A Relational View on Stochastics

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Abstract

As a contribution to the Festschrift on the occasion of the 60th birthday of José N. Oliveira, the author opens a glimpse of his relational scrapbook. Earlier work on relational measures, preference aggregation [4, 7, 5], and social choice [6] motivates to approach also questions of stochastics relationally. This requires some nontrivial work in function spaces. Several examples are computed with different measuring spaces as well as different types of measures, using the relational language TituRel.

Keywords: relational mathematics, relation algebra, stochastics, measure, plausibility, possibility, belief, existential image, power transpose

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1. Introduction

This is an attempt to discover a relational basis in probability, transferring some definitions to a relational level. Probability theory is dominated by the fact that from the very beginning questions of summability and convergence are involved and restrictions to σ-algebras have to be taken into consideration. Notationally, the typing distinction between a set, its powerset, and the set in which to measure is often not made sufficiently clear; in particular, no use seems to be made of the by now well-established algebraic interrelation via existential images or power transposes. Lattice properties of the measuring space are rarely made use of.

A remark seems necessary: The unit interval $[0, 1]$ is a complete lattice with regard to the traditional ordering "$\leq$" accompanied by "$\vee \approx \texttt{lub} \approx$ least upper bound" and "$\wedge \approx \texttt{glb} \approx$ greatest lower bound". It is, however, nearly exclusively used in stochastics with "$+, *$", which seems highly natural, but doesn't lead to a lattice. The absorption law, namely, is violated, viz.

$$a \wedge (a \vee b) = a, \quad \text{but} \quad 0.3 * (0.3 + 0.5) = 0.24 < 0.3.$$

Stochastics is in particular concerned with the aggregation of probabilities, measuring on the linear scale $[0, 1]$ already mentioned. The broadly known work on fuzzy sets by Lotfi Zadeh can also partly be seen from this point of view. Here information is no longer required to be crisp and may be fuzzy. Michio Sugeno [13] defined a method of integration in the fuzzy environment.

Others have investigated how trust and belief may even be dealt with when no longer $p(\overline{X}) = 1 - p(X)$ is required and $p(X) + p(\overline{X}) < 1$ may occur. Arthur Dempster and Glenn Shafer introduced a theory of reasoning about belief and trust that are not just expressed as probabilities, [2, 12]. This brought forward an alternative theory that has widely been applied in engineering and artificial intelligence. It is concentrated around the belief and the plausibility measure to be explained later in this text.

Yet another question is whether one has to restrict to a linear scale. Diverging criteria may obviously better be expressed valuating in a lattice. It requires the ability of aggregating also in this case, which the famous French mathematician Gustave Choquet has already provided with his theory of capacities and integration, [1].

It is impossible to include Choquet's theory of capacities and integration, Sugeno integration and Dempster-Shafer theory in the present text. Rather, we must refer the potential reader to the original sources mentioned.

Following such attempts, stochastics has been approached from the relational side. It inherently allows non-linearity and is best suited for work with trust and belief as already shown via relational integration in [4, 7, 5]. It is, however, difficult to convince people that such an approach may be useful; so a study is needed to compute examples and to visualize how the results using the different techniques fit to what we expect. Such work has here been executed using the TituRel system.

The latter system consists of the relational language TituRel and a supporting tool kit. The language offers constructs to work with relational operations, e.g. with union, intersection (meet), composition, converse, etc. It has, in addition, a sophisticated type control concerning the category aspect and some conceptually new constructs using dependent types. These include projection operations around direct products, injections for direct sums, membership relations in connection with direct powers. Even more intricate are those to handle natural projections wrt. to an equivalence, natural injections of a subset into its superset, and the otherwise completely unknown 'target permutation' according to a bijective mapping.

Slightly adapting the system, introducing relations with non-finite target but finite codomain, and not least introducing operators to cope with binary mappings, it became possible to work out the examples. We have chosen them as lattices, related them to the interval $[0, 1]$, and made visible their effects. We have also been eager to cover all the different measures known as belief, plausibility, or possibility measure in these simple examples. Any question of complexity has as of yet been ignored.

2. Relation-algebraic preliminaries

The texts [8, 9, 5, 10, 11] are given as general references, since the prerequisites of relation algebra cannot be presented over and over again. We write $R : V \longrightarrow W$ if R is a relation with source V and target W, often conceived as a subset of $V \times W$. If V and W are ordered finite sets and of size m and n, respectively, we may consider R as a Boolean matrix with m rows and n columns.

We assume the reader to be familiar with the basic operations on relations, namely R^{T} (*converse*), $\overline{R}$ (*negation*), $R \cup S$ (*union*), $R \cap S$ (*intersection*), and $R ; S$ (*composition*), the predicate $R \subseteq S$ (*containment*), and the special relations $\perp\!\!\!\perp$ (*empty relations*), $\top\!\!\!\top$ (*universal relations*), and $\mathbb{I}$ (*identities*).

Then a *heterogeneous relation algebra* is a structure that

— is a category with respect to composition " $;$ " and identities $\mathbb{I}$,

— has morphism sets that are complete atomic Boolean lattices with operations resp. predicates $\cup, \cap, ^{-}, \perp\!\!\!\perp, \top\!\!\!\top, \subseteq$,

— obeys rules for transposition $^{\mathsf{T}}$ in connection with the latter two that may be stated in either one of the following two ways:

$$\begin{array}{lll} \text{Dedekind} & R;S \cap Q \subseteq (R \cap Q;S^{\mathsf{T}});(S \cap R^{\mathsf{T}};Q) & \text{or} \\ \text{Schröder} & R;S \subseteq Q \iff R^{\mathsf{T}};\overline{Q} \subseteq \overline{S} \iff \overline{Q};S^{\mathsf{T}} \subseteq \overline{R}. & \end{array}$$

When a multiplication is given, one looks for right *residuals* $A;B \subseteq C \iff A \subseteq \overline{\overline{C};B^{\mathsf{T}}} =: C/B$. A left variant is $A;B \subseteq C \iff B \subseteq \overline{A^{\mathsf{T}};\overline{C}} =: A\backslash C$. Intersecting such residuals in $\mathtt{syq}(R,S) := \overline{R^{\mathsf{T}};\overline{S}} \cap \overline{\overline{R}^{\mathsf{T}};S}$, the *symmetric quotient* $\mathtt{syq}(R,S) : W \longrightarrow Z$ of two relations $R : V \longrightarrow W$ and $S : V \longrightarrow Z$ is introduced. Symmetric quotients serve the purpose of spotting where columns coincide. Given an ordering relation E and some subset or vector U, one may, thus, determine the least upper bound column-wise as

$$\mathtt{lub}_E(U) := \mathtt{syq}(E^{\mathsf{T}}, \overline{\overline{E}^{\mathsf{T}};U});$$

see, e.g., [8, 9, 5]: Look where the majorant or upper bound set $\mathtt{ubd}_E(U) := \overline{\overline{E}^{\mathsf{T}};U}$ of U equals the majorant set of some point. Given a relation X, it is also possible to form $\mathtt{lubR}_E(X) := \left[\mathtt{lub}_E(X^{\mathsf{T}})\right]^{\mathsf{T}}$, i.e., to obtain the least upper bound row-wise.

We will use *membership-relations* $\varepsilon : V \longrightarrow \mathbf{2}^V$ between a set V and its powerset $\mathbf{2}^V$. They can be universally characterized via the symmetric quotient. Given a membership relation, the powerset ordering is easily described as the residual $\Omega = \overline{\varepsilon^{\mathsf{T}};\overline{\varepsilon}}$.

If a mapping $f : X \longrightarrow Y$ between sets ordered by E_X, E_Y is given, we will call it *(lattice-)continuous* when $f^{\mathsf{T}};\mathtt{lub}_{E_X}(U) = \mathtt{lub}_{E_Y}(f^{\mathsf{T}};U)$ for all $U \subseteq X$. This combines being “additive” with sending least element to least element. The ordering E represents a *complete* lattice if $\mathtt{lub}_E(X)$ is surjective for every X. It is also possible to relationally characterize a *Boolean* lattice.

When two sets X, Y are considered with direct product $X \times Y$ and projections π, ρ, as well as relations from and to other products, one defines the Kronecker product and the strict fork (when sources coincide) as

$$(R \otimes S) := \pi{:}R{:}\pi'^{\mathsf{T}} \cap \rho{:}S{:}\rho'^{\mathsf{T}} \qquad (R \otimes S) := R{:}\pi'^{\mathsf{T}} \cap S{:}\rho'^{\mathsf{T}}.$$

3. Power operations

There is a highly useful interrelationship from relations to their counterparts between the corresponding powersets. It offers the possibility to work algebraically at situations where this has so far not been the classical approach; some has already been collected in [5].

3.1 Definition. Let any relation $R : X \longrightarrow Y$ be given together with membership relations $\varepsilon : X \longrightarrow \mathbf{2}^X$ and $\varepsilon' : Y \longrightarrow \mathbf{2}^Y$. Then the corresponding **existential image mapping** is defined as $\vartheta_R := \mathtt{syq}(R^{\mathsf{T}}{:}\varepsilon, \varepsilon')$. One may also study the **inverse image mapping** defined as $\vartheta_{R^{\mathsf{T}}} = \mathtt{syq}(R{:}\varepsilon', \varepsilon)$. The construct $\Lambda_R := \mathtt{syq}(R^{\mathsf{T}}, \varepsilon') : X \longrightarrow \mathbf{2}^Y$ is called the **power transpose** of R. □

We recall an important fact concerning the existential image, see [5], namely

$$\varepsilon^{\mathsf{T}}{:}R = \vartheta_R{:}\varepsilon'^{\mathsf{T}},$$

which allows us to work algebraically. Correspondingly, an application to R^{T} instead of R reads

$$\varepsilon'^{\mathsf{T}}{:}R^{\mathsf{T}} = \vartheta_{R^{\mathsf{T}}}{:}\varepsilon^{\mathsf{T}}, \qquad \text{or else} \qquad R{:}\varepsilon' = \varepsilon{:}\vartheta^{\mathsf{T}}_{R^{\mathsf{T}}}.$$

The existential image and the inverse image also satisfy formulae with respect to the powerset orderings:

3.2 Proposition.

i) $\Omega'{:}\vartheta_{f^{\mathsf{T}}} \subseteq \vartheta_{f^{\mathsf{T}}}{:}\Omega$ if f is a mapping,

ii) $\Omega{:}\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}} = \vartheta_f{:}\Omega'$ if f is a mapping. □

A proof may be found in Prop. 5.2 of [10]. Another rule, Prop. 5.3 of [10], combines the inverse image with the singleton injection (see an example in Fig. 4.2).

3.3 Proposition. i) Any relation $R : X \longrightarrow Y$ with the singleton injections $\sigma : X \longrightarrow \mathbf{2}^X$, $\sigma' : Y \longrightarrow \mathbf{2}^Y$ satisfies

$$\sigma{:}\vartheta^{\mathsf{T}}_{R^{\mathsf{T}}}{:}\sigma'^{\mathsf{T}} \subseteq R \quad \text{and} \quad \varepsilon{:}\vartheta^{\mathsf{T}}_{R^{\mathsf{T}}}{:}\sigma'^{\mathsf{T}} = R.$$

ii) When f is a mapping, this sharpens to $\sigma;\vartheta_f = f;\sigma'$. □

We will need the meet and join forming with respect to E also as relations.

