SCRIPTA MATHEMATICA, Volume XXVII, No. 3.

IN MEMORY OF LUDWIG BERWALD

BY MAX PINL[1]

Translated by

PETER G. BERGMANN AND MIRIAM LIEBER GRUMET

"The nineteenth century dislike of realism is the age of Caliban seeing his own face in a glass."
O. WILDE

IN the heart of Europe lies the land of Bohemia, the geographically isolated "Bohemian Basin". In the heart of the Bohemian Basin lies the capital of the country, "The Golden City of Prague".

It was in this city that Charles IV, King of Bohemia, Germany, and Luxemburg created the famous University of Prague well over six hundred years ago. The efforts of the scholars of the University of Prague have been guided for these past six centuries by the precept: "non sordidi lucri causa nec ad vanam captandam gloriam, sed quo magis veritas propagetur et lux eius qua salus humani generis continetur clarius effulgeat".[2]

In Prague, as elsewhere, the scholar's dedication to these lofty principles has been thwarted time and again. According to *Ortega y Gasset*, it is not truths but beliefs that determine the course of world history. So has it been in our time. In the following, we shall show that the fate of Dr. Ludwig Berwald, professor-in-ordinary of Mathematics of the University of Prague has been no exception.

Beginning and Rise

For many years the most important and renowned bookshop in Prague was the Andresche Buchhandlung (Andre's Bookshop). It was located in the center of town in the neighborhood of the "Pulverturm" (Powder Tower). Max Berwald, Professor Berwald's father, was the proprietor of this bookshop prior to World War I. While Max Berwald came from East Prussia his wife Friedericke Fischel Berwald was a native of Prague. Three children were born of this marriage, two brothers and one sister.

[1] Department of Mathematics, University of Idaho, Moscow, Idaho: as of spring 1964.

[2] At this point, I should like to take the opportunity to express my thanks to all those devoted people who made it possible for me to write this obituary. Above all, this applies to my former colleague from my days in Prague, Dr. H. Lowig (University of Edmonton, Canada), also Mrs. Dr. Ilona Adler, Mrs. Jana Meisl, and Mr. Norman (all in Prague), also Dr. E. Berwald (Bochum) and Professor Dr. P. Funk (Vienna). I should further like to express my thanks for the help of Mr. R. Epstein (London) and Mr. Goldschmied (Prague). I am much indebted to all of them. The manuscript for this obituary was prepared in the summer of 1948.

Received by the editors, May 6, 1964.

Ludwig Berwald was born on December 8, 1883, in Prague I, House No. 300. In his early youth he already was interested in mathematics. He entered the Ia class of the former "Imperial-Royal State High School of Prague-Neustadt", Graben, in the beginning of the school year 1893–94. Berwald always listed his religion in the school catalog as Jewish and his mother-tongue was entered as German. We can follow his school years until the sixth grade in the Grabengymnasium. The Berwald family then moved to the historical Hotel "Zum Blauen Stern", Prague II, Graben, Nr. 864, opposite the Pulverturn, the famous landmark of the city. The peace treaty between Prussia and Austria had been signed in the Hotel "Zum Blauen Stern" in 1866. At that time the director of the Grabengymnasium was Dr. Josef Walter and his head teachers were Nikolaus Komma and Karl Kaplan. Berwald's mathematics teachers were Josef Guckla and later on Prokop Knothe. Near the turn of the century Berwald's father sold his bookshop and moved with his family to Munich. Berwald then started his studies in the higher classes of the Munich Luitpoldgymnasium. Following his graduation from the gymnasium he entered the Royal Ludwig–Maximilian University (Munich) in the Fall of 1902, where he pursued his studies in mathematics and physics. He attended the lectures of the following professors and lecturers: V. Baeyer, Bauer, v. Braunmühl, Cornelius, Doehlemann, Fischer, Goetz, Graf, v. Groth, Korn, v. d. Leyen, Lindemann, Pringsheim, Ranke, v. Reinhardstätter, Röntgen, Voss, and Wasserrab. In December of 1908 he earned his doctorate under Professor Auriel Voss with his paper "Über die Krümmungseigenschaften der Brennflachen eines geradlinigen Strahlsystems und der in ihm enthaltenen Regelflächen". While an assistant and student of Burckhardt, he wanted to become a lecturer (Privatdozent) in Munich. However, because of a lung illness which confined him to a sanitarium for three years he was only able to do private tutoring.

