Homomorphism and Isomorphism Theorems Generalized from a Relational Perspective

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Abstract. The homomorphism and isomorphism theorems traditionally taught to students in a group theory or linear algebra lecture are by no means theorems of group theory. They are for a long time seen as general concepts of universal algebra. This article goes even further and identifies them as relational properties which to study does not even require the concept of an algebra. In addition it is shown how the homomorphism and isomorphism theorems generalize to not necessarily algebraic and thus relational structures.

Keywords: homomorphism theorem, isomorphism theorem, relation algebra, congruence, multi-covering.

1 Introduction

Relation algebra has received increasing interest during the last years. Many areas have been reconsidered from the relational point of view, which often provided additional insight. Here, the classical homomorphism and isomorphism theorems (see [1], e.g.) are reviewed from a relational perspective, thereby simplifying and slightly generalizing them.

The paper is organized as follows. First we recall the notion of a heterogeneous relation algebra and some of the very basic rules governing work with relations. With these, function and equivalence properties may be formulated concisely. The relational concept of homomorphism is defined as well as the concept of a congruence which is related with the concept of a multi-covering, which have connections with topology, complex analysis, and with the equivalence problem for flow-chart programs. We deal with the relationship between mappings and equivalence relations. The topics include the so-called substitution property and the forming of quotients.

Homomorphisms may be used to give universal characterizations of domain constructions. Starting from sets, further sets may be obtained by construction, as pair sets (direct product), as variant sets (direct sum), as power sets (direct power), or as the quotient of a set modulo some equivalence. Another

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construction that is not so easily identified as such is subset extrusion. It serves to promote a subset of a set, which needs the larger one to exist, to a set of its own right.

Using the so-called dependent types, quotient set and subset extrusion, we then formulate the homomorphism and isomorphism theorems and prove them in a fully algebraic style. The paper ends with hints on coverings with locally univalent outgoing fans.

2 Homogeneous and Heterogeneous Relation Algebras

A **homogeneous relation algebra** $(\mathcal{R}, \cup, \cap, ^{-}, ;, ^{\mathsf{T}})$ consists of a set $\mathcal{R} \neq \emptyset$, whose elements are called relations, such that $(\mathcal{R}, \cup, \cap, ^{-})$ is a complete, atomic boolean algebra with zero element $\perp\!\!\!\perp$, universal element $\top\!\!\!\top$, and ordering $\subseteq$, that $(\mathcal{R}, ;)$ is a semigroup with precisely one unit element $\mathbb{I}$, and, finally, the Schröder equivalences $Q;R \subseteq S \iff Q^{\mathsf{T}};\overline{S} \subseteq \overline{R} \iff \overline{S};R^{\mathsf{T}} \subseteq \overline{Q}$ are satisfied.

One may switch to heterogeneous relation algebra, which has been proposed in, e.g., [2,3]. A **heterogeneous relation algebra** is a category $\mathcal{R}$ consisting of a set $\mathcal{O}$ of objects and sets $\mathtt{Mor}(A, B)$ of morphisms, where $A, B \in \mathcal{O}$. Composition is denoted by $;$ while identities are denoted by $\mathbb{I}_A \in \mathtt{Mor}(A, A)$. In addition, there is a totally defined unary operation $^{\mathsf{T}}_{A,B} : \mathtt{Mor}(A, B) \longrightarrow \mathtt{Mor}(B, A)$ between morphism sets. Every set $\mathtt{Mor}(A, B)$ carries the structure of a complete, atomic boolean algebra with operations $\cup, \cap, ^{-}$, zero element $\perp\!\!\!\perp_{A,B}$, universal element $\top\!\!\!\top_{A,B}$ (the latter two non-equal), and inclusion ordering $\subseteq$. The Schröder equivalences—where the definedness of one of the three formulae implies that of the other two—are postulated to hold.

Most of the indices of elements and operations are usually omitted for brevity and can easily be reinvented. For the purpose of self-containedness, we recall the following computational rules; see, e.g., [4,5].

2.1 Proposition.

i) $\perp\!\!\!\perp;R = R;\perp\!\!\!\perp = \perp\!\!\!\perp$;

ii) $R \subseteq S \implies Q;R \subseteq Q;S,\ R;Q \subseteq S;Q$;

iii) $Q;(R \cap S) \subseteq Q;R \cap Q;S, \quad (R \cap S);Q \subseteq R;Q \cap S;Q$

$Q;(R \cup S) = Q;R \cup Q;S, \quad (R \cup S);Q = R;Q \cup S;Q$

iv) $(R^{\mathsf{T}})^{\mathsf{T}} = R$;

v) $(R;S)^{\mathsf{T}} = S^{\mathsf{T}};R^{\mathsf{T}}$;

vi) $R \subseteq S \iff R^{\mathsf{T}} \subseteq S^{\mathsf{T}}$;

vii) $\overline{R}^{\mathsf{T}} = \overline{R^{\mathsf{T}}}$;

viii) $(R \cup S)^{\mathsf{T}} = R^{\mathsf{T}} \cup S^{\mathsf{T}}$;

$(R \cap S)^{\mathsf{T}} = R^{\mathsf{T}} \cap S^{\mathsf{T}}$;

ix) $Q;R \cap S \subseteq (Q \cap S;R^{\mathsf{T}});(R \cap Q^{\mathsf{T}};S)$. (Dedekind rule)

A relation R is called **univalent** (or a partial function) if $R^{\mathsf{T}};R \subseteq \mathbb{I}$. When R satisfies $\mathbb{I} \subseteq R;R^{\mathsf{T}}$ (or equivalently if $\top\!\!\!\top \subseteq R;\top\!\!\!\top$), then R is said to be **total**. If both these requirements are satisfied, i.e., if R resembles a total and univalent function, we shall often speak of a **mapping**. A relation R is called injective, surjective and bijective, if R^{T} is univalent, total, or both, respectively. Furthermore

$$R \subseteq Q,\ Q \text{ univalent},\ R;\mathbb{T} \supseteq Q;\mathbb{T} \quad \Longrightarrow \quad R = Q \qquad (*)$$

The following basic properties are mainly recalled from [4,5].

2.2 Proposition (*Row and column masks*)**.** The following formulae hold for arbitrary relations $P : V \longrightarrow W, Q : U \longrightarrow V, R : U \longrightarrow W, S : V \longrightarrow W$, provided the constructs are defined.

i) $(Q \cap R;\mathbb{T}_{WV});S = Q;S \cap R;\mathbb{T}_{WW}$;

ii) $(Q \cap (P;\mathbb{T}_{WU})^{\mathsf{T}});S = Q;(S \cap P;\mathbb{T}_{WW})$. □

We now recall a rule which is useful for calculations involving equivalence relations; it deals with the effect of composition with an equivalence relation with regard to intersection. For a proof see [4,5].

2.3 Proposition. Let Θ be an equivalence and let A, B be arbitrary relations.

$$(A;\Theta \cap B);\Theta = A;\Theta \cap B;\Theta = (A \cap B;\Theta);\Theta$$ □

It is sometimes useful to consider a vector, which is what has at other occasions been called a right ideal element. It is characterized by $U = U;\mathbb{T}$ and thus corresponds to a subset or a predicate. One may, however, also use a partial diagonal to characterize a subset. There is a one-to-one correspondence between the two concepts. Of course, $p \subseteq \mathbb{I} \implies p^2 = p^{\mathsf{T}} = p$. The **symmetric quotient** has been applied in various applications:

$$\mathtt{syq}(A, B) := \overline{A^{\mathsf{T}};\overline{B}} \cap \overline{\overline{A}^{\mathsf{T}};B}$$

3 Homomorphisms

We recall the concept of homomorphism for relational structures with Fig. 3.1. Structure and mappings shall commute, however, not as an equality but just as containment.

