Contact Relations with Applications

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Abstract. Using relation algebra, we generalize Aumann's notion of a contact relation and that of a closure operation from powersets to general membership relations and their induced partial orders. We also investigate the relationship between contacts and closures in this general setting and use contacts to establish a one-to-one correspondence between the column space and the row space of a relation.

1 Introduction

Forming closures of subsets of a set X is a very basic technique in various disciplines. Typically this is combined with some predicate that holds for X and is $\cap$-hereditary, like "being transitive" or "being convex". Such predicates lead to closure systems, i.e., subsets $\mathfrak{C}$ of the powerset 2^X of X that contain X and any intersection of subsets collected in $\mathfrak{C}$. It is well known that there is a one-to-one correspondence between the set of closure systems of 2^X and the set of extensive, monotone, and idempotent functions on 2^X (the closure operations on 2^X).

According to G. Aumann, [1], closures always come with a relation, namely a contact. When introducing this concept, one intention was to formalize the essential properties of a contact between objects and sets of objects, mainly to obtain for beginners a more suggestive access to topology than "traditional" axiom systems provide. In the introduction of his paper, Aumann also mentions sociological applications as motivation, but in fact all examples of [1] are from mathematics. A main result of [1] is that, like closure systems and closure operations, also closure operations and contact relations are cryptomorphic mathematical structures in the sense of [6].

In this paper, we generalize Aumann's concept of a contact between sets and their powersets to contacts given by an (almost) arbitrary relation M, that may be interpreted as "individual is a member of a group of individuals". Such an approach allows to treat also examples from sociology, political science and so forth. As we will show, each group membership relation M induces a partial order Ω_M on the groups of individuals. With respect to Ω_M, we consider a notion of closure operation that directly arises out of the original one by replacing set inclusion by Ω_M. In this very general setting, we investigate contacts, their properties, and a construction similar to the lower/upper-derivative construction of formal concept analysis. The latter leads to a fixed point description of the set of contacts. Guided by Aumann's main result, we also study the relationship between general M-contacts and Ω_M-closures. Finally, we use contacts to establish a one-to-one correspondence between the column space and the row space of a relation (or a Boolean matrix).

To carry out our investigations, we use abstract relation algebra in the sense of [13, 12]. This allows very concise and precise specifications and algebraic proofs that drastically reduce the danger of making mistakes. To give an example, when constructing closures from contacts, a subtle definedness condition plays a decisive role that easily can be overlooked when using the customary approach with closures being functions. Relation-algebraic specifications also allow to use tool support. For obtaining the results of this paper, the use of the RelView tool (see [3]) for computing contacts and closures, testing properties, experimenting with concepts etc. was very helpful.

2 Relation-Algebraic Preliminaries

We denote the set (or type) of relations with domain X and range Y by $[X \leftrightarrow Y]$ and write $R : X \leftrightarrow Y$ instead of $R \in [X \leftrightarrow Y]$. If the sets X and Y are finite, we may consider R as a Boolean matrix. Since this interpretation is well suited for many purposes, we will often use matrix notation and terminology in this paper. In particular, we talk about rows, columns and entries of relations, and write $R_{x,y}$ instead of $\langle x, y\rangle \in R$ or $x\,R\,y$.

We assume the reader to be familiar with the basic operations on relations, viz. R^{T} (transposition), $\overline{R}$ (complement), $R \cup S$ (join), $R \cap S$ (meet), $R;S$ (composition), the predicate indicating $R \subseteq S$ (inclusion), and the special relations O (empty relation), L (universal relation) and I (identity relation). Each type $[X \leftrightarrow Y]$ with the operations $\overline{}$, $\cup$, $\cap$, the ordering $\subseteq$ and the constants O and L forms a complete Boolean lattice. Further well-known rules are, e.g., $R^{\mathsf{T}^{\mathsf{T}}} = R$, $\overline{R^{\mathsf{T}}} = \overline{R}^{\mathsf{T}}$ and that $R \subseteq S$ implies $R^{\mathsf{T}} \subseteq S^{\mathsf{T}}$. The theoretical framework for these rules and many others to hold is that of an (axiomatic) relation algebra. The axioms of a relation algebra are those of a complete Boolean lattice for the Boolean part, the associativity and neutrality of identity relations for composition, the equivalence of $Q;R \subseteq S$, $Q^{\mathsf{T}};\overline{S} \subseteq \overline{R}$, and $\overline{S};R^{\mathsf{T}} \subseteq \overline{Q}$ (*Schröder rule*), and that $R \neq \mathsf{O}$ implies $\mathsf{L};R;\mathsf{L} = \mathsf{L}$ (*Tarski rule*).

Furthermore, we assume the reader to be familiar with relation-algebraic specifications of the most fundamental properties of relations, like univalence $R^{\mathsf{T}};R \subseteq \mathsf{I}$, totality $R;\mathsf{L} = \mathsf{L}$, transitivity $R;R \subseteq R$, and the symmetric quotient construction $\mathrm{syq}(R, S) := \overline{R^{\mathsf{T}};\overline{S}} \cap \overline{\overline{R}^{\mathsf{T}};S}$ together with its main properties like the following ones.

$$\mathrm{syq}(R, S) = \mathrm{syq}(\overline{R}, \overline{S}) \qquad\qquad [\mathrm{syq}(R, S)]^{\mathsf{T}} = \mathrm{syq}(S, R) \tag{1}$$

$$R;\mathrm{syq}(R, R) = R \qquad\qquad \mathrm{syq}(Q, R);\mathrm{syq}(R, S)] \subseteq \mathrm{syq}(Q, S) \tag{2}$$

Otherwise, he may consult e.g., [12], Sections 3.1, 4.2, and 4.4.

The set-theoretic symbol $\in$ gives rise to powerset relations $\varepsilon : X \leftrightarrow 2^X$ that relate $x \in X$ and $Y \in 2^X$ iff $x \in Y$. In [4, 5] it is shown that for ε the formulae of (3) hold and these even characterize the powerset relation ε up to isomorphism.

$$\mathrm{syq}(\varepsilon, \varepsilon) = \mathsf{I} \qquad\qquad \forall\, R : \mathsf{L}; \mathrm{syq}(\varepsilon, R) = \mathsf{L} \tag{3}$$

Based on (3), a lot of further set-theoretic constructions can be formalized in terms of relation algebra. In this paper, we need the following.

$$\imath := \mathrm{syq}(\mathsf{I}, \varepsilon) : X \leftrightarrow 2^X \qquad\qquad \Omega := \overline{\varepsilon^{\mathsf{T}}; \overline{\varepsilon}} \; : 2^X \leftrightarrow 2^X \tag{4}$$

The relation $\imath$ is called singleton-set former, since it associates $x \in X$ with $Y \in 2^X$ iff $Y = \{x\}$. The relation Ω specifies the inclusion order on sets. Based on (3) and (4), the following properties are shown in [4]:

Lemma 2.1. *If $\varepsilon : X \leftrightarrow 2^X$ is a powerset relation, then $\imath : X \leftrightarrow 2^X$ is an injective mapping*[1], *$\Omega : 2^X \leftrightarrow 2^X$ is a partial order, and $\imath; \Omega = \varepsilon = \varepsilon; \Omega$.* □

The construction used in the definition of Ω can be generalized to arbitrary relations $R : X \leftrightarrow Y$. Then $\Omega_R := \overline{R^{\mathsf{T}}; \overline{R}} \; : Y \leftrightarrow Y$ is reflexive and transitive due to the Schröder rule; it shows the "column-is-contained-preorder". In case of $\mathrm{syq}(R, R) = \mathsf{I}$, i.e., without multiple columns, it is even antisymmetric and, thus, a partial order. Besides these partial order properties, we will apply the following fact.

