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Shock Waves of a Continuous Model of Traffic Flow

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With 11 Figures

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Zusammenfassung

Stoßwellen eines kontinuierlichen Verkehrsflußmodells. Ziel dieser Untersuchung ist die Entwicklung eines Programms zum Zeichnen der Stromlinienbilder von Verkehrsflüssen unter Verwendung eines kontinuierlichen Modells von LIGHTHILL. Es handelt sich dabei um eindimensionale Strömungen mit geschwindigkeitsabhängiger Dichte $\rho(v)$. Die Kontinuitätsgleichung führt zu der quasilinearen partiellen Differentialgleichung $v_t + v_x \cdot (v + \rho(v)/\rho'(v)) = 0$. In [3] sind geeignete Randbedingungen dafür angegeben, und es wird gezeigt, daß man aus dem Fahrverhalten $x = x(t; 0)$ des Führungsfahrzeugs einer Kolonne die Trajektorien $x = x(t; c)$, also das Fahrverhalten der folgenden Fahrzeuge, explizit erhalten kann.

Hier wird die Theorie der Verdichtungslinien entwickelt. Bei einem Spezialfall des kontinuierlichen Modells, beim P-Modell, lassen sich die Verdichtungslinien explizit bestimmen. Das P-Modell kann man für eine approximative Behandlung allgemeinerer Fälle verwenden. Zum Abschluß wird eine Standardaufgabe der Simulation von Verkehrsflüssen, die durch eine Folge von Verkehrsampeln geregelt sind, formuliert und ein Programm beschrieben, das anhand des kontinuierlichen Modells Stromlinienbilder davon errechnet.

1. Summary

The purpose of this investigation is the development of a program for the construction of stream-line pictures of traffic flow using a continuous model. In a continuous model traffic flow is treated as if it were a flow of a fluid. The model used here is due to LIGHTHILL and represents a one-dimensional flow with a density $\rho(v)$ dependent on the velocity v. The continuity equation leads to the quasilinear partial differential equation $v_t + v_x\,(v + \rho(v)/\rho'(v)) = 0$. In [3] the author has given adequate boundary conditions. Furthermore it is shown there, that $x = x\,(t;\,0)$ being the path of the leading vehicle of a column, the trajectories $x = x\,(t;\,c)$, i.e., the paths of the following vehicles in the sense of the continuous model, can be obtained explicitly.

The theory of shock waves in this flow is developed here. As a special case for which shock waves can be determined explicitly the P-model is introduced. It is described how general flows of the continuous model can be approximated by means of the P-model.

A standard task of the simulation of a traffic flow regulated by traffic lights is formulated and a description is given of a program constructing stream-line pictures of it.

2. The Continuous Model

2.1. Differential Equation and Leading Condition

We will investigate a one-dimensional flow with a density depending on the velocity in a given manner. The equation of motion of the "leading vehicle" serves as a boundary condition and certain trajectories $x = x(t; c)$ of the flow will be interpreted as the equations of motion of the "following vehicles". Let the reciprocal of the density, which is dependent on the velocity, be given by $a(v)$.

From the continuity equation

$$\rho_t + v\rho_x + \rho v_x = 0 \text{ and } \rho = \frac{1}{a(v)}$$

a quasilinear, partial differential equation of first order for $v(t,x)$ follows:

$$v_t + v_x\left(v - \frac{a(v)}{a'(v)}\right) = 0.$$

Additionally, we need the

initial condition $\quad v(0,X) = V(X)$ for $X \leq 0$

and the

leading condition $v(T, X(T)) = \dot{X}(T)$ for $T \geq 0$.

The initial condition describes the velocity distribution $V(X)$ of the column at the time $t = 0$. The position of the leading vehicle at this time shall be $x = 0$. The leading condition is an adequate boundary condition, if the path $X(T)$ of the leading vehicle is given.

In general, integration of the differential equation leads to the following implicit, local description of the velocity field $v(t,x)$, which contains the arbitrary function $g(v)$:

$$x = \left(v - \frac{a(v)}{a'(v)}\right) \cdot t + g(v).$$

From this we recognize that a velocity v occurring in the initial or in the leading condition will be propagated along a straight isoclinal line with slope $m(v) := v - a(v)/a'(v)$. The flux density $M(v) := \rho \cdot v := v/a(v)$ is maximal if and only if the function $m(v)$ vanishes. If we assume that $a(v)$ has a continuous positive second derivative, with the exception of points of discontinuity, then $m(v)$ will be continuous and monotonically increasing with v.

2.2. Integration and Domains of Determinacy

Replacing v with $\dot{x}$ we interpret the velocity field as a differential equation for the trajectories $x = x(t; c)$

$$x = \left(\dot{x} - \frac{a(\dot{x})}{a'(\dot{x})}\right) \cdot t + g(\dot{x}). \tag{2.2.1}$$

Under certain assumptions, the equation

$$x - X(T) = (t - T) \cdot \left(\dot{X}(T) - \frac{a(\dot{X}(T))}{a'(\dot{X}(T))}\right)$$

describes a family of straight isoclinal lines, parametrised by T, starting from the leading curve and spread over a schlicht domain of determinacy. This family has an envelope. Assuming $\ddot{X}(T) > 0$, the envelope is situated to the left of the leading curve, i.e. outside the domain of determinacy. Given $\ddot{X}(T) < 0$, the envelope is a boundary of the domain of determinacy to the right of the leading curve (see Fig. 1).