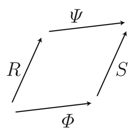

Fig. 3.1. Relational homomorphism

3.1 Definition. Given two relations R, S, we call the pair (Φ, Ψ) of relations a **homomorphism** from R to S, if Φ, Ψ are mappings satisfying

$$R;\Psi \subseteq \Phi;S.$$ □

The homomorphism condition has four variants

$$R;\Psi \subseteq \Phi;S \iff R \subseteq \Phi;S;\Psi^\mathsf{T} \iff \Phi^\mathsf{T};R \subseteq S;\Psi^\mathsf{T} \iff \Phi^\mathsf{T};R;\Psi \subseteq S$$

which may be used interchangeably. This is easily recognized applying the mapping properties

$$\Phi^\mathsf{T};\Phi \subseteq \mathbb{I}, \quad \mathbb{I} \subseteq \Phi;\Phi^\mathsf{T}, \quad \Psi^\mathsf{T};\Psi \subseteq \mathbb{I}, \quad \mathbb{I} \subseteq \Psi;\Psi^\mathsf{T}$$

As usual, also isomorphisms are introduced.

3.2 Definition. We call (Φ, Ψ) an **isomorphism** between the two relations R, S, if it is a homomorphism from R to S and if $(\Phi^\mathsf{T}, \Psi^\mathsf{T})$ is a homomorphism from S to R. □

The following lemma will sometimes help in identifying an isomorphism.

3.3 Lemma. Let relations R, S be given together with a homomorphism (Φ, Ψ) from R to S such that

Φ, Ψ are bijective mappings and $R;\Psi = \Phi;S$.

Then (Φ, Ψ) is an isomorphism.

Proof. $S;\Psi^\mathsf{T} = \Phi^\mathsf{T};\Phi;S;\Psi^\mathsf{T} = \Phi^\mathsf{T};R;\Psi;\Psi^\mathsf{T} = \Phi^\mathsf{T};R.$ □

4 Universal Characterizations

Given a mathematical structure, one is immediately interested in homomorphisms, substructures, and congruences. When handling these, there is a characteristic difference between algebraic and relational structures.

Algebraic structures are defined by composition laws such as a binary multiplication $mult: A \times A \longrightarrow A$ or the unary operation of forming the inverse $inv: A \longrightarrow A$. These operations can, of course, be interpreted as relations. The first example furnishes a "ternary" relation $R_{mult} : (A \times A) \longrightarrow A$, the second, a binary relation $R_{inv} : A \longrightarrow A$, and both are univalent and total.

Relational structures are also defined by certain relations, but these need no longer be univalent or total. Purely relational structures are orders, strictorders, equivalences, and graphs. Typically, however, mixed structures with both, algebraic and relational, features occur, such as ordered fields, for example.

4.1 Standard Domain Constructions

The *direct product* resembling the pair set construction is given via two generic relations π, ρ, the left and the right projection, satisfying

$$\pi^\mathsf{T};\pi = \mathbb{I}, \quad \rho^\mathsf{T};\rho = \mathbb{I}, \quad \pi;\pi^\mathsf{T} \cap \rho;\rho^\mathsf{T} = \mathbb{I}, \quad \pi^\mathsf{T};\rho = \mathbb{T}$$

Whenever a second pair π_1, ρ_1 of relations with these properties should be presented, one may construct the isomorphism $\Phi := \pi;\pi_1^\mathsf{T} \cap \rho;\rho_1^\mathsf{T}$, thus showing that the direct product is defined uniquely up to isomorphism.

The *direct sum* resembling variant set forming (disjoint union) is given via two generic relations ι, κ, the left and the right injection, satisfying

$$\iota;\iota^\mathsf{T} = \mathbb{I}, \quad \kappa;\kappa^\mathsf{T} = \mathbb{I}, \quad \iota^\mathsf{T};\iota \cup \kappa^\mathsf{T};\kappa = \mathbb{I}, \quad \iota;\kappa^\mathsf{T} = \mathbb{⊥}$$

Whenever a second pair ι_1, κ_1 of relations with these properties should be presented, one may construct the isomorphism $\Phi := \iota^\mathsf{T};\iota_1 \cup \kappa^\mathsf{T};\kappa_1$, thus showing that the direct sum is defined uniquely up to isomorphism.

The *direct power* resembling powerset construction is given via a generic relation ε, the membership relation, satisfying

$$\mathtt{syq}(\varepsilon, \varepsilon) \subseteq \mathbb{I} \text{ and that } \mathtt{syq}(\varepsilon, X) \text{ is surjective for every relation } X$$

Should a second membership relation ε_1 with these properties be presented, one may construct the isomorphism $\Phi := \mathtt{syq}(\varepsilon, \varepsilon_1)$, thus showing that the direct power is defined uniquely up to isomorphism. These constructions are by now standard; proofs may be found in [4,5].

4.2 Quotient Forming and Subset Extrusion

In addition to these, other domain constructions are possible which are usually not handled as such. Although relatively simple, they need a bit of care. Known as *dependent types* they do not just start with a domain or two, but with an additional construct, namely an equivalence or a subset.

4.1 Proposition (*Quotient set*)**.** Given an equivalence Ξ on the set V, one may generically define the quotient set V_Ξ together with the natural projection $\eta : V \longrightarrow V_\Xi$ postulating both to satisfy

$$\Xi = \eta;\eta^\mathsf{T}, \qquad \eta^\mathsf{T};\eta = \mathbb{I}_{V_\Xi}.$$

The natural projection η is uniquely determined up to isomorphism: should a second natural projection η_1 be presented, the isomorphism is $(\mathbb{I}, \eta^\mathsf{T};\eta_1)$.

Proof. Assume two such projections $V_\Xi \xleftarrow{\eta} V \xrightarrow{\eta_1} W_\Xi$, for which therefore

$$\Xi = \eta_1;\eta_1^\mathsf{T}, \qquad \eta_1^\mathsf{T};\eta_1 = \mathbb{I}_{W_\Xi}.$$

Looking at this setting, the only way to relate V_Ξ with W_Ξ is to define $\Phi := \eta^\mathsf{T};\eta_1$ and proceed showing

$$\begin{aligned}
\Phi^\mathsf{T};\Phi &= (\eta_1^\mathsf{T};\eta);(\eta^\mathsf{T};\eta_1) && \text{by definition of } \Phi\\
&= \eta_1^\mathsf{T};(\eta;\eta^\mathsf{T});\eta_1 && \text{associative}\\
&= \eta_1^\mathsf{T};\Xi;\eta_1 && \text{as } \Xi = \eta;\eta^\mathsf{T}\\
&= \eta_1^\mathsf{T};(\eta_1;\eta_1^\mathsf{T});\eta_1 && \text{as } \Xi = \eta_1;\eta_1^\mathsf{T}\\
&= (\eta_1^\mathsf{T};\eta_1);(\eta_1^\mathsf{T};\eta_1) && \text{associative}\\
&= \mathbb{I}_{W_\Xi};\mathbb{I}_{W_\Xi} && \text{since } \eta_1^\mathsf{T};\eta_1 = \mathbb{I}_{W_\Xi}\\
&= \mathbb{I}_{W_\Xi} && \text{since } \mathbb{I}_{W_\Xi};\mathbb{I}_{W_\Xi} = \mathbb{I}_{W_\Xi}
\end{aligned}$$

$\Phi;\Phi^\mathsf{T} = \mathbb{I}_{V_\Xi}$ is shown analogously. Furthermore, $(\mathbb{I}, \Phi)$ satisfies the property of an isomorphism between η and η_1 following Lemma 3.3:

$$\eta;\Phi = \eta;\eta^\mathsf{T};\eta_1 = \Xi;\eta_1 = \eta_1;\eta_1^\mathsf{T};\eta_1 = \eta_1;\mathbb{I}_{W_\Xi} = \eta_1 \qquad \square$$

Not least when working on a computer, one is interested in such quotients as the quotient set is usually smaller and may be handled more efficiently. The same reason leads us to consider subset extrusion in a very formal way.