Lemma 2.2. *For all relations $R : X \leftrightarrow Y$ we have $R; \Omega_R = R$.*

Proof. The inclusion $R \subseteq R; \Omega_R$ follows from the reflexivity of Ω_R, and with the help of the Schröder rule $R; \Omega_R \subseteq R$ is shown by

$$R^{\mathsf{T}}; \overline{R} \subseteq R^{\mathsf{T}}; \overline{R} \iff R; \overline{R^{\mathsf{T}}; \overline{R}} \subseteq R. \qquad \square$$

As a last construction, we need the canonical epimorphism $\eta_E : X \leftrightarrow X/E$ induced by an equivalence relation $E : X \leftrightarrow X$. It relates each element $x \in X$ to the equivalence class $c \in X/E$ it belongs to. The following properties are immediate consequences of this component-wise specification; it can even be shown that they characterize canonical epimorphisms up to isomorphism.

$$\eta_E; {\eta_E}^{\mathsf{T}} = E \qquad\qquad {\eta_E}^{\mathsf{T}}; \eta_E = \mathsf{I} \tag{5}$$

In Sections 3 and 5 we will apply canonical epimorphisms induced by the two equivalence relations $\Psi_R := \mathrm{syq}(R, R)$ and $\Phi_R := \mathrm{syq}(R^{\mathsf{T}}, R^{\mathsf{T}})$, respectively. In this context, the following additional property will be used.

Lemma 2.3. *For all $R : X \leftrightarrow Y$, the canonical epimorphism $\eta_{\Psi_R} : Y \leftrightarrow Y/\Psi_R$ induced by Ψ_R fulfils $\overline{R; \eta_{\Psi_R}} = \overline{R}; \eta_{\Psi_R}$.*

[1] ... in the relational sense of Def. 4.2.1 of [12].

Proof. We abbreviate η_{Ψ_R} by η. Then, inclusion "$\subseteq$" follows from

$$\eta^{\mathsf{T}} \text{ total } \implies \overline{\eta^{\mathsf{T}};R^{\mathsf{T}}} \subseteq \eta^{\mathsf{T}};\overline{R^{\mathsf{T}}} \iff \overline{R;\eta} \subseteq \overline{R};\eta$$

using Prop. 4.2.4.i of [12], and inclusion "$\supseteq$" from

$$R \subseteq R \iff R;\mathrm{syq}(R,R) \subseteq R \iff R;\eta;\eta^{\mathsf{T}} \subseteq R \iff \overline{R};\eta \subseteq \overline{R;\eta}$$

using the first rule of (2), the first axiom of (5), and the Schröder rule. □

3 Contact Relations

If we formulate Aumann's original definition of a contact relation given in [1] in our notation, then a relation $A : X \leftrightarrow 2^X$ is an *(Aumann) contact relation* if the following conditions hold.

$$\begin{array}{ll}
(\mathrm{A}_1) & \forall\, x : A_{x,\{x\}} \\
(\mathrm{A}_2) & \forall\, x, Y, Z : A_{x,Y} \wedge Y \subseteq Z \rightarrow A_{x,Z} \\
(\mathrm{A}_3) & \forall\, x, Y, Z : A_{x,Y} \wedge (\forall\, y : y \in Y \rightarrow A_{y,Z}) \rightarrow A_{x,Z}
\end{array}$$

Our aim is to investigate contact relations by relation-algebraic means and supporting tools (like the manipulation system RelView [3]), thereby generalizing Aumann's original approach by replacing the powerset by a set G (of groups of individuals, political parties, alliances, organizations, ...) and the set-theoretic membership relation $\varepsilon : X \leftrightarrow 2^X$ by a generalized membership relation $M : X \leftrightarrow G$ with regard to G. The latter point not only allows to treat mathematical examples for contact relationships as [1] does, but also examples from sociology, political science and so forth. In the following theorem, we present relation-algebraic versions of the above axioms. The proof of their correspondence consists of step-wise transformations of (A_1) to (A_3) into point-free versions using well-known correspondences between logical and relation-algebraic constructions. Doing so, (A_1) leads to a singleton-former $\imath$ and (A_2) to an inclusion order Ω as specified in (4).

Theorem 3.1. *A relation $A : X \leftrightarrow 2^X$ is an Aumann contact relation iff $\imath \subseteq A$, $A;\Omega \subseteq A$, and $A;\overline{\varepsilon^{\mathsf{T}};\overline{A}} \subseteq A$.*

Proof. We only show the equivalence of (A_3) and $A;\overline{\varepsilon^{\mathsf{T}};\overline{A}} \subseteq A$; the other equivalences are calculated in quite a similar way.

$$\begin{array}{ll}
 & \forall\, x, Y, Z : A_{x,Y} \wedge (\forall\, y : y \in Y \rightarrow A_{y,Z}) \rightarrow A_{x,Z} \\
\iff & \forall\, x, Y, Z : A_{x,Y} \wedge \neg(\exists\, y : y \in Y \wedge \overline{A}_{y,Z}) \rightarrow A_{x,Z} \\
\iff & \forall\, x, Y, Z : A_{x,Y} \wedge \overline{\varepsilon^{\mathsf{T}};\overline{A}}_{Y,Z} \rightarrow A_{x,Z} \\
\iff & \forall\, x, Z : (\exists\, Y : A_{x,Y} \wedge \overline{\varepsilon^{\mathsf{T}};\overline{A}}_{Y,Z}) \rightarrow A_{x,Z} \\
\iff & \forall\, x, Z : (A;\overline{\varepsilon^{\mathsf{T}};\overline{A}})_{x,Z} \rightarrow A_{x,Z} \\
\iff & A;\overline{\varepsilon^{\mathsf{T}};\overline{A}} \subseteq A
\end{array}$$

□

The relation-algebraic characterization of contacts just developed does not yet allow the generalization intended. We still have to remove the singleton-former, since such a construct need not exist in the general case of membership we want to deal with. The next theorem shows how this is possible.

Theorem 3.2. *A relation $A : X \leftrightarrow 2^X$ is an Aumann contact relation iff $\varepsilon \subseteq A$ and $A^\mathsf{T}; \overline{A} \subseteq \varepsilon^\mathsf{T}; \overline{A}$.*

Proof. We show that the relation-algebraic specification of an Aumann contact relation of Theorem 3.1 is equivalent to $\varepsilon \subseteq A$ and $A^\mathsf{T}; \overline{A} \subseteq \varepsilon^\mathsf{T}; \overline{A}$. Starting with "$\Longrightarrow$", we use Lemma 2.1 to show $\varepsilon \subseteq A$ by

$$\imath \subseteq A \implies \imath; \Omega \subseteq A; \Omega \iff \varepsilon \subseteq A; \Omega \implies \varepsilon \subseteq A.$$

Because of the Schröder rule, $A^\mathsf{T}; \overline{A} \subseteq \varepsilon^\mathsf{T}; \overline{A}$ is equivalent with $A; \overline{\varepsilon^\mathsf{T}; \overline{A}} \subseteq A$. In the case "$\Longleftarrow$", property $\imath \subseteq A$ follows from $\imath \subseteq \varepsilon$ and $\varepsilon \subseteq A$. Using the Schröder rule, we obtain $A; \Omega \subseteq A$ from

$$A^\mathsf{T}; \overline{A} \subseteq \varepsilon^\mathsf{T}; \overline{A} \subseteq \varepsilon^\mathsf{T}; \overline{\varepsilon} = \overline{\Omega}.$$

For the last property, cf. the proof of "$\Longrightarrow$". □

Hence, we have that membership implies contact and for all $Y, Z \in 2^X$ from the existence of an element that is in contact with Y but not in contact with Z it follows that even a member of Y is not in contact with Z. In the literature such relations are also known as dependence or entailment relations and in particular considered in combination with so-called exchange properties. See [7, 6] for example. And here is our generalization of Aumann's concept of a contact.

