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Arranging binary relations nicely

— A Guide —

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Abstract

Over the years a multitude of structure detecting and structure visualizing programs has been developed. This report shall inform on what is possible with binary relations — refraining from mentioning too much theory. The pictorial presentation may persuade one or the other researcher to use relational methods more often.

Keywords relational mathematics, relation algebra, structure recognition, factorization, algebraic visualization, semiorder, weakorder, intervalorder

Contents

1 Introduction

This shall be conceived as a guide to get acquainted with "pretty-arranging" relations as a tool in structure recognition. Behind it exists the monograph [Sch11], which contains all the theoretical background. While this report gives plenty of illustrations mentioning the respective algebraic formulae so that visualization is readily understood and verified by the respective example, the reference is always the book mentioned where also all the detailed proofs may be found.

This report should be readable by nearly everybody — even with only a moderate mathematical literacy.

2 Prerequisites

Relational methods in principle need a detailed introduction, not least from [SS89, SS93, Sch11, SW14]. But the elementary operations on relations are broadly known, so that here — with the plethora of examples — the ideas will probably be grasped.

2.1 Preliminaries

When given two relations $R, S : X \longrightarrow Y$ between the same two sets, one may "unite" or "intersect" them to $R \cup S$ and $R \cap S$, e.g. We have, thus, that

$$\begin{array}{lll} R \cup S & \text{means} & (R \cup S)_{xy} = R_{xy} \vee S_{xy} \quad \text{for all } x, y, \\ R \cap S & \text{means} & (R \cap S)_{xy} = R_{xy} \wedge S_{xy} \quad \text{for all } x, y, \\ R \subseteq S & \text{means} & (R \subseteq S)_{xy} = R_{xy} \rightarrow S_{xy} \quad \text{for all } x, y, \\ \overline{R} & \text{means} & \overline{R}_{xy} = \neg R_{xy} \quad \text{for all } x, y. \end{array}$$

In this way, the Boolean operations are conceived elementwise. The least and the greatest relations are denoted as $\mathbb{⊥}, \mathbb{T} : X \longrightarrow Y$, resp.

It must be made clear that we distinguish source X and target Y of a relation — which may possibly coincide. We stress that we have not just homogeneous relations in mind, relations "on a set". Also heterogeneous relations, i.e., between different sets shall be envisaged. Proceeding this way requires some control concerning from where to where a relation leads and by such type control gives some added security.

The operation[1] of relational composition is introduced

$$R;S, \quad \text{meaning} \quad (R;S)_{xy} = \exists z : R_{xz} \wedge S_{z,y} \quad \text{for all } x, y,$$

together with the identities $\mathbb{I}$, and finally transposition or conversion

$$R^{\mathsf{T}} \quad \text{meaning} \quad R^{\mathsf{T}}_{xy} = R_{yx} \quad \text{for all } x, y.$$

The most immediate interpretation is that of Boolean matrices, i.e., **0** , **1** -matrices; therefore we often explain effects via rows, columns, and diagonals.

The most well-known properties of a relation Q are being *univalent*, i.e., a (possibly partial) *function*, ($Q^{\mathsf{T}};Q \subseteq \mathbb{I}$), being *injective* (when Q^{T} is univalent), being *total* ($\mathbb{I} \subseteq Q;Q^{\mathsf{T}}$ or, equivalently, $Q;\mathbb{T} = \mathbb{T}$), being *surjective* (when Q^{T} is total), and finally being a *mapping* (when univalent as well as total). The latter word is here reserved for a totally defined function.

[1] Composition of real-valued matrices is denoted without any operation symbol, simply by juxtaposition. Composition of relations has traditionally been expressed by a normal-size semicolon. The tiny symbol chosen here is a mixture of both of these.

2.2 Working on relation input

Relations occurring in practice appear in diverse representations, e.g. as indicated in Fig. 2.1. In such a situation one will highly estimate a transformation to the possibly more convenient version. This is, certainly, rather trivial programming work, but extremely helpful when generally available.

$$\begin{array}{c|c} & \text{red}\ \text{green}\ \text{blue}\ \text{orange} \\ \begin{array}{c}\spadesuit\\ \heartsuit\\ \diamondsuit\\ \clubsuit\end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

BridgeSuit	Colors
♠	{1}
♡	{1, 4}
♢	{}
♣	{2, 4}

BridgeSuit	Colors
♠	{red}
♡	{red,orange}
♢	{}
♣	{green,orange}

Relation R from
[♠,♡,♢,♣]
to
[red,green,blue,orange]
with pairlist
{(1,1),(2,1),(2,4),(4,2),(4,4)}

$$\begin{array}{r} (\spadesuit,\text{red}) \\ (\heartsuit,\text{red}) \\ (\spadesuit,\text{green}) \\ (\diamondsuit,\text{red}) \\ (\heartsuit,\text{green}) \\ (\spadesuit,\text{blue}) \\ (\clubsuit,\text{red}) \\ (\diamondsuit,\text{green}) \\ (\heartsuit,\text{blue}) \\ (\spadesuit,\text{orange}) \\ (\clubsuit,\text{green}) \\ (\diamondsuit,\text{blue}) \\ (\heartsuit,\text{orange}) \\ (\clubsuit,\text{blue}) \\ (\diamondsuit,\text{orange}) \\ (\clubsuit,\text{orange}) \end{array} \begin{pmatrix} 1\\1\\0\\0\\0\\0\\0\\0\\0\\0\\1\\0\\1\\0\\0\\1 \end{pmatrix}$$

Fig. 2.1 Switching between different representations of the same (square but heterogeneous!) relation

The different concepts under the heading of describing a relation include: **0** , **1** -matrices with row and column information and the related versions as predicates, pair sets etc.

```
data Rel = MATRRel BaseSet BaseSet [[Bool]]                  |
           PREDRel BaseSet BaseSet (Int -> Int -> Bool)      |
           SETFRel BaseSet BaseSet (Int -> [Int])            |
           SNAFRel BaseSet BaseSet (Int -> [String])         |
           PALIRel BaseSet BaseSet [(Int,Int)]               |
           VECTRel BaseSet BaseSet [Bool]                    |
           POWERel BaseSet BaseSet [Bool]
```

This means matrix, predicate on pairs, set mapping with integers to indicate the result, set mapping with names to indicate the result, list of related pairs, vector along the direct product and Boolean vector along the powerset. To this diversity correspond these generically available maps for automatic transition:

```
relAsMATRRel    relAsPREDRel    relAsSETFRel
relAsSNAFRel    relAsVECTRel    relAsPALIRel    relAsPOWERel
```

It allows for instance — what one should obviously do only in extremely restricted cases — to say

```
makeTeXRel $ relAsPOWERel $ PALIRel bs2 bsGender [(1,2),(2,1)]
```

and obtain an element or point marking along the powerset of the pairset.

$$\begin{array}{r}
\{\} \\
\{(1,\text{male})\} \\
\{(2,\text{male})\} \\
\{(1,\text{male}),(2,\text{male})\} \\
\{(1,\text{female})\} \\
\{(1,\text{male}),(1,\text{female})\} \\
\{(2,\text{male}),(1,\text{female})\} \\
\{(1,\text{male}),(2,\text{male}),(1,\text{female})\} \\
\{(2,\text{female})\} \\
\{(1,\text{male}),(2,\text{female})\} \\
\{(2,\text{male}),(2,\text{female})\} \\
\{(1,\text{male}),(2,\text{male}),(2,\text{female})\} \\
\{(1,\text{female}),(2,\text{female})\} \\
\{(1,\text{male}),(1,\text{female}),(2,\text{female})\} \\
\{(2,\text{male}),(1,\text{female}),(2,\text{female})\} \\
\{(1,\text{male}),(2,\text{male}),(1,\text{female}),(2,\text{female})\}
\end{array}
\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

We do not elaborate these techniques here any further. We will apply them without much discussion. It should be mentioned, however, that the concept of a set is here specialized to that of an ordered set (called baseset) — as required for the row as well as column inscriptions of matrix representations of a relation.

2.3 Permutations

For technical reasons, not least for the transformations mentioned, permutations have to be handled even before relations are introduced. In early school life, permutations may be given in a variety of forms, as a matrix, a sequence, via cycles, via transpositions, or as a mapping. In the following, it is understood that

- `PS` precedes the sequence of images of `[1..n]`,
- `PT` precedes the number of elements permuted and a set of transpositions,
- `PC` precedes the number of elements and a list of cycles,
- `Pf`[2] precedes the number n of elements and a mapping of `[1..n]`,
- `PM` precedes a Boolean matrix that is a bijective mapping.

Mechanisms to convert between these forms are provided as automatic transformations

`permAsSEQ` `permAsMAT` `permAsCYC` `permAsTRA` `permAsMAP`.

In an example, this looks as follows: A permutation given as transition mapping top/down:

$$\text{Pf}\ \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 3 & 9 & 1 & 6 & 2 & 4 & 5 & 8 \end{matrix}.$$

It is transformed into a version with transpositions (showing their total length first):

PT 9 [(1,7),(7,4),(2,3),(3,9),(9,8),(8,5),(5,6)],

a version with a matrix indicating the permutation:

[2] `PF` has already been used at another occasion.

$$\text{PM}\ \begin{pmatrix} 0&0&0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&1&0 \end{pmatrix},$$

a version with sequences

PS [7,3,9,1,6,2,4,5,8],

and represented by cycles

PC 9 [[1,7,4],[2,3,9,8,5,6]].

Now follows the inverse in all these representations:

$$\text{Pf}\ \begin{array}{ccccccccc} 1&2&3&4&5&6&7&8&9\\ 4&6&2&7&8&5&1&9&3 \end{array}$$

PT 9 [(1,4),(4,7),(2,6),(6,5),(5,8),(8,9),(9,3)]

$$\text{PM}\ \begin{pmatrix} 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&1&0&0&0&0\\ 1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1\\ 0&0&1&0&0&0&0&0&0 \end{pmatrix}$$

PS [4,6,2,7,8,5,1,9,3]

$$\text{Pf}\ \begin{array}{ccccccccc} 1&2&3&4&5&6&7&8&9\\ 4&6&2&7&8&5&1&9&3 \end{array}$$

PC 9 [[1,4,7],[2,6,5,8,9,3]]

Also the possibility is offered to apply a permutation of any such form to an ordered set or a matrix. As an example we apply row and column permutation to a relation with `applyPermsToREL pr pc m`:

$$\text{Pf}\ \begin{array}{ccccccccc} 1&2&3&4&5&6&7&8&9\\ 3&9&2&5&7&1&8&6&4 \end{array}$$

`PC 8 [[1,7,4],[2,3,8,5,6]]`

P_R (above), P_C (below)

$$\begin{array}{c c} & \begin{array}{cccccccc} 1&2&3&4&5&6&7&8 \end{array} \\ \begin{array}{c} a\\b\\c\\d\\e\\f\\g\\h\\i \end{array} & \begin{pmatrix} 1&1&0&0&1&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&1&1&1&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&1&0&0\\ 0&0&0&0&0&1&0&0\\ 1&1&0&1&0&0&1&0\\ 0&0&0&0&1&1&0&1 \end{pmatrix} \end{array}$$

M

$$\begin{array}{c c} & \begin{array}{cccccccc} 4&6&2&7&8&5&1&3 \end{array} \\ \begin{array}{c} f\\c\\a\\i\\d\\h\\e\\g\\b \end{array} & \begin{pmatrix} 0&1&0&0&0&1&0&0\\ 1&1&0&0&0&1&0&1\\ 0&0&1&0&0&1&1&0\\ 0&1&0&0&1&1&0&0\\ 1&0&0&0&0&0&0&0\\ 1&0&1&1&0&0&1&0\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0 \end{pmatrix} \end{array}$$

$P_R^{\mathsf{T}};M;P_C$

Fig. 2.2 Permuting matrix, rows, and columns of a relation

One will find it rather tricky to convince oneself that this is correct. All the more important is it to apply high-level constructs such as `applyPermsToREL` to make such transitions safe. In what follows, we will often make use of permutations that are algebraically generated out of the given example.

3 Determining permutations purposefully to adjust . . .

Now, that we have the mechanisms to perform permutations explicitly, we are going to make use of them. The idea is to have easily perceivable properties of the object after suitable permutations. These in turn will be derived in an algebraic fashion of which we try to give an impression in the next sections.

3.1 Permutations to adjust a bijective mapping

The permutations we aim at are mainly those that provide an intuitive visualization. A most trivial example is that of a bijective mapping, i.e., a relation $R : X \longrightarrow Y$ satisfying $R ; R^{\mathsf{T}} = \mathbb{I}_X$ and $R^{\mathsf{T}} ; R = \mathbb{I}_Y$. These two formulae show directly that the transposed matrix (!) $P := R^{\mathsf{T}}$ manages to bring R to diagonal form. P is, however, not the transposed relation (!) because of the different column names.

$$
\begin{array}{c}
\begin{array}{r}
\\ 1\\2\\3\\4\\5\\6\\7\\8\\9\\10
\end{array}
\begin{array}{c}
\begin{array}{cccccccccc} a&b&c&d&e&f&g&h&i&j \end{array}\\
\begin{pmatrix}
0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&1&0&0
\end{pmatrix}
\end{array}
\\ R
\end{array}
\quad
\begin{array}{c}
\begin{array}{r}
\\ a\\b\\c\\d\\e\\f\\g\\h\\i\\j
\end{array}
\begin{array}{c}
\begin{array}{cccccccccc} b&j&i&e&f&c&a&d&g&h \end{array}\\
\begin{pmatrix}
0&0&0&0&0&0&1&0&0&0\\
1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0\\
0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&1\\
0&0&1&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0
\end{pmatrix}
\end{array}
\\ P
\end{array}
\quad
\begin{array}{c}
\begin{array}{r}
\\ 1\\2\\3\\4\\5\\6\\7\\8\\9\\10
\end{array}
\begin{array}{c}
\begin{array}{cccccccccc} b&j&i&e&f&c&a&d&g&h \end{array}\\
\begin{pmatrix}
1&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&1
\end{pmatrix}
\end{array}
\\ R ; P
\end{array}
$$

Fig. 3.1 Permuting a bijection to diagonal form

We see that R and $R ; P$ describe the same relation. To obtain this effect, the sequence of column inscriptions had to be adapted. In order to arrive at the correct version of R arranged to diagonal form, we had to apply P^{T} to the original list of column inscriptions.