A subset is assumed to exist *relatively to some other set* so that it is not a first-class citizen in our realm of domains. With a bit of formalism, however, it can be managed to convert a subset so as to have it as a set of its own right, a process which one might call a *subset extrusion.*

4.2 Proposition (*Extruded subset*)**.** Given a subset U of some set V, one may generically define the extruded set D_U together with the natural injection $\chi : D_U \longrightarrow V$ postulating both to satisfy

$$\chi;\chi^{\mathsf{T}} = \mathbb{I}_{D_U}, \quad \chi^{\mathsf{T}};\chi = \mathbb{I}_V \cap U;\mathbb{T}_{V,V}.$$

The natural injection χ is uniquely determined up to isomorphism: should a second natural injection χ_1 be presented, the isomorphism is $(\chi;\chi_1^{\mathsf{T}}, \mathbb{I})$.

Proof. We have $D_U \xrightarrow{\chi} V \xleftarrow{\chi_1} D$ with the corresponding properties:

$$\chi_1;\chi_1^{\mathsf{T}} \subseteq \mathbb{I}_D, \quad \chi_1^{\mathsf{T}};\chi_1 = \mathbb{I}_V \cap U;\mathbb{T}_{V,V}$$

and show

$$\Phi^{\mathsf{T}};\Phi = \chi_1;\chi^{\mathsf{T}};\chi;\chi_1^{\mathsf{T}} = \chi_1;(\mathbb{I}_V \cap U;\mathbb{T});\chi_1^{\mathsf{T}} = \chi_1;\chi_1^{\mathsf{T}};\chi_1;\chi_1^{\mathsf{T}} = \mathbb{I}_D;\mathbb{I}_D = \mathbb{I}_D$$

and analogously also $\Phi;\Phi^{\mathsf{T}} = \mathbb{I}_{D_U}$. Furthermore, $(\Phi, \mathbb{I})$ satisfies the property of an isomorphism between χ and χ_1 using Lemma 3.3:

$$\chi;\mathbb{I}_V = \chi = \mathbb{I}_{D_U};\chi = \chi;\chi^{\mathsf{T}};\chi = \chi;(\mathbb{I}_V \cap U;\mathbb{T}) = \chi;\chi_1^{\mathsf{T}};\chi_1 = \Phi;\chi_1 \qquad \square$$

A point to mention is that subset extrusion allows to switch from set-theoretic consideration to an algebraic one. When using a computer and a formula manipulation system or a theorem prover, this means a considerable restriction in expressivity which is honored with much better efficiency.

An important application of extrusion is the concept of *tabulation* introduced by Roger Maddux. It now turns out to be a composite construction; see [6,7], e.g. An arbitrary relation $R : X \longrightarrow Y$ is said to be tabulated by relations (due to the following characterization, they turn out to be mappings) P, Q if

$$P^{\mathsf{T}};Q = R,\ P^{\mathsf{T}};P = \mathbb{I}_X \cap R;\mathbb{T}_{YX},\ Q^{\mathsf{T}};Q = \mathbb{I}_Y \cap R^{\mathsf{T}};\mathbb{T}_{XY},\ P;P^{\mathsf{T}} \cap Q;Q^{\mathsf{T}} = \mathbb{I}_{X\times Y}$$

This may indeed be composed of extruding with $\chi : D_U \longrightarrow X \times Y$ the subset of related pairs out of a direct product

$$\begin{aligned} U &:= (\pi;R \cap \rho);\mathbb{T}_{Y,X\times Y} = (\pi;R \cap \rho);\mathbb{T}_{YX};\pi^{\mathsf{T}} = (\pi;R;\rho^{\mathsf{T}} \cap \mathbb{I});\rho;\mathbb{T}_{YX};\pi^{\mathsf{T}} \\ &= (\pi;R;\rho^{\mathsf{T}} \cap \mathbb{I});\mathbb{T}_{X\times Y,X\times Y} = (\rho;R^{\mathsf{T}};\pi^{\mathsf{T}} \cap \mathbb{I});\mathbb{T}_{X\times Y,X\times Y} \\ &= (\rho;R^{\mathsf{T}};\pi^{\mathsf{T}} \cap \mathbb{I});\pi;\mathbb{T}_{XY};\rho^{\mathsf{T}} = (\rho;R^{\mathsf{T}} \cap \pi);\mathbb{T}_{XY};\rho^{\mathsf{T}} = (\rho;R^{\mathsf{T}} \cap \pi);\mathbb{T}_{X,X\times Y} \end{aligned}$$

and defining $P := \chi;\pi$ and $Q := \chi;\rho$. This is proved quite easily as follows.

$$\begin{aligned} P^{\mathsf{T}};Q &= \pi^{\mathsf{T}};\chi^{\mathsf{T}};\chi;\rho = \pi^{\mathsf{T}};(\pi;R;\rho^{\mathsf{T}} \cap \mathbb{I});\rho \\ &= \pi^{\mathsf{T}};(\pi;R \cap \rho) \\ &= R \cap \pi^{\mathsf{T}};\rho \\ &= R \cap \mathbb{T} = R \end{aligned}$$

$$P^{\mathsf{T}};P = \pi^{\mathsf{T}};\chi^{\mathsf{T}};\chi;\pi = \pi^{\mathsf{T}};(\mathbb{I} \cap (\pi;R \cap \rho);\rho^{\mathsf{T}};\pi;\pi^{\mathsf{T}});\pi$$

$= \pi^\top ; (\pi \cap (\pi ; R \cap \rho) ; \rho^\top ; \pi)$
$= \mathbb{I} \cap \pi^\top ; (\pi ; R \cap \rho) ; \rho^\top ; \pi$
$= \mathbb{I} \cap (R \cap \pi^\top ; \rho) ; \rho^\top ; \pi = \mathbb{I} \cap R ; \mathbb{T}_{YX}$

$Q^\top ; Q$ is handled analogously

$P ; P^\top \cap Q ; Q^\top = \chi ; \pi ; \pi^\top ; \chi^\top \cap \chi ; \rho ; \rho^\top ; \chi^\top = \chi ; (\pi ; \pi^\top \cap \rho ; \rho^\top) ; \chi^\top = \chi ; \mathbb{I} ; \chi^\top = \mathbb{I}$

5 Congruences and Multi-coverings

Whenever equivalences behave well with regard to some other structure, we are accustomed to call them congruences. This is well-known for algebraic structures, i.e., those defined by mappings on some set. We define it correspondingly for the non-algebraic case, including heterogeneous relations; i.e., possibly neither univalent nor total. While the basic idea is known from many application fields, the following general concepts may be a novel abstraction.