Definition 3.1. *A relation $K : X \leftrightarrow G$ is called an* (Aumann) contact relation *with respect to the relation $M : X \leftrightarrow G$ — in short: an M-*contact *— if the following properties hold:*

$$(\mathrm{K}_1) \quad M \subseteq K \qquad\qquad (\mathrm{K}_2) \quad K^\mathsf{T}; \overline{K} \subseteq M^\mathsf{T}; \overline{K}$$

Axiom (K_2) is called the infectivity of a contact. We have chosen this form since it proved to be particularly suitable for relation-algebraic reasoning. For concrete sociological or similar applications, frequently the equivalent version $K; \overline{M^\mathsf{T}; \overline{K}} \subseteq K$ is more appropriate. E.g., in the case of persons and syndicates it says that if a person x is in contact to a syndicate Y_1 all of whose members are in contact to a syndicate Y_2, then also x is in contact to Y_2.

In real life, contacts are frequently established by common interests. As an example, we consider a protesters scene of non-governmental organizations. There exist persons willing to protest against several topics $t \in T$. Then typically a person $x \in X$ will get in touch with an activist group $g \in G$ iff for all topics he is in opposition to, there is at least one supporter for it in the group g. If we formalize the situation in predicate logic and afterwards translate this version into a relation-algebraic expression, we arrive at $\mathrm{mi}_J(\mathrm{ma}_J(M))_{x,g}$, where $M : X \leftrightarrow G$

denotes activist group membership, the complement of the relation $J : X \leftrightarrow T$ specifies the relationship “is in opposition to”, and the functions mi_J and ma_J are defined as follows:

$$\mathrm{mi}_J(R) = \overline{\overline{J}\,;R} \qquad\qquad \mathrm{ma}_J(S) = \overline{\overline{J}^{\mathsf{T}};S} \tag{6}$$

If J is a partial order, then mi_J and ma_J column-wise compute lower bounds and upper bounds, respectively; in the general case, they column-wise compute lower derivatives and upper derivatives, respectively, in the sense of formal concept analysis (see [9]). The next theorem shows that the above construction based on interest-relations J always leads to M-contacts.

Theorem 3.3. *For all relations $M : X \leftrightarrow G$ and $J : X \leftrightarrow T$, we obtain an M-contact K if we define $K := \mathrm{mi}_J(\mathrm{ma}_J(M))$.*

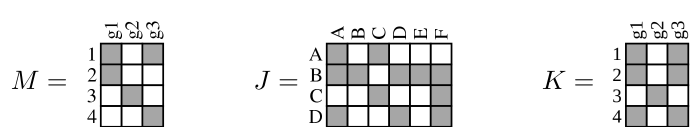
Proof. Property (K_1) follows from

$$\begin{array}{rll}
\overline{J}^{\mathsf{T}};M \subseteq \overline{J}^{\mathsf{T}};M & \iff \overline{J}\,;\,\overline{\overline{J}^{\mathsf{T}};M} \subseteq \overline{M} & \text{Schröder rule} \\
& \iff M \subseteq \overline{\overline{J}\,;\,\overline{\overline{J}^{\mathsf{T}};M}} & \\
& \iff M \subseteq \mathrm{mi}_J(\mathrm{ma}_J(M)) & \text{by (6)} \\
& \iff M \subseteq K, &
\end{array}$$

and property (K_2) from

$$\begin{array}{rll}
& \overline{M^{\mathsf{T}};\overline{J}}\,;\overline{J}^{\mathsf{T}} \subseteq \overline{M^{\mathsf{T}};\overline{J}}\,;\overline{J}^{\mathsf{T}} & \\
\iff & \overline{M^{\mathsf{T}};\overline{J}}\,;\overline{J}^{\mathsf{T}} \subseteq \Big(\overline{J}\,;\,\overline{\overline{J}^{\mathsf{T}};M}\Big)^{\mathsf{T}} & \\
\iff & \overline{\overline{J}\,;\,\overline{\overline{J}^{\mathsf{T}};M}}^{\mathsf{T}}\,;\overline{J} \subseteq M^{\mathsf{T}};\overline{J} & \text{Schröder rule} \\
\iff & \big[\mathrm{mi}_J(\mathrm{ma}_J(M))\big]^{\mathsf{T}};\overline{J} \subseteq M^{\mathsf{T}};\overline{J} & \text{by (6)} \\
\iff & K^{\mathsf{T}};\overline{J} \subseteq M^{\mathsf{T}};\overline{J} & \\
\implies & K^{\mathsf{T}};\overline{J}\,;\,\overline{\overline{J}^{\mathsf{T}};M} \subseteq M^{\mathsf{T}};\overline{J}\,;\,\overline{\overline{J}^{\mathsf{T}};M} & \\
\iff & K^{\mathsf{T}};\overline{\mathrm{mi}_J(\mathrm{ma}_J(M))} \subseteq M^{\mathsf{T}};\overline{\mathrm{mi}_J(\mathrm{ma}_J(M))} & \text{by (6)} \\
\iff & K^{\mathsf{T}};\overline{K} \subseteq M^{\mathsf{T}};\overline{K}\,. & \Box
\end{array}$$

Next, we give a concrete application of the construction of Theorem 3.3. We assume four persons, denoted by the natural numbers 1 to 4, three groups g_1, g_2 and g_3, and six topics A, B, C, D, E and F. If group membership is described by the left-most of the following three RelView-matrices and the persons’ interests by the RelView-matrix in the middle, then these relations lead to the contact specified by the right RelView-matrix.



In these pictures, a black square means 1 and a white square means 0 so that, e.g., the first person is a member of g_1 and g_3. By definition, $M \subseteq K$. In addition, $(4, g_1) \in K$, because wherever *all* persons of the group g_1 are jointly J-interested in a couple of topics (here $\{1,2\}\times\{A\}$), then also person 4 is J-interested in these topics. Also $(2, g_3) \in K$: the rectangle $\{1,4\} \times \{A\}$ indicates that all members of the group are jointly J-interested in topic set $\{A\}$ and so is person 2.

We even can prove completeness of the construction of Theorem 3.3, i.e., that every M-contact K can be represented as an expression $\mathrm{mi}_J(\mathrm{ma}_J(M))$. As the next theorem shows, we only have to take the groups as topics and K itself as interest relation J.

