the topic of Dempster-Shafer belief. It concentrates mostly on work with real numbers and their linear order and applies traditional free-hand mathematics. This makes it sometimes difficult to follow the basic ideas, not least as authors are all too often falling back to probability considerations.

We feel that the point-free relational reformulation of this field and the important generalization accompanying it is a clarification — at least for the strictly growing community of those who do not fear to use relations. Proofs may now be supported by proof systems. The results of this paper have been formulated also in the relational language TituRel [13,14], for which some system support is available making it immediately operational. Not least has it provided computation and representation of the example matrices.

Acknowledgement. The authors are very grateful to the unknown referees one of which also undertook the task of carefully checking the denotational correspondences to classical concepts of measure and integration.

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