3.2 Permutations to adjust a linear strictorder

This was an inherently heterogeneous — while still square — situation. More intricate is it to bring a (homogeneous) linear strictorder to upper triangular form by *simultaneous* permutations. The original relation R of Fig. 3.2 is indeed a linear strictorder. Its Hasse relation H fully indicates the sequence in which to arrange.

$$\begin{array}{cc}
 & \begin{array}{ccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \end{array} \\
\begin{array}{c} 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13 \end{array} &
\begin{pmatrix}
0&1&0&1&1&1&0&1&1&1&0&0&1\\
0&0&0&0&0&1&0&0&0&0&0&0&0\\
1&1&0&1&1&1&1&1&1&1&1&1&1\\
0&1&0&0&1&1&0&1&0&1&0&0&1\\
0&1&0&0&0&1&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&1&1&1&0&1&1&1&0&1&1\\
0&1&0&0&1&1&0&0&0&1&0&0&1\\
0&1&0&1&1&1&0&1&0&1&0&0&1\\
0&1&0&0&1&1&0&0&0&0&0&0&1\\
1&1&0&1&1&1&1&1&1&1&0&1&1\\
1&1&0&1&1&1&0&1&1&1&0&0&1\\
0&1&0&0&0&1&0&0&0&0&0&0&0
\end{pmatrix} \\
 & R
\end{array}
\quad
\begin{array}{c}
\begin{array}{ccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \end{array} \\
\begin{pmatrix}
0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix} \\
H := R \cap \overline{R ; R}
\end{array}
\quad
\begin{array}{c}
\begin{array}{ccccccccccccc} 3 & 11 & 7 & 12 & 1 & 9 & 4 & 8 & 10 & 5 & 13 & 2 & 6 \end{array} \\
\begin{pmatrix}
0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0
\end{pmatrix} \\
P
\end{array}$$

Fig. 3.2 Permuting a linear strictorder R to diagonal form, using its Hasse relation $H := R \cap \overline{R ; R}$

The step from H to P can be made iteratively an algebraic one, sending the H-least to first position, its H-successor to the respective next, and so forth. The rearranged relation in which one immediately recognizes its structure is shown in Fig. 3.3.

$$P^{\mathsf{T}} ; R ; P = \begin{array}{cc}
 & \begin{array}{ccccccccccccc} 3 & 11 & 7 & 12 & 1 & 9 & 4 & 8 & 10 & 5 & 13 & 2 & 6 \end{array} \\
\begin{array}{c} 3\\11\\7\\12\\1\\9\\4\\8\\10\\5\\13\\2\\6 \end{array} &
\begin{pmatrix}
0&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}
\end{array}$$

Fig. 3.3 Permuted linear strictorder of Fig. 3.2

3.3 Permutations to adjust an equivalence

A comparable task is it to arrange an equivalence to block-diagonal form by simultaneous permutations; cf. Fig. 3.4. We postpone discussing the technique used to the next section.

$$\begin{array}{cc}
 & \begin{array}{ccccccccccc} a&b&c&d&e&f&g&h&i&j&k \end{array} \\
\begin{array}{c} a\\b\\c\\d\\e\\f\\g\\h\\i\\j\\k \end{array} &
\begin{pmatrix}
1&0&0&0&0&0&1&0&0&0&0\\
0&1&0&0&0&0&0&0&0&1&0\\
0&0&1&0&1&0&0&1&0&0&0\\
0&0&0&1&0&1&0&0&1&0&1\\
0&0&1&0&1&0&0&1&0&0&0\\
0&0&0&1&0&1&0&0&1&0&1\\
1&0&0&0&0&0&1&0&0&0&0\\
0&0&1&0&1&0&0&1&0&0&0\\
0&0&0&1&0&1&0&0&1&0&1\\
0&1&0&0&0&0&0&0&0&1&0\\
0&0&0&1&0&1&0&0&1&0&1
\end{pmatrix} \\
 & R
\end{array}
\quad
\begin{array}{cc}
 & \begin{array}{ccccccccccc} a&g&b&j&c&e&h&d&f&i&k \end{array} \\
\begin{array}{c} a\\b\\c\\d\\e\\f\\g\\h\\i\\j\\k \end{array} &
\begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0\\
0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix} \\
 & P
\end{array}
\quad
\begin{array}{cc}
 & \begin{array}{ccccccccccc} a&g&b&j&c&e&h&d&f&i&k \end{array} \\
\begin{array}{c} a\\g\\b\\j\\c\\e\\h\\d\\f\\i\\k \end{array} &
\begin{pmatrix}
1&1&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&0&0\\
0&0&1&1&0&0&0&0&0&0&0\\
0&0&1&1&0&0&0&0&0&0&0\\
0&0&0&0&1&1&1&0&0&0&0\\
0&0&0&0&1&1&1&0&0&0&0\\
0&0&0&0&1&1&1&0&0&0&0\\
0&0&0&0&0&0&0&1&1&1&1\\
0&0&0&0&0&0&0&1&1&1&1\\
0&0&0&0&0&0&0&1&1&1&1\\
0&0&0&0&0&0&0&1&1&1&1
\end{pmatrix} \\
 & P^{\mathsf{T}} ; R ; P
\end{array}$$

Fig. 3.4 Permuting an equivalence to block-diagonal form

There are not just diagonal matrices that enjoy to have favourable properties when visualizing some effect. Even if a relation is not homogeneous, i.e. in particular, not square, many of the properties may still be used when clearly formulated in an algebraic fashion.

3.4 Permutations to adjust a symmetric idempotent

The identity relation is often seen as a matrix with just the diagonal filled with **1**'s. For partial identities, i.e. homogeneous relations satisfying $R \subseteq \mathbb{I}$, the diagonal is not necessarily filled completely.

A mixture of partial identities and equivalences are symmetric idempotents, characterized by $R^{\mathsf{T}} \subseteq R$ and $R{:}R \subseteq R$. They may by simultaneous permutation be brought to a partial block-diagonal version.

$$\begin{array}{r@{\;}l} & \begin{array}{cccccccccc} 1&2&3&4&5&6&7&8&9&10 \end{array} \\ \begin{array}{r} 1\\2\\3\\4\\5\\6\\7\\8\\9\\10 \end{array} & \begin{pmatrix} 1&0&0&0&0&1&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&1\\ 0&0&0&1&1&0&0&0&0&0\\ 0&0&0&1&1&0&0&0&0&0\\ 1&0&0&0&0&1&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&1&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&1 \end{pmatrix} \end{array} \qquad \begin{array}{r@{\;}l} & \begin{array}{cccccccccc} 1&6&8&3&10&4&5&2&7&9 \end{array} \\ \begin{array}{r} 1\\2\\3\\4\\5\\6\\7\\8\\9\\10 \end{array} & \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&1&0&0&0&0&0 \end{pmatrix} \end{array} \qquad \begin{array}{r@{\;}l} & \begin{array}{cccccccccc} 1&6&8&3&10&4&5&2&7&9 \end{array} \\ \begin{array}{r} 1\\6\\8\\3\\10\\4\\5\\2\\7\\9 \end{array} & \begin{pmatrix} 1&1&1&0&0&0&0&0&0&0\\ 1&1&1&0&0&0&0&0&0&0\\ 1&1&1&0&0&0&0&0&0&0\\ 0&0&0&1&1&0&0&0&0&0\\ 0&0&0&1&1&0&0&0&0&0\\ 0&0&0&0&0&1&1&0&0&0\\ 0&0&0&0&0&1&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \end{array}$$

$$R \qquad\qquad P \qquad\qquad P^{\mathsf{T}}{:}R{:}P$$

Fig. 3.5 Permuting a symmetric idempotent to block-diagonal form

3.5 Permutations to adjust a matching

Slightly less demanding properties has a matching, a relation $R : X \longrightarrow Y$ that relates any element of the source to at most one on the target side and vice versa; an inherently heterogeneous situation. This is algebraically characterized by saying $R{:}R^{\mathsf{T}} \subseteq \mathbb{I}_X$ and $R^{\mathsf{T}}{:}R \subseteq \mathbb{I}_Y$. Programming this task is mainly the same, but requires some book-keeping with respect to empty rows resp. columns. The decision was to place the empty ones at the end.

$$\begin{array}{r@{\;}l} & \begin{array}{cccccccc} \text{US}&\text{French}&\text{German}&\text{British}&\text{Danish}&\text{Swiss}&\text{Polish}&\text{Italian} \end{array} \\ \begin{array}{r} \text{Clinton}\\\text{Bush}\\\text{Sarkozy}\\\text{Chirac}\\\text{Schmidt}\\\text{Kohl}\\\text{Merkel}\\\text{Obama}\\\text{Major}\\\text{Blair}\\\text{Rajoy} \end{array} & \begin{pmatrix} 0&1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \end{array} \qquad \begin{array}{c} \begin{array}{ccccccccccc} \text{Clinton}&\text{Bush}&\text{Sarkozy}&\text{Schmidt}&\text{Kohl}&\text{Obama}&\text{Chirac}&\text{Merkel}&\text{Major}&\text{Blair}&\text{Rajoy} \end{array} \\ \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&1 \end{pmatrix} \end{array}$$

$$\begin{array}{c} \begin{array}{cccccccc} \text{French}&\text{US}&\text{British}&\text{German}&\text{Polish}&\text{Danish}&\text{Swiss}&\text{Italian} \end{array} \\ \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \end{array} \qquad \begin{array}{r@{\;}l} & \begin{array}{cccccccc} \text{French}&\text{US}&\text{British}&\text{German}&\text{Polish}&\text{Danish}&\text{Swiss}&\text{Italian} \end{array} \\ \begin{array}{r} \text{US}\\\text{French}\\\text{German}\\\text{British}\\\text{Danish}\\\text{Swiss}\\\text{Polish}\\\text{Italian} \end{array} & \begin{pmatrix} 0&1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix} \end{array}$$

$$R \qquad\qquad P_R \qquad\qquad P_R^{\mathsf{T}}{:}R{:}P_C \qquad\qquad P_C$$

Fig. 3.6 Permuting a matching to diagonal form

What may not have been clear from the initial form of this relation is made immediately visible with $P_R^{\mathsf{T}}{:}R{:}P_C$.

3.6 Permutations to adjust a difunctional relation

Difunctional relations can directly be compared with matchings. They may, however, possibly have universal sub-relation blocks instead of single elements. A relation $R : X \longrightarrow Y$ is called difunctional when it satisfies $R;R^{\mathsf{T}};R \subseteq R$ — meaning in fact "=" since "⊇" holds trivially.

One will not easily recognize that the R on the left of Fig. 3.7 is difunctional. However, when the algebraically derived permutations have been executed, difunctionality is evident on the right.

$$\begin{array}{r@{\;}c} & \begin{array}{cccccccccccccccccc} a&b&c&d&e&f&g&h&i&j&k&l&m&n&o&p&q&r \end{array}\\ \begin{array}{r} 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14 \end{array} & \left(\begin{array}{cccccccccccccccccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0\\
0&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&1&0&0&0&1&0&1&0\\
0&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&1&0&1&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&1&0&1&0&0&0&0&0&1&0&0\\
0&0&1&0&0&0&0&0&0&0&1&0&0&0&1&0&1&0\\
0&1&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&1&0&0&0&1&0&1&0\\
0&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0
\end{array}\right)\end{array}$$

$$R$$

$$\begin{array}{r@{\;}c} & \begin{array}{cccccccccccccccccc} b&l&n&e&i&c&k&o&q&f&h&j&p&a&d&g&m&r \end{array}\\ \begin{array}{r} 2\\12\\3\\6\\9\\14\\5\\11\\13\\7\\10\\1\\4\\8 \end{array} & \left(\begin{array}{cccccccccccccccccc}
1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}\right)\end{array}$$

$$P_R^{\mathsf{T}};R;P_C$$

Fig. 3.7 A difunctional relation and its rearrangement

When one has already been enabled to determine the necessary permutations for a matching, one should obviously be able to rearrange a difunctional relation. One should, namely, with some additional technicalities not to be presented here, simply rearrange the relation on the quotient sets — which is a matching.

$$\begin{array}{r@{\;}c} & \begin{array}{cccccccccccccc} 2&12&3&6&9&14&5&11&13&7&10&1&4&8 \end{array}\\ \begin{array}{r} 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14 \end{array} & \left(\begin{array}{cccccccccccccc}
0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0
\end{array}\right)\end{array}$$

$$P_R$$

$$\begin{array}{r@{\;}c} & \begin{array}{cccccccccccccccccc} b&l&n&e&i&c&k&o&q&f&h&j&p&a&d&g&m&r \end{array}\\ \begin{array}{r} a\\b\\c\\d\\e\\f\\g\\h\\i\\j\\k\\l\\m\\n\\o\\p\\q\\r \end{array} & \left(\begin{array}{cccccccccccccccccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{array}\right)\end{array}$$

$$P_C$$

Fig. 3.8 The permutations for rows and columns of Fig. 3.7 necessary for rearrangement

$$P_R^{\mathsf{T}} ; R ; P_C = \begin{array}{cc} & \begin{array}{ccc|cc|cccc|cccc|ccccc} b & l & n & e & i & c & k & o & q & f & h & j & p & a & d & g & m & r \end{array} \\ \begin{array}{r} 2 \\ 12 \\ 3 \\ 6 \\ 9 \\ 14 \\ 5 \\ 11 \\ 13 \\ 7 \\ 10 \\ 1 \\ 4 \\ 8 \end{array} & \left(\begin{array}{ccc|cc|cccc|cccc|ccccc} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

Fig. 3.9 Rearranged difunctional relation with subdivisions

The subdivisions presented in the last matrix don't seem to bring relevant additional information. This will change considerably when we consider relations R arranged according to their difunctional closure.