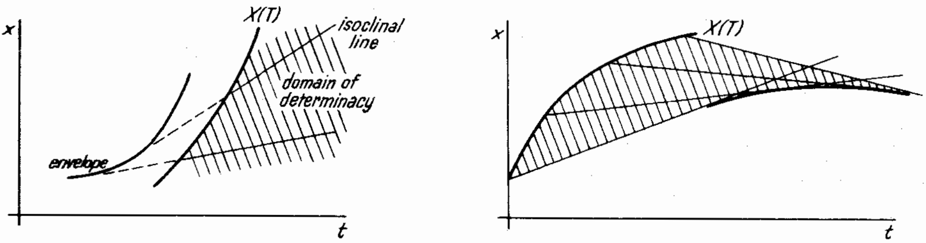


Fig. 1

In [3] the following results are established:

(2.2.2) Let $X(T)$ be a smooth monotonically increasing leading curve with a non-negative second derivative, which is continuous with the exception of a finite number of points.

a) The domain of determinacy is bounded by the leading curve and the isoclinal lines starting from its starting point and end point.

b) For $c \geq 0$ the trajectories are obtained in the parametric representation with parameter T

$$t = T + c \cdot a'(\dot{X}(T))$$

$$x = X(T) + c \cdot a'(\dot{X}(T)) \cdot \left(\dot{X}(T) - \frac{a(\dot{X}(T))}{a'(\dot{X}(T))}\right).$$

Obviously, we get the leading curve itself for $c=0$. The parameter c gives the *number of vehicles* M between the leading curve and the c-th trajectory.

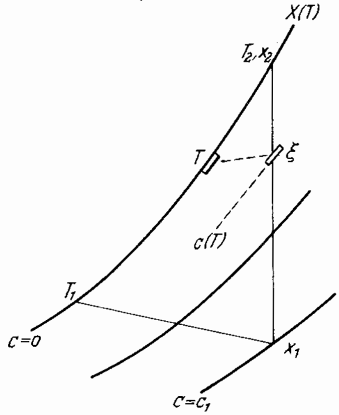


Fig. 2

This is seen as follows.

$$M=\int_{x_1}^{x_2}\frac{1}{a\left(v\left(T_2,\xi\right)\right)}\,d\xi.$$

Using (2.2.2) we substitute
$\xi:=X(T)+c(T)\cdot a'\left(\dot{X}(T)\right)\dot{X}(T)-c(T)\,a\left(\dot{X}(T)\right)$, with $c(T)=(T_2-T)/a'\left(\dot{X}(T)\right)$,
and obtain

$$M=\int_{T_1}^{T_2}\left(\frac{1}{a'}+(T_2-T)\cdot\frac{a''\ddot{X}}{a'^2}\right)dT=-\left[(T_2-T)\cdot\frac{1}{a'}\right]_{T_1}^{T_2}=\frac{T_2-T_1}{a'\left(\dot{X}(T_1)\right)}=c(T_1)=c_1.$$

(2.2.3) Let $X(T)$ be a leading curve as in (2.2.2) but with a negative second derivative.

a) The domain of determinacy is additionally bounded by the envelope of the isoclinal lines.

b) The trajectories are given by (2.2.2.b) with the additional restriction $c\leq\dfrac{-1}{a''\left(\dot{X}(T)\right)\cdot\ddot{X}(T)}$ imposed on c.

Furthermore, we mention that corresponding results can be obtained for the initial condition.

(2.2.4) Let the initial condition be given by a non-negative, continuous function $V(X)$ with a non-negative derivative, which is continuous with the exception of only a finite number of points.

a) The domain of determinacy is bounded by the definition interval $[X_0, 0]$ of $V(X)$ on the x-axis and the isoclinal lines starting from the points $(0, X_0)$ and $(0,0)$.

b) The trajectories are obtained in the parametric representation with parameter X

$$t=a'(V(X))\cdot\left(\int_0^X \frac{d\xi}{a(V(\xi))}+c\right)$$

$$x=X+a'(V(X))\cdot\left(V(X)-\frac{a(V(X))}{a'(V(X))}\right)\cdot\left(\int_0^X \frac{d\xi}{a(V(\xi))}+c\right)$$

(c) For $0\leq c\leq\int_{X_0}^{0} \frac{d\xi}{a(V(\xi))}$ the trajectories start from points of the interval $[X_0, 0]$, and on the parameters X and c the condition $\int_0^X \frac{d\xi}{a(V(\xi))}+c\geq 0$ is imposed.)

(2.2.5) Under the assumptions of (2.2.4), but with $V'(X)<0$, the domain of determinacy is additionally bounded by the envelope of the isoclinal lines.

If general initial and leading conditions are given, they may be investigated separately for the domains of determinacy of the monotonic regions of $V(X)$ and $\dot{X}(T)$ respectively. The transition from one domain of determinacy to the other is usually along *shock waves* dealt with in section 3.

2.3. P-Diagrams

Naturally, $a(v)$ is a convex, monotonically increasing function. The simplest functions with this property are quadratic polynomials $a(v)=$ $=a_0+a_1 v+a_2 v^2$ with $a_i>0$ for $i=0,1,2$. With this consideration in mind, if we admit only parabolas $X(T)=A_0+A_1 T+A_2 T^2$ for leading curves, the envelope of the isoclinal lines degenerates to the single point (t_E, x_E),

$$t_E=-\frac{a_1+2a_2 A_1}{4a_2 A_2},\quad x_E=A_0-\frac{a_0-a_2 A_1^2}{4a_2 A_2}.$$

The slope of an isoclinal line belonging to the velocity v is given by $m(v)=$ $=(-a_0+a_2 v^2)/(a_1+2a_2 v)$.

We have

(2.3.1) The trajectories are

$$t=c(a_1+2a_2 A_1)+T\cdot(1+4a_2 A_2 c)$$

$$x=A_0+c(a_2 A_1^2-a_0)+(A_1 T+A_2 T^2)\cdot(1+4a_2 A_2 c),$$

furthermore:

a) The trajectories are parabolas, which become broader monotonically with increasing c.

b) The vertices of the parabolas are situated on a straight line of slope $m(0)=-a_0/a_1$ through $(t_E,\ x_E)$.

It is a remarkable fact, that not only the leading curve $X(T)$ but all the trajectories are parabolas. Since every trajectory in turn can be regarded as the leading curve for the following trajectories, it is possible to draw the trajectories of all parabolas $X(T)$ in one single diagram. Such a diagram shall be called a parabola- or P-diagram. The special continuous model is called the P-model. In Fig. 3 a P-diagram is shown.