5.1 Definition. Let B be a relation and Ξ, Θ equivalences. The pair (Ξ, Θ) is called a B-**congruence** if $\Xi ; B \subseteq B ; \Theta$. □

If B were an operation on a given set and we had $\Xi = \Theta$, we would say that B "has the substitution property with regard to Ξ". The concept of congruence is related to the concept of a multi-covering.

5.2 Definition. A homomorphism (Φ, Ψ) from B to B' is called a **multi-covering**, provided the functions are surjective and satisfy $\Phi ; B' \subseteq B ; \Psi$ in addition to being a homomorphism. □

The relationship between congruences and multi-coverings is close and seems not to have been pointed out yet.

5.3 Theorem.

i) If (Φ, Ψ) is a multi-covering from B to B', then $(\Xi, \Theta) := (\Phi ; \Phi^\top, \Psi ; \Psi^\top)$ is a B-congruence.
ii) If the pair (Ξ, Θ) is a B-congruence, then there exists up to isomorphism at most one multi-covering (Φ, Ψ) satisfying $\Xi = \Phi ; \Phi^\top$ and $\Theta = \Psi ; \Psi^\top$.

Proof. i) Ξ is certainly reflexive and transitive, as Φ is total and univalent. In the same way, Θ is reflexive and transitive. The relation $\Xi = \Phi ; \Phi^\top$ is symmetric by construction and so is Θ. Now we prove

$\Xi ; B = \Phi ; \Phi^\top ; B \subseteq \Phi ; B' ; \Psi^\top \subseteq B ; \Psi ; \Psi^\top = B ; \Theta$

applying one after the other the definition of Ξ, one of the four homomorphism definitions, the multi-covering condition, and the definition of Θ.

ii) Let (Φ_i, Ψ_i) be a multi-covering from B to B_i, $i = 1, 2$. Then

$B_i \subseteq \Phi_i^\top ; \Phi_i ; B_i \subseteq \Phi_i^\top ; B ; \Psi_i \subseteq B_i$, and therefore everywhere "=", applying surjectivity, the multi-covering property and one of the homomorphism conditions. Now we indicate how to prove that $(\xi,\vartheta) := (\Phi_1^{\mathsf{T}};\Phi_2, \Psi_1^{\mathsf{T}};\Psi_2)$ is a homomorphism from B_1 onto B_2 — which is then of course also an isomorphism.

$$\xi^{\mathsf{T}};\xi = \Phi_2^{\mathsf{T}};\Phi_1;\Phi_1^{\mathsf{T}};\Phi_2 = \Phi_2^{\mathsf{T}};\Xi;\Phi_2 = \Phi_2^{\mathsf{T}};\Phi_2;\Phi_2^{\mathsf{T}};\Phi_2 = \mathbb{I};\mathbb{I} = \mathbb{I}$$
$$B_1;\vartheta = \Phi_1^{\mathsf{T}};B;\Psi_1;\Psi_1^{\mathsf{T}};\Psi_2 = \Phi_1^{\mathsf{T}};B;\Theta;\Psi_2 = \Phi_1^{\mathsf{T}};B;\Psi_2;\Psi_2^{\mathsf{T}};\Psi_2$$
$$\subseteq \Phi_1^{\mathsf{T}};\Phi_2;B_2;\mathbb{I} = \xi;B_2$$
□

The multi-covering (Φ,Ψ) for some given congruences Ξ,Θ need not exist in the given relation algebra. It may, however, be constructed by setting Φ,Ψ to be the quotient mappings according to the two equivalences Ξ,Θ together with $R' := \Phi^{\mathsf{T}};R;\Psi$.

A multi-covering between relational structures most closely resembles a homomorphism on algebraic structures:

5.4 Proposition. A homomorphism between algebraic structures is necessarily a multi-covering.

Proof. Assume two mappings B, B' and the homomorphism (Ψ,Φ) from B to B', so that $B;\Psi \subseteq \Phi;B'$. The relation $\Phi;B'$ is univalent, since Φ and B' are mappings. The domains $B;\Psi;\mathbb{T} = \mathbb{T} = \Phi;B';\mathbb{T}$ of $B;\Psi$ and $\Phi;B'$ coincide, because all the relations are mappings and, therefore, are total. So we may use (*) and obtain $B;\Psi = \Phi;B'$. □

6 Homomorphism and Isomorphism Theorems

Now we study the homomorphism and isomorphism theorems (see [1], e.g.) traditionally offered in a course on group theory or on universal algebra from the relational point of view. In the courses mentioned, R, S are often n-ary mappings such as addition and multiplication. In Fig. 6.1, we are more general allowing them to be relations, i.e., not necessarily mappings. The algebraic laws they satisfy in the algebra are completely irrelevant.

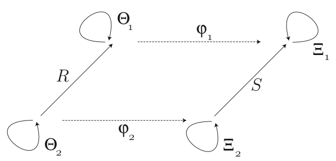

Fig. 6.1. Basic situation of the homomorphism theorem

6.1 Proposition (*Homomorphism Theorem*). Let a relation R be given with an R-congruence (Θ_2,Θ_1) as well as a relation S together with an S-congruence (Ξ_2,Ξ_1). Assume a multi-covering (φ_2,φ_1) from R to S such that at the same time we have $\Theta_i = \varphi_i;\Xi_i;\varphi_i^{\mathsf{T}}$ for $i = 1, 2$; see Fig. 6.1. Introducing the natural projections η_i for Θ_i as well as δ_i for Ξ_i, one has that $\psi_i := \eta_i^{\mathsf{T}};\varphi_i;\delta_i$, $i = 1, 2$, establish an isomorphism from $R' := \eta_2^{\mathsf{T}};R;\eta_1$ to $S' := \delta_2^{\mathsf{T}};S;\delta_1$.

Proof. The equivalences (Θ_2, Θ_1) satisfy $\Theta_2;R \subseteq R;\Theta_1$ while (Ξ_2, Ξ_1) satisfy $\Xi_2;S \subseteq S;\Xi_1$. Furthermore, we have that (φ_2, φ_1) are surjective mappings satisfying $R;\varphi_1 \subseteq \varphi_2;S$ for homomorphism and $R;\varphi_1 \supseteq \varphi_2;S$ for multi-covering.

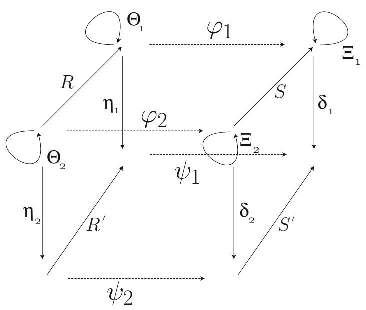

Fig. 6.2. Natural projections added to Fig. 6.1

The ψ_i are bijective mappings, which we prove omitting indices:

$$\begin{aligned}
\psi^{\mathsf{T}};\psi &= (\eta^{\mathsf{T}};\varphi;\delta)^{\mathsf{T}};\eta^{\mathsf{T}};\varphi;\delta && \text{by definition}\\
&= \delta^{\mathsf{T}};\varphi^{\mathsf{T}};\eta;\eta^{\mathsf{T}};\varphi;\delta && \text{executing transposition}\\
&= \delta^{\mathsf{T}};\varphi^{\mathsf{T}};\Theta;\varphi;\delta && \text{natural projection } \eta\\
&= \delta^{\mathsf{T}};\varphi^{\mathsf{T}};\varphi;\Xi;\delta && \text{multi-covering}\\
&= \delta^{\mathsf{T}};\Xi;\delta && \text{as } \varphi \text{ is surjective and univalent}\\
&= \delta^{\mathsf{T}};\delta;\delta^{\mathsf{T}};\delta = \mathbb{I};\mathbb{I} = \mathbb{I} && \text{natural projection } \delta
\end{aligned}$$