3.7 Permutations to adjust according to the difunctional closure

Everybody is acquainted with the transitive closure of a matrix/graph/relation. In a similar way, every — possibly heterogeneous — relation R has $R ; (R^{\mathsf{T}} ; R)^*$ as its difunctional closure, the least difunctional relation containing it. Fig. 3.10 shows an "arbitrary" heterogeneous relation R together with its difunctional closure.

$$R = \begin{array}{cc} & \begin{array}{ccccccccccccc} a & b & c & d & e & f & g & h & i & j & k & l & m \end{array} \\ \begin{array}{r} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \\ 16 \\ 17 \end{array} & \left(\begin{array}{ccccccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array} \qquad R ; (R^{\mathsf{T}} ; R)^* = \begin{array}{cc} & \begin{array}{ccccccccccccc} a & b & c & d & e & f & g & h & i & j & k & l & m \end{array} \\ \begin{array}{r} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \\ 16 \\ 17 \end{array} & \left(\begin{array}{ccccccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

Fig. 3.10 An arbitrary relation together with its difunctional closure

It is, of course, possible to rearrange this closure with appropriately determined permutations. But then, we may also rearrange the original relation, thus revealing and thereby visualizing a lot of its structure.

$$P_R = \begin{array}{cc}
 & \begin{array}{ccccccccccccccccc} 2 & 11 & 17 & 5 & 9 & 10 & 12 & 13 & 15 & 6 & 8 & 14 & 16 & 1 & 3 & 4 & 7 \end{array} \\
\begin{array}{r} 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15\\16\\17 \end{array} &
\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}
\end{array}
\qquad
P_C = \begin{array}{cc}
 & \begin{array}{ccccccccccccc} c & d & e & h & g & i & k & l & a & f & m & b & j \end{array} \\
\begin{array}{r} a\\b\\c\\d\\e\\f\\g\\h\\i\\j\\k\\l\\m \end{array} &
\begin{pmatrix}
0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0
\end{pmatrix}
\end{array}$$

Fig. 3.11 Row and column permutations algebraically derived from the closure

Again, it seems reasonable to include the subdivisions in order to visualize in some more detail, thus indicating the diagonal blocks obtained.

$$P_R^{\mathsf{T}} ; R ; P_C = \begin{array}{cc}
 & \begin{array}{ccccccccccccc} c & d & e & h & g & i & k & l & a & f & m & b & j \end{array} \\
\begin{array}{r} 2\\11\\17\\5\\9\\10\\12\\13\\15\\6\\8\\14\\16\\1\\3\\4\\7 \end{array} &
\begin{pmatrix}
0&0&1&1&0&0&0&0&0&0&0&0&0\\
1&0&1&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&1&0&0&0&0&0\\
0&0&0&0&1&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&1&1&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&1&1&0&0&0\\
0&0&0&0&0&0&0&0&1&0&1&0&0\\
0&0&0&0&0&0&0&0&1&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}
\end{array}
\qquad
\begin{array}{cc}
 & \begin{array}{ccccccccccccc} c & d & e & h & g & i & k & l & a & f & m & b & j \end{array} \\
\begin{array}{r} 2\\11\\17\\5\\9\\10\\12\\13\\15\\6\\8\\14\\16\\1\\3\\4\\7 \end{array} &
\left(\begin{array}{cccc|cccc|ccc|cc}
0&0&1&1&0&0&0&0&0&0&0&0&0\\
1&0&1&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&0&0&0&0\\ \hline
0&0&0&0&1&0&0&1&0&0&0&0&0\\
0&0&0&0&1&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&1&1&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0\\ \hline
0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&1&1&0&0&0\\
0&0&0&0&0&0&0&0&1&0&1&0&0\\
0&0&0&0&0&0&0&0&1&0&1&0&0\\ \hline
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}\right)
\end{array}$$

Fig. 3.12 Original relation rearranged and with auxiliary subdivisions

The diagonal blocks have the specific property to be chainable: When thinking of a rook in chess, every **1** can be reached from every other **1** in such a block by a rook move with changing direction allowed only on **1** 's. The chaining property is hard to see when no subdividing lines are available.

3.8 Permutations to adjust according to the strongly connected components

Again, a special version arises when one requires the permutations to be applied *simultaneously* for rows and columns. This is necessary for the widely known concept of strongly connected components. We start with a relation given as set function relation, proceed to its matrix representation and the reflexive transitive closure thereof.

set17	$\mathbf{2}^{\text{set17}}$
1	$\{4,8\}$
2	$\{14,11\}$
3	$\{\}$
4	$\{17,12\}$
5	$\{6,14\}$
6	$\{4,13\}$
7	$\{1\}$
8	$\{7\}$
9	$\{3,16\}$
10	$\{17\}$
11	$\{9,14\}$
12	$\{3,4\}$
13	$\{5,6\}$
14	$\{15\}$
15	$\{2,3,11\}$
16	$\{17,10\}$
17	$\{9,10\}$

R

$$\begin{array}{r@{\;}c}
 & \begin{array}{ccccccccccccccccc} 1&2&3&4&5&6&7&8&9&10&11&12&13&14&15&16&17 \end{array} \\
\begin{array}{r} 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15\\16\\17 \end{array} &
\left(\begin{array}{ccccccccccccccccc}
0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1\\
0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0\\
0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&1&1&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0
\end{array}\right)
\end{array}$$

R as matrix

$$\begin{array}{c}
\begin{array}{ccccccccccccccccc} 1&2&3&4&5&6&7&8&9&10&11&12&13&14&15&16&17 \end{array} \\
\left(\begin{array}{ccccccccccccccccc}
1&0&1&1&0&0&1&1&1&1&0&1&0&0&0&1&1\\
0&1&1&0&0&0&0&0&1&1&1&0&0&1&1&1&1\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&1&0&0&0&0&1&1&0&1&0&0&0&1&1\\
0&1&1&1&1&1&0&0&1&1&1&1&1&1&1&1&1\\
0&1&1&1&1&1&0&0&1&1&1&1&1&1&1&1&1\\
1&0&1&1&0&0&1&1&1&1&0&1&0&0&0&1&1\\
1&0&1&1&0&0&1&1&1&1&0&1&0&0&0&1&1\\
0&0&1&0&0&0&0&0&1&1&0&0&0&0&0&1&1\\
0&0&1&0&0&0&0&0&1&1&0&0&0&0&0&1&1\\
0&1&1&0&0&0&0&0&1&1&1&0&0&1&1&1&1\\
0&0&1&1&0&0&0&0&1&1&0&1&0&0&0&1&1\\
0&1&1&1&1&1&0&0&1&1&1&1&1&1&1&1&1\\
0&1&1&0&0&0&0&0&1&1&1&0&0&1&1&1&1\\
0&1&1&0&0&0&0&0&1&1&1&0&0&1&1&1&1\\
0&0&1&0&0&0&0&0&1&1&0&0&0&0&0&1&1\\
0&0&1&0&0&0&0&0&1&1&0&0&0&0&0&1&1
\end{array}\right)
\end{array}$$

$(\mathbb{I} \cup R)^*$

Fig. 3.13 Original relation, its matrix version and its reflexive-transitive closure

The underlying structure becomes immediately visible in Fig. 3.14.

$$\begin{array}{r@{\;}c}
 & \begin{array}{ccccccccccccccccc} 5&6&13&2&11&14&15&1&7&8&4&12&9&10&16&17&3 \end{array} \\
\begin{array}{r} 5\\6\\13\\2\\11\\14\\15\\1\\7\\8\\4\\12\\9\\10\\16\\17\\3 \end{array} &
\left(\begin{array}{ccc|cccc|ccc|cc|cccc|c}
1&1&1&1&1&1&1&0&0&0&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&0&0&0&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&0&0&0&1&1&1&1&1&1&1\\ \hline
0&0&0&1&1&1&1&0&0&0&0&0&1&1&1&1&1\\
0&0&0&1&1&1&1&0&0&0&0&0&1&1&1&1&1\\
0&0&0&1&1&1&1&0&0&0&0&0&1&1&1&1&1\\
0&0&0&1&1&1&1&0&0&0&0&0&1&1&1&1&1\\ \hline
0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1\\ \hline
0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1\\ \hline
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1\\ \hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{array}\right)
\end{array}$$

$P^{\mathsf{T}}; (\mathbb{I} \cup R)^*; P$

$$\begin{array}{r@{\;}c}
 & \begin{array}{ccccccccccccccccc} 5&6&13&2&11&14&15&1&7&8&4&12&9&10&16&17&3 \end{array} \\
\begin{array}{r} 5\\6\\13\\2\\11\\14\\15\\1\\7\\8\\4\\12\\9\\10\\16\\17\\3 \end{array} &
\left(\begin{array}{ccc|cccc|ccc|cc|cccc|c}
0&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ \hline
0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&1\\ \hline
0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\ \hline
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1\\ \hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0\\ \hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}\right)
\end{array}$$

$P^{\mathsf{T}}; R; P$

Fig. 3.14 Rearranged reflexive-transitive closure and original arranged accordingly

Chainability reduces here to reachability: Every diagonal block is strongly connected.

3.9 Permutations to adjust according to the fringe of a relation

The concept of a fringe is important, but widely unknown. While a pair (x, y) for which the relation R holds may be contained in several maximal — i.e. non-enlargeable — rectangles, the fringe collects all those points contained in just one.

Consider as an example Fig. 3.14 on the left: The entry $(11, 14)$ has just one non-enlargable rectangle

$$\{5, 6, 13, 2, 11, 14, 15\} \times \{2, 11, 14, 15, 9, 10, 16, 17, 3\},$$

while $(12, 17)$ has two, namely running horizontally first

$$\{5,6,13,1,7,8,4,12\} \times \{4,12,9,10,16,17,3\}$$

and running vertically first

$$\{5,6,13,2,11,14,15,1,7,8,4,12,9,10,16,17\} \times \{9,10,16,17,3\}.$$

In this perfectly arranged relation the fringe consists precisely of all the diagonal blocks.

The important and useful fact is that the fringe has a simple algebraic characterization, namely as $\nabla := \nabla_R := R \cap \overline{R;\overline{R}^{\mathsf{T}};R}$. A fringe is in many a respect comparable with a partial diagonal although in a possibly heterogeneous relation and with possibly rectangular diagonal blocks. This fringe is always difunctional, so that the already known permutations can be applied to permute it to the diagonal form already mentioned. The fringe may, however, be rather small, or may even vanish. In any case, it has remarkable properties that may be helpful in analyzing a relation.

We consider the "arbitrary" relation R in Fig. 3.15 together with its fringe. Both are obviously not yet arranged nicely, and one is probably unable to anticipate what will be going on.

$$R = \begin{array}{rc} & \begin{array}{ccccccccccccccccc} a&b&c&d&e&f&g&h&i&j&k&l&m&n&o&p&q \end{array} \\ \begin{array}{r} 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15\\16\\17\\18\\19\\20\\21 \end{array} & \begin{pmatrix} 0&0&0&0&1&0&0&0&0&0&1&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 1&1&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0\\ 1&1&0&1&1&0&0&0&1&0&0&1&0&0&0&1&0\\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1\\ 1&1&0&1&0&1&1&0&0&0&0&0&0&1&1&1&0\\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&1\\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&1&1&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&1&0&0&0&0&0&0&1&0&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 1&1&0&1&0&1&1&0&0&0&0&0&0&1&1&1&0\\ 1&1&0&1&0&1&1&0&0&0&0&0&0&1&1&1&0\\ 0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 1&1&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&1&0&0&0&0&0&1&1&0&0&0&0&0 \end{pmatrix} \end{array} \qquad \nabla = \begin{array}{rc} & \begin{array}{ccccccccccccccccc} a&b&c&d&e&f&g&h&i&j&k&l&m&n&o&p&q \end{array} \\ \begin{array}{r} 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15\\16\\17\\18\\19\\20\\21 \end{array} & \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 1&1&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&1&1&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&1&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&1&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 1&1&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \end{array}$$

Fig. 3.15 A relation R and its fringe ∇

For the purpose intended, it is interesting to investigate the symmetric idempotents $\nabla;\nabla^{\mathsf{T}}$ and $\nabla^{\mathsf{T}};\nabla$ derived from the fringe and shown in Fig. 3.16. Their nature of being symmetric idempotents is immediately recognized. In Sect. 3.6, these have already been arranged nicely, and we apply this again here.

$$\begin{array}{cc}
 & \begin{array}{ccccccccccccccccccccc}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15&16&17&18&19&20&21\end{array} \\
\begin{array}{c}1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15\\16\\17\\18\\19\\20\\21\end{array} &
\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}
\end{array}$$

$$\begin{array}{cc}
 & \begin{array}{ccccccccccccccccc}a&b&c&d&e&f&g&h&i&j&k&l&m&n&o&p&q\end{array} \\
\begin{array}{c}a\\b\\c\\d\\e\\f\\g\\h\\i\\j\\k\\l\\m\\n\\o\\p\\q\end{array} &
\begin{pmatrix}
1&1&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0\\
1&1&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&1&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&1&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&1&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1
\end{pmatrix}
\end{array}$$

$\nabla;\nabla^{\mathsf{T}}$ $\qquad$ $\nabla^{\mathsf{T}};\nabla$

Fig. 3.16 Symmetric idempotents derived from the fringe ∇

These symmetric idempotents are then compared with the row equivalence $\Xi(\nabla) := \mathtt{syq}(\nabla^{\mathsf{T}}, \nabla^{\mathsf{T}})$ and the column equivalence $\Psi(\nabla) := \mathtt{syq}(\nabla, \nabla)$ of ∇. One will observe that they coincide with $\nabla;\nabla^{\mathsf{T}}$ resp. $\nabla^{\mathsf{T}};\nabla$ shown above up to a big dispersed diagonal square made up of empty rows $\times$ empty rows, resp. empty columns $\times$ empty columns.

$$\begin{array}{cc}
 & \begin{array}{ccccccccccccccccccccc}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15&16&17&18&19&20&21\end{array} \\
\begin{array}{c}1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15\\16\\17\\18\\19\\20\\21\end{array} &
\begin{pmatrix}
1&1&0&1&0&0&1&0&1&1&1&0&1&1&0&0&1&1&1&0&1\\
1&1&0&1&0&0&1&0&1&1&1&0&1&1&0&0&1&1&1&0&1\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
1&1&0&1&0&0&1&0&1&1&1&0&1&1&0&0&1&1&1&0&1\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0\\
1&1&0&1&0&0&1&0&1&1&1&0&1&1&0&0&1&1&1&0&1\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&1&0&0&1&0&1&1&1&0&1&1&0&0&1&1&1&0&1\\
1&1&0&1&0&0&1&0&1&1&1&0&1&1&0&0&1&1&1&0&1\\
1&1&0&1&0&0&1&0&1&1&1&0&1&1&0&0&1&1&1&0&1\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
1&1&0&1&0&0&1&0&1&1&1&0&1&1&0&0&1&1&1&0&1\\
1&1&0&1&0&0&1&0&1&1&1&0&1&1&0&0&1&1&1&0&1\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0\\
1&1&0&1&0&0&1&0&1&1&1&0&1&1&0&0&1&1&1&0&1\\
1&1&0&1&0&0&1&0&1&1&1&0&1&1&0&0&1&1&1&0&1\\
1&1&0&1&0&0&1&0&1&1&1&0&1&1&0&0&1&1&1&0&1\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
1&1&0&1&0&0&1&0&1&1&1&0&1&1&0&0&1&1&1&0&1
\end{pmatrix}
\end{array}$$

$$\begin{array}{cc}
 & \begin{array}{ccccccccccccccccc}a&b&c&d&e&f&g&h&i&j&k&l&m&n&o&p&q\end{array} \\
\begin{array}{c}a\\b\\c\\d\\e\\f\\g\\h\\i\\j\\k\\l\\m\\n\\o\\p\\q\end{array} &
\begin{pmatrix}
1&1&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0\\
1&1&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&1&0&0&0&1&0&1&1&1&0&1&0&0\\
0&0&0&0&0&1&1&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&1&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1\\
0&0&0&0&1&0&0&0&1&0&1&1&1&0&1&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&1&0&1&1&1&0&1&0&0\\
0&0&0&0&1&0&0&0&1&0&1&1&1&0&1&0&0\\
0&0&0&0&1&0&0&0&1&0&1&1&1&0&1&0&0\\
0&0&0&0&0&1&1&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&1&0&0&0&1&0&1&1&1&0&1&0&0\\
1&1&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1
\end{pmatrix}
\end{array}$$