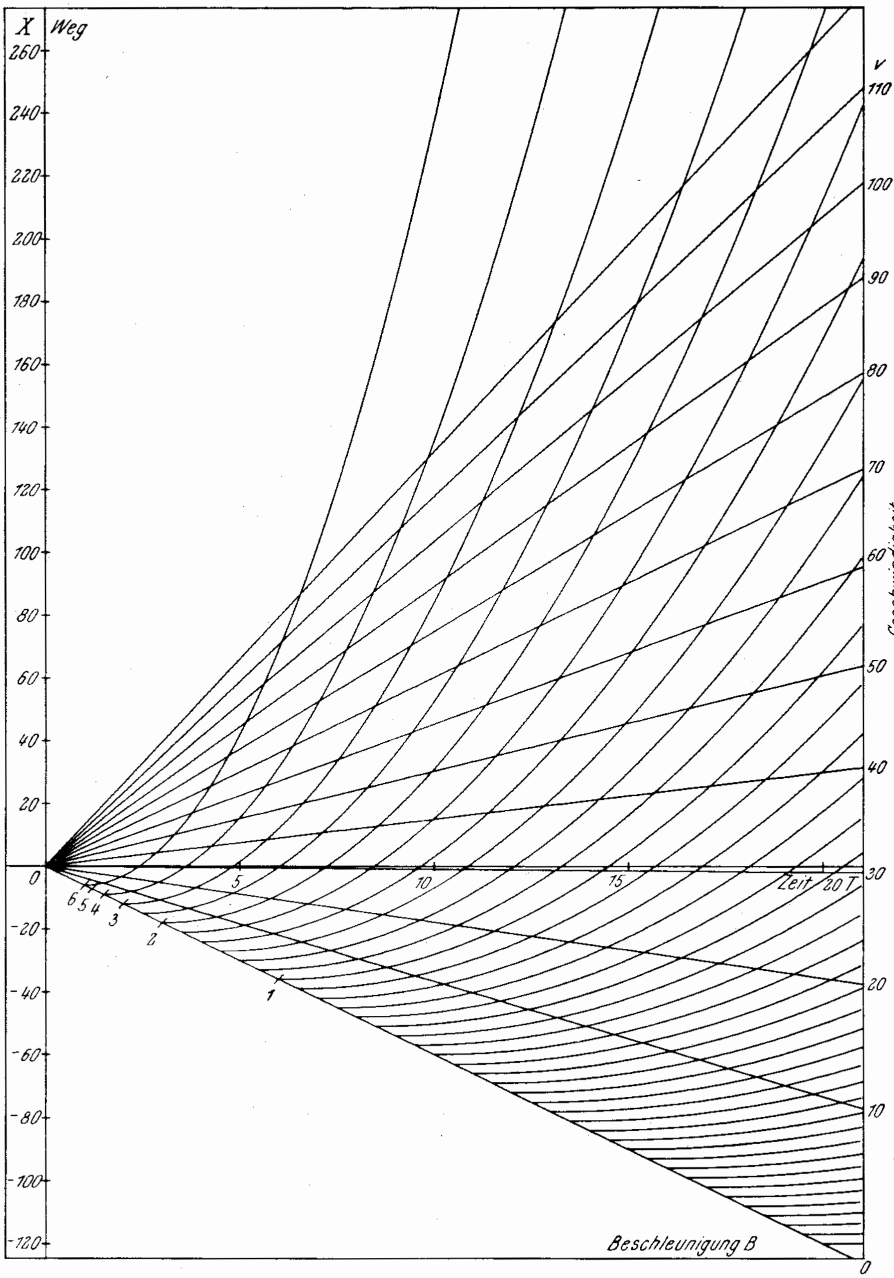


Fig. 3

If a flow problem with arbitrary functions $a(v)$ and $X(T)$ is given, it is natural to try a local approximate treatment by means of P-diagrams. We must therefore approximate the functions $a(v)$ and $X(T)$ by smooth functions $a_p(v)$ and $X_p(T)$ respectively, which are composites of parabolic arcs.

If

$$a(v)=\begin{cases} a_{01}+a_{11}\,v+a_{21}\,v^2 \text{ for } 0\le v\le v_1, \\ a_{02}+a_{12}\,v+a_{22}\,v^2 \text{ for } v_1\le v \end{cases}$$

with $a_{11}+2a_{21}\,v_1=a_{12}+2a_{22}\,v_1$ is such a function, the behavior of trajectories is described by two P-diagrams, the first of which is valid in the sector bounded by the isoclinal lines with slopes $m(0)$ and $m(v_1)$, while the second is valid outside.

3. Shock Waves

3.1. System of Differential Equations for Shock Waves

Let the leading curve $X(T)$ be given in composite form

$$\text{(3.1.1)}\quad X(T):=\begin{cases} X_1(T) \text{ for } T\le T_s \\ X_2(T) \text{ for } T\ge T_s \end{cases}$$

with $X_1(T_s)=X_2(T_s)=:X_s$. Only the case $\dot{X}_1(T_s)\ge \dot{X}_2(T_s)$ is admitted. The functions $X_i(T)$, $i=1,2$, are assumed to be smooth, convex or concave, with a second derivative existing everywhere except possibly at a finite number of points. Since $m(v)$ is a monotonic increasing function, the domains of determinacy belonging to the leading curves $X_i(T)$ overlap. (They would not if $\dot{X}_1(T_s)$ were less $\dot{X}_2(T_s)$ thus leaving a region of indeterminacy.)

Within the intersection of these two domains the shock wave is to be determined, i.e. the line along which the jump from the first domain of determinacy to the second occurs. The condition is that the c-th trajectory of the first domain has to pass over to the c-th trajectory of the second domain.

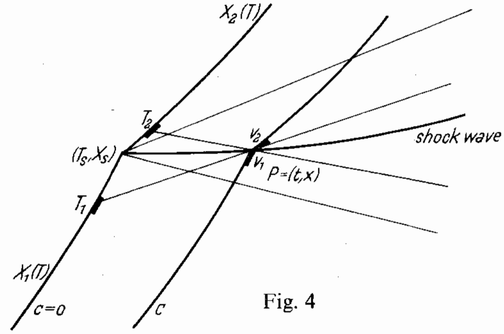


Fig. 4

We take seven parameters, $t, x, v_1, v_2, T_1, T_2, c$ to characterize a point P of the shock wave. The quantities t, x are the space coordinates, and v_1, v_2 are the velocities in P on either side of the shock wave, which are induced by isoclinal lines starting from the points $(T_i, X_i(T_i))$, $i=1,2$, of the leading curve. Finally, c is the parameter of the trajectory through P. Obviously v_i does depend on T_i according to $v_i = \dot{X}_i(T_i)$. In spite of this, we use v_i, too, for the purpose of abbreviation.