During this time he lived in Stockdorf and Grünwald, near Munich. On September 12, 1915, Berwald married Hedwig Adler (born on September 12, 1875, in Prague). As a result of his marriage Berwald frequently visited Prague where he made the acquaintance of Gerhard Kowalewski and Georg Pick. With their encouragement he became a lecturer at the German University.[3] He was promoted to an associate professorship (extraordinarius) on March 24, 1922, and became a full professor in 1924. In 1929 G. Pick[4] retired and Berwald became the Chairman of the Mathematics Department of the University. During the school year 1931–32 he was "Seine Spektabilität der Dekan"; ten years later he was No. 816 of Transport C, which was composed of Jews slated for death in the dreaded concentration camps.

[3] In those days, there existed in Prague a Czech and a German University. In 1939 the Czech University was liquidated by the Germans, and in 1945 the German University by the Czech authorities.

[4] During the German occupation of Bohemia Dr. G. Pick was turned over to the German Secret Police by the German army, and at the age of 80 was deported to the Ghetto of Theresienstadt, where he died on July 26, 1942.

Berwald became acquainted with the author of this obituary in the summer of 1926. He was primarily known as a typical scholar, pointedly reserved, choosing his words carefully and precisely, always serious and cautious, of somewhat skeptical outlook, but convincing and determined if need be. On close acquaintance one discovered behind the scientist a sensitive, artistic personality, exceptionally musical and, on rare occasions, even poetically inclined. Berwald was tall and slender in stature. Because of his nearsightedness he usually wore glasses. As a result of his early serious illness Berwald was very prone to colds. There prevailed, even in winter, an above-normal temperature in his study which was the center of the house, that "subtropical Berwald Climate" that was so well known and beloved by the mathematical and physical circles in Prague. Shortly after the beginning of a conversation, quietly and unnoticeably a little table with tea and excellent "Prager Salzgebäck" appeared before the visitor. If another visitor should then present himself he too received his own little table and the discussion would continue. The conversation was seldom limited to mathematical topics, a second topic of equal importance being music. Berwald was an excellent pianist and also played chamber music with a certain well-ordered regularity that enabled him to study a broad range of musical literature together with his fellow musicians, among them Professor Pick who played the violin. In the course of the years, as he moved his apartment one floor down, this musical activity succeeded in disrupting the radio craze of a neighbor—like most music lovers, Berwald did not own his own radio set. He would describe to me in detail, in his own humorous way, how he would outsmart his neighbor's blaring radio with his Bach Preludes and Fugues. I could just imagine how the furniture in the house danced on these occasions. One could make Berwald very happy if one turned the conversation to his Dalmatian trips—how he loved the hot sun of this stony country. How he loved to describe his travelling experiences and show his numerous pictures that he had collected. Berwald's endurance and his astounding diligence will be shown in the report of his scientific work. This diligence combined with an extraordinary sense of duty, did not deter him from doing work that was beneath his level. It has remained an unforgettable incident to the author of this obituary how Berwald, on the occasion of the indisposition of an assistant of the Institute, undertook, and carried out by himself the cumbersome work of writing out by hand a catalog of the library of the Institute. Berwald was in no way uninformed and disinterested in political and historical matters. But none of us during those wonderful years in Prague could, by any stretch of the imagination, know what really was to come.

Numbers 2793/816, 817, Transport C

On October 22, 1941, at the collecting point located next to the Messepalast (exhibition building) in Prague, the third transport of Jews to be deported to the Ghetto in Lodz, Poland, was assembled by order of the German Secret