$\Xi := \Xi(\nabla)$ $\qquad$ $\Psi := \Psi(\nabla)$

Fig. 3.17 Row and column equivalence of ∇ turn out to be the same as in 3.16 — up to the universal relations between empty rows, resp. columns

It is now advisable to concentrate on the classes with respect to these equivalences. Between these classes holds the bijection

$$\psi := \begin{array}{r} \\ [3\!\rightarrow] \\ [5\!\rightarrow] \\ [6\!\rightarrow] \\ [8\!\rightarrow] \\ [12\!\rightarrow] \end{array} \begin{array}{c} \begin{array}{ccccc} [a\!\rightarrow] & [c\!\rightarrow] & [f\!\rightarrow] & [h\!\rightarrow] & [j\!\rightarrow] \end{array} \\ \left(\begin{array}{ccccc} 1&0&0&0&0 \\ 0&0&0&1&0 \\ 0&0&1&0&0 \\ 0&0&0&0&1 \\ 0&1&0&0&0 \end{array}\right) \end{array}.$$

We have already been able to nicely arrange a bijective mapping and apply this technique here, however, played back to the classes. This will be clear when rearranging the fringe in Fig. 3.18 on the left. The right side of this figure shows the so-called block-transitive kernel of the original relation R in an already rearranged form.

$$\begin{array}{r} \\ 12\\8\\6\\15\\16\\5\\3\\20\\1\\2\\4\\7\\9\\10\\11\\13\\14\\17\\18\\19\\21 \end{array} \begin{array}{c} \begin{array}{ccccccccccccccccc} c&j&f&g&n&h&q&a&b&d&p&e&i&k&l&m&o \end{array} \\ \left(\begin{array}{c|c|ccc|cc|cccc|cccccc} 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ \hline 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ \hline 0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ \hline 0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0 \\ \hline 0&0&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0 \\ \hline 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{array}\right) \end{array} \quad \begin{array}{r} \\ 12\\8\\6\\15\\16\\5\\3\\20\\1\\2\\4\\7\\9\\10\\11\\13\\14\\17\\18\\19\\21 \end{array} \begin{array}{c} \begin{array}{ccccccccccccccccc} c&j&f&g&n&h&q&a&b&d&p&e&i&k&l&m&o \end{array} \\ \left(\begin{array}{c|c|ccc|cc|cccc|cccccc} 1&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0 \\ \hline 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ \hline 0&0&1&1&1&0&0&1&1&1&1&0&0&0&0&0&0 \\ 0&0&1&1&1&0&0&1&1&1&1&0&0&0&0&0&0 \\ 0&0&1&1&1&0&0&1&1&1&1&0&0&0&0&0&0 \\ \hline 0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0 \\ \hline 0&0&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0 \\ \hline 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{array}\right) \end{array}$$

Fig. 3.18 Fringe ∇ and block-transitive kernel of R rearranged and shown with subdivisions

The remarkable fact is that the block-transitive kernel (when shrinking rectangular blocks to single entries) is transitive and arranged to upper triangle. The block-triangular zone below the fringe and inside the rectangular frame defined by the fringe is necessarily empty. The rest of the rearranged original may appear rather arbitrary as shown in Fig. 3.19.

This situation gave rise to defining a relation R to be **block-transitive** when the fringe ∇ "frames" the relation, to be expressed either by $R \subseteq \nabla;\mathbb{T}$, $R \subseteq \mathbb{T};\nabla$ or else by $R \subseteq \nabla;\mathbb{T};\nabla$; see Fig. 3.18 on the right.

$$\begin{array}{cc}
 & \begin{array}{ccccccccccccccccc} c & j & f & g & n & h & q & a & b & d & p & e & i & k & l & m & o \end{array} \\
\begin{array}{r} 12\\8\\6\\15\\16\\5\\3\\20\\1\\2\\4\\7\\9\\10\\11\\13\\14\\17\\18\\19\\21 \end{array} &
\left(\begin{array}{c|c|ccc|cc|cccc|cccccc}
1&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&1\\ \hline
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ \hline
0&0&1&1&1&0&0&1&1&1&1&0&0&0&0&0&1\\
0&0&1&1&1&0&0&1&1&1&1&0&0&0&0&0&1\\
0&0&1&1&1&0&0&1&1&1&1&0&0&0&0&0&1\\ \hline
0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0\\ \hline
0&0&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0\\ \hline
0&0&0&0&0&0&0&0&0&0&0&1&0&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&1&1&1&1&1&0&1&0&0\\
0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&1\\
0&1&0&0&0&0&0&0&0&0&0&0&0&1&1&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&1&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&1&1&0&0
\end{array}\right)
\end{array}
\qquad
\begin{array}{cc}
 & \begin{array}{ccccccccccc} c\uparrow & j\uparrow & f\uparrow & g\uparrow & n\uparrow & h\uparrow & q\uparrow & a\uparrow & b\uparrow & d\uparrow & p\uparrow \end{array} \\
\begin{array}{r} 12\rightarrow\\8\rightarrow\\6\rightarrow\\15\rightarrow\\16\rightarrow\\5\rightarrow\\3\rightarrow\\20\rightarrow \end{array} &
\left(\begin{array}{ccccccccccc}
1&0&0&0&0&1&1&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0\\
0&0&1&1&1&0&0&1&1&1&1\\
0&0&1&1&1&0&0&1&1&1&1\\
0&0&1&1&1&0&0&1&1&1&1\\
0&0&0&0&0&1&1&0&0&0&0\\
0&0&0&0&0&0&0&1&1&1&1\\
0&0&0&0&0&0&0&1&1&1&1
\end{array}\right)
\end{array}$$

Fig. 3.19 Original R rearranged and shown with subdivisions

Considering this, it may be interesting to study the so-called block-transitive kernel — shown in an extruded form on the right of Fig. 3.19 — separately.

4 Rearranging orderings

Structurally near to diagonal matrices are upper triangular ones. The following examples demonstrate convincingly that an order may enjoy nice algebraic properties which, however, become really obvious and give intuition no earlier than when represented adequately by algebraic visualization.

4.1 Classifying orders

It is sometimes helpful to be able to classify an arbitrary homogeneous relation as to possible order properties. When we attribute to the typical order structures their properties, it looks like the left table in Fig. 12.5 of [Sch11], which we here reproduce.

		linear strictorder	weakorder	semiorder	interval-order	strictorder
$R;R \subseteq R$	transitive	$\bullet \mid \circ$	$\bullet$	$\bullet$	$\bullet$	$\bullet \mid \circ$
$R \subseteq \overline{R}^{\mathsf{T}}$	asymmetric	$\bullet \mid$	$\bullet \mid$	$\bullet \quad \circ$	$\bullet \quad \circ$	$\bullet \mid$
$R \subseteq \overline{\mathbb{I}}$	irreflexive	$\bullet \quad \circ$	$\bullet$	$\bullet \mid$	$\bullet \mid$	$\bullet \quad \circ$
$R;\overline{R}^{\mathsf{T}};R \subseteq R$	Ferrers	$\bullet$	$\bullet$	$\bullet \mid \circ$	$\bullet \mid \circ$	—
$R;R;\overline{R}^{\mathsf{T}} \subseteq R$	semi-transitive	$\bullet$	$\bullet$	$\bullet \mid \circ$	—	—
$\overline{R};\overline{R} \subseteq \overline{R}$	negatively transitive	$\bullet$	$\bullet \mid$	—	—	—
$\overline{\mathbb{I}} \subseteq R \cup R^{\mathsf{T}}$	semi-connex	$\bullet \mid \circ$	—	—	—	—

Fig. 4.1 Types of strictorders with 'spanning' subsets of their properties — indicated by "|", resp. "∘"

Given a relation R, these properties are readily checked, and one has immediately the qualification of a given relation R to be a linear strictorder, a weakorder, a semiorder, an intervalorder or a strictorder.

Each of these five types offers different visualizations that are not so commonly known. They may, however, now be bound together in one program with a differentiating case distinction cascade as a start. This avoids treating a linear strictorder as a semiorder, e.g. — which it, of course, also is.

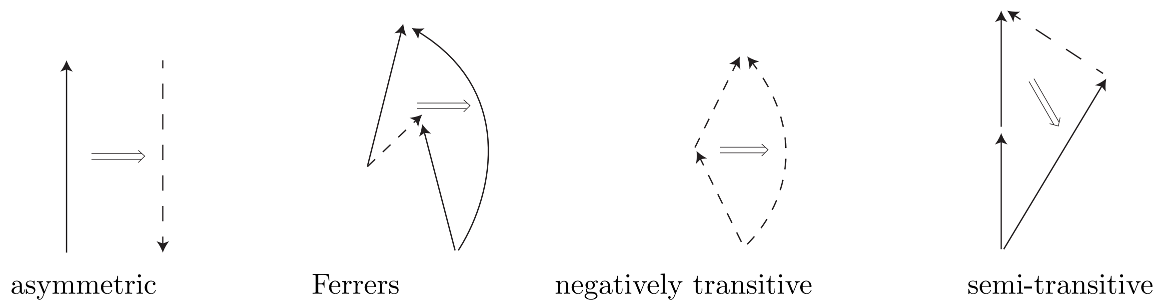


Fig. 4.2 Strictorder properties (dashed lines indicate negation)

4.2 Rearranging a linear strictorder

The task announced is trivial at first sight; every programmer will be able to apply some sorting — available on every computer or programming platform. It is less obvious how this might be integrated in an algebraic environment that allows reasoning as well as computing, which, however, shall not be presented here. Not least is it necessary to also permute the row and column inscriptions to obtain — after algebraic visualization — the same relation though presented differently.

The relation R of Fig. 4.3 is a linear strictorder. It may be arranged by an algebraically derived simultaneous permutation P of rows and columns to upper triangle as

$$R = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array} \qquad P = \begin{array}{c c} & \begin{array}{cccc} 3 & 2 & 4 & 1 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array} \qquad P^{\mathsf{T}} ; R ; P = \begin{array}{c c} & \begin{array}{cccc} 3 & 2 & 4 & 1 \end{array} \\ \begin{array}{c} 3 \\ 2 \\ 4 \\ 1 \end{array} & \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Fig. 4.3 Rearranging a linear strictorder

See the earlier discussion in Sect. 3.2.

4.3 Rearranging a weakorder

One is sometimes confronted with a relation like R of Fig. 4.4, that turns out to be a weakorder. In such a case it may be helpful to obtain a re-arranged version of that relation.

To rearrange a weakorder to upper triangle form may also be achieved by many programmers via standard techniques. But one has to apply the respective permutation to the row and columns inscriptions which may be more tricky. The basic idea is to embed the weakorder with an arbitrary linear strictorder on that set into a linear strictorder and then rearrange this — in the already known way.

One may use an arbitrary linear strictorder L_0, e.g., that on the rows or its reverse form. In any case, $L := R \cup (L_0 \cap \overline{R}^{\mathsf{T}}) \supseteq R$ will be an embedding of R into a linear strictorder. The relation R of Fig. 4.4 is a weakorder that is not just a linear strictorder. It may be arranged by an algebraically derived simultaneous permutation P of rows and columns to upper triangle as shown.

$$
\begin{array}{rc}
 & \begin{array}{ccccccccccc}1&2&3&4&5&6&7&8&9&10&11\end{array} \\
\begin{array}{r}1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\end{array} &
\left(\begin{array}{ccccccccccc}
0&1&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&1&1&1&1&1&0&1&1\\
1&1&0&0&1&1&1&0&0&0&1\\
1&1&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&1&0&0&0&0&0\\
1&1&0&0&1&1&1&0&0&0&1\\
1&1&0&1&1&1&1&1&0&1&1\\
1&1&0&0&1&1&1&0&0&0&1\\
1&1&0&0&0&1&0&0&0&0&0
\end{array}\right)
\end{array}
\quad
\begin{array}{rc}
 & \begin{array}{ccccccccccc}3&9&4&8&10&5&7&11&1&2&6\end{array} \\
\begin{array}{r}1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\end{array} &
\left(\begin{array}{ccccccccccc}
0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&1&0\\
1&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0
\end{array}\right)
\end{array}
\quad
\begin{array}{rc}
 & \begin{array}{ccccccccccc}3&9&4&8&10&5&7&11&1&2&6\end{array} \\
\begin{array}{r}3\\9\\4\\8\\10\\5\\7\\11\\1\\2\\6\end{array} &
\left(\begin{array}{cc|ccc|ccc|c|cc}
0&0&1&1&1&1&1&1&1&1&1\\
0&0&1&1&1&1&1&1&1&1&1\\ \hline
0&0&0&0&0&1&1&1&1&1&1\\
0&0&0&0&0&1&1&1&1&1&1\\
0&0&0&0&0&1&1&1&1&1&1\\ \hline
0&0&0&0&0&0&0&0&1&1&1\\
0&0&0&0&0&0&0&0&1&1&1\\
0&0&0&0&0&0&0&0&1&1&1\\ \hline
0&0&0&0&0&0&0&0&0&1&1\\ \hline
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0
\end{array}\right)
\end{array}
$$

Fig. 4.4 Weakorder R with permutation P and rearranged version $P^{\mathsf{T}};R;P$

It is characteristic for a weakorder that the upper triangle "fully touches" the diagonal of square blocks — different from what happens for the semiorder and intervalorder yet to come.

One will have observed that a weakorder is characterized by the fact that it may rearranged to "blockwise" linear strictorder. The diagonal blocks are square. Modulo the equivalence provided by the diagonal squares, a weakorder reduces to a linear strictorder on the respective classes; see Fig.4.5.

$$
\begin{array}{rc}
 & \begin{array}{ccccccccccc}1&2&3&4&5&6&7&8&9&10&11\end{array} \\
\begin{array}{r}1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\end{array} &
\left(\begin{array}{ccccccccccc}
1&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&1&0&0&0&0&0\\
0&0&1&0&0&0&0&0&1&0&0\\
0&0&0&1&0&0&0&1&0&1&0\\
0&0&0&0&1&0&1&0&0&0&1\\
0&1&0&0&0&1&0&0&0&0&0\\
0&0&0&0&1&0&1&0&0&0&1\\
0&0&0&1&0&0&0&1&0&1&0\\
0&0&1&0&0&0&0&0&1&0&0\\
0&0&0&1&0&0&0&1&0&1&0\\
0&0&0&0&1&0&1&0&0&0&1
\end{array}\right)
\end{array}
\quad
\begin{array}{rc}
 & \begin{array}{ccccc}[1]&[2]&[3]&[4]&[5]\end{array} \\
\begin{array}{r}1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\end{array} &
\left(\begin{array}{ccccc}
1&0&0&0&0\\
0&1&0&0&0\\
0&0&1&0&0\\
0&0&0&1&0\\
0&0&0&0&1\\
0&1&0&0&0\\
0&0&0&0&1\\
0&0&0&1&0\\
0&0&1&0&0\\
0&0&0&1&0\\
0&0&0&0&1
\end{array}\right)
\end{array}
\quad
\begin{array}{rc}
 & \begin{array}{ccccc}[1]&[2]&[3]&[4]&[5]\end{array} \\
\begin{array}{r}[1]\\ [2]\\ [3]\\ [4]\\ [5]\end{array} &
\left(\begin{array}{ccccc}
0&1&0&0&0\\
0&0&0&0&0\\
1&1&0&1&1\\
1&1&0&0&1\\
1&1&0&0&0
\end{array}\right)
\end{array}
\quad
\begin{array}{rc}
 & \begin{array}{ccccc}[3]&[4]&[5]&[1]&[2]\end{array} \\
\begin{array}{r}[3]\\ [4]\\ [5]\\ [1]\\ [2]\end{array} &
\left(\begin{array}{ccccc}
0&1&1&1&1\\
0&0&1&1&1\\
0&0&0&1&1\\
0&0&0&0&1\\
0&0&0&0&0
\end{array}\right)
\end{array}
$$

Fig. 4.5 Row equivalence of the weakorder with natural projection η and linear strictorder $\eta^{\mathsf{T}};R;\eta$ of classes

For a weakorder, row equivalence coincides with column equivalence.