For the points (t,x) of the intersection of the domains of determinacy the following equations are valid.

$$t = T_i + c \cdot a'(\dot{X}_i(T_i))$$
$$x = X_i(T_i) + c \cdot a'(\dot{X}_i(T_i)) \cdot \left\{ \dot{X}_i(T_i) - \frac{a(\dot{X}_i(T_i))}{a'(\dot{X}_i(T_i))} \right\}. \qquad (i=1,2)$$

In order to obtain the equation of the shock wave, we have to eliminate c, T_1 and T_2.

This leads to four differential equations

$$\text{(3.1.2)} \qquad \begin{aligned} dt &= (1 + c \cdot a'(v_i))\, dT_i + a'(v_i)\, dc \\ dx &= (1 + c \cdot a'(v_i))\, v_i\, dT_i + a'(v_i) \left\{ v_i - \frac{a(v_i)}{a'(v_i)} \right\} dc. \end{aligned} \qquad (i=1,2)$$

If we take c as a parameter for the shock wave we obtain the following system of differential equations of first order for the six functions $t(c)$, $x(c)$, $v_i(c)$ and $T_i(c)$

$$dt = \frac{a(v_1) - a(v_2)}{v_1 - v_2}\, dc$$

$$dx = \frac{v_2 a(v_1) - v_1 a(v_2)}{v_1 - v_2}\, dc$$

$$\text{(3.1.3)} \qquad \left.\begin{aligned} dv_i &= \frac{\ddot{X}_i(T_i)}{1 + c a''(v_i)\, \ddot{X}_i(T_i)} \left\{ \frac{a(v_1) - a(v_2)}{v_1 - v_2} - a'(v_i) \right\} dc \\ dT_i &= \frac{1}{1 + c a''(v_i)\, \ddot{X}_i(T_i)} \left\{ \frac{a(v_1) - a(v_2)}{v_1 - v_2} - a'(v_i) \right\} dc \end{aligned} \right\} \quad i=1,2.$$

The initial values are $t(0) = T_s$, $x(0) = X_s$, $v_i(0) = \dot{X}_i(T_s)$ and $T_i(0) = T_s$.

3.2. Spontaneous Formation of Shock Waves

The slope of the shock wave in P is

$$m(v_1, v_2) := \frac{v_2 a(v_1) - v_1 a(v_2)}{a(v_1) - a(v_2)} = v_1 - \frac{a(v_1) \cdot (v_1 - v_2)}{a(v_1) - a(v_2)}.$$

If v_1 approaches $v := v_2$ then $m(v_1, v_2)$ will approach the slope $m(v)$ of isoclinal lines belonging to the velocity v.

We now discuss a smooth transition $(\dot{X}_1(T_s) = \dot{X}_2(T_s) =: V_s)$ from $X_1(T)$ to $X_2(T)$ at (T_s, X_s) with the assumption that $\ddot{X}_1(T) > 0$ and $\ddot{X}_2(T) < 0$. The two domains of determinacy intersect between the isoclinal line through (T_s, X_s)

and the envelope belonging to $X_2(T)$ starting from $Q=(t_A, x_A)$. Therefore a spontaneous formation of a shock wave can be expected. For $0 \le c < c_A$ we have $dv_i = 0$ in (3.1.3.). Since $1 + c a''(\dot{X}_2(T)) \cdot \ddot{X}_2(T) = 0$ is valid along the envelope dv_2 becomes indeterminate for $c = c_A$ at Q.

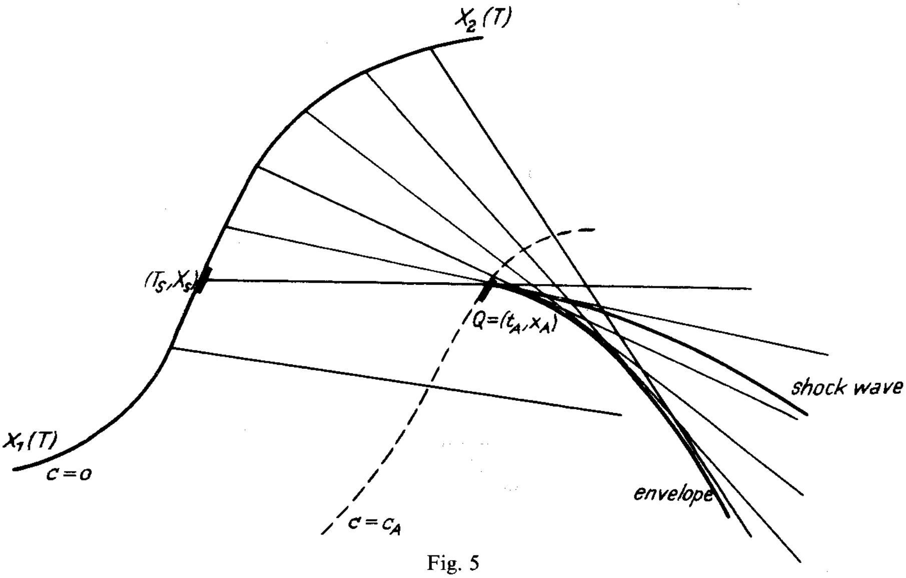


Fig. 5

The initial conditions for the shock wave are

$$t(c_A) = T_s + c_A \cdot a'(V_s) = : t_A$$

$$x(c_A) = X_s + c_A\, a'(V_s) \cdot \left\{ V_s - \frac{a(V_s)}{a'(V_s)} \right\} = : x_A$$

$$v_i(c_A) = V_s, \quad T_i(c_A) = T_s \qquad\qquad i = 1, 2$$

together with

$$c_A = \frac{-1}{a''(V_s) \cdot \ddot{X}_2(T_s)}.$$

If we take $a(v) = a_0 + a_1(v - V_s) + a_2(v - V_s)^2 + a_3(v - V_s)^3 + \ldots$ in the neighborhood of $v = V_s$ and