and

$$\begin{aligned}
\psi;\psi^{\mathsf{T}} &= \eta^{\mathsf{T}};\varphi;\delta;(\eta^{\mathsf{T}};\varphi;\delta)^{\mathsf{T}} && \text{by definition}\\
&= \eta^{\mathsf{T}};\varphi;\delta;\delta^{\mathsf{T}};\varphi^{\mathsf{T}};\eta && \text{transposing}\\
&= \eta^{\mathsf{T}};\varphi;\Xi;\varphi^{\mathsf{T}};\eta && \text{natural projection } \delta\\
&= \eta^{\mathsf{T}};\Theta;\eta && \text{property of } \varphi \text{ wrt. } \Theta, \Xi\\
&= \eta^{\mathsf{T}};\eta;\eta^{\mathsf{T}};\eta = \mathbb{I};\mathbb{I} = \mathbb{I} && \text{natural projection } \eta
\end{aligned}$$

Proof of the isomorphism property:

$$\begin{aligned}
R';\psi_1 &= \eta_2^{\mathsf{T}};R;\eta_1;\eta_1^{\mathsf{T}};\varphi_1;\delta_1 && \text{by definition}\\
&= \eta_2^{\mathsf{T}};R;\Theta_1;\varphi_1;\delta_1 && \text{natural projection } \eta_1\\
&= \eta_2^{\mathsf{T}};R;\varphi_1;\Xi_1;\varphi_1^{\mathsf{T}};\varphi_1;\delta_1 && \text{property of } \varphi \text{ wrt. } \Theta, \Xi\\
&= \eta_2^{\mathsf{T}};R;\varphi_1;\Xi_1;\delta_1 && \text{as } \varphi_1 \text{ is surjective and univalent}\\
&= \eta_2^{\mathsf{T}};\varphi_2;S;\Xi_1;\delta_1 && \text{multi-covering}\\
&= \eta_2^{\mathsf{T}};\varphi_2;\Xi_2;S;\Xi_1;\delta_1 && S;\Xi_1 \subseteq \Xi_2;S;\Xi_1 \subseteq S;\Xi_1;\Xi_1 = S;\Xi_1\\
&= \eta_2^{\mathsf{T}};\varphi_2;\delta_2;\delta_2^{\mathsf{T}};S;\delta_1;\delta_1^{\mathsf{T}};\delta_1 && \text{natural projections}\\
&= \eta_2^{\mathsf{T}};\varphi_2;\delta_2;S';\delta_1^{\mathsf{T}};\delta_1 && \text{definition of } S'\\
&= \eta_2^{\mathsf{T}};\varphi_2;\delta_2;S' && \text{as } \delta_1 \text{ is surjective and univalent}\\
&= \psi_2;S' && \text{definition of } \psi_2
\end{aligned}$$

According to Lemma 3.3, this suffices for an isomorphism. ◻

One should bear in mind that this proposition was in several respects slightly more general than the classical homomorphism theorem: R, S need not be mappings, nor need they be homogeneous relations, Ξ was not confined to be the identity congruence, and not least does relation algebra admit non-standard models.

6.2 Proposition (*First Isomorphism Theorem*)**.** Let a homogeneous relation R on X together with an equivalence Ξ and a non-empty subset U. Assume that U is contracted by R and that Ξ is an R-congruence:

$$R^\mathsf{T}; U \subseteq U \quad \text{and} \quad \Xi; R \subseteq R; \Xi.$$

Now extrude both, U and its Ξ-saturation $\Xi; U$ so as to obtain natural injections

$$\iota : Y \longrightarrow X \text{ and } \lambda : Z \longrightarrow X,$$

universally characterized by (see Fig. 6.3)

$$\iota^\mathsf{T}; \iota = \mathbb{I}_X \cap U; \mathbb{T}, \qquad \iota; \iota^\mathsf{T} = \mathbb{I}_Y,$$
$$\lambda^\mathsf{T}; \lambda = \mathbb{I}_X \cap \Xi; U; \mathbb{T}, \quad \lambda; \lambda^\mathsf{T} = \mathbb{I}_Z.$$

On Y and Z, we consider the derived equivalences $\Xi_Y := \iota; \Xi; \iota^\mathsf{T}$ and $\Xi_Z := \lambda; \Xi; \lambda^\mathsf{T}$ and in addition their natural projections $\eta : Y \longrightarrow Y_\Xi$ and $\delta : Z \longrightarrow Z_\Xi$. In a standard way, restrictions of R may be defined, namely

$$S := \eta^\mathsf{T}; \iota; R; \iota^\mathsf{T}; \eta \text{ and } T := \delta^\mathsf{T}; \lambda; R; \lambda^\mathsf{T}; \delta.$$

In this setting, $\varphi := \delta^\mathsf{T}; \lambda; \iota^\mathsf{T}; \eta$ gives an isomorphism (φ, φ) between S and T.

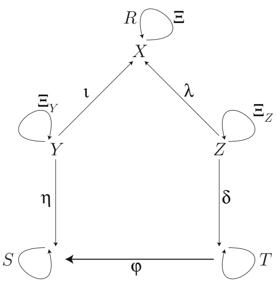

Fig. 6.3. Situation of the First Isomorphism Theorem

Proof. We prove several results in advance, namely

$$\Xi; \iota^\mathsf{T}; \iota; \Xi = \Xi; \lambda^\mathsf{T}; \lambda; \Xi, \tag{1}$$

proved using rules for composition of equivalences:

$\Xi; \iota^\mathsf{T}; \iota; \Xi$	$=$	$\Xi; (\mathbb{I} \cap U; \mathbb{T}); \Xi$	definition of natural injection ι
	$=$	$\Xi; \Xi; (\mathbb{I} \cap U; \mathbb{T}; \Xi); \Xi; \Xi$	Ξ surjective and an equivalence
	$=$	$\Xi; (\mathbb{I} \cap \Xi; U; \mathbb{T}); \Xi$	several applications of Prop. 2.3
	$=$	$\Xi; \lambda^\mathsf{T}; \lambda; \Xi$	definition of natural injection λ

In a similar way follow

$$\iota; \lambda^\mathsf{T}; \lambda = \iota \qquad \iota; R; \iota^\mathsf{T}; \iota = \iota; R \tag{2}$$

The left identity is proved with

$$\begin{aligned}
\iota;\lambda^{\mathsf{T}};\lambda &= \iota;\iota^{\mathsf{T}};\iota;\lambda^{\mathsf{T}};\lambda && \iota \text{ is injective and total}\\
&= \iota;(\mathbb{I}\cap U;\mathbb{T});(\mathbb{I}\cap \Xi;U;\mathbb{T}) && \text{definition of natural injections}\\
&= \iota;(\mathbb{I}\cap U;\mathbb{T}\cap \Xi;U;\mathbb{T}) && \text{intersecting partial identities}\\
&= \iota;(\mathbb{I}\cap U;\mathbb{T}) = \iota;\iota^{\mathsf{T}} = \iota
\end{aligned}$$