Theorem 3.4. *For all relations $M : X \leftrightarrow G$ and all M-contacts $K : X \leftrightarrow G$ the equation $K = \mathrm{mi}_K(\mathrm{ma}_K(M))$ holds.*

Proof. "$\subseteq$": This inclusion is equivalent to property (K_2), since

$$\begin{aligned}
K \subseteq \mathrm{mi}_K(\mathrm{ma}_K(M)) &\iff K \subseteq \overline{\overline{K}\,;\,\overline{\overline{K}^{\mathsf{T}};M}} && \text{by (6)}\\
&\iff \overline{K}\,;\,\overline{\overline{K}^{\mathsf{T}};M} \subseteq \overline{K} \\
&\iff \overline{K}^{\mathsf{T}};K \subseteq \overline{K}^{\mathsf{T}};M && \text{Schröder rule}\\
&\iff K^{\mathsf{T}};\overline{K} \subseteq M^{\mathsf{T}};\overline{K}\,.
\end{aligned}$$

"$\supseteq$": Starting with (K_1), we get the result by

$$\begin{aligned}
M \subseteq K &\iff \overline{K};\mathrm{I} \subseteq \overline{M} \\
&\iff \overline{K}^{\mathsf{T}};M \subseteq \overline{\mathrm{I}} && \text{Schröder rule}\\
&\iff \mathrm{I} \subseteq \overline{\overline{K}^{\mathsf{T}};M} \\
&\implies \overline{K} \subseteq \overline{K}\,;\,\overline{\overline{K}^{\mathsf{T}};M} \\
&\iff \overline{\overline{K}\,;\,\overline{\overline{K}^{\mathsf{T}};M}} \subseteq K \\
&\iff \mathrm{mi}_K(\mathrm{ma}_K(M)) \subseteq K && \text{by (6).}
\end{aligned}$$

□

From the Theorems 3.3 and 3.4, we immediately obtain a fixed point characterization of the set of M-contacts.

Theorem 3.5. *Assume a generalized membership relation $M : X \leftrightarrow G$ to be given and consider all relations $R : X \leftrightarrow T$ for some set T. Then the function*

$$\tau_M : [X \leftrightarrow T] \to [X \leftrightarrow G] \qquad \tau_M(R) = \mathrm{mi}_R(\mathrm{ma}_R(M)),$$

will always produce an M-contact. The set $\mathfrak{K}_M$ of all M-contacts equals the set of fixed points of τ_M in case $T = G$. □

Using relational fixed point enumeration techniques (cf. [2]), this property can be used to compute for small relations M all M-contacts by a tool like RelView.

Since the underlying relation M is contained in each M-contact K, normally in K a lot of columns coincide. The column equivalence relation $\Psi_K = \mathrm{syq}(K, K)$

relates two groups iff the corresponding columns of K are equal. Hence, we can remove duplicates of columns of K by multiplying it with the canonical epimorphism η_{Ψ_K} induced by Ψ_K from the right. In the next theorem we prove that in the construction $\mathrm{mi}_K(\mathrm{ma}_K(M))$, instead of K also its revised form can be used.

Theorem 3.6. *For all relations $M : X \leftrightarrow G$ and all M-contacts $K : X \leftrightarrow G$ we have that $K = \mathrm{mi}_{K;\eta_{\Psi_K}}(\mathrm{ma}_{K;\eta_{\Psi_K}}(M))$.*

Proof. In the following calculation we abbreviate η_{Ψ_K} by η.

$$\begin{aligned}
\mathrm{mi}_{K;\eta}(\mathrm{ma}_{K;\eta}(M)) &= \overline{\overline{K;\eta};\,\overline{\overline{K;\eta}^{\mathsf{T}};M}} && \text{by (6)}\\
&= \overline{\overline{K};\eta;\,\overline{(\overline{K};\eta)^{\mathsf{T}};M}} && \text{Lemma 2.3}\\
&= \overline{\overline{K};\eta;\,\overline{\eta^{\mathsf{T}};\overline{K}^{\mathsf{T}};M}} && \\
&= \overline{\overline{K};\,\overline{\eta;\eta^{\mathsf{T}};\overline{K}^{\mathsf{T}};M}} && \text{[12] Prop. 4.2.4.ii}\\
&= \overline{\overline{K};\,\overline{\mathrm{syq}(K,K);\overline{K}^{\mathsf{T}};M}} && \text{by (5)}\\
&= \overline{\overline{K};\,\overline{[\overline{K};\mathrm{syq}(\overline{K},\overline{K})]^{\mathsf{T}};M}} && \text{by (1)}\\
&= \overline{\overline{K};\,\overline{\overline{K}^{\mathsf{T}};M}} && \text{by (2)}\\
&= K && \text{Theorem 3.4} \qquad \square
\end{aligned}$$

4 Contacts and Closures

Closure operations appear in many fields in computer science and mathematics. Usually, they are defined as extensive, monotone, and idempotent functions on powersets, i.e., functions $h : 2^X \to 2^X$ such that the following conditions hold.

$$\begin{array}{ll}
(\mathrm{H}_1) & \forall Y : Y \subseteq h(Y)\\
(\mathrm{H}_2) & \forall Y, Z : Y \subseteq Z \to h(Y) \subseteq h(Z)\\
(\mathrm{H}_3) & \forall Y : h(h(Y)) \subseteq h(Y)
\end{array}$$

As in the case of Aumann contact relations, we start our investigations with a relation-algebraic characterization of closure operations. In the next theorem, the relation Ω denotes set inclusion on the powerset 2^X as specified in (4).

Theorem 4.1. *A mapping $H : 2^X \leftrightarrow 2^X$ is a closure operation iff $H \subseteq \Omega$, $\Omega \subseteq H;\Omega;H^{\mathsf{T}}$, and $H;H \subseteq H$.*

Proof. As in the case of Theorem 3.1, we only treat one case, viz. the equivalence of (H_3) and $H;H \subseteq H$. To enhance readability, in the following calculations, we apply the common notation of function application also for H.

$$\begin{aligned}
& \forall Y : H(H(Y)) \subseteq H(Y)\\
\iff{}& \forall Y, Z, U : H(Y) = U \wedge H(U) = Z \to (\exists W : H(Y) = W \supseteq Z)\\
\iff{}& \forall Y, Z, U : H_{Y,U} \wedge H_{U,Z} \to (\exists W : H_{Y,W} \wedge \Omega_{Z,W})\\
\iff{}& \forall Y, Z : (\exists U : H_{Y,U} \wedge H_{U,Z}) \to (\exists W : H_{Y,W} \wedge \Omega^{\mathsf{T}}_{W,Z})\\
\iff{}& \forall Y, Z : (H;H)_{Y,Z} \to (H;\Omega^{\mathsf{T}})_{Y,Z}\\
\iff{}& H;H \subseteq H;\Omega^{\mathsf{T}}
\end{aligned}$$

(H_1) equals $H \subseteq \Omega$, so that with antisymmetry and univalency of H we get

$$H;H \subseteq H;\Omega \cap H;\Omega^{\mathsf{T}} = H;(\Omega \cap \Omega^{\mathsf{T}}) \subseteq H;\mathsf{I} = H. \qquad \square$$

A simple relation-algebraic reasoning shows that $H;H \subseteq H$ in fact is equivalent to the equation $H;H = H$ when $H \subseteq \Omega \subseteq H;\Omega;H^{\mathsf{T}}$. This corresponds to the well-known property that in (H_3), due to (H_1) and (H_2), even equality holds.

Because of Theorem 4.1, we are able to generalize the concept of a closure operation from powerset lattices to arbitrary partial order relations within the language of relation algebra as follows.