Police. The Berwalds were included in this transport. Their registration numbers were 2793/816 and 817. The preceding day, Berwald had distributed his last mathematical manuscripts. On October 22nd his scientific work came to a close. It would still have been possible, through a medical certificate, to withdraw Nos. 2793/816 and 817 from Transport C. However, these numbers wanted no part of such a maneuver and chose to put their lot in with the one thousand other people of this transport, thereby assuring themselves of certain death. When Transport C arrived in Lodz, the "native" Jews there told the newcomers that they had been in the Ghetto already for two years. In fact, the Ghetto of Lodz had been created shortly after the German invasion in 1939. The Berwalds were placed in an incompleted one-story schoolhouse whose raw bricks were not yet covered, in the former Marenšinska then called Siegfriedstrasse 48. There were no beds, straw pallets, or even piles of straw for the occupants to lie on. People simply lay down next to each other on the bare floor. Some were also placed in the attic. Naturally, everybody slept in his day-time clothes. One simply covered oneself with those clothes and with any blankets that one had brought along. There was one advantage in being placed in a building that had not been previously occupied. The inhabitants remained free of vermin and lice for a longer period, the Berwalds probably so until their deaths. Fifty-five people slept in one room which was approximately twenty by twenty feet. Families were permitted to stay together. This meant, of course, that men, women and children slept in the same room. There was no heating in the building. However, as a result of so many people being put together in one room the temperature was always high enough in winter.[5]

There were no lavatories provided within the buildings; rather the inhabitants had to relieve themselves in a primitive out-house set in the yard. Under these circumstances many of the occupants, especially the older ones did not wash themselves when it got very cold. Electric lighting was available. The typical diet consisted of $\frac{1}{4}$ kilogram of bread daily; in addition to this they received black "coffee" twice daily, and, once a day, a miserable soup. The Jews were allowed to move about freely within the Ghetto. The younger people were drafted to perform labor. One group of young people were later transported to Posen where they died of typhus. The mass transportation of Jews to extermination camps from Lodz began in 1942. In May 1942, 12,000 elderly and unemployable people were transported to Majdanek; 20,000 in the Fall of 1942. There they were murdered en masse by methods even more brutal than those used in Auschwitz. Since Berwald did not work he most probably would have been murdered in Majdanek if he had lived until May or certainly by the Fall of 1942. However, Mrs Berwald died on March 27, 1942, and Professor Ludwig Berwald died a few weeks later on April 20th, an

[5] According to my inquiries, which I owe to Dr. O. Seibert (Gladbeck), it is reasonable to assume that the room temperature was approximately 65°F., while the outside temperature was −40°F.

infamous day in German history. There are, astonishingly enough, Polish documents relating to his death.[6]

Scientific Work

From the appended bibliography, it appears that Berwald's contributions have been overwhelmingly in the area of differential geometry. Only a small fraction of his investigations (1, 6, 15, 30, 33, 34, 35, 36, 38, 49) refer to other areas, in particular algebraic and purely analytic problems.

The numerous articles in differential geometry which Berwald published in the years from 1909 to 1949, provide a clear picture of the flowering of differential geometry, particularly in Germany, which began with the work of E. Study about the turn of the century. In Bonn, E. Study had taken over the spiritual inheritance of J. Plücker. It is there that the invariant-theoretical methods for the treatment of analytic curves and surfaces originated, which were both rigorous and geometrically fruitful, and which made it possible to subject these structures to a unified treatment without necessitating a separate approach to geometric objects. In the development of the differential geometry during these years, certain singular curves and surfaces (which were unavoidably imaginary because of the Euclidean metric of the embedding space) became preferred objects of investigation. Item No. 3, of the list below, will serve as an example of this type of investigation; this paper is concerned with surfaces containing a single congruence of mutually crossing minimal straight lines. This paper is concerned with the non-cylindrical surfaces which were first examined by G. Monge, on which the two congruences of curvature lines coincide. L. Berwald obtained a simple method for generating these surfaces with the help of a semi-isotropic surface. Depending on the choice of this subsidiary surface as an isotropic cone, or an isotropic torse (tangential surface of an isotropic curve), there results a classification of the "Monge surfaces" into those of the first and those of the second kind. There are no Monge surfaces of the first kind with constant Gaussian curvature. Their position in the classification scheme is occupied by the spheres. Monge surfaces of the second kind with constant Gaussian curvature are *Serret* surfaces. Following

[6] In English translation they read as follows:

Polish Republic. Registry Office in Lodz. Lodz, November 28, 1946. Certificate. The Civil Authority of Lodz declares that in the Registry of Death, III in Lodz-Ghetto for the year 1942, recorded under No. 7163, is Dr. Berwald, Ludwig, resident of Street "40", Nr. 48, born on 8.12.1883 in Prague, widower, professor. Died in Lodz on 20.4.1942 at 11:00 A.M. Cause of death: Intestinal catarrh, heart failure.