3.4 Proposition. Whenever an ordering $E : X \longrightarrow X$ is given, the corresponding binary meet and join are the functions $\mathfrak{M}, \mathfrak{J} : X \times X \longrightarrow X$ (possibly partial) obtained as

$$\mathfrak{M} := \mathtt{syq}((E \otimes E), E), \qquad \mathfrak{J} := \mathtt{syq}((E^{\mathsf{T}} \otimes E^{\mathsf{T}}), E^{\mathsf{T}}). \qquad \square$$

Should E happen to be a powerset ordering $\overline{\varepsilon^{\mathsf{T}};\overline{\varepsilon}}$, more can be said according to Prop. 9.1 of [10]:

3.5 Proposition. Assuming a membership relation $\varepsilon : X \longrightarrow \mathbf{2}^X$, binary meet and join with respect to the powerset ordering may be expressed as

$$\mathfrak{M} = \mathtt{syq}((\varepsilon \otimes \varepsilon), \varepsilon) \qquad \mathfrak{J} = \mathtt{syq}((\overline{\varepsilon} \otimes \overline{\varepsilon}), \overline{\varepsilon}). \qquad \square$$

Least and greatest elements of a lattice will be denoted, respectively, by

$$0_E = \mathtt{glb}_E(\mathbb{T}) = \mathtt{syq}(E, \overline{\overline{E};\mathbb{T}}), \quad 1_E = \mathtt{lub}_E(\mathbb{T}) = \mathtt{syq}(E^{\mathsf{T}}, \overline{\overline{E}^{\mathsf{T}};\mathbb{T}}).$$

When using TituRel in examples, writing down such terms produces the respective map as a relation that may be processed further.

4. Basics of stochastics interpreted relationally

Typically, one starts from a set $\mathcal{D}$, the membership relation $\varepsilon : \mathcal{D} \longrightarrow \mathbf{2}^{\mathcal{D}}$ with its powerset, the powerset ordering $\Omega : \mathbf{2}^{\mathcal{D}} \longrightarrow \mathbf{2}^{\mathcal{D}}$, and the singleton injection $\sigma := \mathtt{syq}(\mathbb{I}, \varepsilon) : \mathcal{D} \longrightarrow \mathbf{2}^{\mathcal{D}}$. The standard definition assumes then that valuation takes place in $[0,1]$. It calls a mapping $m : \mathcal{D} \longrightarrow [0,1]$ a **probability vector**, provided

$$\forall d \in \mathcal{D} : m(d) \geq 0 \quad \text{and} \quad \sum_{d\in\mathcal{D}} m(d) = 1.$$

The corresponding **probability measure** is in this case characterized by the mapping $\mu : \mathbf{2}^{\mathcal{D}} \longrightarrow [0,1]$, defining for every $D \subseteq \mathcal{D}$

$$\mu(D) := \sum_{d\in D} m(d).$$

$$\mu = \varepsilon^{\mathsf{T}} \cdot m = \begin{array}{r} \\ \{\} \\ \{\text{red}\} \\ \{\text{gre}\} \\ \{\text{red,gre}\} \\ \{\text{blu}\} \\ \{\text{red,blu}\} \\ \{\text{gre,blu}\} \\ \{\text{red,gre,blu}\} \\ \{\text{ora}\} \\ \{\text{red,ora}\} \\ \{\text{gre,ora}\} \\ \{\text{red,gre,ora}\} \\ \{\text{blu,ora}\} \\ \{\text{red,blu,ora}\} \\ \{\text{gre,blu,ora}\} \\ \{\text{red,gre,blu,ora}\} \end{array} \begin{array}{c} \text{red}\ \text{gre}\ \text{blu}\ \text{ora} \\ \begin{pmatrix} 0&0&0&0 \\ 1&0&0&0 \\ 0&1&0&0 \\ 1&1&0&0 \\ 0&0&1&0 \\ 1&0&1&0 \\ 0&1&1&0 \\ 1&1&1&0 \\ 0&0&0&1 \\ 1&0&0&1 \\ 0&1&0&1 \\ 1&1&0&1 \\ 0&0&1&1 \\ 1&0&1&1 \\ 0&1&1&1 \\ 1&1&1&1 \end{pmatrix} \end{array} \cdot \begin{array}{r} \text{red} \\ \text{gre} \\ \text{blu} \\ \text{ora} \end{array} \begin{pmatrix} 0.5 \\ 0.0 \\ 0.4 \\ 0.1 \end{pmatrix} = \begin{array}{r} \{\} \\ \{\text{red}\} \\ \{\text{gre}\} \\ \{\text{red,gre}\} \\ \{\text{blu}\} \\ \{\text{red,blu}\} \\ \{\text{gre,blu}\} \\ \{\text{red,gre,blu}\} \\ \{\text{ora}\} \\ \{\text{red,ora}\} \\ \{\text{gre,ora}\} \\ \{\text{red,gre,ora}\} \\ \{\text{blu,ora}\} \\ \{\text{red,blu,ora}\} \\ \{\text{gre,blu,ora}\} \\ \{\text{red,gre,blu,ora}\} \end{array} \begin{pmatrix} 0.0 \\ 0.5 \\ 0.0 \\ 0.5 \\ 0.4 \\ 0.9 \\ 0.4 \\ 0.9 \\ 0.1 \\ 0.6 \\ 0.1 \\ 0.6 \\ 0.5 \\ 1.0 \\ 0.5 \\ 1.0 \end{pmatrix}$$

Fig. 4.1 Probability vector m leading to a probability measure μ

How to proceed from the probability vector $m : \text{Colors} \longrightarrow [0,1]$ to the probability measure μ is shown with an already slightly relational touch in the example of Fig. 4.1, using the membership relation ε.

There has been used an obvious but not completely formal mixed operation between a (Boolean-valued) relation ε and a real-valued vector m.

Already this step could, thus, somehow be seen from a relational perspective. For the relation ε with `True/False`- or **1** / **0** -entries, one had to proceed to its obvious real $1,0$-matrix equivalent and multiply the matrix with a vector in the usual way. Then one has a tiny result as follows, where σ denotes the singleton injection into the powerset. Observe the multiplication with "$\cdot$" and the application of the rule $a ; (b \cdot c) = (a ; b) \cdot c$ invented just for this purpose.

$$\sigma = \begin{array}{r} \\ \text{red} \\ \text{gre} \\ \text{blu} \\ \text{ora} \end{array} \begin{array}{c} \{\}\ \{\text{red}\}\ \{\text{gre}\}\ \{\text{red,gre}\}\ \{\text{blu}\}\ \{\text{red,blu}\}\ \{\text{gre,blu}\}\ \{\text{red,gre,blu}\}\ \{\text{ora}\}\ \{\text{red,ora}\}\ \{\text{gre,ora}\}\ \{\text{red,gre,ora}\}\ \{\text{blu,ora}\}\ \{\text{red,blu,ora}\}\ \{\text{gre,blu,ora}\}\ \{\text{red,gre,blu,ora}\} \\ \begin{pmatrix} 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0 \end{pmatrix} \end{array}$$

Fig. 4.2 Singleton injection $\sigma \subseteq \varepsilon$

4.1 Proposition. Given any relation $m : \mathcal{D} \longrightarrow [0,1]$, the relation

$$\mu := \varepsilon^{\mathsf{T}} \cdot m : \mathbf{2}^{\mathcal{D}} \longrightarrow [0,1]$$

will satisfy $m = \sigma ; \mu$.

Proof: $\sigma ; \mu = \sigma ; \varepsilon^{\mathsf{T}} \cdot m = \mathbb{I} \cdot m = m$ $\square$

We will pursue this basic idea even further: staying as long as possible on the relational side and only finally switching to real arithmetics. Using the system TituRel to execute several examples, this means not least to consider $*$ and $+$ as function parameters in an enveloping program. The system — written in Haskell — had to be adapted only slightly so as to be able to apply lattice intersection $\mathfrak{M}$, e.g., as a binary operator much in the same way as $*$.

The preceding has long been successfully re-modeled with the idea of trust and belief in the Dempster-Shafer theory [2, 12], i.e., admitting also non-additive valuations that allow $p(X) + p(\overline{X}) < 1$ to occur. It has later been brought to relational form with valuation in some lattice $\mathcal{L}$ by [4, 7].

4.2 Definition. Given the powerset ordering $\Omega : \mathbf{2}^{\mathcal{D}} \longrightarrow \mathbf{2}^{\mathcal{D}}$ and some ordering of a complete lattice $E : \mathcal{L} \longrightarrow \mathcal{L}$, the mapping $\mu : \mathbf{2}^{\mathcal{D}} \longrightarrow \mathcal{L}$ will be called a **relational measure**, provided

i) $\Omega ; \mu \subseteq \mu ; E$,

ii) $\mu^{\mathsf{T}} ; 0_{\Omega} = 0_E$,

iii) $\mu^{\mathsf{T}} ; 1_{\Omega} = 1_E$.

When μ is also (lattice-)continuous, it is called a **Bayesian measure**. $\square$

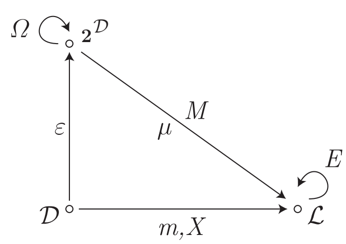

Fig. 4.3 Elementary situation of relational stochastics

This definition refrains from mentioning real numbers, but postulates monotony (not necessarily continuity) together with some gauging, "least to least and greatest to greatest element". A special example for $\mathcal{L}$ is $[0,1] \subseteq \mathbb{R}$ with its traditional ordering $E \approx \leq$ and $0, 1$, however with meet $\mathfrak{M}$ and join $\mathfrak{J}$ instead of $*, +$. With this definition, we did not deviate too far from the traditional case.

When using the mapping m of Fig. 4.1 as a relation in Fig. 4.4, however, $\varepsilon^{\mathsf{T}};m$ is by no means univalent. What we can do is to remind us of the ordering E on $\mathcal{L}$ and form the least upper bound row-wise. This definitely deviates from addition, but will produce a mapping; see Fig. 4.4.

Of course, should m be so defined as to attach always the least element 0_E of E, the least upper bound `lubR` would do the same and, thus, violate the gauging requirement $\mu^{\mathsf{T}};1_\Omega = 1_E$.