Definition 4.1. *Given a partial order $P : X \leftrightarrow X$, a mapping $H : X \leftrightarrow X$ is called a* closure operation with respect to P *— in short: a P-*closure *— if the following conditions hold:*

$$(\mathrm{C}_1)\quad H \subseteq P \qquad\qquad (\mathrm{C}_2)\quad P \subseteq H;P;H^{\mathsf{T}} \qquad\qquad (\mathrm{C}_3)\quad H;H \subseteq H$$

In [1] it is shown that there is a one-to-one correspondence between the set of all Aumann contact relations between X and 2^X and the set of all closure operations on 2^X. Without proof and reference to its origin, this correspondence is also mentioned in [6]. In the remainder of this section, we investigate the relationship between contact relations and closure operations in our general setting, i.e., in conjunction with M-contacts and Ω_M-closures, and using relation-algebraic means. As the only basic prerequisite on the relation $M : X \leftrightarrow G$ we assume $\mathrm{syq}(M,M) = \mathsf{I}$, i.e., pairwise different columns, to ensure that Ω_M is a partial order (see Section 2). (Even this is not a really essential requirement.)

How to obtain M-contacts from Ω_M-closures is shown in the following theorem. In words, the theorem states that $x \in X$ is in contact with $g \in G$ iff x is a member of the closure of g.

Theorem 4.2. *For all relations $M : X \leftrightarrow G$ such that* $\mathrm{syq}(M,M) = \mathsf{I}$ *and all Ω_M-closures $H : G \leftrightarrow G$, the relation $K := M;H^{\mathsf{T}} : X \leftrightarrow G$ is an M-contact.*

Proof. For proving (K_1), we use (C_1) and Prop. 4.2.3 of [12] in

$$M;\Omega_M \subseteq M \implies M;H \subseteq M \iff M \subseteq M;H^{\mathsf{T}} \iff M \subseteq K.$$

Now, Lemma 2.2 yields the result. The verification of property (K_2) bases on the following calculation.

$$\begin{aligned}
K;\overline{M^{\mathsf{T}};\overline{K}} &= M;H^{\mathsf{T}};\overline{M^{\mathsf{T}};\overline{M;H^{\mathsf{T}}}} && \\
&= M;H^{\mathsf{T}};\overline{M^{\mathsf{T}};\overline{M}}\,;H^{\mathsf{T}} && \text{[12] Prop. 4.2.4.iii} \\
&= M;H^{\mathsf{T}};\Omega_M;H^{\mathsf{T}} && \\
&\subseteq M;\Omega_M;H^{\mathsf{T}};H^{\mathsf{T}} && \text{by } (\mathrm{C}_2) \text{ (cf. [12] p. 143)} \\
&\subseteq M;\Omega_M;H^{\mathsf{T}} && \text{by } (\mathrm{C}_3) \\
&= M;H^{\mathsf{T}} && \text{Lemma 2.2} \\
&= K
\end{aligned}$$

An application of the Schröder rule to this inclusion completes the proof. $\square$

To obtain a closure operation h from a contact relation A, in [1] the closure $h(Y)$ of a set Y is defined as the set of elements being in contact with Y. Relation-algebraically, this leads to the expression $\mathrm{syq}(A,\varepsilon)$ for the closure operation. Contrary to the transition from closure operations to contact relations, which also works in our general setting, the transition from M-contacts K to Ω_M-closures is problematic. The reason is that $\mathrm{syq}(K,M)$ may be non-total. But if $\mathrm{syq}(K,M)$ is total, it is indeed an Ω_M-closure as the following theorem shows.

Theorem 4.3. *For all relations $M : X \leftrightarrow G$ such that $\mathrm{syq}(M,M) = \mathsf{I}$ and all M-contacts $K : X \leftrightarrow G$, the relation $H := \mathrm{syq}(K,M) : G \leftrightarrow G$ is an Ω_M-closure provided it is total.*

Proof. Since totality of H has been assumed as a prerequisite, we show univalence to establish H as a mapping:

$$\begin{aligned} H^{\mathsf{T}};H &= [\mathrm{syq}(K,M)]^{\mathsf{T}};\mathrm{syq}(K,M) && \\ &= \mathrm{syq}(M,K);\mathrm{syq}(K,M) && \text{by (1)} \\ &\subseteq \mathrm{syq}(M,M) && \text{by (2)} \\ &= \mathsf{I}. && \end{aligned}$$

Property (C_1) follows from (K_1), since

$$H = \mathrm{syq}(K,M) \subseteq \overline{K^{\mathsf{T}};\overline{M}} \subseteq \overline{M^{\mathsf{T}};\overline{M}} = \Omega_M.$$

In the proof of (C_2) we use that totality of $\mathrm{syq}(K,M)$ implies surjectivity of $\mathrm{syq}(M,K) = \mathrm{syq}(\overline{M},\overline{K})$ (cf. Prop. 4.4.1.i,ii of [12]). We start with

$$\begin{aligned} H;\overline{\Omega_M};H^{\mathsf{T}} &= [\mathrm{syq}(M,K)]^{\mathsf{T}};M^{\mathsf{T}};\overline{M};\mathrm{syq}(M,K) && \text{by (1)} \\ &= [M;\mathrm{syq}(M,K)]^{\mathsf{T}};\overline{M};\mathrm{syq}(\overline{M},\overline{K}) && \text{by (1)} \\ &= K^{\mathsf{T}};\overline{K} && \text{[12] Prop. 4.4.2.ii} \\ &\subseteq M^{\mathsf{T}};\overline{K} && \text{by } (\mathrm{K}_2) \\ &\subseteq M^{\mathsf{T}};\overline{M} && \text{by } (\mathrm{K}_1). \end{aligned}$$

Using that H is a mapping, we get from this $\overline{H;\Omega_M;H^{\mathsf{T}}} \subseteq \overline{\Omega_M}$, i.e., the desired inclusion $\Omega_M \subseteq H;\Omega_M;H^{\mathsf{T}}$. Also the first two calculations of the subsequent proof of property (C_3) use the surjectivity of $\mathrm{syq}(M,K) = \mathrm{syq}(\overline{M},\overline{K})$. From (1) and Prop. 4.4.2.ii of [12] and (K_1) we get

$$K^{\mathsf{T}};\overline{M};\mathrm{syq}(M,K) = K^{\mathsf{T}};\overline{M};\mathrm{syq}(\overline{M},\overline{K}) = K^{\mathsf{T}};\overline{K} \subseteq K^{\mathsf{T}};\overline{M}$$

and Prop. 4.4.2.ii of [12] and (K_2) yield

$$\overline{K}^{\mathsf{T}};M;\mathrm{syq}(M,K) = \overline{K}^{\mathsf{T}};K = (K^{\mathsf{T}};\overline{K})^{\mathsf{T}} \subseteq (M^{\mathsf{T}};\overline{K})^{\mathsf{T}} = \overline{K}^{\mathsf{T}};M.$$

Putting these inclusions together, we obtain

$$(K^{\mathsf{T}};\overline{M} \cup \overline{K}^{\mathsf{T}};M);\mathrm{syq}(M,K) \subseteq K^{\mathsf{T}};\overline{M} \cup \overline{K}^{\mathsf{T}};M$$

that, due to the definition of $\mathrm{syq}(K,M)$ and (1), holds iff

$$\overline{\mathrm{syq}(K,M)}\,;[\mathrm{syq}(K,M)]^{\mathsf{T}} \subseteq \overline{\mathrm{syq}(K,M)}\,.$$

An application of the Schröder rule to this result followed by the definition of H, finally, shows $H;H \subseteq H$. □

Combining the last two theorems, we obtain for our general setting an injective embedding of the Ω_M-closures into the M-contacts.