The Registrar: Felicja Poznanska m.p.
"Felicja Poznanska L.S.".

The Civil Authority of Lodz declares that in the Registry of Death in Lodz-Ghetto for the year 1942 recorded under No. 5673 is Berwaldova, Hedwika, resident of Street "9", Nr. 49, daughter of Emanuel and Friedericke, born on 12.9.1875 in Prague, married, unemployed. Died in Lodz-Ghetto on 27.3.1942 at 15:00 o'clock. Cause of death: Blocked arteries.

The Registrar: Felicja Poznanska m.p.
"Felicja Poznanska L.S.".

the analytic representation Berwald succeeds in this paper in determining all algebraic Monge surfaces, including the simplest of this type, as Monge surfaces of the third order. The simplest Serret surfaces turn out to be of the fourth order. Of particular interest are the "automorphic" Monge surfaces, which permit the group of rotations about the vertex of the corresponding isotropic cone. Papers listed as Nos. 4 and 5 below, also belong to this class of problems that originated at Bonn. The paper on the invariants of motion and on the elementary geometry in a minimal plane shows that the method initiated by E. Study to set up all the integral irreducible invariants of the motion in an unbounded system of points of straight lines, as well as the irreducible relationships between these invariants for the geometry of a Euclidean plane, is also capable of accomplishing the same with respect to the automorphic similarity transformations of a minimal plane (isotropic plane). This topic, which was then worked on intensively also by H. Beck,[7] has recently attracted again considerable interest through the investigations by K. Strubecker and W. Vogel on the geometry of the isotropic space (minimal space).

In Prague, L. Berwald's creative work came under new stimuli, which are associated primarily with the names of G. Pick and P. Funk. P. Funk, who had studied with Hilbert, brought to Prague the most up-to-date variational tools from Göttingen, and G. Pick developed at the same time a continuation of the "Erlanger Program" for the elaboration of the differential geometries of transformation groups which are not motion-invariant, in particular as the most obvious application, the idea of an affine differential geometry. It is well known how this program was carried out under the scientific and organizational leadership of W. Blaschke. In these investigations L. Berwald was one of the chief participants. This was the time when the so-called "Prager Kränzchen" (the Prague circle) was born on the shores of the Moldau River, that organization of mathematicians and physicists at Prague which had such outstanding scientific and personal qualities (the designation "Kränzchen" is by W. Blaschke, who incidentally reduced the statutes and by-laws of that organization to a set of zero measure). During this period we should like to mention in particular items 8–14 of the below listing. Whereas papers 8 and 9 rather represent the beginnings of a series of investigations on the geometry of general (chiefly Finsler) metrics, which achieved their full significance later, papers 10–14 were of a more definitive character and have been incorporated in large measure into the second volume of W. Blaschke's textbook of differential geometry.[8] The papers Nos. 8 and 9 may be considered as the precursors of a series of papers of great importance, whose broader conception begins in papers Nos. 17 and 18, and whose continuation have occupied

[7] Cf. H. Beck, "Zur Geometrie in der Minimalebene," *Sber. berl. math. Ges.*, **12** (1912/13), pp. 14–30.

[8] In W. Blaschke's preface, we read (Springer, Berlin, 1923) "Kindest friendly greetings to the mathematical circle of Prague! In 1916 G. Pick had published jointly with one of us the first investigations concerned with the theory of affine surfaces. Later on A. Winternitz and L. Berwald had joined the affine society; we owe thanks in particular to L. Berwald for having contributed much to this volume" (cf. in particular Secs. 65, 66).