This is the reason to employ the vacuous belief μ_0 in order to gauge the measure. Vacuous belief[1] doesn't believe anything, i.e. assigns the least element — up to the greatest to the greatest, just due to gauging. Observe that μ_1 is always Bayesian while μ_0 is not.

$$
\begin{array}{r}
\\ \{\} \\ \{\text{red}\} \\ \{\text{gre}\} \\ \{\text{red,gre}\} \\ \{\text{blu}\} \\ \{\text{red,blu}\} \\ \{\text{gre,blu}\} \\ \{\text{r,g,b}\} \\ \{\text{ora}\} \\ \{\text{red,ora}\} \\ \{\text{gre,ora}\} \\ \{\text{r,g,o}\} \\ \{\text{blu,ora}\} \\ \{\text{r,b,o}\} \\ \{\text{g,b,o}\} \\ \text{all}
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\begin{pmatrix}
0&0&0&0\\ 1&0&0&0\\ 0&1&0&0\\ 1&1&0&0\\ 0&0&1&0\\ 1&0&1&0\\ 0&1&1&0\\ 1&1&1&0\\ 0&0&0&1\\ 1&0&0&1\\ 0&1&0&1\\ 1&1&0&1\\ 0&0&1&1\\ 1&0&1&1\\ 0&1&1&1\\ 1&1&1&1
\end{pmatrix}
\end{array}
= \varepsilon^{\mathsf{T}};m
$$

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\begin{array}{c|ccccc}
 & 0.0 & 0.1 & 0.4 & 0.5 & \text{rest} \\ \hline
\text{red} & 0&0&0&1&0 \\
\text{gre} & 1&0&0&0&0 \\
\text{blu} & 0&0&1&0&0 \\
\text{ora} & 0&1&0&0&0
\end{array}
\qquad
=
\begin{array}{c}
\begin{array}{ccccc} 0.0 & 0.1 & 0.4 & 0.5 & \text{rest of } [0,1] \end{array} \\
\begin{pmatrix}
0&0&0&0&0\\ 0&0&0&1&0\\ 1&0&0&0&0\\ 1&0&0&1&0\\ 0&0&1&0&0\\ 0&0&1&1&0\\ 1&0&1&0&0\\ 1&0&1&1&0\\ 0&1&0&0&0\\ 0&1&0&1&0\\ 1&1&0&0&0\\ 1&1&0&1&0\\ 0&1&1&0&0\\ 0&1&1&1&0\\ 1&1&1&0&0\\ 1&1&1&1&0
\end{pmatrix}
\end{array}
\qquad
\texttt{lubR}_E(\varepsilon^{\mathsf{T}};m) \longrightarrow
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\begin{array}{ccccc} 0.0 & 0.1 & 0.4 & 0.5 & \text{rest of } [0,1] \end{array} \\
\begin{pmatrix}
1&0&0&0&0\\ 0&0&0&1&0\\ 1&0&0&0&0\\ 0&0&0&1&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&1&0&0&0\\ 0&0&0&1&0\\ 0&1&0&0&0\\ 0&0&0&1&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&1&0&0\\ 0&0&0&1&0
\end{pmatrix}
\end{array}
$$

Fig. 4.4 Probability vector m leading over $\varepsilon^{\mathsf{T}};m$ to a probability measure μ

This observation illustrates the following result that may be found in [4, 7, 5].

4.3 Proposition. Whenever a mapping $m : \mathcal{D} \longrightarrow \mathcal{L}$ is given together with the vacuous belief μ_0, the following is a Bayesian measure:

$$\mu := \texttt{lubR}_E(\mu_0 \cup \varepsilon^{\mathsf{T}};m) \qquad \square$$

[1] A schematic example is shown in Fig. 4.5 together with a light-minded belief μ_1; it refers to Fig. 5.2.

We do not recall the proof, since more can be proved; see Prop. 4.4. Let us compare the traditional real-valued case: That the empty set is mapped to 0 is clear when specializing

$$\mu(U) := \sum_{d\in U} m(d) \qquad \text{to} \qquad \mu(\emptyset) := \sum_{d\in\emptyset} m(d) = 0.$$

That the full set $\mathcal{D}$ is mapped to $\sum_{d\in\mathcal{D}} m(d) = 1$, is also a separate requirement.

We improve this, recalling Prop. 8.1 from [4, 7]. The forthcoming Fig. 6.9 shows an example.

4.4 Proposition. Given any relation $m : \mathcal{D} \to \mathcal{L}$, the construct

$$\mu_m := \mathtt{lubR}_E(\mu_0 \cup \varepsilon^{\mathsf{T}};m),$$

is a measure, the so-called **possibility measure** $\mu_m : \mathbf{2}^{\mathcal{D}} \longrightarrow \mathcal{L}$ for m.

Should m be a mapping[2], μ will be Bayesian and satisfy $m = \sigma;\mu_m$.

$$\begin{array}{cc}
 & \begin{array}{cccccccccccc} \text{(low,lazy)} & \text{(low,fair)} & \text{(low,good)} & \text{(low,bulldozer)} & \text{(medium,lazy)} & \text{(medium,fair)} & \text{(medium,good)} & \text{(medium,bulldozer)} & \text{(high,lazy)} & \text{(high,fair)} & \text{(high,good)} & \text{(high,bulldozer)} \end{array} \\
\begin{array}{r} \{\} \\ \{a\} \\ \{b\} \\ \{a,b\} \\ \{c\} \\ \{a,c\} \\ \{b,c\} \\ \{a,b,c\} \end{array} &
\begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0\\
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1&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix}
\end{array}
\qquad
\begin{array}{c}
\begin{array}{cccccccccccc} \text{(low,lazy)} & \text{(low,fair)} & \text{(low,good)} & \text{(low,bulldozer)} & \text{(medium,lazy)} & \text{(medium,fair)} & \text{(medium,good)} & \text{(medium,bulldozer)} & \text{(high,lazy)} & \text{(high,fair)} & \text{(high,good)} & \text{(high,bulldozer)} \end{array} \\
\begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&1\\
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0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix}
\end{array}$$

Fig. 4.5 Vacuous as opposed to light-minded belief: μ_0, μ_1

Proof: The proof is based on the easy to prove fact that the antitone

$$\sigma(a) := \overline{\varepsilon^{\mathsf{T}};a;\overline{E}} \qquad \text{and} \qquad \pi(b) := \overline{\varepsilon;b;\overline{E}^{\mathsf{T}}}$$

constitute a Galois connection, i.e. satisfy

$$a \subseteq \pi(b) \quad \Longleftrightarrow \quad b \subseteq \sigma(a).$$

We restrict to demonstrating how the greatest lower bound of $\sigma(m)$ leads to the least upper bound of $\varepsilon^{\mathsf{T}};m$.

$$\mathtt{glbR}_E(\sigma(m)) = \mathtt{glbR}_E(\overline{\varepsilon^{\mathsf{T}};m;\overline{E}}) = \left[\mathtt{glb}_E(\overline{\overline{E}^{\mathsf{T}};m^{\mathsf{T}};\varepsilon})\right]^{\mathsf{T}}$$

[2]as it occurs with the probability vector in Prop. 4.3

$= \left[\texttt{lbd}_E(\overline{\overline{E}^\mathsf{T}; m^\mathsf{T}; \varepsilon}) \cap \texttt{ubd}_E(\texttt{lbd}_E(\overline{\overline{E}^\mathsf{T}; m^\mathsf{T}; \varepsilon}))\right]^\mathsf{T}$ definition of $\texttt{glb}$
$= \left[\texttt{lbd}_E(\texttt{ubd}_E(m^\mathsf{T}; \varepsilon)) \cap \texttt{ubd}_E(\texttt{lbd}_E(\texttt{ubd}_E(m^\mathsf{T}; \varepsilon)))\right]^\mathsf{T}$
$= \left[\texttt{lbd}_E(\texttt{ubd}_E(m^\mathsf{T}; \varepsilon)) \cap \texttt{ubd}_E(m^\mathsf{T}; \varepsilon)\right]^\mathsf{T}$ since $\texttt{ubd}(\texttt{lbd}(\texttt{ubd}(\texttt{x}))) = \texttt{ubd}(\texttt{x})$
$= \left[\texttt{lub}_E(m^\mathsf{T}; \varepsilon)\right]^\mathsf{T} = \texttt{lubR}_E(\varepsilon^\mathsf{T}; m)$ definition of $\texttt{lub}$ □

Therefore, the classical probability measure resulting from a probability vector m more or less subsumes to the idea of a relational measure of Def. 4.2.

In a similar way as in Prop. 4.4, we may derive relational measures out of an arbitrary relation $M : \mathbf{2}^\mathcal{D} \longrightarrow \mathcal{L}$, the so-called body of evidence. The relation M is restricted only by the requirement that $M^\mathsf{T}; 0_\Omega \subseteq 0_E$, i.e., when M should relate the empty subset somehow, then necessarily to the least element. There exist two relational measures closely resembling M, namely the

belief measure $\mu_{\text{belief}}(M) := \texttt{lubR}_E(\mu_0 \cup \Omega^\mathsf{T}; M)$ and the

plausibility measure $\mu_{\text{plausi}}(M) := \texttt{lubR}_E(\mu_0 \cup \Omega^\mathsf{T}; (\Omega \cap \overline{\Omega}; \mathbb{T}); M)$.

Proofs can be found in [4, 7, 5]. In general, the belief measure assigns values below those of the plausibility measure, i.e.,

$\mu_{\text{belief}}(M) \subseteq \mu_{\text{plausi}}(M); E^\mathsf{T}$.

Visualizations will follow in the examples to come. The belief measure accumulates with least upper bound what all subsets together deliver as measure. The plausibility measure instead accumulates what might flow into a set when all intersecting sets contribute with their total values. (In addition, gauging takes place.)

But also the stochastic concepts of product probability and independence seem to find their analogues on the relational side.

4.5 Proposition. Given two probability vectors $m_i : \mathcal{D}_i \longrightarrow \mathcal{L}$ with common measuring lattice $\mathcal{L}$, one obtains as the **product probability vector**

$(m_1 \otimes m_2); \mathfrak{M}_E : \mathcal{D}_1 \times \mathcal{D}_2 \longrightarrow \mathcal{L}$.

In adddition, a **product probability measure** for given probability measures $\mu_i : \mathbf{2}^{\mathcal{D}_i} \longrightarrow \mathcal{L}$ may be defined as

$(\mu_1 \otimes \mu_2); \mathfrak{M}_E : \mathbf{2}^{\mathcal{D}_1} \times \mathbf{2}^{\mathcal{D}_2} \longrightarrow \mathcal{L}$.

Proof: The first claim is trivial. For the second, we only prove monotony:

$(\Omega_1 \otimes \Omega_2); (\mu_1 \otimes \mu_2); \mathfrak{M}_E \subseteq (\Omega_1; \mu_1 \otimes \Omega_2; \mu_2); \mathfrak{M}_E$
$\subseteq (\mu_1; E \otimes \mu_2; E); \mathfrak{M}_E \subseteq (\mu_1 \otimes \mu_2); (E \otimes E); \mathfrak{M}_E = (\mu_1 \otimes \mu_2); E$ □

Probability vector and measure above are *not* related as in Prop. 4.3, Prop. 4.4, and Fig. 4.3, which one might have expected in the first place. When observing strict typing discipline, the much bigger $\mu_{(m_1 \bigotimes m_2);\mathfrak{M}_E} : \mathbf{2}^{\mathcal{D}_1 \times \mathcal{D}_2} \longrightarrow \mathcal{L}$ should here have been taken. This is, however, traditionally neglected in stochastics, which leads to the observation that only rectangular sets will get a measure. For such pairs of subsets one has a well-known concept:

4.6 Definition. Assume a probability measure, defined as $\mu : \mathbf{2}^{\mathcal{D}} \longrightarrow \mathcal{L}$. We say that a pair of subsets $U_1, U_2 \subseteq \mathcal{D}$, or correspondingly the pair of points $u_i := \mathtt{syq}(\varepsilon, U_i)$ in the powerset, is **independent** when it belongs to the set

$$\big[(\mu \bigotimes \mu);\mathfrak{M}_E \cap \mathfrak{M}_{\mathcal{D}};\mu\big];\mathbb{T}. \qquad \square$$

Here, the mappings $(\mu \bigotimes \mu);\mathfrak{M}_E$ and $\mathfrak{M}_{\mathcal{D}};\mu$ are considered as to where they agree. The first term assigns the respective two measure values and proceeds to their greatest lower bound. The second in turn delivers the measure of the intersection of the two.