$$X_i(T) = X_s + V_s(T - T_s) + B_i(T - T_s)^2 + R_i(T - T_s)^3 + \ldots \qquad (i = 1, 2)$$

in the left respectively right neighborhood of $T = T_s$ we obtain

$$t = t_A + a_1 \cdot (c - c_A) + \ldots$$

$$x = x_A + (a_1 V_s - a_0) \cdot (c - c_A) + \ldots$$

$$v_1 = V_s - 4 a_2^3 \frac{B_1 \cdot B_2^4}{(B_1 - B_2)} \cdot \frac{1}{(2 a_3 B_2^2 + a_2 R_2)} \cdot (c - c_A)^2 + \ldots$$

$$v_2 = V_s + 4 a_2^2 \frac{B_2^3}{(2 a_3 B_2^2 + a_2 R_2)} \cdot (c - c_A) + \ldots$$

$$T_1 = T_s - 2a_2^3 \frac{B_2^4}{(B_1 - B_2)} \cdot \frac{1}{(2a_3 B_2^2 + a_2 R_2)} \cdot (c - c_A)^2 + \dots$$

$$T_2 = T_s + 2a_2^2 \frac{B_2^2}{(2a_3 B_2^2 + a_2 R_2)} \cdot (c - c_A) + \dots$$

as the beginning of the Taylor series for the parameters of the shock wave. The slopes of the envelope and the shock wave in Q coincide.

3.3. Determination of Shock Waves for the P-Model

With the assumptions from the P-model the system (3.1.3) of differential equations can be integrated explicitly. Setting $X_s := T_s := 0$, $X_1(T) := A_1 T + A_2 T^2$, $X_2(T) := B_1 T + B_2 T^2$ with $A_1 \geq B_1$ we obtain from (3.1.3)

$$\begin{aligned} dt &= (a_1 + a_2 [v_1 + v_2])\, dc \\ dx &= (a_2 v_1 v_2 - a_0)\, dc \\ dv_1 &= -\frac{2a_2 A_2}{1 + 4a_2 A_2 c} (v_1 - v_2)\, dc \\ dv_2 &= \frac{2a_2 B_2}{1 + 4a_2 B_2 c} (v_1 - v_2)\, dc \end{aligned} \tag{3.3.1}$$

and $v_1 = A_1 + 2A_2 T_1$, $v_2 = B_1 + 2B_2 T_2$. The initial conditions are $t(0) = x(0) = 0$, $v_1(0) = A_1$, $v_2(0) = B_1$. We introduce $w := v_1 - v_2$ satisfying the differential equation

$$\frac{dw}{w} = -\left[\frac{2a_2 A_2}{1 + 4a_2 A_2 c} + \frac{2a_2 B_2}{1 + 4a_2 B_2 c}\right] dc$$

and obtain

$$v_1 - v_2 = w = \frac{\alpha}{\sqrt{1 + 4a_2 A_2 c}\,\sqrt{1 + 4a_2 B_2 c}}.$$

Now, v_1 and v_2 can be determined,

$$v_1 = \frac{\alpha A_2}{A_2 - B_2} \cdot \frac{\sqrt{1 + 4a_2 B_2 c}}{\sqrt{1 + 4a_2 A_2 c}} + \beta$$

$$v_2 = \frac{\alpha B_2}{A_2 - B_2} \cdot \frac{\sqrt{1 + 4a_2 A_2 c}}{\sqrt{1 + 4a_2 B_2 c}} + \beta,$$

as well as the constants of integration

$$\alpha = A_1 - B_1, \quad \beta = \frac{B_1 A_2 - A_1 B_2}{A_2 - B_2}.$$

Finally, this leads to a parametric representation of the shock wave with parameter c

$$t=c\,(a_1+2a_2\beta)+\frac{\alpha}{2\,(A_2-B_2)}\,\{\sqrt{1+4a_2A_2c}\,\sqrt{1+4a_2B_2c}-1\}$$
$$x=c\left(-a_0+a_2\beta^2+a_2\,\frac{\alpha^2A_2B_2}{(A_2-B_2)^2}\right)+ \tag{3.3.2}$$
$$+\frac{\alpha\beta}{2\,(A_2-B_2)}\,\{\sqrt{1+4a_2A_2c}\,\sqrt{1+4a_2B_2c}-1\}.$$

Consideration of the limit case $A_2=B_2$ yields

$$t=c\,(a_1+a_2\,(A_1+B_1))$$
$$x=c\,(-a_0+a_2A_1B_1)+\frac{4a_2^2A_2c^2\,(A_1-B_1)^2}{1+4a_2A_2c}. \tag{3.3.3}$$

3.4. Discussion of the Shock Waves of the P-Model

In this section we will ascertain the main geometric properties of the shock waves. The equations

$$t=A\tau+B\xi,\ x=D\tau+E\xi$$
$$\left(A=a_1+2a_2\beta,\ B=\frac{\alpha}{2\,(A_2-B_2)},\ D=-a_0+a_2\beta^2+a_2\,\frac{\alpha^2A_2B_2}{(A_2-B_2)^2},\ E=\frac{\alpha\beta}{2\,(A_2-B_2)}\right)$$

describe an affine transformation of the conic K given by $\tau=c$,

$$\xi=\sqrt{1+4a_2A_2c}\cdot\sqrt{1+4a_2B_2c}-1$$

in the (τ,ξ)-plane. The curve K is an ellipse (hyperbola, parabola) for $A_2\cdot B_2<0$ ($A_2\cdot B_2>0$, $A_2\cdot B_2=0$), the determinant of the affine transformation being

$$M:=\begin{vmatrix}A & B\\ D & E\end{vmatrix}=\frac{A_1-B_1}{2\,(A_2-B_2)^2}\cdot(A_2a\,(B_1)-B_2a\,(A_1)).$$

We obtain the discriminant δ of the shock wave (3.3.2) by $\delta=-16a_2^2A_2B_2M^2$ and the second invariant Δ by $\Delta=4a_2^2(A_2-B_2)^2M^4$. Therefore we have

(3.4.1) In the P-model with $a(v)=a_0+a_1v+a_2v^2$ the leading curve (3.1.1) produces a shock wave, which is a

straight line if $A_1=B_1$ or $A_2a\,(B_1)=B_2a\,(A_1)$; otherwise an
ellipse if $A_2\cdot B_2<0$
hyperbola if $A_2\cdot B_2>0$
parabola if $A_2\cdot B_2=0$

In general, a jump of the velocity when passing over from an accelerating (decelerating) leading curve $X_1(T)$ to an accelerating (decelerating) leading curve $X_2(T)$ will yield a shock hyperbola. If the signs of the accelerations for $X_1(T)$ and $X_2(T)$ are opposite, a shock ellipse will be formed. There are some additional cases of degeneration.