Corollary 4.1. *Assume a relation* $M : X \leftrightarrow G$ *such that* $\mathrm{syq}(M,M) = \mathsf{I}$ *and let* $\mathfrak{K}_M$ *and* $\mathfrak{H}_{\Omega_M}$ *denote the set of* M*-contacts and* Ω_M*-closures, respectively. Then the function* $\mathrm{con}_M : \mathfrak{H}_{\Omega_M} \to \mathfrak{K}_M$*, where* $\mathrm{con}_M(H) = M;H^{\mathsf{T}}$*, is injective.*

Proof. First we show that $\mathrm{syq}(\mathrm{con}_M(H),M)$ is total for all $H \in \mathfrak{H}_{\Omega_M}$.

$$\begin{aligned}
\mathrm{syq}(\mathrm{con}_M(H),M);\mathsf{L} &= \mathrm{syq}(M;H^{\mathsf{T}},M);\mathsf{L} && \text{definition of } \mathrm{con}_M(H)\\
&= H;\mathrm{syq}(M,M);\mathsf{L} && \text{[12] Prop. 4.4.1.vi}\\
&= H;\mathsf{L} && \text{since } \mathrm{syq}(M,M) = \mathsf{I}\\
&= \mathsf{L} && H \text{ total}
\end{aligned}$$

Hence, $\mathrm{syq}(\mathrm{con}_M(H),M)$ is an Ω_M-closure due to Theorems 4.2 and 4.3. The above calculation, furthermore, shows that the function

$$\mathrm{clo}_M : \mathrm{con}_M(\mathfrak{H}_{\Omega_M}) \to \mathfrak{H}_{\Omega_M} \qquad \mathrm{clo}_M(K) = \mathrm{syq}(K,M)$$

fulfils $\mathrm{clo}_M(\mathrm{con}_M(H)) = H$ for all $H \in \mathfrak{H}_{\Omega_M}$, and we are done. □

Specifying the point-wise ordering of mappings relation-algebraically, we obtain for $H_1, H_2 \in \mathfrak{H}_{\Omega_M}$ that $H_1 \leq H_2$ iff $H_1 \subseteq H_2;{\Omega_M}^{\mathsf{T}}$. In respect thereof, the following theorem shows that the function con_M is even an order embedding from the ordered set $(\mathfrak{H}_{\Omega_M}, \leq)$ into the ordered set $(\mathfrak{K}_M, \subseteq)$.

Theorem 4.4. *Under the assumptions of Corollary 4.1 we have* $H_1 \subseteq H_2;{\Omega_M}^{\mathsf{T}}$ *iff* $M;{H_1}^{\mathsf{T}} \subseteq M;{H_2}^{\mathsf{T}}$.

Proof. In the following calculation we combine the fact that H_1 and H_2 are mappings with Prop. 4.2.4.iii of [12].

$$\begin{aligned}
H_1 \subseteq H_2;{\Omega_M}^{\mathsf{T}} &\iff H_1 \subseteq H_2;\overline{M^{\mathsf{T}};\overline{M}}^{\mathsf{T}} &&\\
&\iff H_1 \subseteq H_2;\overline{\overline{M}^{\mathsf{T}};M} &&\\
&\iff H_1 \subseteq \overline{\overline{H_2;M^{\mathsf{T}}};M} && \text{Prop. 4.2.4.ii of [12]}\\
&\iff \overline{H_2;M^{\mathsf{T}}};M \subseteq \overline{H_1} &&\\
&\iff H_1;M^{\mathsf{T}} \subseteq H_2;M^{\mathsf{T}} && \text{Schröder rule}
\end{aligned}$$

□

A little reflection shows that $(\mathfrak{K}_M, \subseteq)$ is a complete lattice. For the ordered set $(\mathfrak{H}_{\Omega_M}, \leq)$ this is not true in general. It is, however, true if the underlying set G on which the closure operations work is finite [10]. In general, we are not able to establish a one-to-one correspondence between contact relations and closure operations in our general setting without further assumptions on the underlying relation $M : X \leftrightarrow G$. For instance, for the example of Section 3, RelView computed for the membership relation M and M-contact K given there the following matrices for Ω_M and $\mathrm{syq}(K, M)$.

$$\Omega_M = \begin{array}{c|ccc} & g1 & g2 & g3 \\ \hline g1 & \blacksquare & \square & \square \\ g2 & \square & \blacksquare & \square \\ g3 & \square & \square & \blacksquare \end{array} \qquad \mathrm{syq}(K, M) = \begin{array}{c|ccc} & g1 & g2 & g3 \\ \hline g1 & \square & \square & \square \\ g2 & \square & \blacksquare & \square \\ g3 & \square & \square & \square \end{array}$$

The relation Ω may be described as being the column-is-contained-preorder for M, while $\mathrm{syq}(K, M)$ compares columns of K and M for being identical. Furthermore, the tool ascertained that there exist exactly 128 relations containing M and exactly 66 of them are M-contacts. Since Ω_M is the identity relation, however, there exists only one Ω_M-closure, viz. Ω_M.

In matrix terminology, totality of $\mathrm{syq}(K, M)$ means that each column of $K : X \leftrightarrow G$ appears also as a column of M. Hence, this property should hold for G being a powerset 2^X and M being the powerset relation $\varepsilon : X \leftrightarrow 2^X$. And, in fact, totality of $\mathrm{syq}(K, \varepsilon)$ can be shown so that, together with the already obtained results, we are able to give not only a completely relation-algebraic proof of the above mentioned result of Aumann but also to show that the sets are isomorphic complete lattices.

Corollary 4.2. *For all powerset relations $\varepsilon : X \leftrightarrow 2^X$, the ordered sets $(\mathfrak{K}_\varepsilon, \subseteq)$ and $(\mathfrak{H}_\Omega, \leq)$ are isomorphic via the function $\mathrm{con}_\varepsilon : \mathfrak{H}_\Omega \to \mathfrak{K}_\varepsilon$ of Corollary 4.1 and its inverse function $\mathrm{clo}_\varepsilon : \mathfrak{K}_\varepsilon \to \mathfrak{H}_\Omega$, where $\mathrm{clo}_\varepsilon(K) = \mathrm{syq}(K, \varepsilon)$.*

Proof. For each $K \in \mathfrak{K}_\varepsilon$, (1) and the second axiom of (3) imply

$$\mathrm{syq}(K, \varepsilon); \mathsf{L} = \big(\mathsf{L}; \mathrm{syq}(K, \varepsilon)^{\mathsf{T}}\big)^{\mathsf{T}} = \big(\mathsf{L}; \mathrm{syq}(\varepsilon, K)\big)^{\mathsf{T}} = \mathsf{L}.$$

Because of Theorem 4.3, therefore, $\mathrm{clo}_\varepsilon(K)$ is defined for all $K \in \mathfrak{K}_\varepsilon$. From the proof of Corollary 4.1 we know already that

$$\mathrm{clo}_\varepsilon(\mathrm{con}_\varepsilon(H)) = H$$

holds for all $H \in \mathfrak{H}_\Omega$. Furthermore, we obtain for all $K \in \mathfrak{K}_\varepsilon$ the equation

$$\mathrm{con}_\varepsilon(\mathrm{clo}_\varepsilon(K)) = \varepsilon; \mathrm{syq}(K, \varepsilon)^{\mathsf{T}} = \varepsilon; \mathrm{syq}(\varepsilon, K) = K$$