Berwald's interest right until the time of his deportation, as may be gathered from the posthumous paper No. 52. In the theory of parallel transport and curvature in so-called Finsler spaces (general metric spaces) the principal task consists of generalizing the results of the theory of Riemannian manifolds to the case that an arbitrary function of the coordinates and their differentials (homogeneous of the first order in those differentials) assumes the role of the line element. If it is possible to express this basic function through a coordinate transformation into a form which depends only on the differentials, one then speaks of a Minkowski space. L. Berwald succeeded in constructing the theory of these general metric spaces in close analogy to the theory of Riemannian spaces (cf. Nos. 17, 18, 23, and 26). The second derivatives of the square of the basic functions with respect to the differentials form the so-called Finsler fundamental tensor. With its help L. Berwald defines the length of an arbitrary vector at a point with respect to an arbitrary direction at that point, in brief its length "with respect to an arbitrary line element", and the volume of an n-dimensional domain with respect to a congruence of curves. For the theory of parallel transport, L. Berwald uses the formalism of parallel transport originated by Emmy Noether, which he modifies by solving corresponding Euler–Lagrange equations with respect to the second derivatives of the coordinates. The coefficients of these solutions form a system of functions homogeneous in the first derivatives of the coordinates of the second degree, and the second derivatives of these functions with respect to the first derivatives of the coordinates form the components of parallel transport in general metric spaces. The curvature tensor of the space and the other curvature quantities are obtained from these components. In view of the fact that the tensors of this geometry are functions of the line elements, there exist, in addition to the covariant derivatives of Ricci, a second co-variant differentiation, that is the ordinary partial derivative with respect to the differentials, and hence also two kinds of commutators for the second derivatives, of which one enters into the expression for the curvature tensor, and the second into the so-called asymmetry tensor of the affine connection. The norm of a vector will generally change if the vector is transported parallel to itself around a closed curve, that is to say, there exists a curvature of length. With these concepts L. Berwald obtains the following classification of general metric spaces:

(1) Riemannian spaces, in which the components of the Finsler fundamental tensor are functions of the spatial coordinates only;
(2) affine spaces which are characterized by a vanishing asymmetry tensor;
(3) Landsberg spaces, in which the Finsler fundamental tensor is covariantly constant;
(4) spaces with zero curvature of length; and
(5) spaces with non-vanishing curvature of length ("Streckenkrümung").

Having completed the development of the general theory, L. Berwald began to attack more specialized questions within the theory of general metric spaces. In this connection papers Nos. 27 and 28 deserve mention. In the paper "On the n-dimensional geometries of constant curvature in which the straight lines are the shortest curves" he generalized the characteristics which P. Funk had established for the two-dimensional Hilbert geometries to the general n-dimensional case. In particular he finds in this paper that there can be no general spaces with straight-line extremals whose curvature scalar is a non-constant function of position, and a generalization of a theorem proved by P. Funk for $n = 2$: Minkowski's geometry and the geometry of specific measure are the only geometries with straight-line extremals having the property that the transverse hyperplanes belonging to the same straight-line map are parallel to each other.

A third set of investigations on the theory of general metric spaces begins with the excerpt of a lecture (No. 39) delivered in 1935 after the theory of these spaces had been given a definitive form by E. Cartan in the meantime. L. Berwald connects these investigations by Cartan with previous work going back to L. Koschmieder concerned with the invariant normal forms of the second variation of a parameter-invariant $(n - 1)$-fold surface integral, which he introduces together with the basic integral, in the sense of Cartan's theory, for a Cartan geometry; the Cartan spaces are assumed to be regular. These investigations are partly complementary and parallel to the theories of hypersurfaces in Finsler spaces due to J. M. Wegener. It is possible to bring the second variation of a surface integral which extends over an extreme hypersurface with fixed $(n - 2)$-dimensional boundary into a normal form which is identical with that of Koschmieder. Hereby the invariant that enters into Koschmieder's normal form is represented by the torsion tensor, by the curvature forms of the Cartan space, and by certain quantities belonging to the hypersurface.

In 1935 Blaschke had begun the systematic investigation of the so-called integral geometry. L. Berwald participated in this work in his papers Nos. 42 and 43, in which *inter alia* he obtains interesting generalizations of the concept of the mixed volume of two ovals and the appropriate integral formulas. The solids of constant luminosity (i.e., those whose normal projections all have the same area) permit other integral geometric characteristics as follows: They turn out to be identical with the oval of the constant supporting function which appears as a supporting function of an oval which is centered on the point in question.