4.4 Rearranging a semiorder

When proceeding to semiorders, one will allow not necessarily square diagonal blocks under the same regime, i.e., with the upper triangle filled. In this case, row equivalence is different from column equivalence.

The study of this situation is fairly difficult. The basic idea is again the same: We have learned to bring a weakorder to upper triangle form embedding it into some linear strictorder. Now, the situation is greatly facilitated by the fact that every semiorder R has a uniquely determined weakorder in which it acts as some sort of a threshold — and we had already been successful in rearranging a weakorder properly. This embedding is achieved by simply defining $W := \overline{R}^{\mathsf{T}};R \cup R;\overline{R}^{\mathsf{T}}$.

$$
\begin{array}{c@{}c}
 & \begin{array}{ccccccccccccc} a & b & c & d & e & f & g & h & i & j & k & l & m \end{array} \\
\begin{array}{c} a\\ b\\ c\\ d\\ e\\ f\\ g\\ h\\ i\\ j\\ k\\ l\\ m \end{array} &
\begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0
\end{pmatrix}
\end{array}
$$

R

$$
\begin{array}{c@{}c}
 & \begin{array}{ccccccccccccc} i & m & a & f & j & b & e & l & c & g & h & d & k \end{array} \\
\begin{array}{c} a\\ b\\ c\\ d\\ e\\ f\\ g\\ h\\ i\\ j\\ k\\ l\\ m \end{array} &
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

P

$$
\begin{array}{c@{}c}
 & \begin{array}{ccccccccccccc} i & m & a & f & j & b & e & l & c & g & h & d & k \end{array} \\
\begin{array}{c} i\\ m\\ a\\ f\\ j\\ b\\ e\\ l\\ c\\ g\\ h\\ d\\ k \end{array} &
\left(\begin{array}{cc|ccc|ccc|c|cc|cc}
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
\end{array}
$$

$P^{\mathsf{T}};R;P$

Fig. 4.6 Semiorder R with arrangement P to subdivided upper triangle

When we look at W and the fringes ∇_R, ∇_W in Fig. 4.7, we will not recognize too much similarity.

$$
\begin{array}{c@{}c}
 & \begin{array}{ccccccccccccc} a & b & c & d & e & f & g & h & i & j & k & l & m \end{array} \\
\begin{array}{c} a\\ b\\ c\\ d\\ e\\ f\\ g\\ h\\ i\\ j\\ k\\ l\\ m \end{array} &
\begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0
\end{pmatrix}
\end{array}
$$

W

$$
\begin{array}{c@{}c}
 & \begin{array}{ccccccccccccc} a & b & c & d & e & f & g & h & i & j & k & l & m \end{array} \\
\begin{array}{c} a\\ b\\ c\\ d\\ e\\ f\\ g\\ h\\ i\\ j\\ k\\ l\\ m \end{array} &
\begin{pmatrix}
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

∇_R

$$
\begin{array}{c@{}c}
 & \begin{array}{ccccccccccccc} a & b & c & d & e & f & g & h & i & j & k & l & m \end{array} \\
\begin{array}{c} a\\ b\\ c\\ d\\ e\\ f\\ g\\ h\\ i\\ j\\ k\\ l\\ m \end{array} &
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

∇_W

Fig. 4.7 Weakorder $W \supseteq R$ and fringes of R and W

But now we apply the permutation P, which we have derived from W as demonstrated earlier and have a better chance to observe how everything fits together.

$$
\begin{array}{c@{}c}
 & \begin{array}{ccccccccccccc} i & m & a & f & j & b & e & l & c & g & h & d & k \end{array} \\
\begin{array}{c} i\\ m\\ a\\ f\\ j\\ b\\ e\\ l\\ c\\ g\\ h\\ d\\ k \end{array} &
\begin{pmatrix}
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

W

$$
\begin{array}{c@{}c}
 & \begin{array}{ccccccccccccc} i & m & a & f & j & b & e & l & c & g & h & d & k \end{array} \\
\begin{array}{c} i\\ m\\ a\\ f\\ j\\ b\\ e\\ l\\ c\\ g\\ h\\ d\\ k \end{array} &
\begin{pmatrix}
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

∇_R

$$
\begin{array}{c@{}c}
 & \begin{array}{ccccccccccccc} i & m & a & f & j & b & e & l & c & g & h & d & k \end{array} \\
\begin{array}{c} i\\ m\\ a\\ f\\ j\\ b\\ e\\ l\\ c\\ g\\ h\\ d\\ k \end{array} &
\begin{pmatrix}
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

∇_W

Fig. 4.8 Weakorder and fringes rearranged via P

4.5 Rearranging an intervalorder

A weakorder was characterized by a diagonal of square matrices. The semiorder was slightly more liberal admitting a diagonal of possibly rectangular matrices. In both cases, it was possible to rearrange by simultaneous permutation to upper triangle. For an intervalorder, the condition requires even less, admitting *independent* permutation of rows and columns. In all cases we can via quotient forming — be it for rows and columns independently — end up with a linear strictorder.

The following R_0 is indeed an intervalorder that is not just a semiorder:

$$R_0 = \begin{array}{r@{}l} & \begin{array}{ccccccccccccc} 1&2&3&4&5&6&7&8&9&10&11&12&13 \end{array} \\ \begin{array}{r} 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13 \end{array} & \left(\begin{array}{ccccccccccccc}
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&1&0&0&1&0&0&1\\
0&1&0&1&1&1&1&1&1&1&0&1&1\\
0&0&0&0&0&1&1&0&0&0&0&0&1\\
0&0&0&1&0&1&1&0&0&1&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&1\\
0&1&0&1&0&1&1&0&1&1&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&1&0&1&1&1&1&1&1&1&0&1&1\\
0&1&0&1&0&1&1&0&0&1&0&0&1\\
0&0&0&0&0&1&0&0&0&0&0&0&0
\end{array}\right) \end{array}
\qquad
R := P_R^\mathsf{T};R;P_C = \begin{array}{r@{}l} & \begin{array}{ccccccccccccc} 1&3&11&5&8&12&9&2&4&10&7&13&6 \end{array} \\ \begin{array}{r} 3\\11\\8\\12\\5\\2\\4\\7\\10\\13\\1\\6\\9 \end{array} & \left(\begin{array}{ccc|ccc|c|c|c|c|c|c|c}
0&0&0&1&1&1&1&1&1&1&1&1&1\\
0&0&0&1&1&1&1&1&1&1&1&1&1\\ \hline
0&0&0&0&0&0&1&1&1&1&1&1&1\\ \hline
0&0&0&0&0&0&0&1&1&1&1&1&1\\ \hline
0&0&0&0&0&0&0&0&1&1&1&1&1\\ \hline
0&0&0&0&0&0&0&0&0&1&1&1&1\\ \hline
0&0&0&0&0&0&0&0&0&0&1&1&1\\ \hline
0&0&0&0&0&0&0&0&0&0&0&1&1\\ \hline
0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&1\\ \hline
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}\right) \end{array}$$

$$P_R = \begin{array}{r@{}l} & \begin{array}{ccccccccccccc} 3&11&8&12&5&2&4&7&10&13&1&6&9 \end{array} \\ \begin{array}{r} 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13 \end{array} & \left(\begin{array}{ccccccccccccc}
0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0
\end{array}\right) \end{array}
\qquad
P_C = \begin{array}{r@{}l} & \begin{array}{ccccccccccccc} 1&3&11&5&8&12&9&2&4&10&7&13&6 \end{array} \\ \begin{array}{r} 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13 \end{array} & \left(\begin{array}{ccccccccccccc}
1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0
\end{array}\right) \end{array}$$

Fig. 4.9 Rearranging an intervalorder

It is an essential property of such an intervalorder that it may be rearranged — permuting rows and columns independently with P_R, P_C — to a still square but now heterogeneous version R, which is also presented subdivided according to the equivalences to exhibit the weakorder on row and column equivalences as the basis for determining P_R, P_C. In what follows, we show what happens to R, which is far easier to perceive, but means the action on R_0.

For $\overline{R}$ we determine the (heterogeneous) fringe $\nabla := \overline{R} \cap \overline{\overline{R};R^\mathsf{T};\overline{R}}$ that appears somehow as a bijection of possibly rectangular universal blocks simulating a diagonal in order to demonstrate how the corresponding pair $\Xi_R := \nabla;\nabla^\mathsf{T}$, $\Xi_C = \nabla^\mathsf{T};\nabla$ of row and column equivalence acts.

$$
\begin{array}{rc}
 & \begin{matrix} 1&3&11&5&8&12&9&2&4&10&7&13&6 \end{matrix} \\
\begin{matrix} 3\\11\\8\\12\\5\\2\\4\\7\\10\\13\\1\\6\\9 \end{matrix} &
\begin{pmatrix}
1&1&1&0&0&0&0&0&0&0&0&0&0\\
1&1&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&1&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix}
\end{array}
\quad
\begin{array}{c}
\begin{matrix} 3&11&8&12&5&2&4&7&10&13&1&6&9 \end{matrix} \\
\begin{pmatrix}
1&1&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&0&0&0\\
0&0&0&0&0&0&0&0&1&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&1\\
0&0&0&0&0&0&0&0&0&0&1&1&1\\
0&0&0&0&0&0&0&0&0&0&1&1&1
\end{pmatrix}
\end{array}
\quad
\begin{array}{c}
\begin{matrix} 1&3&11&5&8&12&9&2&4&10&7&13&6 \end{matrix} \\
\begin{pmatrix}
1&1&1&0&0&0&0&0&0&0&0&0&0\\
1&1&1&0&0&0&0&0&0&0&0&0&0\\
1&1&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&1&1&0&0&0&0&0&0&0\\
0&0&0&1&1&1&0&0&0&0&0&0&0\\
0&0&0&1&1&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix}
\end{array}
$$

$$\nabla \qquad\qquad \Xi_R \qquad\qquad \Xi_C$$

Fig. 4.10 Fringe and its row and column equivalence — already rearranged

We observe a bijection of classes resembling the fringe as well as a linear strictordering derived from R on these classes (presented only for the column classes).

$$
\beta := \eta_1^{\mathsf{T}};\nabla;\eta_2 =
\begin{array}{rc}
 & \begin{matrix} [1]&[5]&[9]&[2]&[4]&[10]&[7]&[13]&[6] \end{matrix} \\
\begin{matrix} [3]\\ [8]\\ [12]\\ [5]\\ [2]\\ [4]\\ [7]\\ [10]\\ [1] \end{matrix} &
\begin{pmatrix}
1&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&1
\end{pmatrix}
\end{array}
\qquad
C := \beta^{\mathsf{T}};\eta_1^{\mathsf{T}};R;\eta_2 =
\begin{array}{rc}
 & \begin{matrix} [1]&[5]&[9]&[2]&[4]&[10]&[7]&[13]&[6] \end{matrix} \\
\begin{matrix} [1]\\ [5]\\ [9]\\ [2]\\ [4]\\ [10]\\ [7]\\ [13]\\ [6] \end{matrix} &
\begin{pmatrix}
0&1&1&1&1&1&1&1&1\\
0&0&1&1&1&1&1&1&1\\
0&0&0&1&1&1&1&1&1\\
0&0&0&0&1&1&1&1&1\\
0&0&0&0&0&1&1&1&1\\
0&0&0&0&0&0&1&1&1\\
0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0
\end{pmatrix}
\end{array}
$$

The linear strictorder on column classes may be directly derived from the original intervalorder since R is by definition a Ferrers relation.

Using C, we are in a position to factorize the permuted version R of the given relation as $R = f;C;g^{\mathsf{T}}$ with f, g that satisfy $g^{\mathsf{T}};f \subseteq \mathbb{I} \cup C$, as can be seen from

$$
\begin{array}{rc}
 & \begin{matrix} [1]&[5]&[9]&[2]&[4]&[10]&[7]&[13]&[6] \end{matrix} \\
\begin{matrix} 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13 \end{matrix} &
\begin{pmatrix}
0&0&0&0&0&0&0&0&1\\
0&0&0&0&1&0&0&0&0\\
1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0\\
0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&1&0&0\\
0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&1&0\\
1&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0
\end{pmatrix} \\
 & f := P_R;\eta_1;\beta
\end{array}
\quad
\begin{array}{rc}
 & \begin{matrix} [1]&[5]&[9]&[2]&[4]&[10]&[7]&[13]&[6] \end{matrix} \\
\begin{matrix} 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13 \end{matrix} &
\begin{pmatrix}
1&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0\\
0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&1&0&0\\
0&1&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0\\
1&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0
\end{pmatrix} \\
 & g := P_C;\eta_2
\end{array}
\quad
\begin{array}{rc}
 & \begin{matrix} [1]&[5]&[9]&[2]&[4]&[10]&[7]&[13]&[6] \end{matrix} \\
\begin{matrix} [1]\\ [5]\\ [9]\\ [2]\\ [4]\\ [10]\\ [7]\\ [13]\\ [6] \end{matrix} &
\begin{pmatrix}
1&0&0&0&0&0&0&0&1\\
0&1&1&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1\\
0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&1
\end{pmatrix} \\
 & g^{\mathsf{T}};f
\end{array}
$$

Fig. 4.11 Factors f, g of an intervalorder

From here on, we again work on the still homogeneous original R_0 and determine the interval intersection relation

$$B := \overline{R_0 \cup R_0^{\mathsf{T}}} = \begin{array}{r} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \end{array} \begin{array}{c} \begin{array}{ccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

and its maxcliques factorization $B =: M^{\mathsf{T}}; M$ with

$$M = \begin{array}{r} \\ \{1,2,4,9\}\to \\ \{1,2,5,9\}\to \\ \{1,6,9\}\to \\ \{1,4,9,10\}\to \\ \{1,7,9,10\}\to \\ \{1,3,11\}\to \\ \{1,5,8,12\}\to \\ \{1,5,9,12\}\to \\ \{1,9,10,13\}\to \end{array} \begin{array}{c} \begin{array}{ccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