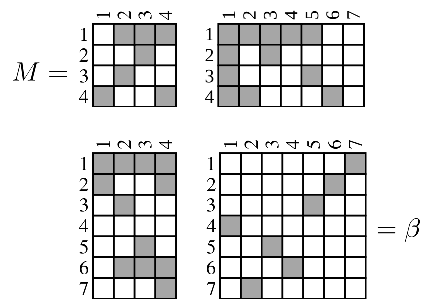
using the second axiom of (3) in combination with Prop. 4.4.2.ii of [12]. These two properties show that the functions are bijective and mutually inverses. That the two mappings are order isomorphisms follows from Theorem 4.4. $\square$

One might conjecture that in the case $\mathrm{syq}(M, M) = \mathsf{I}$ from an isomorphism between the sets $\mathfrak{K}_M$ and $\mathfrak{H}_{\Omega_M}$ also the second axiom of (3) follows, i.e., M is essentially a powerset relation. Unfortunately, this speculation is false, as the simple example with a single group, i.e., $G := \mathbf{1}$, and M as $\mathsf{L} : X \leftrightarrow \mathbf{1}$ shows.

5 Linking Column and Row Types of a Relation

Considering a relation $M : X \leftrightarrow Y$ as a Boolean matrix, rows and columns may be joined or intersected in much the same way as one may form sums of rows or columns of real-valued matrices. In comparison with the vector space spanned by the real-valued rows, on will then obtain unions of rows, or intersections, respectively. Unions of rows of M may, of course, also be considered as complements of intersections of complemented rows. For the following, we decide to treat mainly intersections. Although this looks more complicated introducing complements, it gives better guidance along residuation.

By the following four RelView-pictures we want to decribe the situation. We consider a 4×4 Boolean matrix M. The 4×7 matrix right besides M shows all possible intersections of sets of columns of M. Each of the seven results represented by a column of the matrix. Note, that the full vector is obtained by intersecting the empty set of columns. In the same way the 7×4 matrix below M enumerates all intersections of sets of rows of M. Again we have seven different results, now represented by the matrix's rows. Finally, the 7×7 matrix β bijectively links the column intersections and the row intersections of M.

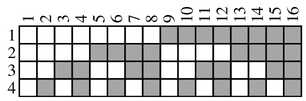


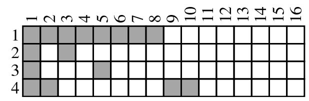
It is evident that several combinations of rows may produce the same union. When considering $\varepsilon_X^{\mathsf{T}}$, multiplied from the left, where $\varepsilon_X : X \leftrightarrow 2^X$ is the powerset relation of X, one will probably obtain many identical unions of rows. In order to eliminate multiply occurring unions, one may, of course, wish to identify them. A little reflection shows in an analogous way that all intersections of rows of M are given by the rows of $R := \overline{\varepsilon_X^{\mathsf{T}}; \overline{M}} : 2^X \leftrightarrow X$. The elimination of multiple rows of R is obtained via $\eta_\Xi^{\mathsf{T}}; R : 2^X/\Xi \leftrightarrow Y$, where $\eta_\Xi : 2^X \leftrightarrow 2^X/\Xi$ is the canonical epimorphism induced by the row equivalence relation $\Xi := \operatorname{syq}(R^{\mathsf{T}}, R^{\mathsf{T}}) : 2^X \leftrightarrow 2^X$. Equivalence classes of rows so obtained will be called row types.

We will use contacts for linking the row types of a relation with its column types. The corresponding reflection, namely, shows that all intersections of columns of M are given by the columns of $C := \overline{\overline{M}; \varepsilon_Y} : X \leftrightarrow 2^Y$, where $\varepsilon_Y : Y \leftrightarrow 2^Y$ is the powerset relation of Y, so that we proceed with

Definition 5.1. *Given $M : X \leftrightarrow Y$, $C := \overline{\overline{M}; \varepsilon_Y}$, and $R := \overline{\varepsilon_X^{\mathsf{T}}; \overline{M}}$, we define the* column intersection types relation *as $C; \eta_\Psi : X \leftrightarrow 2^Y/\Psi$ and the* row intersection types relation *as $\eta_\Xi^{\mathsf{T}}; R : 2^X/\Xi \leftrightarrow Y$.*

To visualize the constructions, we consider again the 4×4 matrix M of the above example. The following RelView-matrices represent the membership relation ε_Y and the relation C, respectively.

$\varepsilon_Y =$ 

$C =$ 

If we transform C into the column intersection types relation $C;\eta_\Psi$ by the elimination of all multiple occurrences of columns, we exactly obtain the result already shown above.

It is a remarkable fact that there exists a close connection between the row and the column types relation. By the following bijection, one may feel reminded that for a real-valued matrix the row rank equals the column rank. Some ideas from the approach stem from real valued matrices as presented e.g., in [11]. For the proof we need that symmetric quotients are difunctional in the sense that

$$\mathrm{syq}(P,Q);\left[\mathrm{syq}(P,Q)\right]^{\mathsf{T}};\mathrm{syq}(P,Q) \subseteq \mathrm{syq}(P,Q), \tag{7}$$

which immediately follows from (1), (2) and Prop. 4.4.1.iv of [12].

Theorem 5.1. *Given a relation $M : X \leftrightarrow Y$ together with the derived relations $C := \overline{\overline{M};\varepsilon_Y}$, $R := \overline{\varepsilon_X^{\mathsf{T}};\overline{M}}$, $\Psi := \mathrm{syq}(C,C)$, and $\Xi := \mathrm{syq}(R^{\mathsf{T}},R^{\mathsf{T}})$, there exists a bijective mapping (in the relational sense) of type $[2^X/\Xi \leftrightarrow 2^Y/\Psi]$.*

Proof. The idea is to compare the contact relation $\mathrm{mi}_M(\mathrm{ma}_M(\varepsilon_X)) = \mathrm{mi}_M(R^{\mathsf{T}})$ and the lower derivative $\mathrm{mi}_M(\varepsilon_Y) = C$ via a symmetric quotient construction; so we define (equality of the two versions is easy to prove by expansion):

$$A := \mathrm{syq}(\mathrm{mi}_M(R^{\mathsf{T}}), C) = \mathrm{syq}(R^{\mathsf{T}}, \mathrm{ma}_M(C)) : 2^X \leftrightarrow 2^Y$$

The relation A is total and surjective. For totality, we calculate

$$\begin{aligned} A &= \mathrm{syq}(\mathrm{mi}_M(R^{\mathsf{T}}), C) \\ &= \mathrm{syq}(\, \overline{\overline{M}\,;\, \overline{\overline{M}^{\mathsf{T}};\varepsilon_X}} \,,\, \overline{\overline{M}\,;\varepsilon_Y} \,) \\ &= \mathrm{syq}(\overline{M}\,;\, \overline{\overline{M}^{\mathsf{T}};\varepsilon_X} \,,\, \overline{M}\,;\varepsilon_Y) && \text{by (1)} \\ &\supseteq \mathrm{syq}(\, \overline{\overline{M}^{\mathsf{T}};\varepsilon_X} \,,\varepsilon_Y) && \text{[12] Prop. 4.4.1.v} \end{aligned}$$

and apply then that $\mathrm{syq}(\, \overline{\overline{M}^{\mathsf{T}};\varepsilon_X} \,,\varepsilon_Y)$ is total by (3) and (1) . To prove surjectivity, we reason in the same way, but use the other variant of A.