L. Berwald was interested in convex solids quite apart from his systematic series of investigations, as shown, for instance, by the posthumous papers Nos. 49, 53, and 54. These papers are concerned with certain functional equations which lead to generalizations of the mean value theorem by J. Favard for positive concave functions, as well as for several variables. Among all the remaining papers outside of the series of comprehensive works, we should like to mention, in particular, paper No. 32, whose topic has been elaborated further by L. Koschmieder and H. Gericke. Finally, we should call attention to the article No. 21, a contribution to the Encyclopedia, which makes evident both Berwald's comprehensive knowledge of the literature and his sustained industry and enormous perseverance. Precisely because this contribution was meant to be a review article, one can appreciate Berwald's scientific thoroughness, who does not omit to call attention to a hidden but remarkable gap in the theory of quadratic differential forms, calling for an existence proof of Schläfli's conjecture that all Riemannian manifolds can be embedded in Euclidian spaces of lowest dimensional number. As a matter of fact, this gap was closed soon thereafter by E. Cartan and M. Janet.[9]

[9] Cf. M. Janet, *Annls Soc. pol. Math.*, **5** (1926), pp. 38–73; E. Cartan, *ibid*, **6** (1927), pp. 1–7; G. Vranceanu, *Vorlesungen über Differentialgeometrie I*, Anhang III, Berlin 1961.

BIBLIOGRAPHY

L. Berwald hat die mathematische Wissenschaft durch folgende Abhandlungen und wissenschaftliche Untersuchungen bereichert:

1. "Vereinfachte Herleitung unharmonischer trigonometrischer Reihen," *Proceedings of Königlich Bayrische Akademie der Wissenschaften* (Mathematics and Physics Division), 1909 Yearbook, Section 5, 19 pp.

2. "Krümmungseigenschaften der Brennflächen eines geradlinigen Strahlensystems und der in ihnen enthaltenen Regelflächen," *Dissertation*, Munich, 1909, 67 pp.

3. "Über die Flächen mit einer einzigen Schar zu einander windschiefer Minimalgeraden," *Proceedings of Königliche Bayrische Akademie der Wissenschaften* (Mathematics and Physics Division), 1913 Yearbook, pp. 143–211.

4. "Über Bewegungsinvarianten und elementare Geometrie in einer Minimalebene," *Monatshefte für Mathematik und Physik*, **26** (1915), pp. 211–28.

5. "Über die algebraisch rektifizierbaren Kurven im Nichteukldischen Raum," *Proceedings of Königlich Bayrische Akademie der Wissenschaften* (Mathematics and Physics Division), 1916 Yearbook, pp. 1–18.

6. "Über einige Minimumsätze der Dreicks-und Tetraedergeometrie," *Archiv. der Mathematik und Physik* (3) **25** (1916), pp. 97–103.

7. "Zur Geometrie in einer speziellen Kongruenz erster Ordnung erster Klasse," *Proceedings of Akademie der Wissenschaften zu Wien* (Mathematics and Scientific Division), **128** (1919), pp. 1403–51.

8. (With P. Funk), "Flächeninhalt und Winkel in der Variationsrechnung," *Lotos* (Prag), **67/68** (1919), pp. 45–49.

9. "Über die erste Krümmung der Kurven bei allgemeiner Maßbestimmung," *Lotos* (Prag), **67/68** (1919), pp. 52–56.

10. "Über affine Geometrie: XXVII—Liesche F_2, Affinnormale und mittlere Affinkrümmung," *Mathematische Zeitschrift*, **8** (1920), pp. 63–78.

11. "Über affine Geometrie: XXX—Die oskulierenden Flächen zweiter Ordung in der affinen Flächentheorie," *Mathematische Zeitschrift*, **10** (1921), pp. 160–72.

12. "Zur Geometrie einer n-dimensionalen Riemannschen Mannigfaltigkeit im $(n + 1)$-dimensionalen euklidischen Raum," *Jahresbericht der deutschen Mathematikervereinigung*, **30** (1921) p. 76.

13. "Zur Geometrie einer n-dimensionalen Riemannschen Mannigfaltigkeit im $(n + 1)$-dimensionalen euklidischen Raum," *Jahresbericht der deutschen Mathematikervereinigung*, **31** (1922), pp. 162–70.