Relational integration with such measures is not easy to capture. We therefore recall the formula, referring to [4, 7, 5], and then present examples. These will be given with different measuring spaces and all the types of measures already mentioned.

4.7 Definition. Given a relational measure μ and some relation X indicating the values attributed to certain criteria, the **relational integral** is defined as

$$(R)\!\int X \circ \mu := \mathtt{lubR}_E(\mathbb{T};\mathtt{glbR}_E[X \cup \mathtt{syq}(X;E^{\mathsf{T}};X^{\mathsf{T}},\varepsilon);\mu]). \qquad \square$$

This follows in a pointfree formulation the general scheme: summation ($\mathtt{lub}$) over products ($\mathtt{glb}$). The relational integral, written as a term for TituRel, may in particular be used to aggregate preferences.

5. Examples of valuation spaces

Many of the assessments in everyday life are fully based on a linear scale. This may have its origin in the omnipresent work with money, for which numbers seem adequate. When, however, deciding which company shall build a complex technical building, autobahn, airport terminal, subway station, etc., it doesn't seem wise to just orient oneself at price alone. Quality of material, architectural beauty, costs to follow periodically, and many others are criteria at least as important.

A generally accepted way of working with a multitude of criteria seems not yet to exist; when one is about to respect these, it will most frequently be via endless debates — as opposed to clear rational reasoning. We have decided for four different examples of valuation spaces $\mathcal{L}$. The first three are complete lattices while the last is not a lattice. The third lattice fails to be modular.

To handle these four as long as possible completely in parallel, we always look at meet $\mathfrak{M}$ and join $\mathfrak{J}$, derived from the given ordering E. They shall be handled in the same way as at other occasions $*$ and $+$. Every single example will, thus, consist of giving an $\mathcal{L}$ via the respective ordering E, as well as meet and join $\mathfrak{M}, \mathfrak{J}$ computed thereof in TituRel applying the terms shown earlier.

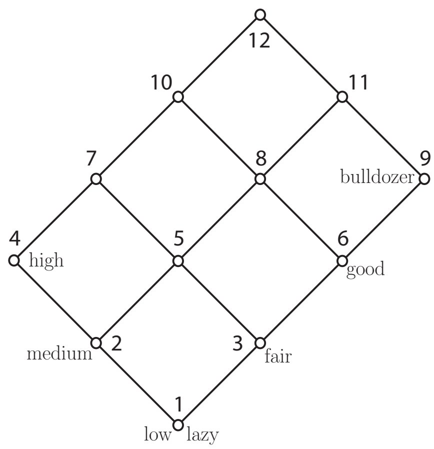
5.1. First example of a valuation lattice

The lattice of Fig. 5.2 (for space reasons it is shown on the next page) might allow a fine grained — not just linear — assessment of a person, with respect to its intellectual capability and the intensity of work; see [4, 7, 5].

$$
\begin{array}{r}
\\ 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccc} 1&2&3&4&5&6&7&8&9&10&11&12 \end{array}\\
\begin{pmatrix}
1&1&1&1&1&1&1&1&1&1&1&1\\
1&2&1&2&2&1&2&2&1&2&2&2\\
1&1&3&1&3&3&3&3&3&3&3&3\\
1&2&1&4&2&1&4&2&1&4&2&4\\
1&2&3&2&5&3&5&5&3&5&5&5\\
1&1&3&1&3&6&3&6&6&6&6&6\\
1&2&3&4&5&3&7&5&3&7&5&7\\
1&2&3&2&5&6&5&8&6&8&8&8\\
1&1&3&1&3&6&3&6&9&6&9&9\\
1&2&3&4&5&6&7&8&6&10&8&10\\
1&2&3&2&5&6&5&8&9&8&11&11\\
1&2&3&4&5&6&7&8&9&10&11&12
\end{pmatrix}
\end{array}
\quad
\begin{array}{c}
\begin{array}{cccccccccccc} 1&2&3&4&5&6&7&8&9&10&11&12 \end{array}\\
\begin{pmatrix}
1&2&3&4&5&6&7&8&9&10&11&12\\
2&2&5&4&5&8&7&8&11&10&11&12\\
3&5&3&7&5&6&7&8&9&10&11&12\\
4&4&7&4&7&10&7&10&12&10&12&12\\
5&5&5&7&5&8&7&8&11&10&11&12\\
6&8&6&10&8&6&10&8&9&10&11&12\\
7&7&7&7&7&10&7&10&12&10&12&12\\
8&8&8&10&8&8&10&8&11&10&11&12\\
9&11&9&12&11&9&12&11&9&12&11&12\\
10&10&10&10&10&10&10&10&12&10&12&12\\
11&11&11&12&11&11&12&11&11&12&11&12\\
12&12&12&12&12&12&12&12&12&12&12&12
\end{pmatrix}
\end{array}
$$

Fig. 5.1 Mappings $\mathfrak{M}, \mathfrak{J}$ of Fig. 5.2 represented in tabular form

The qualifying pairs of words are for space reasons also abbreviated by numbers. Only the first rows of the 144×12-relation $\mathfrak{M}$ can be shown. In addition, its tabular form is presented as Fig. 5.1.

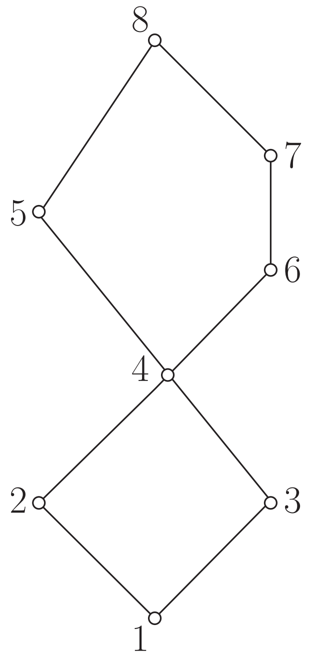


$$
\begin{array}{c|c}
 & \begin{array}{cccccccccccc} 1&2&3&4&5&6&7&8&9&10&11&12 \end{array} \\
\begin{array}{c} 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12 \end{array} &
\begin{pmatrix}
1&1&1&1&1&1&1&1&1&1&1&1\\
0&1&0&1&1&0&1&1&0&1&1&1\\
0&0&1&0&1&1&1&1&1&1&1&1\\
0&0&0&1&0&0&1&0&0&1&0&1\\
0&0&0&0&1&0&1&1&0&1&1&1\\
0&0&0&0&0&1&0&1&1&1&1&1\\
0&0&0&0&0&0&1&0&0&1&0&1\\
0&0&0&0&0&0&0&1&0&1&1&1\\
0&0&0&0&0&0&0&0&1&0&1&1\\
0&0&0&0&0&0&0&0&0&1&0&1\\
0&0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix}
\end{array}
$$

$$
\begin{array}{c|c}
 & \begin{array}{cccccccccccc} 1&2&3&4&5&6&7&8&9&10&11&12 \end{array} \\
\begin{array}{c} (1,1)\\(1,2)\\(1,3)\\(1,4)\\(1,5)\\(1,6)\\(1,7)\\(1,8)\\(1,9)\\(1,10)\\(1,11)\\(1,12)\\(2,1)\\(2,2)\\(2,3)\\(2,4)\\(2,5)\\(2,6)\\(2,7)\\(2,8)\\(2,9)\\(2,10)\\(2,11)\\(2,12)\\(3,1)\\(3,2)\\(3,3)\\(3,4)\\(3,5)\\(3,6)\\(3,7)\\(3,8)\\(3,9)\\(3,10)\\(3,11)\\(3,12)\\(4,1)\\(4,2)\\(4,3)\\(4,4)\\(4,5)\\(4,6)\\(4,7) \end{array} &
\begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0
\end{pmatrix}
\end{array}
$$

Fig. 5.2 Complete lattice E together with first rows of meet forming relation $\mathfrak{M}$

The triple E, $\mathfrak{M}$, $\mathfrak{J}$ resembles the computation in the measuring lattice. Given a valuation lattice in this way, we will later provide a set $\mathcal{D}$ of — in this example — persons together with some measure $\mu : \mathbf{2}^{\mathcal{D}} \longrightarrow \mathcal{L}$. We have already discussed different methods how to obtain such a measure μ.



$$\begin{array}{cc} & \begin{matrix}1&2&3&4&5&6&7&8\end{matrix} \\ \begin{matrix}1\\2\\3\\4\\5\\6\\7\\8\end{matrix} & \begin{pmatrix}1&1&1&1&1&1&1&1\\0&1&0&1&1&1&1&1\\0&0&1&1&1&1&1&1\\0&0&0&1&1&1&1&1\\0&0&0&0&1&0&0&1\\0&0&0&0&0&1&1&1\\0&0&0&0&0&0&1&1\\0&0&0&0&0&0&0&1\end{pmatrix}\end{array}$$

$$\begin{array}{ccc} & \begin{matrix}1&2&3&4&5&6&7&8\end{matrix} & \begin{matrix}1&2&3&4&5&6&7&8\end{matrix} \\ \begin{matrix}(1,1)\\(1,2)\\(1,3)\\(1,4)\\(1,5)\\(1,6)\\(1,7)\\(1,8)\\(2,1)\\(2,2)\\(2,3)\\(2,4)\\(2,5)\\(2,6)\\(2,7)\\(2,8)\\(3,1)\\(3,2)\\(3,3)\\(3,4)\\(3,5)\\(3,6)\\(3,7)\\(3,8)\\(4,1)\\(4,2)\\(4,3)\\(4,4)\\(4,5)\\(4,6)\\(4,7)\\(4,8)\\(5,1)\\(5,2)\\(5,3)\end{matrix} & \begin{pmatrix}
1&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
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1&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
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1&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
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0&0&0&1&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&1&0&0&0&0\\
1&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0
\end{pmatrix} & \begin{pmatrix}
1&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&1\\
0&1&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&1\\
0&0&1&0&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&1&0&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&1\\
0&0&0&1&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&1\\
0&0&0&0&1&0&0&0\\
0&0&0&0&1&0&0&0\\
0&0&0&0&1&0&0&0
\end{pmatrix}\end{array}$$

Fig. 5.3 Complete lattice E with first rows of meet and join forming relations $\mathfrak{M}$, $\mathfrak{J}$

This non-modular lattice could express different attitudes around a medium case.

$$
\begin{array}{cc}
 & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} &
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 2 & 1 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 3 & 3 & 3 & 3 & 3 & 3 \\
1 & 2 & 3 & 4 & 4 & 4 & 4 & 4 \\
1 & 2 & 3 & 4 & 5 & 4 & 4 & 5 \\
1 & 2 & 3 & 4 & 4 & 6 & 6 & 6 \\
1 & 2 & 3 & 4 & 4 & 6 & 7 & 7 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8
\end{pmatrix}
\end{array}
\qquad
\begin{array}{cc}
 & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} &
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
2 & 2 & 4 & 4 & 5 & 6 & 7 & 8 \\
3 & 4 & 3 & 4 & 5 & 6 & 7 & 8 \\
4 & 4 & 4 & 4 & 5 & 6 & 7 & 8 \\
5 & 5 & 5 & 5 & 5 & 8 & 8 & 8 \\
6 & 6 & 6 & 6 & 8 & 6 & 7 & 8 \\
7 & 7 & 7 & 7 & 8 & 7 & 7 & 8 \\
8 & 8 & 8 & 8 & 8 & 8 & 8 & 8
\end{pmatrix}
\end{array}
$$

Fig. 5.4 Mappings $\mathfrak{M}$, $\mathfrak{J}$ of Fig. 5.3 represented in tabular form

The relations $\mathfrak{M}$, $\mathfrak{J}$ are produced by the term mentioned in Def. 3.4. The transition from E to the tables of Figs. 5.3 and 5.4 occurs completely on the representation side as provided with TituRel.