The center of the shock conic, if it exists, is

$$t_m = -\frac{A_2+B_2}{8a_2A_2B_2}\left(a_1+2a_2\frac{A_1B_2+A_2B_1}{A_2+B_2}\right)$$

$$x_m = -\frac{A_2+B_2}{8a_2A_2B_2}\left(-a_0+a_2\frac{A_1^2B_2+A_2B_1^2}{A_2+B_2}\right).$$

The point (t_m, x_m) bisects the line segment between the origins of the P-diagrams belonging to $X_1(T)$ and $X_2(T)$ respectively.

If we introduce

$$w_1 := \frac{B_1\sqrt{|A_2|}+A_1\sqrt{|B_2|}}{\sqrt{|A_2|}+\sqrt{|B_2|}}, \quad w_2 := \frac{B_1\sqrt{|A_2|}-A_1\sqrt{|B_2|}}{\sqrt{|A_2|}-\sqrt{|B_2|}}$$

as weighted means of the velocities A_1 and B_1, we have $A_1 \le w_1 \le B_1$ and $m(A_1) \le m(w_1) \le m(B_1)$. The slopes of the asymptotes of a shock hyperbola are $(D \pm E \cdot 4a_2\sqrt{A_2B_2})/(A \pm B \cdot 4a_2\sqrt{A_2B_2})$ and it can be seen that they are identical with $m(w_1)$ and $m(w_2)$. Obviously, the slope of the asymptote of the branch of the shock hyperbola, running in the intersection of the domains of determinacy is $m\ (w_1)$.

3.5. Intersection of Shock Waves

If two shock waves intersect, the intersection is the starting point of a new shock wave. For the P-model, the intersection points can be obtained explicitly.

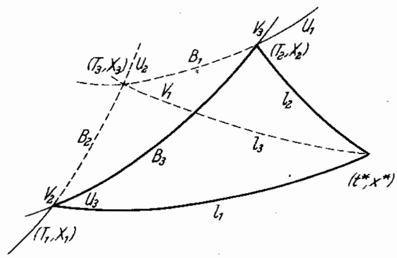


Fig. 6

Let the two shock waves l_1 and l_2 start from the points (T_1, X_1) and (T_2, X_2), where they are generated by the jump of a parabola with acceleration $2 \cdot B_2$ (respectively $2 \cdot B_3$) and velocity $V_2\ (V_3)$ to a parabola with acceleration $2 \cdot B_3$ $(2 \cdot B_1)$ and velocity U_3 (U_1). We deal with the general case only; other cases are to be treated similarly.

For convenience, we introduce the intersection point (T_3, X_3) of the first and third parabola and choose cyclic denotations with indexes to be reduced modulo 3. The shock wave starting from the intersection point (t^*, x^*) is identical with the shock wave l_3, which would be generated in (T_3, X_3).

Obviously, the following geometric relations are valid:

$$X_i+U_{i+2}(T_{i+1}-T_i)+B_{i+2}(T_{i+1}-T_i)^2=X_{i+1}$$
$$U_i+2B_i(T_{i+2}-T_{i+1})=V_i. \qquad (i=1,2,3)$$

Following (3.3.2), the equations of the three shock waves considered, are

$$t=P_i\tau+Q_i\xi_i+T_i$$
$$x=R_i\tau+S_i\xi_i+X_i$$
$$\tau=c$$
$$\xi_i=\sqrt{1+4a_2(B_{i+1}+B_{i+2})c+16a_2^2B_{i+1}B_{i+2}c^2}-1$$

with

$$P_i=a_1+2a_2\beta_i, \qquad Q_i=\frac{\alpha_i}{2(B_{i+1}-B_{i+2})},$$
$$R_i=-a_0+a_2\beta_i^2+a_2\frac{\alpha_i^2B_{i+1}B_{i+2}}{(B_{i+1}-B_{i+2})^2}, \quad S_i=\frac{\alpha_i\beta_i}{2(B_{i+1}-B_{i+2})}.$$

It is necessary to evaluate the point (t^*,x^*) common to the shock waves l_1, l_2, l_3. We determine the parameter $c=c^*$ belonging to this point, using the system of four linear equations for τ,ξ_1,ξ_2,ξ_3.

$$(P_1-P_2)\tau+Q_1\xi_1-Q_2\xi_2+T_1-T_2=0$$
$$(P_2-P_3)\tau+Q_2\xi_2-Q_3\xi_3+T_2-T_3=0$$
$$(R_1-R_2)\tau+S_1\xi_1-S_2\xi_2+X_1-X_2=0$$
$$(R_2-R_3)\tau+S_2\xi_2-S_3\xi_3+X_2-X_3=0.$$

The values of τ,ξ_1,ξ_2,ξ_3 can easily be determined. With extensive calculations, it can be shown that the four quantities thus obtained satisfy the relations $(\xi_i+1)^2=1+4a_2(B_{i+1}+B_{i+2})\tau+16a_2^2B_{i+1}B_{i+2}\tau^2$, ascerting that the points (τ,ξ_i) are situated on the conics in the (τ,ξ_i)-plane discussed in 3.4. Therefore we have $c^*=\tau$.

4. Simulation

We have used a one-dimensional flow with a density depending on the velocity as a basis for a continuous model of traffic flow. In the preceding paragraphs the theory has been developed to the extent that all the important effects can be evaluated exactly or at least approximately. We have to discuss methods for applying this theory to concrete situations. We will formulate a standard problem of simulation of traffic flow regulated by traffic lights.