14. "Die Grundgleichungen der Hyperflächen im euklidischen R_{n+1} gegenüber den inhaltstreuen Affinitäten," *Monatshefte für Mathematik und Physik*, **32** (1922), pp. 89–106.

15. "Über eine Ungleichheit für bestimmte Integrale," *The Tôhoko Mathematical Journal*, **24** (1924), pp. 88–94.

16. (With P. Frank), "Über eine kovariante Gestalt der Differentialgleichungen der Bahnkurven allgemeiner mechanischer Systeme," *Mathematische Zeitschrift*, **21** (1924), pp. 154–59.

17. "Über Parallelübertragung in Räumen mit allgemeiner Maßbestimmung," *Jahresbericht der deutschen Mathematikervereinigung*, **34** (1925), pp. 213–20.

18. "Untersuchung der Krümmung allgemeiner metrischer Räume auf Grund des in ihnen herrschenden Parallelismus," *Mathematische Zeitschrift*, **25** (1) (1926), pp. 40–73.

19. "Zur Geometrie ebener Variationsprobleme," *Lotos* (Prag), **74** (1926), pp. 43–52.
20. "Sulle trasformazioni puntuali a di contatto nell piano," *Bolletino della Unione Matematica Italiana*, **6** (1927), pp. 241–50.
21. "Differentialinvarianten in der Geometrie. Riemannsche Mannigfaltigkeiten und ihre Verallgemeinerungen," *Encyklopädie der Mathematischen Wissenschaften mit Einschluß ihrer Anwendungen*, vol. III, D (1927), p. 109.
22. "Sui differenziali secondi covarianti," *Atti della Reale Accademia Nazionale dei Lincei* (6). Rendiconti, Classe di Scienze Fisiche, Matematische e Naturali, **5** (1927), pp. 763–68.
23. "Review of 'Untersuchung der Krümmung allgemeiner metrischer Räume auf Grund des in ihnen herrschenden Parallelismus,'" *Mathematische Zeitschrift*, **26** (1927), pp. 176.
24. "Über zweidimensionale allgemeine metrische Räume—I, II," *Journal für reine und angewandte Mathematik*, **156** (1927), pp. 191–222.
25. "Una forma normale invariante della seconda variazione," *Atti della Reale Accademia Nazionale dei Lincei* (6). Rendiconti, Classe die Scienze Fisiche, Matematische e Naturali, **7** (1928), pp. 301–06.
26. "Parallelübertragung in allgemeinen Räumen," *Atti del Congresso Internazionale dei Matematici, Bologna*, 1928, pp. 263–70.
27. "Über die n-dimensionalen Geometrien konstanter Krümmung, in welchen die Geraden die Kürzesten sind," *Mathematische Zeitschrift*, **30** (3) (1929), pp. 449–69.
28. "Über eine charakteristische Eigenschaft der allgemeinen Räume konstanter Krümmung mit geradlinigen Extremalen," *Monatshefte für Mathematik und Physik*, **36** (1929), pp. 315–30.
29. "Kleine Bemerkungen zur natürlichen Geometrie ebener Transformationsgruppen," *Jahresbericht der deutschen Mathematikervereinigung*, **39** (1930), pp. 59–60.
30. "Verallgemeinerte Matrizen-Zirkulanten," *Jahresbericht der deutschen Mathematikervereinigung*, **40** (1930), pp. 2–3.
31. "Kleine Bemerkungen zur Theorie der ebenen Transformationsgruppen," *Jahresbericht der deutschen Mathematikervereinigung*, **40** (1931), pp. 167–76.
32. "Über adjungierte Variationsprobleme und adjungierte Extremalflächen," *Monatshefte für Mathematik und Physik*, **38** (1931), pp. 89–108.
33. "Lösung einer Aufgabe von Nikola Obreschkoff," *Jahresbericht der deutschen Mathematikervereinigung*, **42** (1932), pp. 83–84.
34. "Über einige mit dem Satz von Kakeya verwandte Sätze," *Mathematische Zeitschrift*, **37** (1933), p. 768.
35. "Remark regarding the paper 'Über einige mit dem Satz von Kakeya verwandte Sätze'" (*Diese Zeitschrift*, **37** (1933), pp. 61–76), *Mathematische Zeitschrift*, **37** (1933), p. 768.
36. "Über die Lage der Nullstellen von Linearkombinationen eines Polynoms und seiner Ableitung in Bezug auf einen Punkt," *The Tôhoku Mathematical Journal*, **37** (1933), pp. 52–68.
37. "Lösung einer Aufgabe von H. Liebmann," *Jahresbericht der deutschen Mathematikervereinigung*, **43** (1933), pp. 25–26.
38. "Elementare Sätze über die Abgrenzung der Wurzeln einer algebraischen Gleichung," *Acta Litterarum ac Scientiarum Regiae Universitatis Hungariae Francisco-Josephinae Szeged*, **6** (1933), pp. 209–21.
39. "Über Finslersche und verwandte Räume," *Comptes Rendus du deuxième congrès des mathématiciens des pays slaves* (Prague), 1934 pp. 1–16; *Časopis pro pestovani mathematiky a fysiky*, **64** (1935), pp. 1–16.
40. "Über die Hauptkrümmungen einer Fläche im dreidimensionalen Finslerschen Raum," *Monatshefte für Mathematik und Physik*, **43** (1936), pp. 1–14.
41. "On the projective geometry of paths," *Annals of Mathematics* (2) **37** (1937), pp. 879–98.
42. (With O. Varga), "Integralgeometrie 24. Über die Schiebungen im Raum," *Mathematische Zeitschrift*, **42** (5) (1937), pp. 710–36.
43. "Integralgeometrie 25. Über die Körper konstanter Helligkeit" (Comments on the preceding item), *Mathematische Zeitschrift*, **42** (5) (1937), pp. 737–38.
44. "Über Finslersche und Cartansche Geometrie: II (G. Pick zum 80 Geburtstag)—Invarianten beig der Variation vielfacher Integrale und Parallelflächen in Cartanschen Räumen," *Compositio mathematica*, **7** (1939), pp. 141–76.
45. "Über die n-dimensionalen Cartanischen Räume und eine Normalform der zweiten Variation eines $(n - 1)$-fachen Oberflächenintegrals (É. Cartan zum 70 Geburtstag)," *Acta mathematica*, **71** (1939), pp. 191–248.
46. "Über Finslersche und Cartansche Geometrie: I—Geometrische Erklärung der Krümmung und des Hauptskalars eines zweidimensionalen Finslerschen Raumes," *Mathematica, Cluj* **16** (1940), pp. 34–58.
47. "On Finsler and Cartan geometries: III—Two-dimensional Finsler spaces with rectilinear extremals," *Annals of Mathematics* (2), **42** (1) (1941), pp. 84–112.
48. "Über die Beziehungen zwischen den Theorien der Parallelübertragung in Finslerschen Räumen (aus dem Nachlaß)," *Nederl. Akad. Wetensch. Proc.*, **49**, 642–47 (Indagationes Math., 8, 401–06 (1946)).