5.3. Third example of a valuation lattice: the unit interval

Presented continuously, the unit interval with operations `glb` and `lub` looks as in Fig. 5.5.

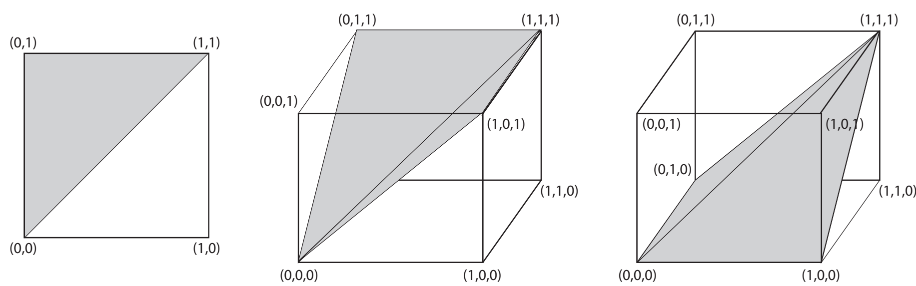

Fig. 5.5 Corresponding to Fig. 5.6: relation $x \leq y$, and mappings `lub` (x, y), `glb` (x, y) on the unit interval $[0, 1]$

The following shows the unit interval, however, granulated down to steps of $\frac{1}{8}$. Every such interval is represented by its middle value which is then abbreviated again due to space limitations. Thus, it appears still as a finite one.

$$\begin{array}{rc}
 & \begin{array}{cccccccc}0.06&0.19&0.31&0.44&0.56&0.69&0.81&0.94\end{array} \\
\begin{array}{r}0.06\\0.19\\0.31\\0.44\\0.56\\0.69\\0.81\\0.94\end{array} & \left(\begin{array}{cccccccc}
1&1&1&1&1&1&1&1\\
0&1&1&1&1&1&1&1\\
0&0&1&1&1&1&1&1\\
0&0&0&1&1&1&1&1\\
0&0&0&0&1&1&1&1\\
0&0&0&0&0&1&1&1\\
0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&1
\end{array}\right)
\end{array}$$

$$\begin{array}{rcc}
 & \begin{array}{cccccccc}0.06&0.19&0.31&0.44&0.56&0.69&0.81&0.94\end{array} & \begin{array}{cccccccc}0.06&0.19&0.31&0.44&0.56&0.69&0.81&0.94\end{array} \\
\begin{array}{r}
(0.06,0.06)\\(0.06,0.19)\\(0.06,0.31)\\(0.06,0.44)\\(0.06,0.56)\\(0.06,0.69)\\(0.06,0.81)\\(0.06,0.94)\\
(0.19,0.06)\\(0.19,0.19)\\(0.19,0.31)\\(0.19,0.44)\\(0.19,0.56)\\(0.19,0.69)\\(0.19,0.81)\\(0.19,0.94)\\
(0.31,0.06)\\(0.31,0.19)\\(0.31,0.31)\\(0.31,0.44)\\(0.31,0.56)\\(0.31,0.69)\\(0.31,0.81)\\(0.31,0.94)\\
(0.44,0.06)\\(0.44,0.19)\\(0.44,0.31)\\(0.44,0.44)\\(0.44,0.56)\\(0.44,0.69)\\(0.44,0.81)\\(0.44,0.94)\\
(0.56,0.06)\\(0.56,0.19)\\(0.56,0.31)\\(0.56,0.44)\\(0.56,0.56)\\(0.56,0.69)
\end{array} & \left(\begin{array}{cccccccc}
1&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
1&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&1&0&0&0&0\\
1&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&0&1&0&0&0\\
0&0&0&0&1&0&0&0
\end{array}\right) & \left(\begin{array}{cccccccc}
1&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&1\\
0&1&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&1\\
0&0&1&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&1\\
0&0&0&1&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&1\\
0&0&0&0&1&0&0&0\\
0&0&0&0&1&0&0&0\\
0&0&0&0&1&0&0&0\\
0&0&0&0&1&0&0&0\\
0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0
\end{array}\right)
\end{array}$$

Fig. 5.6 Complete lattice E together with first rows of meet and join forming $\mathfrak{M}$, $\mathfrak{J}$

5.4. Standard valuation space: Unit interval with $*$ and $+$

Opposed to the former three, we have the non-lattice on the unit interval formed by the classical $x \leq y$ with mappings $x \mapsto x{*}y$ and $x \mapsto x{+}y$. The corresponding Fig. 5.7 does not look totally different from Fig. 5.5.

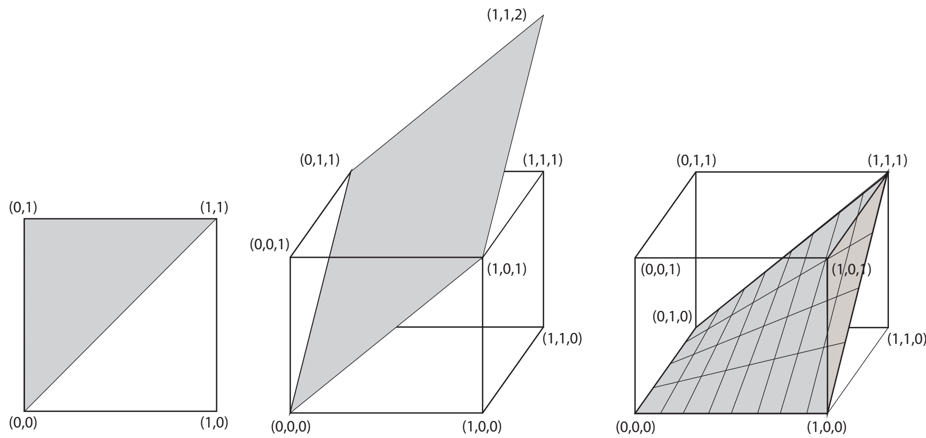

Fig. 5.7 Relation $x \leq y$ with mappings $x \mapsto x + y$ and $x \mapsto x * y$ on the unit interval $[0, 1]$

Multiplication is a tiny part cut out of a hyperboloid, while addition produces a plane. Of course, addition is applied only to the extent that, while measuring, it never exceeds 1 when reasonable problems are dealt with. But this is not axiomatized, and just a well-established traditional way to work.

In total, one or the other mathematically minded person may find the three former measures in some sense more natural.

6. Examples of relational integration

Using these valuation or measuring spaces and differently generated measures, we show several examples of relational integration, stressing the aspects they enjoy in common. We visualize, thus, not least that all the measures lead to reasonable outcomes, in particular when aggregating votings.

6.1. First example of relational integration

Here, we assume the lattice indicated in Figs. 5.1 and 5.2. Then a measuring into this lattice is defined; this time as a belief measure μ via an "arbitrary" body of evidence $M : \mathbf{2}^{\mathcal{D}} \longrightarrow \mathcal{L}$; see Fig. 6.1.

$$
\begin{array}{rcc}
 & \begin{array}{cccccccccccc}\text{(low,lazy)} & \text{(medium,lazy)} & \text{(low,fair)} & \text{(high,lazy)} & \text{(medium,fair)} & \text{(low,good)} & \text{(high,fair)} & \text{(medium,good)} & \text{(low,bulldozer)} & \text{(high,good)} & \text{(medium,bulldozer)} & \text{(high,bulldozer)}\end{array} & \begin{array}{cccccccccccc}\text{(low,lazy)} & \text{(medium,lazy)} & \text{(low,fair)} & \text{(high,lazy)} & \text{(medium,fair)} & \text{(low,good)} & \text{(high,fair)} & \text{(medium,good)} & \text{(low,bulldozer)} & \text{(high,good)} & \text{(medium,bulldozer)} & \text{(high,bulldozer)}\end{array} \\
\begin{array}{r}
\{\} \\ \{\text{red}\} \\ \{\text{gre}\} \\ \{\text{red,gre}\} \\ \{\text{blu}\} \\ \{\text{red,blu}\} \\ \{\text{gre,blu}\} \\ \{\text{red,gre,blu}\} \\ \{\text{ora}\} \\ \{\text{red,ora}\} \\ \{\text{gre,ora}\} \\ \{\text{red,gre,ora}\} \\ \{\text{blu,ora}\} \\ \{\text{red,blu,ora}\} \\ \{\text{gre,blu,ora}\} \\ \{\text{r,g,b,o}\}
\end{array}
&
\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&1&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&1&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&1&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&1&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&1&0&0&0&0&0&0
\end{pmatrix}
&
\begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix}
\end{array}
$$

Fig. 6.1 Body of evidence M with belief measure μ

Over some X, the typing of which has thus already been agreed upon, shall now be integrated. One may take the color as representing the apparel of the person to be assessed. We ask for the overall score of this team.

$$
X = \begin{array}{rc}
 & \begin{array}{cccccccccccc}\text{(low,lazy)} & \text{(medium,lazy)} & \text{(low,fair)} & \text{(high,lazy)} & \text{(medium,fair)} & \text{(low,good)} & \text{(high,fair)} & \text{(medium,good)} & \text{(low,bulldozer)} & \text{(high,good)} & \text{(medium,bulldozer)} & \text{(high,bulldozer)}\end{array} \\
\begin{array}{r}\text{red}\\ \text{gre}\\ \text{blu}\\ \text{ora}\end{array} &
\begin{pmatrix}
0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0
\end{pmatrix}
\end{array}
\qquad
X;E^{\mathsf{T}};X^{\mathsf{T}} = \begin{array}{c}
\begin{array}{cccc}\text{red} & \text{gre} & \text{blu} & \text{ora}\end{array} \\
\begin{pmatrix}
1&0&1&1\\
0&1&1&0\\
0&0&1&0\\
1&0&1&1
\end{pmatrix}
\end{array}
$$

Fig. 6.2 First step of integrating X

The right relation of Fig. 6.2 indicates — as columns — those rows of X that carry values equal or above. By comparison with the membership relation ε — see column inscriptions — one obtains relation s of Fig. 6.3.