A single lane shall be given, on which the traffic flow is regulated by traffic lights. The positions of the traffic lights and their switching plan shall be known as well as the maximum velocity admitted between two consecutive traffic lights. Additionally the standard accelerations shall be prescribed, which a vehicle has to observe in the case that it is the first vehicle behind one of the traffic lights switching to „green". If the distribution of traffic density is known at time $t=0$ the further course of the traffic situation in the continuous model is determined and stream-line pictures are to be drawn.

At the Mathematical Institute of the Technical University of Munich a program has been developed, which allows a stream-line picture for a given problem of this standard type to be drawn.

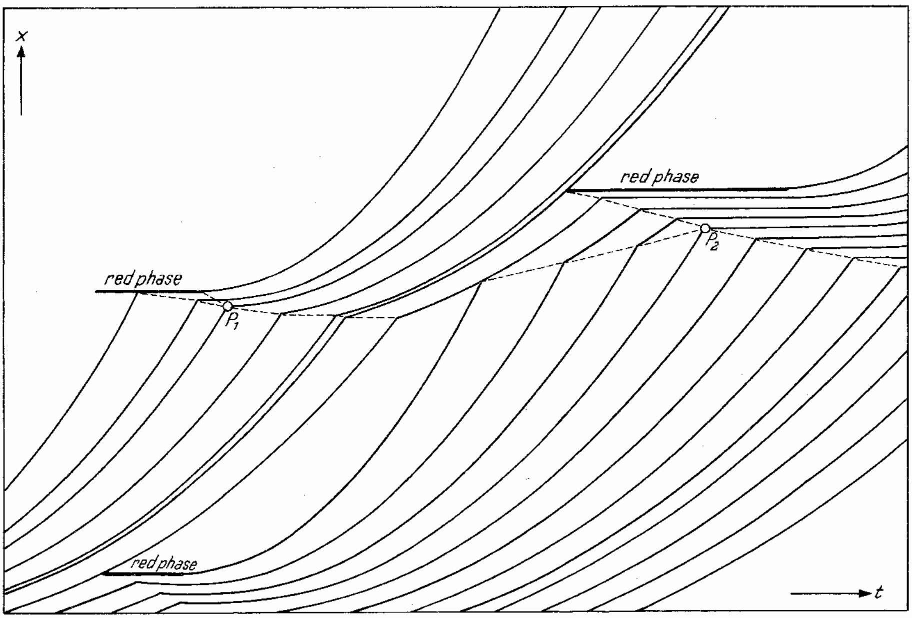
At present the program presumes as an initial distribution of density that a column of a certain length is standing at the first traffic light and that the rest of the street is empty. For the time being, only functions $a(v)$ of the type $a(v)=a_0+a_1 v+a_2 v^2$ are admitted.

The program needs the following items for input

— a_0, a_1, a_2 and some quantities of capacity

— the distances of the traffic lights from the origin, the maximum velocities and the standard accelerations

— for every traffic light a list of times, which are to be interpreted alternately as the beginnings of red and green phases.



P_1, P_2 intersection points of shock waves

Fig. 7

At the beginning, the path of the first vehicle is determined from standard accelerations, maximum velocities and occasional delay by traffic lights. We obtain a sequence of segments of parabolas. Hereafter the number of vehicles remaining in "immediate" connection with the leading vehicle is evaluated. A vehicle is said to be in immediate connection with the leading vehicle if, firstly, it passes all traffic lights within the same green phase and, secondly, it is influenced by every segment of the leading curve. (Two shock waves intersecting for $c=c^*$, the influence of the domain of determinacy enclosed between them vanishes for $c \geq c^*$.)

For the vehicles being in immediate connection with the leading vehicle, the trajectories according to (2.3.1) with respect to the boundary lines of (3.3.2) are drawn by means of a plotter.

After this, two cases are to be distinguished.

A) The remaining flow has been cut off by the red phase of a traffic light. Then the first of the remaining vehicles is treated as mentioned before.

B) An intersection of shock waves has occured. Then the vanishing of a segment of the leading curve has to be managed.

Thus, a new leading curve being obtained, the trajectories of the vehicles which are in immediate connection to the leading vehicle can be drawn, etc.

In Fig. 7 a stream-line picture is given to show formation and intersection of shock waves.

Two periodically switching traffic lights are given in Fig. 8 and Fig. 9. In one picture the switching plan is optimally chosen in relation to the standard acceleration and the function $a(v)$. The lengths of the red phases and the green phases being the same, the shift of the phases causes a stoppage and diminishes the traffic capacity of the lane.