49. "Verallgemeinerung eines Mittelwertsatzes von J. Favard für positive konkave Funktionen (aus dem Nachlaß)," *Acta mathematica*, **79** (1947), pp. 17–37.
50. "Über Haar's Verallgemeinerung des Lemmas von Du Bois-Raymond und verwandte Sätze (aus dem Nachlaß)," *Acta mathematica*, **79** (1947), pp. 39–49.
51. "Über Systeme von gewöhnlichen Differentialgleichungen zweiter Ordnung, deren Integralkurven mit dem System der geraden Linien äquivalent sind (aus dem Nachlaß)," *Annals of Mathematics* (2), **48** (1947), pp. 193–215.
52. "Über Finslersche und Cartansche Geometrie IV. Projektivkrümmung affiner Räume und Finslersche Räume skalarer Krümmung (aus dem Nachlaß)," *Annals of Mathematics* (2), **48** (1947), pp. 755–81.
53. "Über den Schwerpunkt gewisser konvexer Bereiche" (aus dem Nachlaß; not printed until now).
54. "Obere Schranken für das isoperimetrische Defizit bei Eilinien und die entsprechenden Größen bei Eiflächen (aus dem Nachlaß)," *Monatshefte für Mathematik und Physik*, **53** (3) (1949), pp. 202–10.