Next, we have a look at the row equivalence relation $\Xi' := \mathrm{syq}(A^{\mathsf{T}},A^{\mathsf{T}})$ and the column equivalence relation $\Psi' := \mathrm{syq}(A,A)$. It so happens that $\Xi = \Xi'$ and $\Psi = \Psi'$ via a general cancelling rule for symmetric quotients that follows from the laws of [12], Section 4.4. E.g., the second equality is shown by

$$\begin{aligned} \Psi' &= \mathrm{syq}(A,A) \\ &= \mathrm{syq}(\mathrm{syq}(\mathrm{mi}_M(R^{\mathsf{T}}),C), \mathrm{syq}(\mathrm{mi}_M(R^{\mathsf{T}}),C)) \\ &= \mathrm{syq}(C,C) && \text{cancelling} \\ &= \Psi. \end{aligned}$$

Based on $A : 2^X \leftrightarrow 2^Y$ and the canonical epimorphisms $\eta_\Xi : 2^X \leftrightarrow 2^X/\Xi$ and $\eta_\Psi : 2^Y \leftrightarrow 2^Y/\Psi$, now we define the following relation by simple composition:

$$\beta := {\eta_\Xi}^{\mathsf{T}}; A; \eta_\Psi : 2^X/\Xi \leftrightarrow 2^Y/\Psi$$

This is a matching, defined as a relation that is at the same time univalent and injective. Using the Schröder rule, for the proof of univalency we start with

$$A^{\mathsf{T}}; A \subseteq \overline{\overline{A}^{\mathsf{T}}; A} \iff A; \overline{A}^{\mathsf{T}}; A \subseteq \overline{A} \iff A; A^{\mathsf{T}}; \overline{A} \subseteq \overline{A} \iff A; A^{\mathsf{T}}; A \subseteq A.$$

This yields $A^{\mathsf{T}}; A \subseteq \overline{\overline{A}^{\mathsf{T}}; A}$ and, by transposition, also $A^{\mathsf{T}}; A \subseteq \overline{A^{\mathsf{T}}; \overline{A}}$, since symmetric quotients are difunctional due to (7). So, we have $A^{\mathsf{T}}; A \subseteq \mathrm{syq}(A, A)$. If we combine this with $\Xi = \Xi' = \mathrm{syq}(A^{\mathsf{T}}, A^{\mathsf{T}})$ and Prop 4.4.1.iii of [12], we get

$$A^{\mathsf{T}}; \Xi; A = A^{\mathsf{T}}; \mathrm{syq}(A^{\mathsf{T}}, A^{\mathsf{T}}); A = A^{\mathsf{T}}; A \subseteq \mathrm{syq}(A, A) = \Psi' = \Psi.$$

Now, the univalency of the relation β can be shown as follows:

$$\begin{aligned}
\beta^{\mathsf{T}}; \beta &= \left[{\eta_\Xi}^{\mathsf{T}}; A; \eta_\Psi\right]^{\mathsf{T}}; {\eta_\Xi}^{\mathsf{T}}; A; \eta_\Psi \\
&= {\eta_\Psi}^{\mathsf{T}}; A^{\mathsf{T}}; \eta_\Xi; {\eta_\Xi}^{\mathsf{T}}; A; \eta_\Psi \\
&= {\eta_\Psi}^{\mathsf{T}}; A^{\mathsf{T}}; \Xi; A; \eta_\Psi && \text{by (5)} \\
&\subseteq {\eta_\Psi}^{\mathsf{T}}; \Psi; \eta_\Psi && \text{see above} \\
&= {\eta_\Psi}^{\mathsf{T}}; \eta_\Psi; {\eta_\Psi}^{\mathsf{T}}; \eta_\Psi && \text{by (5)} \\
&= \mathsf{I} && \text{by (5)}
\end{aligned}$$

Transpositions of difunctional relations obviously are also difunctional. This implies $A; A^{\mathsf{T}} \subseteq \mathrm{syq}(A^{\mathsf{T}}, A^{\mathsf{T}})$ and from this fact we obtain, analogously to the above calculations, first $A; \Psi; A^{\mathsf{T}} \subseteq \Xi$ and then injectivity $\beta; \beta^{\mathsf{T}} \subseteq \mathsf{I}$.

Since canonical epimorphisms and their transpositions are total and surjective and these properties pass on to compositions, by construction β is also total and surjective, i.e., the bijective mapping we have searched for. □

Let, for $M : X \leftrightarrow Y$ and $y \in Y$, by $M^{(y)} : Y \leftrightarrow \mathbf{1}$ the y-column of M be denoted. Then $M_c^\cap := \{\bigcap_{y \in I} M^{(y)} \mid I \in 2^Y\}$ is the set of all intersections of sets of columns of M and $M_r^\cap := (M^{\mathsf{T}})_c^\cap$ that of all intersections of sets of rows. It is easy to show that $\bigcap_{y \in I} M^{(y)} \mapsto [I]$ is a bijective function from $M_c^\cap$ to $2^Y/\Psi$ in the usual mathematical sense and, hence, $|M_c^\cap| = |2^Y/\Psi|$ and $|M_r^\cap| = |2^X/\Xi|$. Now, from the above theorem we get $|M_c^\cap| = |M_r^\cap|$, as already demonstrated by means of the introductory example of this section.

Note that all constructions of Theorem 5.1 and its proof are relation-algebraic expressions, that is, algorithmic. As a consequence, they immediately can be translated into RelView code, such that the tool can be used to compute for a given relation its column intersection types relation as well as its row intersection types relation and also the mapping that bijectively links the rows of the latter with the columns of the first one. Similar to Definition 5.1 also column union types relations and row union types relations can be introduced and then an analogon of Theorem 5.1 holds for these constructions.

6 Conclusion

At the end of Section 4, we have remarked that a one-to-one correspondence between M-contacts and Ω_M-closures also may exist for M not being (isomorphic to) a set-theoretic membership relation. Presently, we are looking for simple conditions on M which ensure that the sets $\mathfrak{K}_M$ and $\mathfrak{H}_{\Omega_M}$ are isomorphic. In this context, it is also interesting to study whether these conditions imply that Ω_M belongs to a specific class of orders. In respect thereof, a first result is that each relation M that, using matrix terminology, is obtained from a powerset relation ϵ by adding additional rows consisting of 1's only has as many M-contacts as Ω_M-closures and in this case Ω_M is isomorphic to Ω.

Besides Aumann contacts, another concept of contacts is discussed in the literature, mainly for reasoning about spatial regions. In most cases (see e.g.,[8]), the underlying structure is a Boolean lattice, i.e., essentially a powerset ordered by set inclusion. This fact leads in a natural way to the task of detecting the interdependencies between the two concepts (if such are) and whether it is also possible and reasonable to generalize the latter one similar to our generalization of Aumann contacts to M-contacts, a work that is planned for the future. Another future work is the relation-algebraic treatment of other closure objects, like implicational structures, join-congruences, Moore families and so on.

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