$$
s := \mathtt{syq}(X{;}E^{\mathsf{T}}{;}X^{\mathsf{T}}, \varepsilon) =
\begin{array}{r}
\\ \text{red} \\ \text{gre} \\ \text{blu} \\ \text{ora}
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccccccc}
\rotatebox{90}{\{\}} & \rotatebox{90}{\{red\}} & \rotatebox{90}{\{gre\}} & \rotatebox{90}{\{red,gre\}} & \rotatebox{90}{\{blu\}} & \rotatebox{90}{\{red,blu\}} & \rotatebox{90}{\{gre,blu\}} & \rotatebox{90}{\{red,gre,blu\}} & \rotatebox{90}{\{ora\}} & \rotatebox{90}{\{red,ora\}} & \rotatebox{90}{\{gre,ora\}} & \rotatebox{90}{\{red,gre,ora\}} & \rotatebox{90}{\{blu,ora\}} & \rotatebox{90}{\{red,blu,ora\}} & \rotatebox{90}{\{gre,blu,ora\}} & \rotatebox{90}{\{red,gre,blu,ora\}}
\end{array} \\
\begin{pmatrix}
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0
\end{pmatrix}
\end{array}
$$

Fig. 6.3 Second step of integrating over X with measure μ

These sets are now measured by μ and processed further according to Fig. 6.4, thus resembling the definition of the relational integral in Def. 4.7.

$$
\begin{array}{rc}
 & \begin{array}{cccccccccccc}
\rotatebox{90}{(low,lazy)} & \rotatebox{90}{(medium,lazy)} & \rotatebox{90}{(low,fair)} & \rotatebox{90}{(high,lazy)} & \rotatebox{90}{(medium,fair)} & \rotatebox{90}{(low,good)} & \rotatebox{90}{(high,fair)} & \rotatebox{90}{(medium,good)} & \rotatebox{90}{(low,bulldozer)} & \rotatebox{90}{(high,good)} & \rotatebox{90}{(medium,bulldozer)} & \rotatebox{90}{(high,bulldozer)}
\end{array} \\
s{;}\mu = \begin{array}{r}\text{red}\\ \text{gre}\\ \text{blu}\\ \text{ora}\end{array} &
\begin{pmatrix}
0&0&0&0&0&0&0&0&0&1&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&1&0&0
\end{pmatrix} \\
X \cup s{;}\mu = \begin{array}{r}\text{red}\\ \text{gre}\\ \text{blu}\\ \text{ora}\end{array} &
\begin{pmatrix}
0&0&0&0&0&0&0&1&0&1&0&0\\
1&0&0&0&0&0&1&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&1&0&1&0&0
\end{pmatrix} \\
\mathtt{glbR}_E(X \cup s{;}\mu) = \begin{array}{r}\text{red}\\ \text{gre}\\ \text{blu}\\ \text{ora}\end{array} &
\begin{pmatrix}
0&0&0&0&0&0&0&1&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0
\end{pmatrix} \\
\mathbb{T}{;}\mathtt{glbR}_E(X \cup s{;}\mu) = \{\} &
\begin{pmatrix}1&1&0&0&0&0&0&1&0&0&0&0\end{pmatrix} \\
(R)\!\int X \circ \mu = \{\} &
\begin{pmatrix}0&0&0&0&0&0&0&1&0&0&0&0\end{pmatrix}
\end{array}
$$

Fig. 6.4 Relational integration executed in full detail

When one looks at Fig. 5.2 and identifies the values according to X as points in the lattice, the relational integral may be considered a reasonable aggregation. The remarkable fact is that no real numbers have been used and no numerical pseudo-precision has been generated.

In nearly the same way, we start an investigation with the valuation lattice $[0,1]$ according to Fig. 5.5, i.e., using `lub`, `glb`. Our just finitely many elements of the interval are here visualized first on a continuous scale. To avoid any pseudo-precision introduced by real numbers, we have chosen only rational ones that are presented as fractionals. For measuring into this lattice, the possibility measure μ derived from the fairly "arbitrary" $m : \mathcal{D} \longrightarrow [0,1]$ is chosen as in Fig. 6.5.

$m =$

US	4/7
French	3/4
German	1/17
British	2/11
Spanish	17/19

$\mu =$

{}	0/1
{US}	4/7
{French}	3/4
{US,French}	3/4
{German}	1/17
{US,German}	4/7
{French,German}	3/4
{US,French,German}	3/4
{British}	2/11
{US,British}	4/7
{French,British}	3/4
{US,French,British}	3/4
{German,British}	2/11
{US,German,British}	4/7
{French,German,British}	3/4
{US,French,German,British}	3/4
{Spanish}	17/19
{US,Spanish}	17/19
⋮	⋮
{US,German,British,Spanish}	17/19
{French,German,British,Spanish}	17/19
{US,French,German,British,Spanish}	1/1

Fig. 6.5 Possibility measure μ for m

Again, this completes the general setting and fixes the typing. Over some candidate votings X, shall now be integrated.

$$X = \begin{array}{r|l} & 0 \quad\quad\quad 1 \\ \text{US} & 3/7 \\ \text{French} & 0/1 \\ \text{German} & 2/5 \\ \text{British} & 4/7 \\ \text{Spanish} & 1/3 \end{array} \qquad X ; E^\mathsf{T} ; X^\mathsf{T} = \begin{array}{ccccc} \text{US} & \text{French} & \text{German} & \text{British} & \text{Spanish} \end{array}$$

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Fig. 6.6 Candidate voting X and relation to all rows with values equal or below

Remarkable is that $X ; E^\mathsf{T} ; X^\mathsf{T}$ is a relation in the normal sense. Again, $s := \mathtt{syq}(X ; E^\mathsf{T} ; X^\mathsf{T}, \varepsilon)$ identifies the sets according to the columns.

$s := \mathtt{syq}(X ; E^\mathsf{T} ; X^\mathsf{T}, \varepsilon) =$

	…	{British}	{US,British}	{French,British}	{US,French,British}	{German,British}	{US,German,British}	{French,German,British}	{US,French,German,British}	{Spanish}	{US,Spanish}	{French,Spanish}	{US,French,Spanish}	{German,Spanish}	{US,German,Spanish}	{French,German,Spanish}	{US,French,German,Spanish}	{British,Spanish}	{US,British,Spanish}	{French,British,Spanish}	{US,French,British,Spanish}	{German,British,Spanish}	{US,German,British,Spanish}	{French,German,British,Spanish}	{US,French,German,British,Spanish}
US	…	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
French	…	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
German	…	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
British	…	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Spanish	…	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0

Fig. 6.7 Identified subsets according to $X ; E^\mathsf{T} ; X^\mathsf{T}$

These are measured by μ and the results united with X, so as to proceed further according to the definition Def. 4.7 of relational integration:

A remark may be in order concerning the row inscription {} in the two final lines of Figs. 6.4 and 6.8. The row vector $\mathbb{T}$ multiplied from the left has simply a 1-element source, and the most universal denotation for it is "powerset of the empty set".

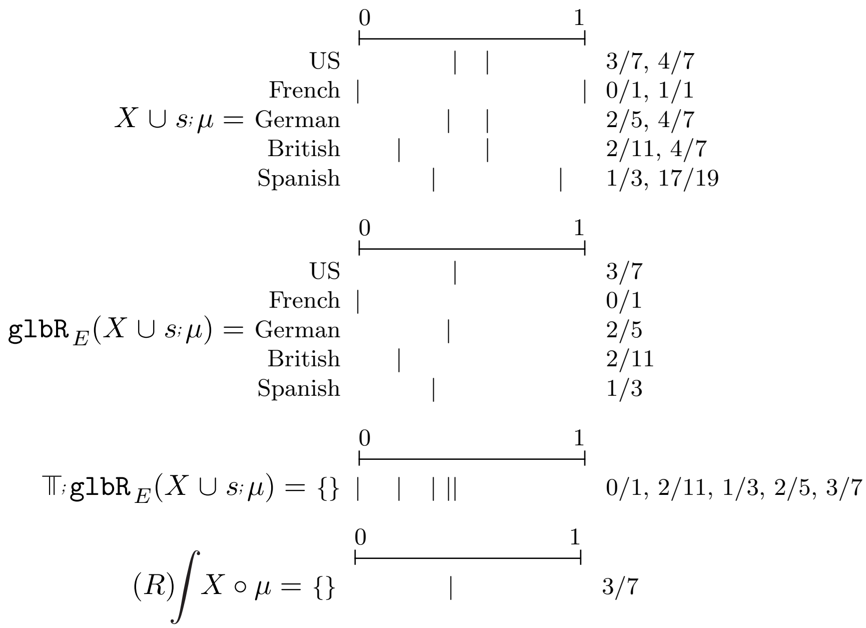

Fig. 6.8 Final steps of relational integration with possibility measure in $[0, 1]$

We arrive at a value inside the codomain of our target valuation space that rests on a clear formal basis and looks quite acceptable. In particular, we do not end somewhere between the values of the codomain simply following numerics.

6.3. Third example of relational integration

This time again a possibility measure shall be derived from the "arbitrary" relation $m_0 : \mathcal{D} \longrightarrow \mathcal{L}$ according to Prop. 4.4.

The relation m_0 is far from a mapping, so that μ is not a Bayesian measure, which may be seen from sets {AA,BB} and {CC,DD} together with their union.

$$
\begin{array}{c|cccccccccccc}
 & \text{(low,lazy)} & \text{(medium,lazy)} & \text{(low,fair)} & \text{(high,lazy)} & \text{(medium,fair)} & \text{(low,good)} & \text{(high,fair)} & \text{(medium,good)} & \text{(low,bulldozer)} & \text{(high,good)} & \text{(medium,bulldozer)} & \text{(high,bulldozer)} \\
\hline
\text{AA} & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\text{BB} & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\text{CC} & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\text{DD} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

$$m_0 = \quad \uparrow$$

$$
\mu = \begin{array}{c|cccccccccccc}
 & \text{(low,lazy)} & \text{(medium,lazy)} & \text{(low,fair)} & \text{(high,lazy)} & \text{(medium,fair)} & \text{(low,good)} & \text{(high,fair)} & \text{(medium,good)} & \text{(low,bulldozer)} & \text{(high,good)} & \text{(medium,bulldozer)} & \text{(high,bulldozer)} \\
\hline
\{\} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{\text{AA}\} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{\text{BB}\} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\{\text{AA,BB}\} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\{\text{CC}\} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{\text{AA,CC}\} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{\text{BB,CC}\} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\{\text{AA,BB,CC}\} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\{\text{DD}\} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{\text{AA,DD}\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\{\text{BB,DD}\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\{\text{AA,BB,DD}\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\{\text{CC,DD}\} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{\text{AA,CC,DD}\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\{\text{BB,CC,DD}\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\{\text{AA,BB,CC,DD}\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

Fig. 6.9 Valuating in a lattice with possibility measure μ according to Prop. 4.4

This completes the general setting. Again, we decide for some X. The typing of X has already been determined. The relational integral shall now be formed according to Def. 4.7.