Then $C' := M; R_0; M^{\mathsf{T}}$ is necessarily a linear strictorder on these maxcliques

$$\begin{array}{r} \\ \{1,2,4,9\}\to \\ \{1,2,5,9\}\to \\ \{1,6,9\}\to \\ \{1,4,9,10\}\to \\ \{1,7,9,10\}\to \\ \{1,3,11\}\to \\ \{1,5,8,12\}\to \\ \{1,5,9,12\}\to \\ \{1,9,10,13\}\to \end{array} \begin{array}{c} \begin{array}{ccccccccc} \{1,2,4,9\}\to & \{1,2,5,9\}\to & \{1,6,9\}\to & \{1,4,9,10\}\to & \{1,7,9,10\}\to & \{1,3,11\}\to & \{1,5,8,12\}\to & \{1,5,9,12\}\to & \{1,9,10,13\}\to \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array},$$

which we can rearrange as

$$C_{\text{rearranged}} = \begin{array}{r} \\ \{1,3,11\}\to \\ \{1,5,8,12\}\to \\ \{1,5,9,12\}\to \\ \{1,2,5,9\}\to \\ \{1,2,4,9\}\to \\ \{1,4,9,10\}\to \\ \{1,7,9,10\}\to \\ \{1,9,10,13\}\to \\ \{1,6,9\}\to \end{array} \begin{array}{c} \begin{array}{ccccccccc} \{1,3,11\}\to & \{1,5,8,12\}\to & \{1,5,9,12\}\to & \{1,2,5,9\}\to & \{1,2,4,9\}\to & \{1,4,9,10\}\to & \{1,7,9,10\}\to & \{1,9,10,13\}\to & \{1,6,9\}\to \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

This enables us to get the so-called consecutive $\mathbf{1}$'s version

$$
C^{\mathsf{T}}_{\text{rearranged}};M = \begin{array}{r} \\ \{1,3,11\}\rightarrow \\ \{1,5,8,12\}\rightarrow \\ \{1,5,9,12\}\rightarrow \\ \{1,2,5,9\}\rightarrow \\ \{1,2,4,9\}\rightarrow \\ \{1,4,9,10\}\rightarrow \\ \{1,7,9,10\}\rightarrow \\ \{1,9,10,13\}\rightarrow \\ \{1,6,9\}\rightarrow \end{array}
\begin{array}{c}
\begin{array}{ccccccccccccc} 1&2&3&4&5&6&7&8&9&10&11&12&13 \end{array} \\
\begin{pmatrix}
1&0&1&0&0&0&0&0&0&0&1&0&0\\
1&0&0&0&1&0&0&1&0&0&0&1&0\\
1&0&0&0&1&0&0&0&1&0&0&1&0\\
1&1&0&0&1&0&0&0&1&0&0&0&0\\
1&1&0&1&0&0&0&0&1&0&0&0&0\\
1&0&0&1&0&0&0&0&1&1&0&0&0\\
1&0&0&0&0&0&1&0&1&1&0&0&0\\
1&0&0&0&0&0&0&0&1&1&0&0&1\\
1&0&0&0&0&1&0&0&1&0&0&0&0
\end{pmatrix}
\end{array}.
$$

It shows indeed that coherent vertical intervals ($\approx$ consecutive **1**'s) of maxcliques of this linear strictorder may be identified, justifying to call this an intervalorder!

4.6 Rearranging a block-transitive strictorder

Before going to the completely general version of a strictorder, we consider the intermediate case of a block-transitive strictorder. It offers two possibilities, namely to be investigated as a strictorder or to be studied as something that allows extension to an intervalorder.

In the former case, it will be used that every strictorder may be sharpened (in an in no way uniquely determined form) to a linear strictorder via the well-known iterative Szpilrajn construction: Spot a pair of points x, y such that neither x is below y nor the other way round and then decide voluntarily for the order of these two — and make the relation transitive again.

The relation

$$
R = \begin{array}{r} \\ 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13 \end{array}
\begin{array}{c}
\begin{array}{ccccccccccccc} 1&2&3&4&5&6&7&8&9&10&11&12&13 \end{array} \\
\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&1&0&0&0\\
0&1&0&1&1&1&1&1&1&1&0&1&1\\
0&0&0&0&0&1&1&0&0&0&0&0&1\\
0&0&0&1&0&1&1&0&0&1&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&1\\
0&1&0&0&0&1&1&0&1&1&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&1&0&1&1&1&1&1&1&1&0&1&1\\
0&1&0&0&0&1&0&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0
\end{pmatrix}
\end{array}
$$

is a block-transitive strictorder that is not just an intervalorder. The first method works simply using the so extended strictorder allowing a simultaneously applied permutation as follows:

$$
P = \begin{array}{r} \\ 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13 \end{array}
\begin{array}{c}
\begin{array}{ccccccccccccc} 1&3&11&8&12&2&5&4&7&10&13&6&9 \end{array} \\
\begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0
\end{pmatrix}
\end{array}
\qquad
P^{\mathsf{T}};R;P = \begin{array}{r} \\ 1\\3\\11\\8\\12\\2\\5\\4\\7\\10\\13\\6\\9 \end{array}
\begin{array}{c}
\begin{array}{ccccccccccccc} 1&3&11&8&12&2&5&4&7&10&13&6&9 \end{array} \\
\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&1&1&1&1&1&1&1&1&1\\
0&0&0&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&1&0&0&1&1&1&1&1\\
0&0&0&0&0&1&0&0&0&1&0&1&0\\
0&0&0&0&0&0&0&0&0&1&0&1&0\\
0&0&0&0&0&0&0&1&1&1&1&1&0\\
0&0&0&0&0&0&0&0&1&0&1&1&0\\
0&0&0&0&0&0&0&0&0&0&1&1&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}
\end{array}.
$$

The second method takes into consideration the Ferrers extension to some intervalorder — working mainly in the same way as a Szpilrain extension, but now for blocks. On then applies independent permutations to rows and columns resulting in Fig. 4.12.

$$P_R = \begin{array}{r@{\,}l} & \begin{array}{ccccccccccccc} 3 & 11 & 5 & 8 & 12 & 2 & 4 & 7 & 10 & 13 & 1 & 6 & 9 \end{array} \\ \begin{array}{r} 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13 \end{array} & \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&1&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0&0 \end{pmatrix} \end{array} \qquad \begin{array}{r@{\,}l} & \begin{array}{ccccccccccccc} 1 & 3 & 11 & 5 & 8 & 12 & 4 & 9 & 2 & 10 & 7 & 13 & 6 \end{array} \\ \begin{array}{r} 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13 \end{array} & \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&1&0 \end{pmatrix} \end{array} = P_C$$

$$P_R^\mathsf{T} ; R ; P_C = \begin{array}{r@{\,}l} & \begin{array}{ccccccccccccc} 1 & 3 & 11 & 5 & 8 & 12 & 4 & 9 & 2 & 10 & 7 & 13 & 6 \end{array} \\ \begin{array}{r} 3\\11\\5\\8\\12\\2\\4\\7\\10\\13\\1\\6\\9 \end{array} & \begin{pmatrix} 0&0&0&1&1&1&1&1&1&1&1&1&1\\ 0&0&0&1&1&1&1&1&1&1&1&1&1\\ 0&0&0&0&0&0&1&0&0&1&1&1&1\\ 0&0&0&0&0&0&0&1&1&1&1&1&1\\ 0&0&0&0&0&0&0&0&1&1&0&0&1\\ 0&0&0&0&0&0&0&0&0&1&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&1&1&1\\ 0&0&0&0&0&0&0&0&0&0&0&1&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \end{array} \subseteq \begin{array}{r@{\,}l} & \begin{array}{ccccccccccccc} 1 & 3 & 11 & 5 & 8 & 12 & 4 & 9 & 2 & 10 & 7 & 13 & 6 \end{array} \\ \begin{array}{r} 3\\11\\5\\8\\12\\2\\4\\7\\10\\13\\1\\6\\9 \end{array} & \begin{pmatrix} 0&0&0&1&1&1&1&1&1&1&1&1&1\\ 0&0&0&1&1&1&1&1&1&1&1&1&1\\ 0&0&0&0&0&0&1&1&1&1&1&1&1\\ 0&0&0&0&0&0&0&1&1&1&1&1&1\\ 0&0&0&0&0&0&0&0&1&1&1&1&1\\ 0&0&0&0&0&0&0&0&0&1&1&1&1\\ 0&0&0&0&0&0&0&0&0&0&1&1&1\\ 0&0&0&0&0&0&0&0&0&0&0&1&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \end{array}$$

Fig. 4.12 Row and column permutations derived from the Ferrers extension

Fig. 4.12 shows the result of the Ferrers extension in the lower right position.

4.7 Rearranging a strictorder

Assume a relation R that is simply a strictorder — and neither a block-transitive strictorder nor an intervalorder. It may be arranged by an algebraically derived simultaneous permutation P of rows and columns to upper triangle as already shown above for the block-transitive strictorder.

5 Factorizing relations

When results of any kind of an investigation are to be presented, one will most often see real numbers, usually arranged in some rectangular form. With minor effort one will distinguish in such matrices a relational structure that may be successfully studied with relational means.

5.1 Diclique factorization

When working in numerical mathematics, one will quite often see that a relation or operator is — at least in principle — factorizable into a product with very helpful properties of the factors. It is this general idea we follow here.

For an arbitrary relation R a diclique factorization is possible as follows, following [Sch11, Prop. 11.4]. The representation leads to a product of two factors, $R = U ; V^\mathsf{T}$ with U, V, satisfying

$$\overline{R};V = \overline{U}, \quad \overline{R}^{\mathsf{T}};U = \overline{V}, \quad \mathtt{syq}(U,U) = \mathbb{I}, \quad \mathtt{syq}(V,V) = \mathbb{I},$$

where the latter two algebraically formulated conditions mean nothing else than that columns of U, resp. V, be pairwise different.

$$R = \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7 \end{array}\!\!\begin{array}{c} \begin{array}{ccccccccc} a&b&c&d&e&f&g&h&i \end{array} \\ \begin{pmatrix} 0&0&0&0&0&0&1&0&1\\ 1&0&1&0&0&0&0&0&0\\ 0&0&0&1&0&1&0&1&0\\ 0&0&1&1&0&1&0&0&0\\ 0&1&0&0&1&0&0&0&0\\ 1&0&0&1&1&0&0&0&0\\ 1&0&0&1&0&0&0&0&0 \end{pmatrix} \end{array}$$

$$U = \begin{array}{c|cccccccccccccc} & \{\}\to & \{a\}\to & \{c\}\to & \{a,c\}\to & \{d\}\to & \{a,d\}\to & \{e\}\to & \{b,e\}\to & \{a,d,e\}\to & \{d,f\}\to & \{c,d,f\}\to & \{d,f,h\}\to & \{g,i\}\to & \{a,b,c,d,e,f,g,h,i\}\to \\ \hline 1 & 1&0&0&0&0&0&0&0&0&0&0&0&1&0\\ 2 & 1&1&1&1&0&0&0&0&0&0&0&0&0&0\\ 3 & 1&0&0&0&1&0&0&0&0&1&0&1&0&0\\ 4 & 1&0&1&0&1&0&0&0&0&1&1&0&0&0\\ 5 & 1&0&0&0&0&0&1&1&0&0&0&0&0&0\\ 6 & 1&1&0&0&1&1&1&0&1&0&0&0&0&0\\ 7 & 1&1&0&0&1&1&0&0&0&0&0&0&0&0 \end{array}$$

$$V = \begin{array}{c|cccccccccccccc} & \{\}\to & \{a\}\to & \{c\}\to & \{a,c\}\to & \{d\}\to & \{a,d\}\to & \{e\}\to & \{b,e\}\to & \{a,d,e\}\to & \{d,f\}\to & \{c,d,f\}\to & \{d,f,h\}\to & \{g,i\}\to & \{a,b,c,d,e,f,g,h,i\}\to \\ \hline a & 0&1&0&1&0&1&0&0&1&0&0&0&0&1\\ b & 0&0&0&0&0&0&0&1&0&0&0&0&0&1\\ c & 0&0&1&1&0&0&0&0&0&0&1&0&0&1\\ d & 0&0&0&0&1&1&0&0&1&1&1&1&0&1\\ e & 0&0&0&0&0&0&1&1&1&0&0&0&0&1\\ f & 0&0&0&0&0&0&0&0&0&1&1&1&0&1\\ g & 0&0&0&0&0&0&0&0&0&0&0&0&1&1\\ h & 0&0&0&0&0&0&0&0&0&0&0&1&0&1\\ i & 0&0&0&0&0&0&0&0&0&0&0&0&1&1 \end{array}$$

Every column of U characterizes a set of columns for which a non-enlargeable rectangle inside R exists, the corresponding rows are uniquely and easily determined. This factorization is essentially unique, meaning that columns of U, V may be simultaneously permuted. The case of $R = \mathbb{L}$ is included; with $U =$ column-less relation with row number of R and $V =$ the column-less relation with row number equal to column number of R.

The easy observation is, that the columns of U as well as of V are precisely the column sets of non-enlargeable rectangles inside R.

5.2 Maxclique factorization

The preceding may also be executed for any reflexive and symmetric relation with slightly different properties of the result: A special variant concentrates on homogeneous relations and in particular reflexive and symmetric ones.

Any given symmetric and reflexive relation B allows a factorization such that

$$B = M^{\mathsf{T}};M, \quad M;\overline{B} = \overline{M}, \quad M;\overline{M}^{\mathsf{T}} = \overline{\mathbb{I}} \quad \text{— or } \mathtt{syq}(M^{\mathsf{T}}, M^{\mathsf{T}}) \subseteq \mathbb{I}.$$

Given the symmetric and reflexive

$$B = \begin{array}{c|cccccc} & 1&2&3&4&5&6 \\ \hline 1 & 1&0&1&1&0&0\\ 2 & 0&1&0&1&1&0\\ 3 & 1&0&1&0&0&0\\ 4 & 1&1&0&1&1&1\\ 5 & 0&1&0&1&1&1\\ 6 & 0&0&0&1&1&1 \end{array},$$

the process of maxclique factorization results in

$$M = \begin{array}{r} \\ \{1,3\}\rightarrow \\ \{1,4\}\rightarrow \\ \{2,4,5\}\rightarrow \\ \{4,5,6\}\rightarrow \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \end{array}.$$

This factorization is unique up to an arbitrary permutation of columns of M. One will observe that the rows of M correspond to diagonal squares of **1** 's contained in B.