The contraction condition $R^{\mathsf{T}};U \subseteq U$ and $\Xi;R \subseteq R;\Xi$ allows to prove the right one for which " $\subseteq$ " is obvious. For " $\supseteq$ ", we apply $\iota;\iota^{\mathsf{T}} = \mathbb{I}$ after having shown

$$\begin{aligned}
\iota^{\mathsf{T}};\iota;R &= (\mathbb{I}\cap U;\mathbb{T});R = U;\mathbb{T};\mathbb{I}\cap R && \text{according to Prop. 2.2}\\
&\subseteq (U;\mathbb{T}\cap R;\mathbb{I}^{\mathsf{T}});(\mathbb{I}\cap (U;\mathbb{T})^{\mathsf{T}};R) && \text{Dedekind}\\
&\subseteq (R\cap U;\mathbb{T});(\mathbb{I}\cap \mathbb{T};U^{\mathsf{T}}) && \text{since } R^{\mathsf{T}};U \subseteq U\\
&= (R\cap U;\mathbb{T});(\mathbb{I}\cap U;\mathbb{T}) && \text{as } Q \subseteq \mathbb{I} \text{ implies } Q = Q^{\mathsf{T}}\\
&= (\mathbb{I}\cap U;\mathbb{T});R;(\mathbb{I}\cap U;\mathbb{T}) && \text{according to Prop. 2.2 again}\\
&= \iota^{\mathsf{T}};\iota;R;\iota^{\mathsf{T}};\iota && \text{definition of natural injection}
\end{aligned}$$

With $R^{\mathsf{T}};\Xi;U \subseteq \Xi;R^{\mathsf{T}};U \subseteq \Xi;U$, we get in a completely similar way

$$\lambda;R;\lambda^{\mathsf{T}};\lambda = \lambda;R \qquad (3)$$

We show that φ is univalent and surjective:

$$\begin{aligned}
\varphi^{\mathsf{T}};\varphi &= \eta^{\mathsf{T}};\iota;\lambda^{\mathsf{T}};\delta;\delta^{\mathsf{T}};\lambda;\iota^{\mathsf{T}};\eta && \text{by definition}\\
&= \eta^{\mathsf{T}};\iota;\lambda^{\mathsf{T}};\Xi_Z;\lambda;\iota^{\mathsf{T}};\eta && \text{natural projection}\\
&= \eta^{\mathsf{T}};\iota;\lambda^{\mathsf{T}};\lambda;\Xi;\lambda^{\mathsf{T}};\lambda;\iota^{\mathsf{T}};\eta && \text{definition of } \Xi_Z\\
&= \eta^{\mathsf{T}};\iota;\Xi;\iota^{\mathsf{T}};\eta && \text{as proved initially}\\
&= \eta^{\mathsf{T}};\Xi_Y;\eta && \text{definition of } \Xi_Y\\
&= \eta^{\mathsf{T}};\eta;\eta^{\mathsf{T}};\eta = \mathbb{I};\mathbb{I} = \mathbb{I} && \text{natural projection}
\end{aligned}$$

To show that φ is injective and total, we start

$$\begin{aligned}
\delta;\varphi;\varphi^{\mathsf{T}};\delta^{\mathsf{T}} &= \delta;\delta^{\mathsf{T}};\lambda;\iota^{\mathsf{T}};\eta;\eta^{\mathsf{T}};\iota;\lambda^{\mathsf{T}};\delta;\delta^{\mathsf{T}} && \text{by definition}\\
&= \Xi_Z;\lambda;\iota^{\mathsf{T}};\Xi_Y;\iota;\lambda^{\mathsf{T}};\Xi_Z && \text{natural projections}\\
&= \lambda;\Xi;\lambda^{\mathsf{T}};\lambda;\iota^{\mathsf{T}};\iota;\Xi;\iota^{\mathsf{T}};\iota;\lambda^{\mathsf{T}};\lambda;\Xi;\lambda^{\mathsf{T}} && \text{by definition of } \Xi_Y, \Xi_Z\\
&= \lambda;\Xi;\iota^{\mathsf{T}};\iota;\Xi;\iota^{\mathsf{T}};\iota;\Xi;\lambda^{\mathsf{T}} && \text{as } \iota;\lambda^{\mathsf{T}};\lambda = \iota\\
&= \lambda;\Xi;\lambda^{\mathsf{T}};\lambda;\Xi;\lambda^{\mathsf{T}};\lambda;\Xi;\lambda^{\mathsf{T}} && \text{see above}\\
&= \Xi_Z;\Xi_Z;\Xi_Z = \Xi_Z && \text{by definition of } \Xi_Z
\end{aligned}$$

so that we may go on with

$$\begin{aligned}
\varphi;\varphi^{\mathsf{T}} &= \delta^{\mathsf{T}};\delta;\varphi;\varphi^{\mathsf{T}};\delta^{\mathsf{T}};\delta && \text{as } \delta \text{ is univalent and surjective}\\
&= \delta^{\mathsf{T}};\Xi_Z;\delta && \text{as before}\\
&= \delta^{\mathsf{T}};\delta;\delta^{\mathsf{T}};\delta = \mathbb{I};\mathbb{I} = \mathbb{I} && \text{natural projection}
\end{aligned}$$

The interplay of subset forming and equivalence classes is visualized in Fig. 6.4.

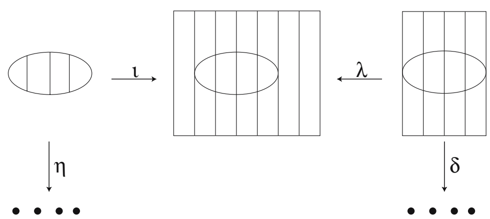

Fig. 6.4. Visualization of the First Isomorphism Theorem

It turns out that Ξ_Y is an R_Y-congruence for $R_Y := \iota;R;\iota^\mathsf{T}$:

$$\begin{array}{rcll}
\Xi_Y;R_Y &=& \iota;\Xi;\iota^\mathsf{T};\iota;R;\iota^\mathsf{T} & \text{by definition}\\
&\subseteq& \iota;\Xi;R;\iota^\mathsf{T} & \iota \text{ is univalent}\\
&\subseteq& \iota;R;\Xi;\iota^\mathsf{T} & \text{congruence}\\
&\subseteq& \iota;R;\iota^\mathsf{T};\iota;\Xi;\iota^\mathsf{T} & (2)\\
&\subseteq& R_Y;\Xi_Y & \text{definition of } R_Y, \Xi_Y
\end{array}$$

The construct $\alpha := \iota;\Xi;\lambda^\mathsf{T};\delta$ is a surjective mapping:

$$\begin{array}{rcll}
\alpha^\mathsf{T};\alpha &=& \delta^\mathsf{T};\lambda;\Xi;\iota^\mathsf{T};\iota;\Xi;\lambda^\mathsf{T};\delta & \text{by the definition just given}\\
&=& \delta^\mathsf{T};\lambda;\Xi;\lambda^\mathsf{T};\lambda;\Xi;\lambda^\mathsf{T};\delta & (1)\\
&=& \delta^\mathsf{T};\Xi_Z;\Xi_Z;\delta & \text{definition of } \Xi_Z\\
&=& \delta^\mathsf{T};\Xi_Z;\delta & \Xi_Z \text{ is indeed an equivalence}\\
&=& \delta^\mathsf{T};\delta;\delta^\mathsf{T};\delta = \mathbb{I};\mathbb{I} = \mathbb{I} & \delta \text{ is natural projection for } \Xi_Z\\
\alpha;\alpha^\mathsf{T} &=& \iota;\Xi;\lambda^\mathsf{T};\delta;\delta^\mathsf{T};\lambda;\Xi;\iota^\mathsf{T} & \text{by definition}\\
&=& \iota;\Xi;\lambda^\mathsf{T};\Xi_Z;\lambda;\Xi;\iota^\mathsf{T} & \delta \text{ is natural projection for } \Xi_Z\\
&=& \iota;\Xi;\lambda^\mathsf{T};\lambda;\Xi;\lambda^\mathsf{T};\lambda;\Xi;\iota^\mathsf{T} & \text{definition of } \Xi_Z\\
&=& \iota;\Xi;\iota^\mathsf{T};\iota;\Xi;\iota^\mathsf{T};\iota;\Xi;\iota^\mathsf{T} & (1)\\
&=& \Xi_Y;\Xi_Y;\Xi_Y = \Xi_Y \supseteq \mathbb{I} & \text{definition of equivalence } \Xi_Y
\end{array}$$