$$
\begin{array}{c|cccccccccccc}
 & \text{(low,lazy)} & \text{(medium,lazy)} & \text{(low,fair)} & \text{(high,lazy)} & \text{(medium,fair)} & \text{(low,good)} & \text{(high,fair)} & \text{(medium,good)} & \text{(low,bulldozer)} & \text{(high,good)} & \text{(medium,bulldozer)} & \text{(high,bulldozer)} \\
\hline
\text{AA} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\text{BB} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\text{CC} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\text{DD} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{array}
\qquad
\begin{array}{c|cccc}
 & \text{AA} & \text{BB} & \text{CC} & \text{DD} \\
\hline
\text{AA} & 1 & 0 & 1 & 1 \\
\text{BB} & 0 & 1 & 1 & 0 \\
\text{CC} & 0 & 0 & 1 & 0 \\
\text{DD} & 1 & 0 & 1 & 1
\end{array}
$$

Fig. 6.10 Relation X with intermediate result $X;E^{\mathsf{T}};X^{\mathsf{T}}$ for relational integration

The following $X;E^{\mathsf{T}};X^{\mathsf{T}}$ indicates as columns the rows of X that carry values equal or above. By comparison with the membership relation ε, one obtains the s of Fig. 6.11.

$$
s := \mathtt{syq}(X{;}E^{\mathsf{T}}{;}X^{\mathsf{T}}, \varepsilon) = \begin{array}{c} \\ \mathrm{AA} \\ \mathrm{BB} \\ \mathrm{CC} \\ \mathrm{DD} \end{array} \begin{array}{c} \begin{array}{cccccccccccccccc} \rotatebox{90}{\{\}} & \rotatebox{90}{\{AA\}} & \rotatebox{90}{\{BB\}} & \rotatebox{90}{\{AA,BB\}} & \rotatebox{90}{\{CC\}} & \rotatebox{90}{\{AA,CC\}} & \rotatebox{90}{\{BB,CC\}} & \rotatebox{90}{\{AA,BB,CC\}} & \rotatebox{90}{\{DD\}} & \rotatebox{90}{\{AA,DD\}} & \rotatebox{90}{\{BB,DD\}} & \rotatebox{90}{\{AA,BB,DD\}} & \rotatebox{90}{\{CC,DD\}} & \rotatebox{90}{\{AA,CC,DD\}} & \rotatebox{90}{\{BB,CC,DD\}} & \rotatebox{90}{\{AA,BB,CC,DD\}} \end{array} \\ \begin{pmatrix} 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \end{pmatrix} \end{array}
$$

$$
s{;}\mu = \begin{array}{c} \\ \mathrm{AA} \\ \mathrm{BB} \\ \mathrm{CC} \\ \mathrm{DD} \end{array} \begin{array}{c} \begin{array}{cccccccccccc} \rotatebox{90}{(low,lazy)} & \rotatebox{90}{(medium,lazy)} & \rotatebox{90}{(low,fair)} & \rotatebox{90}{(high,lazy)} & \rotatebox{90}{(medium,fair)} & \rotatebox{90}{(low,good)} & \rotatebox{90}{(high,fair)} & \rotatebox{90}{(medium,good)} & \rotatebox{90}{(low,bulldozer)} & \rotatebox{90}{(high,good)} & \rotatebox{90}{(medium,bulldozer)} & \rotatebox{90}{(high,bulldozer)} \end{array} \\ \begin{pmatrix} 0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1&0&0&0&0 \end{pmatrix} \end{array}
$$

$$
X \cup s{;}\mu = \begin{array}{c} \mathrm{AA} \\ \mathrm{BB} \\ \mathrm{CC} \\ \mathrm{DD} \end{array} \begin{pmatrix} 0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1&0&0&0&0 \end{pmatrix} \qquad \mathtt{glbR}_E(X \cup s{;}\mu) = \begin{array}{c} \mathrm{AA} \\ \mathrm{BB} \\ \mathrm{CC} \\ \mathrm{DD} \end{array} \begin{pmatrix} 0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0 \end{pmatrix}
$$

$$
\mathbb{T}{;}\mathtt{glbR}_E(X \cup s{;}\mu) = (0\;1\;0\;0\;0\;0\;1\;1\;0\;0\;0\;0)
$$

$$
(R)\!\int X \circ \mu = \mathtt{lubR}_E(\mathbb{T}{;}\mathtt{glbR}(X \cup s{;}\mu)) = (0\;0\;0\;0\;0\;0\;0\;0\;0\;1\;0\;0)
$$

Fig. 6.11 Further steps of relational integration

Again, when having in mind Fig. 5.2 and looking at the X in Fig. 6.10, the result of this aggregation seems reasonable.

6.4. Fourth example of relational integration

Now a measuring into the lattice of Fig. 5.3 is defined; this time as a plausibility measure via a body of evidence M. One may see therein a valuation around a presupposed medium point with positive as well as negative deviations.

$$
\begin{array}{r}
\\
\{\} \\ \{\text{red}\} \\ \{\text{green}\} \\ \{\text{red,green}\} \\ \{\text{blue}\} \\ \{\text{red,blue}\} \\ \{\text{green,blue}\} \\ \{\text{red,green,blue}\} \\ \{\text{orange}\} \\ \{\text{red,orange}\} \\ \{\text{green,orange}\} \\ \{\text{red,green,orange}\} \\ \{\text{blue,orange}\} \\ \{\text{red,blue,orange}\} \\ \{\text{green,blue,orange}\} \\ \{\text{red,green,blue,orange}\}
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} 1&2&3&4&5&6&7&8 \end{array} \\
\begin{pmatrix}
1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0
\end{pmatrix}
\end{array}
\qquad
\begin{array}{c}
\begin{array}{cccccccc} 1&2&3&4&5&6&7&8 \end{array} \\
\begin{pmatrix}
1&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&1
\end{pmatrix}
\end{array}
$$

Fig. 6.12 Body of evidence M with corresponding plausibility measure μ

We explain with an example how the plausibility measure is formed: To red, blue, orange and the set {red,blue} no value is assigned via M. However, subsets as e.g. {red, green} get assigned 3 and others as e.g. {green, blue} a 2. When all these values flow together, the subset {red, green, orange} will get a 4, according to the lattice of Fig. 5.3.

Now we are prepared to integrate over every relation X the typing of which is $X : \mathcal{D} \longrightarrow \mathcal{L}$. Our example shall be the left relation of Fig. 6.13.

$$
X = \begin{array}{r} \\ \text{red} \\ \text{green} \\ \text{blue} \\ \text{orange} \end{array}
\begin{array}{c}
\begin{array}{cccccccc} 1&2&3&4&5&6&7&8 \end{array} \\
\begin{pmatrix}
0&0&0&0&1&0&0&0\\
0&0&1&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0
\end{pmatrix}
\end{array}
\qquad
X{;}E^{\mathsf{T}}{;}X^{\mathsf{T}} = \begin{array}{r} \\ \text{red} \\ \text{green} \\ \text{blue} \\ \text{orange} \end{array}
\begin{array}{c}
\begin{array}{cccc} \text{red}&\text{green}&\text{blue}&\text{orange} \end{array} \\
\begin{pmatrix}
1&1&1&0\\
0&1&0&0\\
0&0&1&0\\
0&1&1&1
\end{pmatrix}
\end{array}
$$

Fig. 6.13 Next example X to be integrated with corresponding $X{;}E^{\mathsf{T}}{;}X^{\mathsf{T}}$

The right side of Fig. 6.13 assigns to the row entries all rows of X that carry values equal or below. One will need to consult Fig. 5.3 rather carefully to be convinced about that. By comparison with the membership relation ε — see column inscriptions — one obtains the s of Fig. 6.14 and everything else evaluated out of it in the by now known form.

$$
s := \mathtt{syq}(X;E^{\mathsf{T}};X^{\mathsf{T}},\varepsilon) = \begin{array}{r}
\\ \text{red} \\ \text{green} \\ \text{blue} \\ \text{orange}
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccccccc}
\rotatebox{90}{\{\}} & \rotatebox{90}{\{red\}} & \rotatebox{90}{\{green\}} & \rotatebox{90}{\{red,green\}} & \rotatebox{90}{\{blue\}} & \rotatebox{90}{\{red,blue\}} & \rotatebox{90}{\{green,blue\}} & \rotatebox{90}{\{red,green,blue\}} & \rotatebox{90}{\{orange\}} & \rotatebox{90}{\{red,orange\}} & \rotatebox{90}{\{green,orange\}} & \rotatebox{90}{\{red,green,orange\}} & \rotatebox{90}{\{blue,orange\}} & \rotatebox{90}{\{red,blue,orange\}} & \rotatebox{90}{\{green,blue,orange\}} & \rotatebox{90}{\{red,green,blue,orange\}}
\end{array} \\
\begin{pmatrix}
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0
\end{pmatrix}
\end{array}
$$

$$
s;\mu = \begin{array}{r|cccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\text{red} & 0&0&0&1&0&0&0&0\\
\text{green} & 0&0&0&1&0&0&0&0\\
\text{blue} & 0&0&0&1&0&0&0&0\\
\text{orange} & 0&1&0&0&0&0&0&0
\end{array}
$$

$$
X \cup s;\mu = \begin{array}{r}\text{red}\\ \text{green}\\ \text{blue}\\ \text{orange}\end{array}
\begin{pmatrix}
0&0&0&1&1&0&0&0\\
0&0&1&1&0&0&0&0\\
0&1&0&1&0&0&0&0\\
0&1&0&0&0&1&0&0
\end{pmatrix}
$$

$$
\mathtt{glbR}_E(X \cup s;\mu) = \begin{array}{r}\text{red}\\ \text{green}\\ \text{blue}\\ \text{orange}\end{array}
\begin{pmatrix}
0&0&0&1&0&0&0&0\\
0&0&1&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0
\end{pmatrix}
$$

$$
\mathbb{T};\mathtt{glbR}_E(X \cup s;\mu) = \ \{\}\begin{pmatrix}0&1&1&1&0&0&0&0\end{pmatrix}
$$

$$
(R)\!\int X \circ \mu = \mathtt{lubR}_E(\mathbb{T};\mathtt{glbR}_E(X \cup s;\mu)) = \ \{\}\begin{pmatrix}0&0&0&1&0&0&0&0\end{pmatrix}
$$

Fig. 6.14 Evaluating relational integral over X of Fig. 6.13 with μ of Fig. 6.12

In view of the arguments, this integral seems intuitively acceptable.

7. Random Variables

Here, we follow the book [3] by Konrad Jacobs on discrete stochastics when defining random variables. He sticks to standard probability measures. The generalization is again that we allow measuring also in lattices $\mathcal{L}$ with `glb` and `lub`, instead of just the unit interval $[0, 1]$. As relational tools we will use the existential image $\vartheta_R = \mathtt{syq}(R^{\mathsf{T}};\varepsilon, \varepsilon')$ of a relation R and its power transpose $\Lambda_R = \mathtt{syq}(R^{\mathsf{T}}, \varepsilon')$.

$$
\begin{tikzcd}
\mathcal{D} \arrow[rr, "f"'] \arrow[d, "\varepsilon"] \arrow[ddr, bend right=60, "m"'] & & X \arrow[d, "\varepsilon_X"'] \arrow[dll, "\Lambda_{f^\mathsf{T}}"] \arrow[ddl, bend left=60, "\Lambda_{f^\mathsf{T}};\mu"] \\
2^{\mathcal{D}} \arrow[dr, "\mu"'] & & 2^X \arrow[ll, "\vartheta_{f^\mathsf{T}}"'] \arrow[dl, "\mu_f"] \\
& \mathcal{L} &
\end{tikzcd}
$$

Fig. 7.1 Schema of a random variable f and the expectation relation $\Lambda_{f^\mathsf{T}};\mu$

When any measure $\mu : \mathbf{2}^{\mathcal{D}} \longrightarrow \mathcal{L}$ is given, a mapping $f : \mathcal{D} \longrightarrow X$ into some set X shall be called a **random variable** with state space X; see Figs. 7.2 and 7.3 for an example.