5.3 Difunctional factorization

A rather simple technique allows to factorize a difunctional relation. As an example consider the relation

$$R = \begin{array}{r} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array} \begin{array}{c} \begin{array}{cccccccccccccc} a & b & c & d & e & f & g & h & i & j & k & l & m & n \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \end{pmatrix} \end{array},$$

which is indeed a difunctional relation. We consider projections to row resp. column equivalence:

$$\eta_\Xi = \begin{array}{r} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array} \begin{array}{c} \begin{array}{cccc} [1] & [2] & [3] & [8] \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array} \qquad \eta_\Psi = \begin{array}{r} \\ a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \\ i \\ j \\ k \\ l \\ m \\ n \end{array} \begin{array}{c} \begin{array}{ccc} [a] & [b] & [f] \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \end{array}$$

Now the relation

$$\beta := \eta_\Xi^\mathsf{T} ; R ; \eta_\Psi = \begin{array}{r} \\ [1] \\ [2] \\ [3] \\ [8] \end{array} \begin{array}{c} \begin{array}{ccc} [a] & [b] & [f] \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

is a matching, allowing us to build

$$f := \eta_{\Xi} ; \beta = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array} \!\!\! \begin{array}{c} \begin{array}{ccc} [a] & [b] & [f] \end{array} \\ \begin{pmatrix} \color{red}{1} & 0 & 0 \\ 0 & \color{red}{1} & 0 \\ 0 & 0 & \color{red}{1} \\ \color{red}{1} & 0 & 0 \\ 0 & \color{red}{1} & 0 \\ \color{red}{1} & 0 & 0 \\ 0 & \color{red}{1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \color{red}{1} \\ \color{red}{1} & 0 & 0 \\ 0 & 0 & \color{red}{1} \\ \color{red}{1} & 0 & 0 \end{pmatrix} \end{array} \qquad g := \eta_{\Psi} = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \\ i \\ j \\ k \\ l \\ m \\ n \end{array} \!\!\! \begin{array}{c} \begin{array}{ccc} [a] & [b] & [f] \end{array} \\ \begin{pmatrix} \color{red}{1} & 0 & 0 \\ 0 & \color{red}{1} & 0 \\ \color{red}{1} & 0 & 0 \\ 0 & \color{red}{1} & 0 \\ \color{red}{1} & 0 & 0 \\ 0 & 0 & \color{red}{1} \\ 0 & \color{red}{1} & 0 \\ \color{red}{1} & 0 & 0 \\ \color{red}{1} & 0 & 0 \\ 0 & \color{red}{1} & 0 \\ \color{red}{1} & 0 & 0 \\ \color{red}{1} & 0 & 0 \\ 0 & 0 & \color{red}{1} \\ \color{red}{1} & 0 & 0 \end{pmatrix} \end{array}$$

Then, indeed $R = f ; g^{\mathsf{T}}$.

6 Sliding cut technique for structure recognition

An important point should be made absolutely clear concerning the following examples that start with real-valued matrices: When one would use randomly generated ones, hardly ever an interesting structure would show up in all the random noise. The detection algorithm, however, is intended to be applied when the researcher has reason to hope for some structure. For the examples with real-valued matrices we have therefore already been starting with the structure to be retrieved! Then some perturbation and distortion has been applied and above that a real-valued matrix was generated randomly according to some cut. Once this matrix was obtained, the algorithm is completely formal and algebraic without human interaction until the final visualization and analysis.

Assume the outcome of whatever an experiment or investigation to appear as a real-valued matrix. How can one recognize a possibly existing structure of this matrix? It would be easy to find out that the matrix is symmetric, e.g. But this is not a really profound insight. We propose a method of abstracting from the mass of reals and fully concentrate on structure by testing whether there might be a favourable cut. In the first examples, we assume a square matrix, for simplicity.

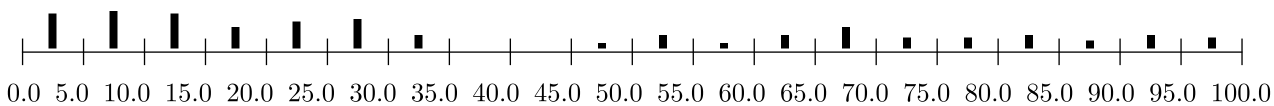
When sliding down with the cut, several situations may occur. First of all, the situation may be compeletely irrelevant, a fact that will soon be recognized. But it may also be the case that some sort of an ordering shows up. For this case measures have been provided installing the case distinction cascade already mentioned. It allows to recognize which sort of an order type has been met. But they also allow to transform the order found to some representation that may easily allow to perceive which order is around.

6.1 A cut showing a weakorder

Assume a real-valued matrix M of percentages [0..100] to be investigated with sliding cut technique for structure recognition. It may look like

$$\begin{pmatrix}
14.31 & 91.31 & 11.58 & 68.87 & 82.34 & 92.50 & 14.56 & 45.53 & 16.87 & 82.06 & 6.75 \\
24.22 & 12.97 & 24.52 & 21.55 & 29.48 & 5.25 & 23.22 & 30.85 & 9.25 & 3.23 & 14.10 \\
87.13 & 53.09 & 3.45 & 83.16 & 66.97 & 54.17 & 95.62 & 77.25 & 99.97 & 81.63 & 17.20 \\
27.93 & 62.65 & 12.32 & 11.76 & 62.62 & 51.72 & 8.00 & 29.03 & 18.33 & 21.24 & 20.23 \\
8.12 & 65.72 & 15.38 & 5.15 & 7.97 & 48.17 & 27.00 & 5.29 & 30.51 & 27.25 & 20.37 \\
15.32 & 31.72 & 27.91 & 7.42 & 1.32 & 31.38 & 2.13 & 14.60 & 14.52 & 28.23 & 24.70 \\
4.15 & 54.74 & 27.11 & 76.51 & 97.83 & 77.91 & 28.62 & 88.86 & 19.97 & 76.72 & 0.26 \\
10.06 & 72.94 & 21.61 & 26.31 & 94.62 & 99.39 & 7.50 & 2.69 & 3.75 & 23.11 & 4.72 \\
27.20 & 83.28 & 13.17 & 63.79 & 71.16 & 58.69 & 3.18 & 69.09 & 9.14 & 69.74 & 9.41 \\
17.84 & 70.96 & 10.96 & 30.57 & 88.57 & 56.72 & 2.08 & 6.18 & 2.94 & 17.14 & 4.00 \\
93.43 & 92.01 & 11.45 & 61.67 & 69.62 & 74.00 & 68.21 & 67.66 & 61.32 & 53.99 & 5.60
\end{pmatrix}$$

A good idea is to take a histogramm that produces



This leads us to make a separating cut at value of 37.00. According to this cut, one will obtain the relation R below which turns out to be a weakorder.

$$R = \begin{array}{cc}
 & \begin{array}{ccccccccccc} 1&2&3&4&5&6&7&8&9&10&11 \end{array} \\
\begin{array}{c} 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11 \end{array} &
\begin{pmatrix}
0&1&0&1&1&1&0&1&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&1&1&1&1&1&1&1&0\\
0&1&0&0&1&1&0&0&0&0&0\\
0&1&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&1&1&1&0&1&0&1&0\\
0&1&0&0&1&1&0&0&0&0&0\\
0&1&0&1&1&1&0&1&0&1&0\\
0&1&0&0&1&1&0&0&0&0&0\\
1&1&0&1&1&1&1&1&1&1&0
\end{pmatrix}
\end{array}$$

It may be arranged by simultaneous permutation to upper triangle as

$$P^{\mathsf{T}};R;P = \begin{array}{cc}
 & \begin{array}{ccccccccccc} 3&11&1&7&9&4&8&10&5&2&6 \end{array} \\
\begin{array}{c} 3\\11\\1\\7\\9\\4\\8\\10\\5\\2\\6 \end{array} &
\begin{pmatrix}
0&0&1&1&1&1&1&1&1&1&1\\
0&0&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&1&1&1&1&1&1\\
0&0&0&0&0&1&1&1&1&1&1\\
0&0&0&0&0&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&1&1&1\\
0&0&0&0&0&0&0&0&1&1&1\\
0&0&0&0&0&0&0&0&1&1&1\\
0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}
\end{array}$$

with

$$P = \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11 \end{array} \begin{array}{c} \begin{array}{ccccccccccc} 3 & 11 & 1 & 7 & 9 & 4 & 8 & 10 & 5 & 2 & 6 \end{array} \\ \begin{pmatrix} 0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0\\ 1&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0 \end{pmatrix} \end{array}$$

and one may additionally indicate the fringe via subdivisions

$$\begin{array}{c} \\ 3\\11\\1\\7\\9\\4\\8\\10\\5\\2\\6 \end{array} \begin{array}{c} \begin{array}{ccccccccccc} 3 & 11 & 1 & 7 & 9 & 4 & 8 & 10 & 5 & 2 & 6 \end{array} \\ \left(\begin{array}{cc|ccc|ccc|c|cc} 0&0&1&1&1&1&1&1&1&1&1\\ 0&0&1&1&1&1&1&1&1&1&1\\ \hline 0&0&0&0&0&1&1&1&1&1&1\\ 0&0&0&0&0&1&1&1&1&1&1\\ 0&0&0&0&0&1&1&1&1&1&1\\ \hline 0&0&0&0&0&0&0&0&1&1&1\\ 0&0&0&0&0&0&0&0&1&1&1\\ 0&0&0&0&0&0&0&0&1&1&1\\ \hline 0&0&0&0&0&0&0&0&0&1&1\\ \hline 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0 \end{array}\right) \end{array}$$

The original matrix M permuted accordingly looks like

$$P^{\mathsf{T}};M;P = \begin{pmatrix} 3.45 & 17.20 & 87.13 & 95.62 & 99.97 & 83.16 & 77.25 & 81.63 & 66.97 & 53.09 & 54.17\\ 11.45 & 5.60 & 93.43 & 68.21 & 61.32 & 61.67 & 67.66 & 53.99 & 69.62 & 92.01 & 74.00\\ 11.58 & 6.75 & 14.31 & 14.56 & 16.87 & 68.87 & 45.53 & 82.06 & 82.34 & 91.31 & 92.50\\ 27.11 & 0.26 & 4.15 & 28.62 & 19.97 & 76.51 & 88.86 & 76.72 & 97.83 & 54.74 & 77.91\\ 13.17 & 9.41 & 27.20 & 3.18 & 9.14 & 63.79 & 69.09 & 69.74 & 71.16 & 83.28 & 58.69\\ 12.32 & 20.23 & 27.93 & 8.00 & 18.33 & 11.76 & 29.03 & 21.24 & 62.62 & 62.65 & 51.72\\ 21.61 & 4.72 & 10.06 & 7.50 & 3.75 & 26.31 & 2.69 & 23.11 & 94.62 & 72.94 & 99.39\\ 10.96 & 4.00 & 17.84 & 2.08 & 2.94 & 30.57 & 6.18 & 17.14 & 88.57 & 70.96 & 56.72\\ 15.38 & 20.37 & 8.12 & 27.00 & 30.51 & 5.15 & 5.29 & 27.25 & 7.97 & 65.72 & 48.17\\ 24.52 & 14.10 & 24.22 & 23.22 & 9.25 & 21.55 & 30.85 & 3.23 & 29.48 & 12.97 & 5.25\\ 27.91 & 24.70 & 15.32 & 2.13 & 14.52 & 7.42 & 14.60 & 28.23 & 1.32 & 31.72 & 31.38 \end{pmatrix}$$

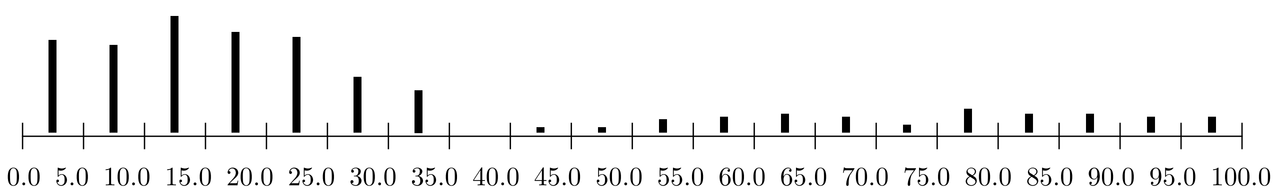
One will find out that the cut may be moved from 31.72 to 45.53 without affecting the relational structure discovered!

6.2 A cut showing an intervalorder

Assume again a real-valued matrix M of percentages [0..100] to be investigated with sliding cut technique for structure recognition. It may look like

$$\begin{pmatrix}
14.31 & 91.31 & 11.58 & 68.87 & 24.75 & 92.50 & 75.71 & 5.54 & 97.33 & 26.67 & 55.71 & 85.96 & 55.35 & 81.40 & 21.55 & 81.81 & 77.03 \\
23.22 & 30.85 & 9.25 & 99.39 & 14.10 & 15.82 & 53.09 & 3.45 & 83.16 & 24.14 & 17.71 & 20.80 & 26.22 & 99.97 & 19.60 & 17.20 & 85.29 \\
62.65 & 97.00 & 11.76 & 62.62 & 31.13 & 76.36 & 59.75 & 18.33 & 79.31 & 50.87 & 75.43 & 65.72 & 89.27 & 51.50 & 7.97 & 48.17 & 81.92 \\
5.29 & 30.51 & 27.25 & 20.37 & 15.32 & 31.72 & 27.91 & 7.42 & 85.54 & 31.38 & 2.13 & 14.60 & 14.52 & 28.23 & 24.70 & 4.15 & 25.32 \\
27.11 & 3.49 & 15.78 & 77.91 & 28.62 & 16.32 & 71.66 & 10.48 & 79.64 & 10.06 & 23.39 & 21.61 & 26.31 & 94.62 & 8.70 & 7.50 & 63.10 \\
3.75 & 23.11 & 4.72 & 27.20 & 14.42 & 13.17 & 63.79 & 3.09 & 58.69 & 3.18 & 3.00 & 9.14 & 13.77 & 9.41 & 17.84 & 3.95 & 10.96 \\
30.57 & 19.75 & 30.53 & 2.08 & 6.18 & 2.94 & 17.14 & 4.00 & 93.43 & 30.52 & 11.45 & 31.33 & 29.32 & 24.61 & 6.09 & 17.18 & 17.08 \\
12.88 & 5.60 & 12.89 & 66.14 & 13.12 & 20.73 & 85.76 & 16.59 & 64.38 & 6.71 & 3.54 & 23.64 & 14.80 & 1.38 & 24.77 & 5.81 & 82.37 \\
31.97 & 19.05 & 9.77 & 13.26 & 7.31 & 15.59 & 27.62 & 19.20 & 18.10 & 21.14 & 8.48 & 10.20 & 19.28 & 24.86 & 14.11 & 13.31 & 24.86 \\
5.21 & 15.84 & 17.64 & 74.56 & 2.29 & 18.66 & 58.90 & 12.36 & 60.45 & 21.13 & 2.10 & 22.40 & 51.27 & 59.34 & 10.92 & 2.25 & 94.26 \\
16.44 & 24.52 & 24.50 & 10.32 & 9.24 & 31.24 & 65.58 & 12.08 & 86.04 & 23.76 & 29.41 & 12.76 & 5.63 & 15.76 & 4.49 & 12.27 & 13.38 \\
14.94 & 2.25 & 9.00 & 4.83 & 12.33 & 28.42 & 15.25 & 18.13 & 60.79 & 14.21 & 1.21 & 5.64 & 1.65 & 4.63 & 21.94 & 5.30 & 6.06 \\
30.29 & 13.15 & 30.70 & 15.50 & 22.34 & 18.04 & 82.77 & 24.04 & 77.03 & 12.87 & 15.92 & 23.04 & 23.88 & 9.83 & 11.13 & 1.69 & 31.12 \\
7.92 & 20.97 & 29.79 & 4.41 & 10.42 & 1.90 & 25.71 & 18.45 & 74.55 & 8.60 & 24.20 & 15.84 & 21.35 & 8.40 & 23.51 & 20.54 & 12.36 \\
5.86 & 23.07 & 1.82 & 69.70 & 7.51 & 12.00 & 96.88 & 15.46 & 82.76 & 4.76 & 9.74 & 23.43 & 91.19 & 87.39 & 28.81 & 18.39 & 75.14 \\
31.90 & 2.37 & 18.75 & 46.07 & 26.91 & 18.57 & 98.89 & 4.97 & 44.23 & 12.24 & 11.71 & 7.38 & 43.71 & 52.51 & 10.22 & 26.79 & 66.57 \\
1.03 & 5.89 & 21.10 & 4.24 & 16.47 & 14.21 & 22.70 & 17.51 & 3.26 & 10.56 & 14.46 & 25.55 & 0.40 & 10.44 & 15.13 & 30.39 & 28.14
\end{pmatrix}$$