With α, we may express S, T in a shorter way:

$$\begin{array}{rcll}
\alpha^\mathsf{T};R_Y;\alpha &=& \delta^\mathsf{T};\lambda;\Xi;\iota^\mathsf{T};R_Y;\iota;\Xi;\lambda^\mathsf{T};\delta & \text{definition of } \alpha\\
&=& \delta^\mathsf{T};\lambda;\Xi;\iota^\mathsf{T};\iota;R;\iota^\mathsf{T};\iota;\Xi;\lambda^\mathsf{T};\delta & \text{definition of } R_Y\\
&=& \delta^\mathsf{T};\lambda;\Xi;\iota^\mathsf{T};\iota;R;\Xi;\lambda^\mathsf{T};\delta & (2)\\
&=& \delta^\mathsf{T};\lambda;\Xi;\iota^\mathsf{T};\iota;\Xi;R;\Xi;\lambda^\mathsf{T};\delta & \Xi;R;\Xi \subseteq R;\Xi;\Xi = R;\Xi \subseteq \Xi;R;\Xi\\
&=& \delta^\mathsf{T};\lambda;\Xi;\lambda^\mathsf{T};\lambda;\Xi;R;\Xi;\lambda^\mathsf{T};\delta & (1)\\
&=& \delta^\mathsf{T};\Xi_Z;\lambda;R;\Xi;\lambda^\mathsf{T};\delta & \text{as before, definition of } \Xi_Z\\
&=& \delta^\mathsf{T};\Xi_Z;\lambda;R;\lambda^\mathsf{T};\lambda;\Xi;\lambda^\mathsf{T};\delta & (3)\\
&=& \delta^\mathsf{T};\Xi_Z;\lambda;R;\lambda^\mathsf{T};\Xi_Z;\delta & \text{definition of } \Xi_Z\\
&=& \delta^\mathsf{T};\delta;\delta^\mathsf{T};\lambda;R;\lambda^\mathsf{T};\delta;\delta^\mathsf{T};\delta & \delta \text{ is natural projection for } \Xi_Z\\
&=& \delta^\mathsf{T};\lambda;R;\lambda^\mathsf{T};\delta = T & \delta \text{ is a surjective mapping}\\
\eta^\mathsf{T};R_Y;\eta &=& \eta^\mathsf{T};\iota;R;\iota^\mathsf{T};\eta & \text{definition of } R_Y\\
&=& S & \text{definition of } S
\end{array}$$

Relations α and φ are closely related:

$$\begin{array}{rll}
\alpha;\varphi &= \iota;\Xi;\lambda^\mathsf{T};\delta;\delta^\mathsf{T};\lambda;\iota^\mathsf{T};\eta & \text{definition of } \alpha, \varphi\\
&= \iota;\Xi;\lambda^\mathsf{T};\Xi_Z;\lambda;\iota^\mathsf{T};\eta & \delta \text{ is natural projection for } \Xi_Z\\
&= \iota;\Xi;\lambda^\mathsf{T};\lambda;\Xi;\lambda^\mathsf{T};\lambda;\iota^\mathsf{T};\eta & \text{definition of } \Xi_Z\\
&= \iota;\Xi;\lambda^\mathsf{T};\lambda;\Xi;\iota^\mathsf{T};\eta & (2)\\
&= \iota;\Xi;\iota^\mathsf{T};\iota;\Xi;\iota^\mathsf{T};\eta & (1)\\
&= \Xi_Y;\Xi_Y;\eta & \text{definition of } \Xi_Y\\
&= \eta;\eta^\mathsf{T};\eta;\eta^\mathsf{T};\eta = \eta & \eta \text{ is natural projection for } \Xi_Y\\
\alpha^\mathsf{T};\eta &= \alpha^\mathsf{T};\alpha;\varphi & \text{see before}\\
&= \varphi & \alpha \text{ is univalent and surjective}
\end{array}$$

This enables us already to prove the homomorphism condition:

$$\begin{array}{rll}
T;\varphi &= \alpha^\mathsf{T};R_Y;\alpha;\alpha^\mathsf{T};\eta & \text{above results on } T, \varphi\\
&= \alpha^\mathsf{T};R_Y;\Xi_Y;\eta & \alpha;\alpha^\mathsf{T} = \Xi_Y, \text{ see above}\\
&= \alpha^\mathsf{T};\Xi_Y;R_Y;\Xi_Y;\eta & \Xi_Y \text{ is an } R_Y\text{-congruence}
\end{array}$$

$$\begin{aligned} &= \alpha^{\mathsf{T}};\eta;\eta^{\mathsf{T}};R_Y;\eta;\eta^{\mathsf{T}};\eta && \eta \text{ is natural projection for } \Xi_Y \\ &= \varphi;\eta^{\mathsf{T}};R_Y;\eta && \eta \text{ is univalent and surjective} \\ &= \varphi;S && \text{see above} \end{aligned}$$

This was an equality, so that it suffices according to Lemma 3.3. □

It will have become clear, that these proofs completely rely on generic constructions and their algebraic laws. When elaborated they seem lengthy. With a supporting system, however, they reduce considerably to a sequence of rules to be applied.

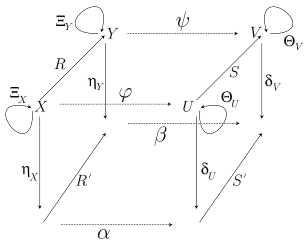

Fig. 6.5. Situation of the Second Isomorphism Theorem

6.3 Proposition (*Second Isomorphism Theorem*)**.** Let a multi-covering (φ, ψ) between any two relations $R : X \longrightarrow Y$ and $S : U \longrightarrow V$ be given as well as an R-congruence (Ξ_X, Ξ_Y) and an S-congruence (Θ_U, Θ_V). Let also the equivalences be related through φ, ψ as $\Xi_Y = \psi;\Theta_V;\psi^{\mathsf{T}}$ and $\Xi_X = \varphi;\Theta_U;\varphi^{\mathsf{T}}$. Given this situation, introduce the natural projections $\eta_X, \eta_Y, \delta_U, \delta_V$ for the equivalences and proceed to relations $R' := \eta_X^{\mathsf{T}};R;\eta_Y$ and $S' := \delta_U^{\mathsf{T}};S;\delta_V$. Then $\alpha := \eta_X^{\mathsf{T}};\varphi;\delta_U$ and $\beta := \eta_Y^{\mathsf{T}};\psi;\delta_V$ constitute an isomorphism from R' to S' (see Fig. 6.5).