7.1 Proposition. For the membership $\varepsilon_X : X \longrightarrow \mathbf{2}^X$ and the random variable f, the relation $\mu_f : \mathbf{2}^X \longrightarrow \mathcal{L}$ defined as $\mu_f := \vartheta_{f^\mathsf{T}};\mu$ is again a measure.

Proof: The main point to be proved is monotony:

$\Omega_X;\mu_f = \Omega_X;\vartheta_{f^\mathsf{T}};\mu \subseteq \vartheta_{f^\mathsf{T}};\Omega;\mu$ Prop. 3.2 (or Prop. 5.2 of [10])
$\subseteq \vartheta_{f^\mathsf{T}};\mu;E$ since the measure μ is monotonic
$= \mu_f;E$

Merely as an exercise, we prove also for the point $0_{\Omega_X} = \mathtt{syq}(\varepsilon_X, \perp\!\!\!\perp)$

$\mu_f^\mathsf{T};0_{\Omega_X} = \mu^\mathsf{T};\vartheta_{f^\mathsf{T}}^\mathsf{T};0_{\Omega_X} = \mu^\mathsf{T};\mathtt{syq}(\varepsilon, f;\varepsilon_X);0_{\Omega_X}$ existential image
$= \mu^\mathsf{T};\mathtt{syq}(\varepsilon, f;\varepsilon_X;0_{\Omega_X}) = \mu^\mathsf{T};\mathtt{syq}(\varepsilon, f;\varepsilon_X;\mathtt{syq}(\varepsilon_X, \perp\!\!\!\perp))$
$= \mu^\mathsf{T};\mathtt{syq}(\varepsilon, f;\perp\!\!\!\perp) = \mu^\mathsf{T};\mathtt{syq}(\varepsilon, \perp\!\!\!\perp) = \mu^\mathsf{T};0_\Omega = 0_E$ due to gauging of μ.

The case of $1_{\Omega_X} = \mathtt{syq}(\varepsilon_X, \top\!\!\!\top)$ is handled completely analogously. □

In the preceding proof there has in no way real arithmetic been involved, so it holds true also when $\mathcal{L}$ should happen to be the unit interval $[0, 1]$ with `glb,lub`.

We are going to visualize Prop. 7.1 along the random variable f of Fig. 7.2, in which the number strings are taken to give the intuition of reals.

$$
f = \begin{array}{r} \\ \text{US} \\ \text{French} \\ \text{German} \\ \text{British} \\ \text{Spanish} \end{array}
\begin{array}{c}
\begin{array}{ccccc} \rotatebox{90}{97.82} & \rotatebox{90}{11.25} & \rotatebox{90}{-45.36} & \rotatebox{90}{61.24} & \rotatebox{90}{42.33} \end{array} \\
\begin{pmatrix}
\mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\
\mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{pmatrix}
\end{array}
$$

Fig. 7.2 Example of a random variable f to be studied with μ of Fig. 7.3

The measure shall be a belief measure μ based on a body of evidence $M : \mathbf{2}^{\mathcal{D}} \longrightarrow \mathcal{L}$ as shown in Fig. 7.3.

	0 — 1	0 — 1
{}	0/1	0/1
{US}	5/7	5/7
{Fr}	5/13	5/13
{US,Fr}	0/1	5/7
{Ge}	0/1	0/1
{US,Ge}	0/1	5/7
{Fr,Ge}	11/17, 23/29	23/29
{US,Fr,Ge}	0/1	23/29
{Br}	0/1	0/1
{US,Br}	0/1	5/7
{Fr,Br}	0/1	5/13
{US,Fr,Br}	0/1	5/7
{Ge,Br}	5/16	5/16
{US,Ge,Br}	0/1	5/7
{Fr,Ge,Br}	0/1	23/29
{US,Fr,Ge,Br}	0/1	23/29
{Sp}	3/7	3/7
{US,Sp}	3/17, 23/27	23/27
{Fr,Sp}	0/1	3/7
{US,Fr,Sp}	0/1	23/27
{Ge,Sp}	0/1	3/7
{US,Ge,Sp}	0/1	23/27
{Fr,Ge,Sp}	0/1	23/29
{US,Fr,Ge,Sp}	0/1	23/27
{Br,Sp}	0/1	3/7
{US,Br,Sp}	0/1	23/27
{Fr,Br,Sp}	0/1	3/7
{US,Fr,Br,Sp}	0/1	23/27
{Ge,Br,Sp}	0/1	3/7
{US,Ge,Br,Sp}	0/1	23/27
{Fr,Ge,Br,Sp}	0/1	23/29
{US,Fr,Ge,Br,Sp}	0/1	1/1

Fig. 7.3 Body of evidence M and belief measure μ

We might even get something similar to a probability vector out of the μ_f thus obtained and define $m := \sigma ; \mu_f$, with σ the singleton injection. However, this m so defined will hardly sum up to 1.

$\mu_f =$

{}	0/1
{97.82}	0/1
{11.25}	3/7
{97.82,11.25}	3/7
{-45.36}	5/7
{97.82,-45.36}	5/7
{11.25,-45.36}	23/27
{97.82,11.25,-45.36}	23/27
{61.24}	5/13
{97.82,61.24}	5/13
{11.25,61.24}	3/7
{97.82,11.25,61.24}	3/7
{-45.36,61.24}	23/29
{97.82,-45.36,61.24}	23/29
{11.25,-45.36,61.24}	1/1
{97.82,11.25,-45.36,61.24}	1/1
{42.33}	0/1
{97.82,42.33}	0/1
{11.25,42.33}	3/7
{97.82,11.25,42.33}	3/7
{-45.36,42.33}	5/7
{97.82,-45.36,42.33}	5/7
{11.25,-45.36,42.33}	23/27
{97.82,11.25,-45.36,42.33}	23/27
{61.24,42.33}	5/13
{97.82,61.24,42.33}	5/13
{11.25,61.24,42.33}	3/7
{97.82,11.25,61.24,42.33}	3/7
{-45.36,61.24,42.33}	23/29
{97.82,-45.36,61.24,42.33}	23/29
{11.25,-45.36,61.24,42.33}	1/1
{97.82,11.25,-45.36,61.24,42.33}	1/1

Fig. 7.4 Measure μ_f derived for random variable f and the μ of Fig. 7.3

Also the well-known concept of an expectation may be found in the present context:

7.2 Definition. Let a measure $\mu : \mathbf{2}^{\mathcal{D}} \longrightarrow \mathcal{L}$ be given and consider any random variable $f : \mathcal{D} \longrightarrow X$. Using the power transpose of f^{T}, we call $\mathcal{E}_{f,\mu} := \Lambda_{f^{\mathsf{T}}} ; \mu$ the **expectation relation** of f under μ. $\square$

Of course, not a real number appears but a relation closely resembling it as we show with the following tiny example. The column inscriptions of Fig. 7.2 had been just a fake, number strings, to make clear how these are related via the measure and how one might now form a scalar product.

$$\Lambda_{f^{\mathsf{T}}} = \begin{array}{r|cccccccccccccccccccccccccccccccc}
 & \rotatebox{90}{\{\}} & \rotatebox{90}{\{US\}} & \rotatebox{90}{\{Fr\}} & \rotatebox{90}{\{US,Fr\}} & \rotatebox{90}{\{Ge\}} & \rotatebox{90}{\{US,Ge\}} & \rotatebox{90}{\{Fr,Ge\}} & \rotatebox{90}{\{US,Fr,Ge\}} & \rotatebox{90}{\{Br\}} & \rotatebox{90}{\{US,Br\}} & \rotatebox{90}{\{Fr,Br\}} & \rotatebox{90}{\{US,Fr,Br\}} & \rotatebox{90}{\{Ge,Br\}} & \rotatebox{90}{\{US,Ge,Br\}} & \rotatebox{90}{\{Fr,Ge,Br\}} & \rotatebox{90}{\{US,Fr,Ge,Br\}} & \rotatebox{90}{\{Sp\}} & \rotatebox{90}{\{US,Sp\}} & \rotatebox{90}{\{Fr,Sp\}} & \rotatebox{90}{\{US,Fr,Sp\}} & \rotatebox{90}{\{Ge,Sp\}} & \rotatebox{90}{\{US,Ge,Sp\}} & \rotatebox{90}{\{Fr,Ge,Sp\}} & \rotatebox{90}{\{US,Fr,Ge,Sp\}} & \rotatebox{90}{\{Br,Sp\}} & \rotatebox{90}{\{US,Br,Sp\}} & \rotatebox{90}{\{Fr,Br,Sp\}} & \rotatebox{90}{\{US,Fr,Br,Sp\}} & \rotatebox{90}{\{Ge,Br,Sp\}} & \rotatebox{90}{\{US,Ge,Br,Sp\}} & \rotatebox{90}{\{Fr,Ge,Br,Sp\}} & \rotatebox{90}{\{US,Fr,Ge,Br,Sp\}} \\
\hline
97.82 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
11.25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-45.36 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
61.24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
42.33 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

$$\Lambda_{f^{\mathsf{T}}} ; \mu = \begin{array}{r|c|l}
 & 0 \quad\quad\quad 1 & \\
97.82 & | & 0/1 \\
11.25 & | & 3/7 \\
-45.36 & | & 5/7 \\
61.24 & | & 5/13 \\
42.33 & | & 0/1
\end{array}$$

Fig. 7.5 Power transpose $\Lambda_{f^{\mathsf{T}}}$ and expectation relation $\Lambda_{f^{\mathsf{T}}} ; \mu$

To capture variance in a similar way seems hardly possible, at least much more delicate.

8. Concluding remarks

This was a structural study. It may lead our minds to conceive some concepts in another way than before. In particular is it a plea not to mix up the measuring space with what has to be measured when indiscriminately using the always present real numbers $\mathbb{R}$. Real numbers are often admired for offering the possibility to express results with high precision. But exactness so obtained is all too often just a virtual one, due to numerical circumstances obfuscating structural effects.

There is a considerable difference between making decisions and computing arithmetic means. When designing decisions, one should devise a measuring or valuation space and stick to it. Decision making is something that should have an outcome of yes/no, resp. inside the scale. It should not depend on overly detailed evaluations of real numbers. Precisely this requirement is frequently violated. There may be reason to change the measuring space. It is unwise to go slidingly to ever more fine granulated reals and base decisions on such results.

One may find interesting in this article also the examples of possibility, belief, and plausibility measures. It is shown how they come into existence as well as how they work.

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