The histogramm produces



and shows many more small values than bigger ones indicating that the relation will have many 0's. According to a cut at 37.00, one will obtain the relation R below which turns out to be an intervalorder.

$$\begin{array}{rc}
 & \begin{array}{ccccccccccccccccc} 1&2&3&4&5&6&7&8&9&10&11&12&13&14&15&16&17 \end{array} \\
\begin{array}{r} 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15\\16\\17 \end{array} &
\begin{pmatrix}
0&1&0&1&0&1&1&0&1&0&1&1&1&1&0&1&1\\
0&0&0&1&0&0&1&0&1&0&0&0&0&1&0&0&1\\
1&1&0&1&0&1&1&0&1&1&1&1&1&1&0&1&1\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&1&0&1&0&0&0&0&1&0&0&1\\
0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&1&0&1&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&1&0&1&0&0&0&1&1&0&0&1\\
0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&1&0&1&0&0&0&1&1&0&0&1\\
0&0&0&1&0&0&1&0&1&0&0&0&1&1&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}
\end{array}$$

It may be arranged to upper triangle permuting rows and columns independently to obtain

$$P_R^{\mathsf{T}} ; R ; P_C = \begin{array}{cc}
 & \begin{array}{ccccccccccccccccc} 3&5&8&15&1&10&2&6&11&12&16&13&14&4&17&7&9 \end{array} \\
\begin{array}{c} 3\\1\\10\\15\\16\\2\\5\\8\\6\\11\\13\\4\\7\\12\\14\\9\\17 \end{array} &
\left(\begin{array}{ccccccccccccccccc}
0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}\right)
\end{array}$$

$$P_R = \begin{array}{cc}
 & \begin{array}{ccccccccccccccccc} 3&1&10&15&16&2&5&8&6&11&13&4&7&12&14&9&17 \end{array} \\
\begin{array}{c} 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15\\16\\17 \end{array} &
\left(\begin{array}{ccccccccccccccccc}
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{array}\right)
\end{array}
\qquad
P_C = \begin{array}{c}
\begin{array}{ccccccccccccccccc} 3&5&8&15&1&10&2&6&11&12&16&13&14&4&17&7&9 \end{array} \\
\left(\begin{array}{ccccccccccccccccc}
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0
\end{array}\right)
\end{array}$$

and one may additionally indicate the fringe via subdivisions

$$\begin{array}{cc}
 & \begin{array}{ccccccccccccccccc} 3&5&8&15&1&10&2&6&11&12&16&13&14&4&17&7&9 \end{array} \\
\begin{array}{c} 3\\1\\10\\15\\16\\2\\5\\8\\6\\11\\13\\4\\7\\12\\14\\9\\17 \end{array} &
\left(\begin{array}{cccc|cc|ccccc|c|c|cc|c|c}
0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1\\ \hline
0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1\\ \hline
0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1\\ \hline
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1\\ \hline
0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1\\ \hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\ \hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\ \hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}\right)
\end{array}$$

The original matrix M permuted accordingly looks like

$P_R^\mathsf{T};M;P_C =$

$$\begin{pmatrix}
11.76 & 31.13 & 18.33 & 7.97 & 62.65 & 50.87 & 97.00 & 76.36 & 75.43 & 65.72 & 48.17 & 89.27 & 51.50 & 62.62 & 81.92 & 59.75 & 79.31\\
11.58 & 24.75 & 5.54 & 21.55 & 14.31 & 26.67 & 91.31 & 92.50 & 55.71 & 85.96 & 81.81 & 55.35 & 81.40 & 68.87 & 77.03 & 75.71 & 97.33\\
17.64 & 2.29 & 12.36 & 10.92 & 5.21 & 21.13 & 15.84 & 18.66 & 2.10 & 22.40 & 2.25 & 51.27 & 59.34 & 74.56 & 94.26 & 58.90 & 60.45\\
1.82 & 7.51 & 15.46 & 28.81 & 5.86 & 4.76 & 23.07 & 12.00 & 9.74 & 23.43 & 18.39 & 91.19 & 87.39 & 69.70 & 75.14 & 96.88 & 82.76\\
18.75 & 26.91 & 4.97 & 10.22 & 31.90 & 12.24 & 2.37 & 18.57 & 11.71 & 7.38 & 26.79 & 43.71 & 52.51 & 46.07 & 66.57 & 98.89 & 44.23\\
9.25 & 14.10 & 3.45 & 19.60 & 23.22 & 24.14 & 30.85 & 15.82 & 17.71 & 20.80 & 17.20 & 26.22 & 99.97 & 99.39 & 85.29 & 53.09 & 83.16\\
15.78 & 28.62 & 10.48 & 8.70 & 27.11 & 10.06 & 3.49 & 16.32 & 23.39 & 21.61 & 7.50 & 26.31 & 94.62 & 77.91 & 63.10 & 71.66 & 79.64\\
12.89 & 13.12 & 16.59 & 24.77 & 12.88 & 6.71 & 5.60 & 20.73 & 3.54 & 23.64 & 5.81 & 14.80 & 1.38 & 66.14 & 82.37 & 85.76 & 64.38\\
4.72 & 14.42 & 3.09 & 17.84 & 3.75 & 3.18 & 23.11 & 13.17 & 3.00 & 9.14 & 3.95 & 13.77 & 9.41 & 27.20 & 10.96 & 63.79 & 58.69\\
24.50 & 9.24 & 12.08 & 4.49 & 16.44 & 23.76 & 24.52 & 31.24 & 29.41 & 12.76 & 12.27 & 5.63 & 15.76 & 10.32 & 13.38 & 65.58 & 86.04\\
30.70 & 22.34 & 24.04 & 11.13 & 30.29 & 12.87 & 13.15 & 18.04 & 15.92 & 23.04 & 1.69 & 23.88 & 9.83 & 15.50 & 31.12 & 82.77 & 77.03\\
27.25 & 15.32 & 7.42 & 24.70 & 5.29 & 31.38 & 30.51 & 31.72 & 2.13 & 14.60 & 4.15 & 14.52 & 28.23 & 20.37 & 25.32 & 27.91 & 85.54\\
30.53 & 6.18 & 4.00 & 6.09 & 30.57 & 30.52 & 19.75 & 2.94 & 11.45 & 31.33 & 17.18 & 29.32 & 24.61 & 2.08 & 17.08 & 17.14 & 93.43\\
9.00 & 12.33 & 18.13 & 21.94 & 14.94 & 14.21 & 2.25 & 28.42 & 1.21 & 5.64 & 5.30 & 1.65 & 4.63 & 4.83 & 6.06 & 15.25 & 60.79\\
29.79 & 10.42 & 18.45 & 23.51 & 7.92 & 8.60 & 20.97 & 1.90 & 24.20 & 15.84 & 20.54 & 21.35 & 8.40 & 4.41 & 12.36 & 25.71 & 74.55\\
9.77 & 7.31 & 19.20 & 14.11 & 31.97 & 21.14 & 19.05 & 15.59 & 8.48 & 10.20 & 13.31 & 19.28 & 24.86 & 13.26 & 24.86 & 27.62 & 18.10\\
21.10 & 16.47 & 17.51 & 15.13 & 1.03 & 10.56 & 5.89 & 14.21 & 14.46 & 25.55 & 30.39 & 0.40 & 10.44 & 4.24 & 28.14 & 22.70 & 3.26
\end{pmatrix}$$

One will find out that the cut may be moved from 31.97 to 43.71 without affecting the relational structure discovered!

In a different program one will then also discover the interval structure that gives the name to intervalorders. The following R is indeed an intervalorder which is not simply a semiorder!

$$R = \begin{array}{rc}
 & \begin{array}{ccccccccccccc}1&2&3&4&5&6&7&8&9&10&11&12&13\end{array}\\
\begin{array}{r}1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\end{array} &
\begin{pmatrix}
0&1&0&1&0&1&1&0&0&1&1&1&1\\
0&0&0&1&0&0&1&0&0&0&0&0&1\\
1&1&0&1&0&1&1&0&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&1&0&0&0&0&0&1\\
0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&1&0&0&0&0&0&0\\
0&0&0&1&0&0&1&0&0&0&0&1&1\\
0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}
\end{array}$$

One will form the intersection

$$B := \overline{R \cup R^\mathsf{T}} = \begin{array}{rc}
 & \begin{array}{ccccccccccccc}1&2&3&4&5&6&7&8&9&10&11&12&13\end{array}\\
\begin{array}{r}1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\end{array} &
\begin{pmatrix}
1&0&0&0&1&0&0&1&1&0&0&0&0\\
0&1&0&0&1&1&0&1&1&1&1&1&0\\
0&0&1&0&1&0&0&1&0&0&0&0&0\\
0&0&0&1&0&1&1&0&0&1&1&1&1\\
1&1&1&0&1&1&0&1&1&1&1&1&0\\
0&1&0&1&1&1&0&1&1&1&1&1&1\\
0&0&0&1&0&0&1&0&0&0&1&0&1\\
1&1&1&0&1&1&0&1&1&1&1&1&1\\
1&1&0&0&1&1&0&1&1&1&1&0&0\\
0&1&0&1&1&1&0&1&1&1&1&1&1\\
0&1&0&1&1&1&1&1&1&1&1&1&1\\
0&1&0&1&1&1&0&1&0&1&1&1&1\\
0&0&0&1&0&1&1&1&0&1&1&1&1
\end{pmatrix}
\end{array}$$

proceed to the — not so easily obtained — maxcliques factorization $B =: M^\mathsf{T};M$ with

$$M = \begin{array}{r} \\ \{3,5,8\}\to \\ \{1,5,8,9\}\to \\ \{2,5,6,8,9,10,11\}\to \\ \{2,5,6,8,10,11,12\}\to \\ \{4,7,11,13\}\to \\ \{4,6,10,11,12,13\}\to \\ \{6,8,10,11,12,13\}\to \end{array} \begin{array}{c} \begin{array}{ccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array},$$

detect the linear strictorder on these maxcliques

$$C := M; R; M^{\mathsf{T}} = \begin{array}{r} \\ \{3,5,8\}\to \\ \{1,5,8,9\}\to \\ \{2,5,6,8,9,10,11\}\to \\ \{2,5,6,8,10,11,12\}\to \\ \{4,7,11,13\}\to \\ \{4,6,10,11,12,13\}\to \\ \{6,8,10,11,12,13\}\to \end{array} \begin{array}{c} \begin{array}{ccccccc} \{3,5,8\}\to & \{1,5,8,9\}\to & \{2,5,6,8,9,10,11\}\to & \{2,5,6,8,10,11,12\}\to & \{4,7,11,13\}\to & \{4,6,10,11,12,13\}\to & \{6,8,10,11,12,13\}\to \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array},$$

rearrange it by simultaneous permutation to upper triangle

$$C_{\text{rearranged}} = \begin{array}{r} \\ \{3,5,8\}\to \\ \{1,5,8,9\}\to \\ \{2,5,6,8,9,10,11\}\to \\ \{2,5,6,8,10,11,12\}\to \\ \{6,8,10,11,12,13\}\to \\ \{4,6,10,11,12,13\}\to \\ \{4,7,11,13\}\to \end{array} \begin{array}{c} \begin{array}{ccccccc} \{3,5,8\}\to & \{1,5,8,9\}\to & \{2,5,6,8,9,10,11\}\to & \{2,5,6,8,10,11,12\}\to & \{6,8,10,11,12,13\}\to & \{4,6,10,11,12,13\}\to & \{4,7,11,13\}\to \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

and then obtain the consecutive $\mathbf{1}$'s version, i.e.

$$C^{\mathsf{T}}_{\text{rearranged}}; M = \begin{array}{r} \\ \{3,5,8\}\to \\ \{1,5,8,9\}\to \\ \{2,5,6,8,9,10,11\}\to \\ \{2,5,6,8,10,11,12\}\to \\ \{6,8,10,11,12,13\}\to \\ \{4,6,10,11,12,13\}\to \\ \{4,7,11,13\}\to \end{array} \begin{array}{c} \begin{array}{ccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array}.$$

This shows indeed the vertical intervals after which this type of ordering has been named.

7 Concluding Remarks

In the present report, an attempt has been made to bring relational mathematics closer to the attentiveness of researchers of varying disciplines, refraining from too much mathematical formalism and using visualizations instead. Many of the formulae mentioned can be understood immediately out of the examples presented.

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References

[Sch11] Gunther Schmidt. *Relational Mathematics*, volume 132 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, 2011. ISBN 978-0-521-76268-7, 584 pages.

[SS89] Gunther Schmidt and Thomas Ströhlein. *Relationen und Graphen*. Mathematik für Informatiker. Springer-Verlag, 1989. ISBN 3-540-50304-8, ISBN 0-387-50304-8.

[SS93] Gunther Schmidt and Thomas Ströhlein. *Relations and Graphs — Discrete Mathematics for Computer Scientists*. EATCS Monographs on Theoretical Computer Science. Springer-Verlag, 1993. ISBN 3-540-56254-0, ISBN 0-387-56254-0.

[SW14] Gunther Schmidt and Michael Winter. Relational Mathematics Continued. Technical Report 2014-01, Fakultät für Informatik, Universität der Bundeswehr München, April 2014. `http://arxiv.org/abs/1403.6957`.

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