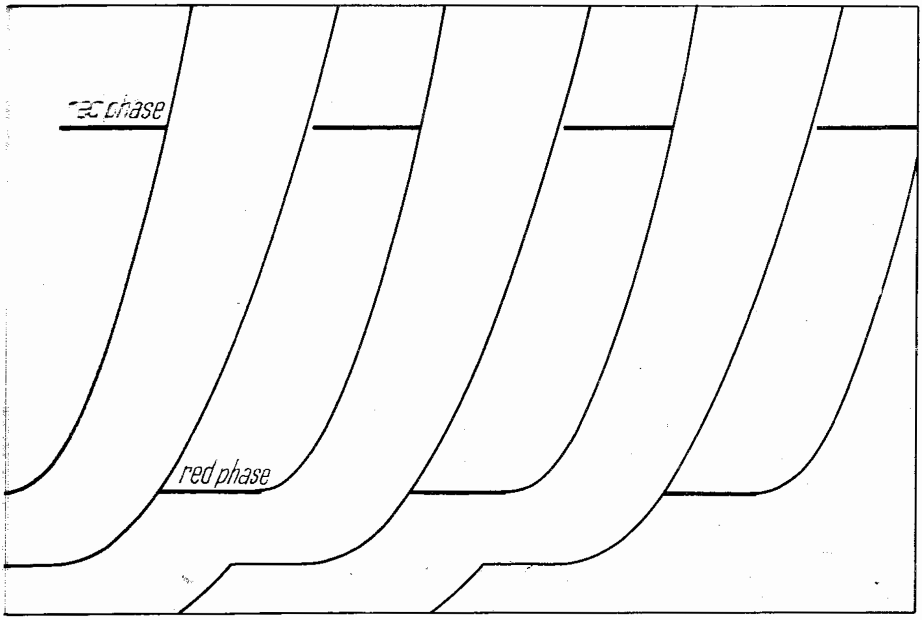


Fig. 8

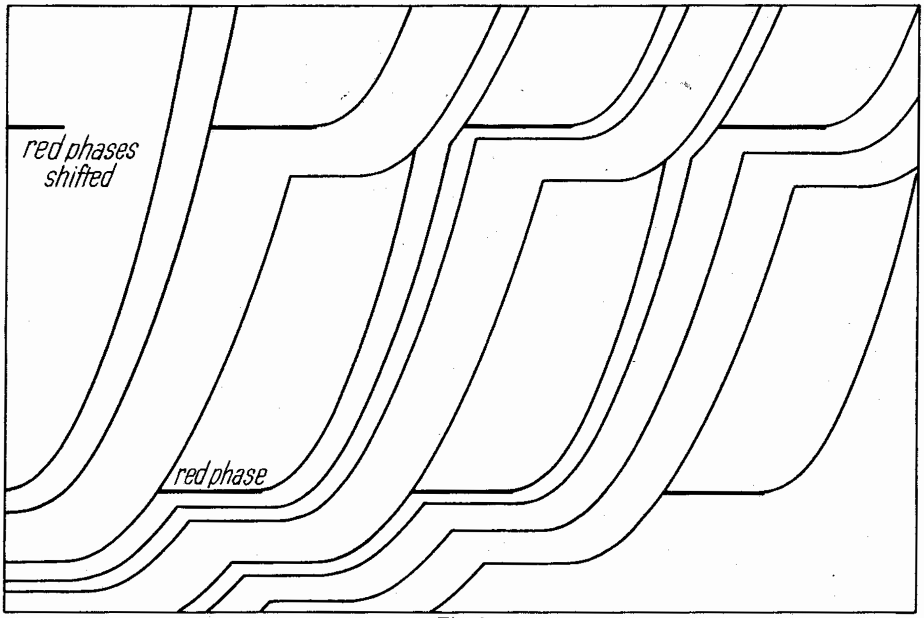


Fig. 9

In Fig. 10 a traffic flow over a lane with four traffic lights is shown. The switching plan of these four traffic lights being unchanged, in Fig. 11 an additional traffic light optimally adjusted to the flow situation is introduced as the last but one. It can be seen that the traffic capacity is increased.

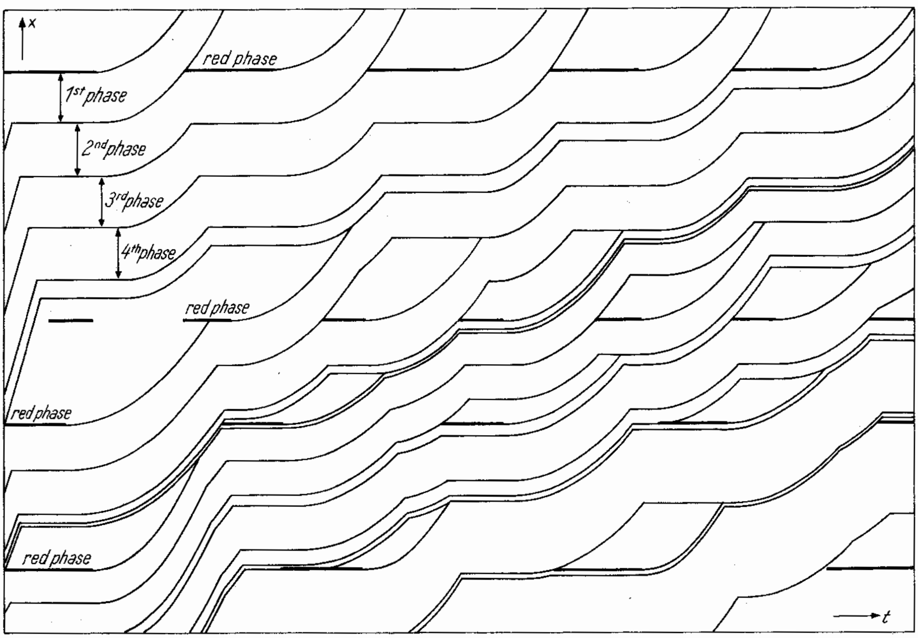


Fig. 10

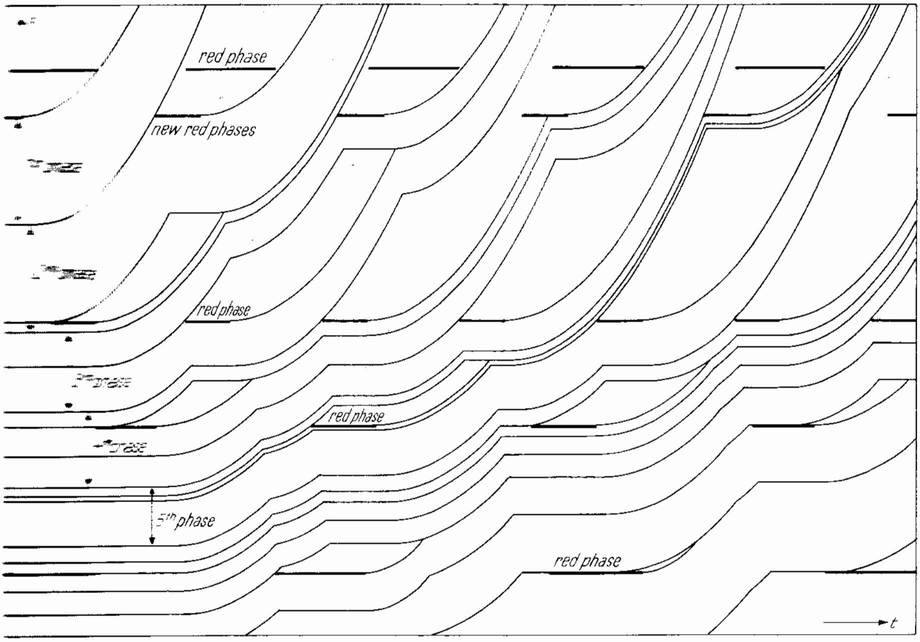


Fig. 11

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