Proof. α is univalent and surjective (β follows completely analogous)

$$\begin{aligned} \alpha^{\mathsf{T}};\alpha &= (\eta_X^{\mathsf{T}};\varphi;\delta_U)^{\mathsf{T}};\eta_X^{\mathsf{T}};\varphi;\delta_U && \text{by definition} \\ &= \delta_U^{\mathsf{T}};\varphi^{\mathsf{T}};\eta_X;\eta_X^{\mathsf{T}};\varphi;\delta_U && \text{transposing} \\ &= \delta_U^{\mathsf{T}};\varphi^{\mathsf{T}};\Xi_X;\varphi;\delta_U && \text{natural projection} \\ &= \delta_U^{\mathsf{T}};\varphi^{\mathsf{T}};\varphi;\Theta_U;\varphi^{\mathsf{T}};\varphi;\delta_U && \text{condition on mapping equivalences} \\ &= \delta_U^{\mathsf{T}};\Theta_U;\delta_U && \text{as } \varphi \text{ is a surjective mapping} \\ &= \delta_U^{\mathsf{T}};\delta_U;\delta_U^{\mathsf{T}};\delta_U && \text{natural projection} \\ &= \mathbb{I};\mathbb{I} = \mathbb{I} \end{aligned}$$

We show that α is total and injective (β follows completely analogous)

$$\begin{aligned} \alpha;\alpha^{\mathsf{T}} &= \eta_X^{\mathsf{T}};\varphi;\delta_U;(\eta_X^{\mathsf{T}};\varphi;\delta_U)^{\mathsf{T}} && \text{by definition} \\ &= \eta_X^{\mathsf{T}};\varphi;\delta_U;\delta_U^{\mathsf{T}};\varphi^{\mathsf{T}};\eta_X && \text{transposing} \\ &= \eta_X^{\mathsf{T}};\varphi;\Theta_U;\varphi^{\mathsf{T}};\eta_X && \text{natural projection} \\ &= \eta_X^{\mathsf{T}};\Xi_X;\eta_X && \text{condition on mapping equivalences} \\ &= \eta_X^{\mathsf{T}};\eta_X;\eta_X^{\mathsf{T}};\eta_X && \text{natural projection} \\ &= \mathbb{I};\mathbb{I} = \mathbb{I} \end{aligned}$$

We show that α, β is a homomorphism:

$$\begin{aligned}
R' ; \beta &= \eta_X^{\mathsf{T}} ; R ; \eta_Y ; \eta_Y^{\mathsf{T}} ; \psi ; \delta_V && \text{by definition} \\
&= \eta_X^{\mathsf{T}} ; R ; \Xi_Y ; \psi ; \delta_V && \text{natural projection} \\
&= \eta_X^{\mathsf{T}} ; R ; \psi ; \Theta_V ; \psi^{\mathsf{T}} ; \psi ; \delta_V && \text{condition on mapping equivalences} \\
&= \eta_X^{\mathsf{T}} ; R ; \psi ; \Theta_V ; \delta_V && \text{as } \psi \text{ is surjective and univalent} \\
&= \eta_X^{\mathsf{T}} ; \varphi ; S ; \Theta_V ; \delta_V && \text{multi-covering} \\
&= \eta_X^{\mathsf{T}} ; \varphi ; \Theta_U ; S ; \Theta_V ; \delta_V && S ; \Theta_V \subseteq \Theta_U ; S ; \Theta_V \subseteq S ; \Theta_V ; \Theta_V = S ; \Theta_V \\
&= \eta_X^{\mathsf{T}} ; \varphi ; \Theta_U ; S ; \delta_V ; \delta_V^{\mathsf{T}} ; \delta_V && \text{natural projection} \\
&= \eta_X^{\mathsf{T}} ; \varphi ; \Theta_U ; S ; \delta_V && \text{as } \delta \text{ is a surjective mapping} \\
&= \eta_X^{\mathsf{T}} ; \varphi ; \delta_U ; \delta_U^{\mathsf{T}} ; S ; \delta_V && \text{natural projection} \\
&= \alpha ; S' && \text{by definition}
\end{aligned}$$

This was an equality, so that it suffices according to Lemma 3.3. □

7 Covering of Graphs and Path Equivalence

There is another point to mention here which has gained considerable interest in an algebraic or topological context, not least for Riemann surfaces.

7.1 Proposition (*Lifting property*). Let a homogeneous relation B be given together with a multi-covering (Φ, Φ) on the relation B'. Let furthermore some rooted graph B_0 with root a_0, i.e., satisfying and $B_0^{\mathsf{T}*} ; a_0 = \mathbb{T}$, be given together with a homomorphism Φ_0 that sends the root a_0 to $a' := \Phi_0^{\mathsf{T}} ; a_0$. If $a \subseteq \Phi^{\mathsf{T}} ; a'$ is some point mapped by Φ to a', there exists always a relation Ψ — not necessarily a mapping — satisfying the properties

$$\Psi^{\mathsf{T}} ; a_0 = a \quad \text{and} \quad B_0 ; \Psi \subseteq \Psi ; B.$$

Idea of proof: Define $\Psi := \inf\{X \mid a_0 ; a^{\mathsf{T}} \cup (B_0^{\mathsf{T}} ; X ; B \cap \Phi_0 ; \Phi^{\mathsf{T}}) \subseteq X\}$. □

The relation Ψ enjoys the homomorphism property but fails to be a mapping in general. In order to make it a mapping, one will choose one of the following two possibilities:

- Firstly, one might follow the recursive definition starting from a_0 and at every stage make an arbitrary choice among the relational images offered, thus choosing a fiber.
- Secondly, one may further restrict the multi-covering condition to "locally univalent" fans in Φ, requiring $B_0^{\mathsf{T}} ; \Psi ; B \cap \Phi_0 ; \Phi^{\mathsf{T}} \subseteq \mathbb{I}$ to hold for it, which leads to a well-developed theory, see [2,3,8].

In both cases, one will find a homomorphism from B_0 to B. The effect of a flow chart diagram is particularly easy to understand when the underlying rooted graph is also a rooted *tree*, so that the view is not blocked by nested circuits which can be traveled several times. When dealing with a rooted graph that does contain such circuits one has to keep track of the possibly infinite number of ways in which the graph can be traversed from its root. To this end there exists a theory of coverings which is based on the notion of homomorphy.

The idea is to unfold circuits. We want to characterize those homomorphisms of a graph that preserve to a certain extent the possibilities of traversal. We shall see that such a homomorphism is surjective and that it carries the successor relation at any point onto that at the image point.

7.2 Definition. A surjective homomorphism $\Phi\colon G \longrightarrow G'$ is called a **covering**, provided that it is a multi-covering satisfying $B^{\mathsf{T}}; B \cap \Phi; \Phi^{\mathsf{T}} \subseteq \mathbb{I}$. □

The multi-covering Φ compares two relations between the points of G and of G' and ensures that for any inverse image point x of some point x' and successor y' of x' there is *at least* one successor y of x which is mapped onto y'. The new condition guarantees that there is *at most* one such y since it requires that the relation "have a common predecessor according to B, and have a common image under Φ" is contained in the identity.

8 Concluding Remark

We have reworked mathematical basics from a relational perspective. First the step from an algebraic to a relational structure has been made. This is so serious a generalization, that one would not expect much of the idea of homomorphism and isomorphism theorems to survive. With the concept of a multi-covering, however, a new and adequate concept seems to have been found. Prop. 5.4 shows that it reduces completely to homomorphisms when going back to the algebraic case. For relational structures, a multi-covering behaves nicely with respect to quotient forming. This relates to earlier papers (see [2,3,8]) where semantics of programs (partial correctness, total correctness, and flow equivalence, even for systems of recursive procedures) has first been given a componentfree relational form.

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